

At 14.15

ο αὐθιγα τέρψασθαι ἄλλῃσιν ὑπὸ τῶν ζῴων· ὁ αὐτὸς δὲ δαΐζαι·
 τὰρ δύο ὡς μοι τέρψασθαι ἄλλῃσιν· οὐκ ἔστιν αὐτῶν τὸ αὐ-
 τὸ κέρτρον· δύο γὰρ ὡς μοι οἱ δὲ β Γ Γ Δ Η τέρψασθαι ἄ-
 λλῃσιν κατὰ τὰ β Γ στήμα· λέγω ὅτι οὐκ ἔστιν αὐτῶν τὸ
 αὐτὸ κέρτρον· ἄν γὰρ διωρατῶν· ὅτι τὸ ε ὑπὸ τῶν δ α χ θ α ἢ ε γ
 ὑπὸ δ ι λ χ θ α ἢ ε ζ η ὡς στήμα· ὑπὸ δ τὸ ε στήμα ὁ κέρτρον
 αὐτοῦ δ β Γ ὡς μοι· ἴση δὲ τῇ ἢ ε γ τῇ ε ζ· παρὶν ὡς τὸ
 ε στήμα ὁ κέρτρον αὐτοῦ γ Δ Η ὡς μοι· ἴση δὲ τῇ ἢ ε γ τῇ
 ε η· δαΐζαι θ δ ε ἢ ε γ ὑπὸ τῇ ε ζ ἴση· ὑπὸ ἢ ε ζ ἄρα τῇ ε η

EUCLID'S ELEMENTS

TRANSLATED FROM THE TEXT OF HEIBERG

WITH INTRODUCTION AND COMMENTARY

BY

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PREFACE

“THERE never has been, and till we see it we never shall believe that there can be, a system of geometry worthy of the name, which has any material departures (we do not speak of *corrections* or *extensions* or *developments*) from the plan laid down by Euclid.” De Morgan wrote thus in October 1848 (*Short supplementary remarks on the first six Books of Euclid's Elements* in the *Companion to the Almanac* for 1849); and I do not think that, if he had been living to-day, he would have seen reason to revise the opinion so deliberately pronounced sixty years ago. It is true that in the interval much valuable work has been done on the continent in the investigation of the first principles, including the formulation and classification of axioms or postulates which are necessary to make good the deficiencies of Euclid's own explicit postulates and axioms and to justify the further assumptions which he tacitly makes in certain propositions, content apparently to let their truth be inferred from observation of the figures as drawn; but, once the first principles are disposed of, the body of doctrine contained in the recent textbooks of elementary geometry does not, and from the nature of the case cannot, show any substantial differences from that set forth in the *Elements*. In England it would seem that far less of scientific value has been done; the efforts of a multitude of writers have rather been directed towards producing alternatives for Euclid which shall be more suitable, that is to say, easier, for schoolboys. It is of course not surprising that, in

these days of short cuts, there should have arisen a movement to get rid of Euclid and to substitute a "royal road to geometry"; the marvel is that a book which was not written for schoolboys but for grown men (as all internal evidence shows, and in particular the essentially theoretical character of the work and its aloofness from anything of the nature of "practical" geometry) should have held its own as a school-book for so long. And now that Euclid's proofs and arrangement are no longer required from candidates at examinations there has been a rush of competitors anxious to be first in the field with a new text-book on the more "practical" lines which now find so much favour. The natural desire of each teacher who writes such a text-book is to give prominence to some special nostrum which he has found successful with pupils. One result is, too often, a loss of a due sense of proportion; and, in any case, it is inevitable that there should be great diversity of treatment. It was with reference to such a danger that Lardner wrote in 1846: "Euclid once superseded, every teacher would esteem his own work the best, and every school would have its own class book. All that rigour and exactitude which have so long excited the admiration of men of science would be at an end. These very words would lose all definite meaning. Every school would have a different standard; matter of assumption in one being matter of demonstration in another; until, at length, GEOMETRY, in the ancient sense of the word, would be altogether frittered away or be only considered as a particular application of Arithmetic and Algebra." It is, perhaps, too early yet to prophesy what will be the ultimate outcome of the new order of things; but it would at least seem possible that history will repeat itself and that, when chaos has come again in geometrical teaching, there will be a return to Euclid more or less complete for the purpose of standardising it once more.

But the case for a new edition of Euclid is independent of any controversies as to how geometry shall be taught to schoolboys. Euclid's work will live long after all the text-books

of the present day are superseded and forgotten. It is one of the noblest monuments of antiquity; no mathematician worthy of the name can afford not to know Euclid, the real Euclid as distinct from any revised or rewritten versions which will serve for schoolboys or engineers. And, to know Euclid, it is necessary to know his language, and, so far as it can be traced, the history of the "elements" which he collected in his immortal work.

This brings me to the *raison d'être* of the present edition. A new translation from the Greek was necessary for two reasons. First, though some time has elapsed since the appearance of Heiberg's definitive text and prolegomena, published between 1883 and 1888, there has not been, so far as I know, any attempt to make a faithful translation from it into English even of the Books which are commonly read. And, secondly, the other Books, VII. to X. and XIII., were not included by Simson and the editors who followed him, or apparently in any English translation since Williamson's (1781—8), so that they are now practically inaccessible to English readers in any form.

In the matter of notes, the edition of the first six Books in Greek and Latin with notes by Camerer and Hauber (Berlin, 1824—5) is a perfect mine of information. It would have been practically impossible to make the notes more exhaustive at the time when they were written. But the researches of the last thirty or forty years into the history of mathematics (I need only mention such names as those of Bretschneider, Hankel, Moritz Cantor, Hultsch, Paul Tannery, Zeuthen, Loria, and Heiberg) have put the whole subject upon a different plane. I have endeavoured in this edition to take account of all the main results of these researches up to the present date. Thus, so far as the geometrical Books are concerned, my notes are intended to form a sort of dictionary of the history of elementary geometry, arranged according to subjects; while the notes on the arithmetical Books VII.—IX. and on Book X. follow the same plan.

I desire to express here my thanks to my brother, Dr R. S. Heath, Vice-Principal of Birmingham University, for suggestions on the proof sheets and, in particular, for the reference to the parallelism between Euclid's definition of proportion and Dedekind's theory of irrationals, to Mr R. D. Hicks for advice on a number of difficult points of translation, to Professor A. A. Bevan for help in the transliteration of Arabic names, and to the Curators and Librarian of the Bodleian Library for permission to reproduce, as frontispiece, a page from the famous Bodleian MS. of the *Elements*. Lastly, my best acknowledgments are due to the Syndics of the Cambridge University Press for their ready acceptance of the work, and for the zealous and efficient coöperation of their staff which has much lightened the labour of seeing the book through the Press.

T. L. H.

November, 1908.

PREFACE TO THE SECOND EDITION

I LIKE to think that the exhaustion of the first edition of this work furnishes a new proof (if such were needed) that Euclid is far from being defunct or even dormant, and that, so long as mathematics is studied, mathematicians will find it necessary and worth while to come back again and again, for one purpose or another, to the twenty-two-centuries-old book which, notwithstanding its imperfections, remains the greatest elementary textbook in mathematics that the world is privileged to possess.

The present edition has been carefully revised throughout, and a number of passages (sometimes whole pages) have been rewritten, with a view to bringing it up to date. Some not inconsiderable additions have also been made, especially in the Excursuses to Volume I, which will, I hope, find interested readers.

Since the date of the first edition little has happened in the domain of geometrical teaching which needs to be chronicled. Two distinct movements however call for notice.

The first is a movement having for its object the mitigation of the difficulties (affecting in different ways students, teachers and examiners) which are found to arise from the multiplicity of the different textbooks and varying systems now in use for the teaching of elementary geometry. These difficulties have evoked a widespread desire among teachers for the establishment of an agreed sequence to be generally adopted in teaching the subject. One proposal to this end has already been made: but the chance of the acceptance of an agreed sequence has in the meantime been prejudiced by a second movement which has arisen in other quarters.

I refer to the movement in favour of reviving, in a modified form, the proposal made by Wallis in 1663 to replace Euclid's Parallel-Postulate by a Postulate of Similarity (as to which see pp. 210—111 of Volume I of this work). The form of Postulate now suggested is an assumption that "Given one triangle, there can be constructed, on any arbitrary base, another triangle equiangular with (or similar to) the given triangle." It may perhaps be held that this assumption has the advantage of not referring, in the statement of it, to the fact that a straight line is of unlimited length; but, on the other hand, as is well known, Saccheri showed (1733) that it involves more than is necessary to enable Euclid's Postulate to be proved. In any case it would seem certain that a scheme based upon the proposed Postulate, if made scientifically sound, must be more difficult than the procedure now generally followed. This being so, and having regard to the facts (1) that the difference between the suggested Postulate and that of Euclid is in effect so slight and (2) that the historic interest of Euclid's Postulate is so great, I am of opinion that the proposal is very much to be deprecated.

T. L. H.

December 1925.

CONTENTS

INTRODUCTION.

FACSIMILE OF A PAGE OF THE BODLEIAN MS. OF THE *ELEMENTS* . . . *Frontispiece*

This is a facsimile of a page (fol. 45 verso) of the famous Bodleian MS. of the *Elements*, D'Orville 301 (formerly X. 1 inf. 2, 30), written in the year 888. The scholium in the margin, not very difficult to decipher, though some letters are almost rubbed out, is one of the scholia Vaticana given by Heiberg (Vol. v. p. 263) as III. No. 15: *Διὰ τοῦ κέντρου οὐσῶν οὐκ ἦν ζητήσεως ἄξιον, εἰ διχα τέμνουσιν ἀλλήλας· τὸ γὰρ κέντρον αὐτῶν ἡ διχοτομία. ὁμοίως καὶ ἡ εἰ τῆς ἑτέρας διὰ τοῦ κέντρου οὐσσης ἡ ἑτέρα μὴ διὰ τοῦ κέντρου εἴη, ὅτι οὐ διχα τέμνεται ἡ διὰ τοῦ κέντρου.* The ἡ before εἰ in the last sentence should be omitted. PFVat. read ἡ without εἰ. The marginal references lower down are of course to propositions quoted, (1) *διὰ τὸ α' τοῦ γ'*, "by III. 1," and (2) *διὰ τὸ γ' τοῦ αὐτοῦ*, "by 3 of the same."

	PAGE
CHAP. I. EUCLID AND THE TRADITIONS ABOUT HIM . . .	1
„ II. EUCLID'S OTHER WORKS	7
„ III. GREEK COMMENTATORS OTHER THAN PROCLUS . . .	19
„ IV. PROCLUS AND HIS SOURCES	29
„ V. THE TEXT	46
„ VI. THE SCHOLIA	64
„ VII. EUCLID IN ARABIA	75
„ VIII. PRINCIPAL TRANSLATIONS AND EDITIONS . . .	91
„ IX. § 1. ON THE NATURE OF ELEMENTS . . .	114
„ § 2. ELEMENTS ANTERIOR TO EUCLID'S . . .	116
„ § 3. FIRST PRINCIPLES: DEFINITIONS, POSTULATES AND AXIOMS	117
„ § 4. THEOREMS AND PROBLEMS	124
„ § 5. THE FORMAL DIVISIONS OF A PROPOSITION . . .	129
„ § 6. OTHER TECHNICAL TERMS	132
„ § 7. THE DEFINITIONS	143

THE *ELEMENTS*.

BOOK I. DEFINITIONS, POSTULATES, COMMON NOTIONS . .	153
„ NOTES ON DEFINITIONS ETC.	155
„ PROPOSITIONS	241
BOOK II. DEFINITIONS	370
„ NOTE ON GEOMETRICAL ALGEBRA	372
„ PROPOSITIONS	375
EXCURSUS I. PYTHAGORAS AND THE PYTHAGOREANS . . .	411
EXCURSUS II. POPULAR NAMES FOR EUCLIDEAN PROPOSITIONS .	415

	PAGE
BOOK III. DEFINITIONS	I
PROPOSITIONS	6
BOOK IV. DEFINITIONS	78
PROPOSITIONS	80
BOOK V. INTRODUCTORY NOTE	112
DEFINITIONS	113
PROPOSITIONS	138
BOOK VI. INTRODUCTORY NOTE	187
DEFINITIONS	188
PROPOSITIONS	191
BOOK VII. DEFINITIONS	277
PROPOSITIONS	296
BOOK VIII.	345
BOOK IX.	384
BOOK X. INTRODUCTORY NOTE	I
DEFINITIONS I.	10
PROPOSITIONS I—47	14-101
DEFINITIONS II.	101
PROPOSITIONS 48—84	102-177
DEFINITIONS III.	177
PROPOSITIONS 85—115	178-254
ANCIENT EXTENSIONS OF THEORY OF BOOK X	255
BOOK XI. DEFINITIONS	260
PROPOSITIONS	272
BOOK XII. HISTORICAL NOTE	365
PROPOSITIONS	369
BOOK XIII. HISTORICAL NOTE	438
PROPOSITIONS	440
APPENDIX. I. THE SO-CALLED "BOOK XIV." (BY HYPsicLES)	512
II. NOTE ON THE SO-CALLED "BOOK XV."	519
GENERAL INDEX: GREEK	521
" " ENGLISH	527

INTRODUCTION

It was with the greatest interest that I heard that the publishers thought of adding to Everyman's Library an edition of the *Elements* of Euclid; for what book in the world could be more suitable for inclusion in the Library than this, the greatest textbook of elementary mathematics that there has ever been or is likely to be, a book which, ever since it was written twenty-two centuries ago, has been read and appealed to as authoritative by mathematicians great and small, from Archimedes and Apollonius of Perga onwards? No textbook, presumably, can ever be without flaw (especially in a subject like geometry, where some first principles, postulates or axioms, have to be assumed without proof, and any number of alternative systems are possible), and flaws there are in Euclid; but it is safe to say that no alternative to the *Elements* has yet been produced which is open to fewer or less serious objections. The only general criticism of it which is deserving of consideration is that it is unsuitable as a textbook for very young boys and girls who are just beginning to learn the first things about geometry. This can be admitted without detracting in the least from the greatness or the permanent value of the book. The simple truth is that it was not written for schoolboys or schoolgirls, but for the grown man who would have the necessary knowledge and judgment to appreciate the highly contentious matters which have to be grappled with in any attempt to set out the essentials of Euclidean geometry as a strictly logical system, and, in particular, the difficulty of making the best selection of unproved postulates or axioms to form the foundation of the subject. My advice would, therefore, be: if you must spoon-feed the very young, do so; but when they have shown a taste for the subject and attained the standard necessary for passing honours examinations, let them *then* be introduced to Euclid in his original form as an antidote to the more or less feeble echoes of him that are to be found in the ordinary school textbooks

of 'geometry.' I should be surprised if such qualified readers, making the acquaintance of Euclid for the first time, did not find it fascinating, a book to be read in bed or on a holiday, a book as difficult as any detective story to lay down when once begun. I know of one actual case, that of an undergraduate at Cambridge suddenly presented with a copy of Euclid, where this happened. This is the true test of such a book. Nor does the reading of it require the 'higher mathematics.' Any intelligent person with a fair recollection of school work in elementary geometry would find it (progressing as it does by gradual and nicely contrived steps) easy reading, and should feel a real thrill in following its development, always assuming that enjoyment of the book is not marred by any prospect of having to pass an examination in it! This is why I applaud the addition of this great classic to Everyman's Library; for everybody ought to read it who *can*, that is, all educated persons except the very few who are constitutionally incapable of mathematics.

When it came to the question of selecting the particular version of Euclid to be reproduced, this might have been difficult but for the fact that the format, size of page, etc. (which had to be copied exactly), had to be similar to those of the other volumes in the series. This restricted the choice considerably; and I think it was a fortunate circumstance that the most suitable edition, from the point of view of format, should be precisely that of Isaac Todhunter, which, as he says in his preface, substantially reproduces that of Robert Simson.

Robert Simson was born in 1687 and was Professor of Mathematics in the University of Glasgow from 1711 to 1761. He was an enthusiastic admirer of the Greek geometers and spent the best part of his life in studying and elucidating them. Steeped in the subject, he even made important attempts, with the aid of indications of content, etc., given in the *Collection* of Pappus of Alexandria, to restore three lost works, the *Porisms* of Euclid (a difficult treatise in higher geometry) and two minor works of Apollonius. The only textbook published by Simson besides his Euclid was on conic sections, *Sectionum conicarum libri*

quinque (Edinburgh 1835); this, except for the definitions of the three conics, was based on Apollonius.

Simson's *Euclid*, which was published simultaneously in Latin and English, did not appear till 1756 (i.e., when he was about 69), so that the edition must have represented his most mature thought on the subject. The merits of Simson, both as interpreter and as critic of Euclid, are very great; and it was mainly due to the excellence of his edition that the words 'Euclid' and 'geometry' became almost synonymous terms in this country. Todhunter, Senior Wrangler in 1848, was the author of a series of mathematical textbooks quite unrivalled in their day; and his notes to *Euclid*, admirably concise and to the point, fully deserve re-impression.

Simson's own edition included the geometrical Books I—VI, XI and XII in full, omitting only Book X (on Irrationals), and Book XIII (on the five Regular Solids); Todhunter's edition includes only 21 propositions of Book XI (omitting 22–40), and two propositions of Book XII (out of 18).

Simson had, it is true, a 'bee in his bonnet.' The title-page of his first editions says that 'in this edition the errors by which Theon or others have long ago vitiated these Books are corrected, and some of Euclid's Demonstrations are restored.' Simson, however, was not in any real sense a competent textual critic; he acted on the simple but uncritical principle that whatever he found in the text which fell short of perfection, whether of form or content, must have been due to alterations made by Theon or 'some unskilful editor.' Apart from this, the security for the general excellence of his version was the fact that it was made from the Latin translation of Commandinus (1572).

Unfortunately, the *editio princeps* of the Greek text by Grynaeus (Basel, 1533) and the various translations from the Greek down to the end of the eighteenth century (including that of Commandinus) depended upon MSS. of the class now known as Theonine, i.e., containing the recension of Euclid by Theon of Alexandria in the fourth century A.D. They purport by their titles to be either 'from the edition of Theon' or 'from the school (i.e., lectures) of Theon.'

Moreover, Theon himself says in a passage of his commentary on the *Syntaxis* of Ptolemy: 'But that sectors in equal circles are to one another as the angles on which they stand has been proved by me in my edition of the Elements at the end of the sixth book,' from which it is plain that the second part of Euclid VI. 33 containing this proof was added by Theon to the original. Now the great Vatican MS., Vat. Gr. 190, discovered by F. Peyrard in or about 1808, was found to contain neither the words about Theon quoted above from the titles to the other MSS., nor the addition by Theon to VI. 33 referred to. Peyrard was, therefore, justified in concluding that in the Vatican MS. we have an edition more ancient than Theon's, and therefore a key to the changes made by Theon. Simson had not the advantage of knowing these facts, which were discovered after his time. But the Vatican MS. has now been given its proper weight, and it is among the main sources of the authoritative text of the *Elements* published, with Latin translation, critical notes, and prolegomena, in five volumes between 1883 and 1888 (Teubner), by J. L. Heiberg.

By way of addition to the bibliography of the subject in Todhunter's notes, pp. 250-1, the following references may be given. A faithful translation of Heiberg's text into English is contained in the present writer's *The Thirteen Books of Euclid's Elements, with Introduction and Commentary*, 3 vols., second edition, 1926 (Cambridge University Press). Those who wish to sample the original Greek text of Euclid, which is well worth while, may be referred to *Euclid in Greek, Book I, with Introduction and Notes* (Cambridge 1920). Of recent histories of Greek mathematics in general we may mention James Gow's *A Short History of Greek Mathematics*, 1884 (now out of print), and the present writer's *History of Greek Mathematics*, 2 vols., 1921, and *Manual of Greek Mathematics*, 1931 (Clarendon Press).

Although the edition of the *Elements* here presented, being Simson's edition as revised by Todhunter, rests on the Theonine recension, there is no objection to Euclid being read in this form. It is only necessary to bear certain things in mind, which shall be briefly noted here.

Simson's text itself suffered changes in later editions for which Simson himself was not responsible; some changes made by editors or publishers were not for the better. In this respect Simson's fate was not unlike Euclid's own. Simson died in 1768, and only two editions were issued under his own supervision, those of 1756 and 1762. To the edition of 1762 he added the *Data* of Euclid, a work recalling some of the subject matter of the *Elements* but in a different aspect; at the same time he revised somewhat his version of the *Elements* if we may judge by the sentence which he added to the preface of the same edition: 'Besides, the translation is much amended by the friendly assistance of a learned gentleman.'

There are some differences of substance between Simson's text and the genuine text of Euclid as restored in Heiberg's edition.

1. Perhaps the most important of these is the difference between the arrangement of the postulates and the axioms in the two editions. Simson's text contains three postulates and twelve axioms. The last two of these axioms Euclid gave, not as axioms (or 'common notions' as he called them), but as postulates. Thus 'Axiom 11,' that all right angles are equal to one another, was Euclid's 'Postulate 4,' while 'Axiom 12' (the well-known parallel-axiom) was 'Postulate 5.' Further, of the first ten 'Axioms' only five can, with any probability, be attributed to Euclid himself (1, 2, 3, 8, and 9 in Todhunter's edition). Thus Euclid did not give, as a common notion, 'Two straight lines cannot enclose a space.' Where this is cited in the text of I. 4, the words are interpolated. Euclid inferred the coincidence of BC with EF immediately from the fact that the ends of the two straight lines coincide respectively; he seems to have regarded the fact that there is only *one* straight joining two given points as implicit in Postulate 1 that 'a straight line can be drawn from any one point to any other point.'

2. In some cases where the text of Euclid as he found it gave two alternative proofs of one and the same proposition, e.g., III. 9 and 10, VI. 20 (and XI. 22), Simson happened to choose the alternative which had the lesser authority. There is nothing to wonder at in this, because the considerations in

his mind might well be similar to those which influenced the author of the interpolated proof.

3. Simson is not responsible for the omission of VI. 27-9. Todhunter omitted them 'as they appear now to be never required and have been condemned as useless by various modern commentators: see Austin, Walker, and Lardner.' Simson, however, had strongly protested against this view. He says of Props. 28 and 29: 'These two problems, to the first of which the 27th proposition is necessary, are the most general and useful of all in the *Elements*, and are most frequently made use of by the ancient geometers in the solution of other problems; and, therefore, are very ignorantly left out by Tacquet and Dechales in their editions of the *Elements*, who pretend that they are scarce of any use.' The important words here are those referring to the ancient geometers. The propositions embody, in fact, the general method known as the 'application of areas,' which was of vital consequence to the Greek geometers, being the geometrical equivalent of the solution of the general quadratic equations $ax \pm \frac{b}{c}x^2 = S$, so far as they have real roots. Nowadays we solve such equations by algebra; the Greek geometers, however, had no algebraical notation, and hence they had to invent a sort of *geometrical algebra*. The simplest case of 'application of areas,' which is equivalent to the solution for x of the simple equation $ax = S$, can be read in this volume (*Eucl.* I. 44, 45); the general method is a marvel of geometrical ingenuity. No wonder that Plutarch mentions a doubt whether it was not the discovery of this method, rather than that of the theorem of the square of the hypotenuse (*Eucl.* I. 47), which was the occasion of the famous sacrifice of oxen supposed to have been made by Pythagoras. In order to make good the omission in Todhunter's edition, the three propositions in question, translated from the original text of Euclid, are here added in an Appendix.

One other note of Todhunter's should be mentioned. He says (p. 288) that the proposition VI. 32 'seems of no use.' Now Euclid was not in the habit of giving any proposition except for use; and this one is, in fact, required and used by

Euclid himself in Prop. 17 of Book XIII, containing the construction and 'comprehension in a sphere' of a regular dodecahedron (the regular solid figure with twelve faces, all of which are equal regular pentagons).

CHAPTER I.

EUCLID AND THE TRADITIONS ABOUT HIM.

AS in the case of the other great mathematicians of Greece, so in Euclid's case, we have only the most meagre particulars of the life and personality of the man.

Most of what we have is contained in the passage of Proclus' summary relating to him, which is as follows¹:

"Not much younger than these (sc. Hermotimus of Colophon and Philippus of Medma) is Euclid, who put together the *Elements*, collecting many of Eudoxus' theorems, perfecting many of Theaetetus', and also bringing to irrefragable demonstration the things which were only somewhat loosely proved by his predecessors. This man lived² in the time of the first Ptolemy. For Archimedes, who came immediately after the first (Ptolemy)³, makes mention of Euclid: and, further, they say that Ptolemy once asked him if there was in geometry any shorter way than that of the elements, and he answered that there was no royal road to geometry⁴. He is then younger than the pupils of Plato but older than Eratosthenes and Archimedes; for the latter were contemporary with one another, as Eratosthenes somewhere says."

This passage shows that even Proclus had no direct knowledge of Euclid's birthplace or of the date of his birth or death. He proceeds by inference. Since Archimedes lived just after the first

¹ Proclus, ed. Friedlein, p. 68, 6—20.

² The word γέγονε must apparently mean "flourished," as Heiberg understands it (*Litterargeschichtliche Studien über Euklid*, 1882, p. 26), not "was born," as Hankel took it: otherwise part of Proclus' argument would lose its cogency.

³ So Heiberg understands ἐπιβαλὼν τῷ πρώτῳ (sc. Πτολεμαίῳ). Friedlein's text has καὶ ἐν ἐπιβαλὼν and τῷ πρώτῳ; and it is right to remark that another reading is καὶ ἐν τῷ πρώτῳ (without ἐπιβαλὼν) which has been translated "in his first book," by which is understood *On the Sphere and Cylinder* 1., where (1) in Prop. 2 are the words "let *BC* be made equal to *D* by the second (proposition) of the first of Euclid's (books)," and (2) in Prop. 6 the words "For these things are handed down in the *Elements*" (without the name of Euclid). Heiberg thinks the former passage is referred to, and that Proclus must therefore have had before him the words "by the second of the first of Euclid": a fair proof that they are genuine, though in themselves they would be somewhat suspicious.

⁴ The same story is told in Stobaeus, *Ecl.* (II. p. 228, 30, ed. Wachsmuth) about Alexander and Menaechmus. Alexander is represented as having asked Menaechmus to teach him geometry concisely, but he replied: "O king, through the country there are royal roads and roads for common citizens, but in geometry there is one road for all."

Ptolemy, and Archimedes mentions Euclid, while there is an anecdote about *some* Ptolemy and Euclid, *therefore* Euclid lived in the time of the first Ptolemy.

We may infer then from Proclus that Euclid was intermediate between the first pupils of Plato and Archimedes. Now Plato died in 347/6, Archimedes lived 287–212, Eratosthenes *c.* 284–204 B.C. Thus Euclid must have flourished *c.* 300 B.C., which date agrees well with the fact that Ptolemy reigned from 306 to 283 B.C.

It is most probable that Euclid received his mathematical training in Athens from the pupils of Plato; for most of the geometers who could have taught him were of that school, and it was in Athens that the older writers of elements, and the other mathematicians on whose works Euclid's *Elements* depend, had lived and taught. He may himself have been a Platonist, but this does not follow from the statements of Proclus on the subject. Proclus says namely that he was of the school of Plato and in close touch with that philosophy¹. But this was only an attempt of a New Platonist to connect Euclid with his philosophy, as is clear from the next words in the same sentence, "for which reason also he set before himself, as the end of the whole *Elements*, the construction of the so-called Platonic figures." It is evident that it was only an idea of Proclus' own to infer that Euclid was a Platonist because his *Elements* end with the investigation of the five regular solids, since a later passage shows him hard put to it to reconcile the view that the construction of the five regular solids was the end and aim of the *Elements* with the obvious fact that they were intended to supply a foundation for the study of geometry in general, "to make perfect the understanding of the learner in regard to the whole of geometry²." To get out of the difficulty he says³ that, if one should ask him what was the aim (σκοπός) of the treatise, he would reply by making a distinction between Euclid's intentions (1) as regards the subjects with which his investigations are concerned, (2) as regards the learner, and would say as regards (1) that "the whole of the geometer's argument is concerned with the cosmic figures." This latter statement is obviously incorrect. It is true that Euclid's *Elements* end with the construction of the five regular solids; but the planimetical portion has no direct relation to them, and the arithmetical no relation at all; the propositions about them are merely the conclusion of the stereometrical division of the work.

One thing is however certain, namely that Euclid taught, and founded a school, at Alexandria. This is clear from the remark of Pappus about Apollonius⁴: "he spent a very long time with the pupils of Euclid at Alexandria, and it was thus that he acquired such a scientific habit of thought."

It is in the same passage that Pappus makes a remark which might, to an unwary reader, seem to throw some light on the

¹ Proclus, p. 68, 20, καὶ τῇ προαιρέσει δὲ Πλατωνικός ἐστι καὶ τῇ φιλοσοφίᾳ ταύτῃ οἰκεῖος.

² *ibid.* p. 71, 8.

³ *ibid.* p. 70, 19 sqq.

⁴ Pappus, VII. p. 678, 10–12, συσχολάσας τοῖς ὑπὸ Εὐκλείδου μαθηταῖς ἐν Ἀλεξανδρείᾳ πλείστον χρόνον, ὅθεν ἔσχε καὶ τὴν τοιαύτην ἔξιν οὐκ ἀμαθῆ.

personality of Euclid. He is speaking about Apollonius' preface to the first book of his *Conics*, where he says that Euclid had not completely worked out the synthesis of the "three- and four-line locus," which in fact was not possible without some theorems first discovered by himself. Pappus says on this¹: "Now Euclid—regarding Aristaeus as deserving credit for the discoveries he had already made in conics, and without anticipating him or wishing to construct anew the same system (such was his scrupulous fairness and his exemplary kindness towards all who could advance mathematical science to however small an extent), being moreover in no wise contentious and, though exact, yet no braggart like the other [Apollonius]—wrote so much about the locus as was possible by means of the conics of Aristaeus, without claiming completeness for his demonstrations." It is however evident, when the passage is examined in its context, that Pappus is not following any tradition in giving this account of Euclid: he was offended by the terms of Apollonius' reference to Euclid, which seemed to him unjust, and he drew a fancy picture of Euclid in order to show Apollonius in a relatively unfavourable light.

Another story is told of Euclid which one would like to believe true. According to Stobaeus², "some one who had begun to read geometry with Euclid, when he had learnt the first theorem, asked Euclid, 'But what shall I get by learning these things?' Euclid called his slave and said 'Give him threepence, since he must make gain out of what he learns.'"

In the middle ages most translators and editors spoke of Euclid as Euclid of *Megara*. This description arose out of a confusion between our Euclid and the philosopher Euclid of Megara who lived about 400 B.C. The first trace of this confusion appears in Valerius Maximus (in the time of Tiberius) who says³ that Plato, on being appealed to for a solution of the problem of doubling the cubical altar, sent the inquirers to "Euclid the geometer." There is no doubt about the reading, although an early commentator on Valerius Maximus wanted to correct "Eucliden" into "*Eudoxum*," and this correction is clearly right. But, if Valerius Maximus took Euclid the geometer for a contemporary of Plato, it could only be through confusing him with Euclid of Megara. The first specific reference to Euclid as Euclid of Megara belongs to the 14th century, occurring in the *ὑπομνηματισμοί* of Theodorus Metochita (d. 1332) who speaks of "Euclid of Megara, the Socratic philosopher, contemporary of Plato," as the author of treatises on plane and solid geometry, data, optics etc.: and a Paris MS. of the 14th century has "Euclidis philosophi Socratici liber elementorum." The misunderstanding was general in the period from Campanus' translation (Venice 1482) to those of Tartaglia (Venice 1565) and Candalla (Paris 1566). But one Constantinus Lascaris (d. about 1493) had already made the proper

¹ Pappus, VII. pp. 676, 25—678, 6. Hultsch, it is true, brackets the whole passage pp. 676, 25—678, 15, but apparently on the ground of the diction only.

² Stobaeus, *l.c.*

³ VIII. 12, ext. 1.

distinction by saying of our Euclid that "he was different from him of Megara of whom Laertius wrote, and who wrote dialogues"¹; and to Commandinus belongs the credit of being the first translator² to put the matter beyond doubt: "Let us then free a number of people from the error by which they have been induced to believe that our Euclid is the same as the philosopher of Megara" etc.

Another idea, that Euclid was born at Gela in Sicily, is due to the same confusion, being based on Diogenes Laertius' description³ of the philosopher Euclid as being "of Megara, or, according to some, of Gela, as Alexander says in the *Διαδοχαί*."

In view of the poverty of Greek tradition on the subject even as early as the time of Proclus (410-485 A.D.), we must necessarily take *cum grano* the apparently circumstantial accounts of Euclid given by Arabian authors; and indeed the origin of their stories can be explained as the result (1) of the Arabian tendency to romance, and (2) of misunderstandings.

We read⁴ that "Euclid, son of Naucrates, grandson of Zenarchus⁵, called the author of geometry, a philosopher of somewhat ancient date, a Greek by nationality domiciled at Damascus, born at Tyre, most learned in the science of geometry, published a most excellent and most useful work entitled the foundation or elements of geometry, a subject in which no more general treatise existed before among the Greeks: nay, there was no one even of later date who did not walk in his footsteps and frankly profess his doctrine. Hence also Greek, Roman and Arabian geometers not a few, who undertook the task of illustrating this work, published commentaries, scholia, and notes upon it, and made an abridgment of the work itself. For this reason the Greek philosophers used to post up on the doors of their schools the well-known notice: 'Let no one come to our school, who has not first learned the elements of Euclid.'" The details at the beginning of this extract cannot be derived from Greek sources, for even Proclus did not know anything about Euclid's father, while it was not the Greek habit to record the names of grandfathers, as the Arabians commonly did. Damascus and Tyre were no doubt brought in to gratify a desire which the Arabians always showed to connect famous Greeks in some way or other with the East. Thus Naṣīraddin, the translator of the *Elements*, who was of Tūs in Khurāsān, actually makes Euclid out to have been "Thusinus" also⁶. The readiness of the Arabians to run away with an idea is illustrated by the last words

¹ Letter to Fernandus Acuna, printed in Maurolycus, *Historia Siciliæ*, fol. 21 r. (see Heiberg, *Euklid-Studien*, pp. 22-3, 25).

² Preface to translation (Pisauri, 1572).

³ Diog. L. II. 106, p. 58 ed. Cobet.

⁴ Casiri, *Bibliotheca Arabico-Hispana Escorialensis*, I. p. 339. Casiri's source is al-Qiftī (d. 1248), the author of the *Tārīkh al-Ḥukamā*, a collection of biographies of philosophers, mathematicians, astronomers etc.

⁵ The *Fihrist* says "son of Naucrates, the son of Berenice (?)" (see Suter's translation in *Abhandlungen zur Gesch. d. Math.* VI. Heft, 1892, p. 16).

⁶ The same predilection made the Arabs describe Pythagoras as a pupil of the wise Salomo, Hipparchus as the exponent of Chaldaean philosophy or as the Chaldaean, Archimedes as an Egyptian etc. (Hājī Khalfa, *Lexicon Bibliographicum*, and Casiri).

of the extract. Everyone knows the story of Plato's inscription over the porch of the Academy: "let no one unversed in geometry enter my doors"; the Arab turned geometry into *Euclid's* geometry, and told the story of Greek philosophers in general and "*their* Academies."

Equally remarkable are the Arabian accounts of the relation of Euclid and Apollonius¹. According to them the *Elements* were originally written, not by Euclid, but by a man whose name was Apollonius, a carpenter, who wrote the work in 15 books or sections². In the course of time some of the work was lost and the rest became disarranged, so that one of the kings at Alexandria who desired to study geometry and to master this treatise in particular first questioned about it certain learned men who visited him and then sent for Euclid who was at that time famous as a geometer, and asked him to revise and complete the work and reduce it to order. Euclid then re-wrote it in 13 books which were thereafter known by his name. (According to another version Euclid composed the 13 books out of commentaries which he had published on two books of Apollonius on conics and out of introductory matter added to the doctrine of the five regular solids.) To the thirteen books were added two more books, the work of others (though some attribute these also to Euclid) which contain several things not mentioned by Apollonius. According to another version Hypsicles, a pupil of Euclid at Alexandria, offered to the king and published Books XIV. and XV., it being also stated that Hypsicles had "discovered" the books, by which it appears to be suggested that Hypsicles had edited them from materials left by Euclid.

We observe here the correct statement that Books XIV. and XV. were not written by Euclid, but along with it the incorrect information that Hypsicles, the author of Book XIV., wrote Book XV. also.

The whole of the fable about Apollonius having preceded Euclid and having written the *Elements* appears to have been evolved out of the preface to Book XIV. by Hypsicles, and in this way; the Book must in early times have been attributed to Euclid, and the inference based upon this assumption was left uncorrected afterwards when it was recognised that Hypsicles was the author. The preface is worth quoting:

"Basilides of Tyre, O Protarchus, when he came to Alexandria and met my father, spent the greater part of his sojourn with him on account of their common interest in mathematics. And once, when

¹ The authorities for these statements quoted by Casiri and Hāji Khalfā are al-Kindī's tract *de instituto libri Euclidis* (al-Kindī died about 873) and a commentary by Qāḍizāde ar-Rūmī (d. about 1440) on a book called *Ashkāl at-ta' sīs* (fundamental propositions) by Ashraf Shamsaddin as-Samarqandī (c. 1276) consisting of elucidations of 35 propositions selected from the first books of Euclid. Naṣīraddin likewise says that Euclid cut out two of 15 books of elements then existing and published the rest under his own name. According to Qāḍizāde the king heard that there was a celebrated geometer named Euclid at Tyre: Naṣīraddin says that he sent for Euclid of Tūs.

² So says the *Fihrist*. Suter (*op. cit.* p. 49) thinks that the author of the *Fihrist* did not suppose Apollonius of Perga to be the writer of the *Elements*, as later Arabian authorities did, but that he distinguished another Apollonius whom he calls "a carpenter." Suter's argument is based on the fact that the *Fihrist's* article on Apollonius (of Perga) says nothing of the *Elements*; and that it gives the three great mathematicians, Euclid, Archimedes and Apollonius, in the correct chronological order.

examining the treatise written by Apollonius about the comparison between the dodecahedron and the icosahedron inscribed in the same sphere, (showing) what ratio they have to one another, they thought that Apollonius had not expounded this matter properly, and accordingly they emended the exposition, as I was able to learn from my father. And I myself, later, fell in with another book published by Apollonius, containing a demonstration relating to the subject, and I was greatly interested in the investigation of the problem. The book published by Apollonius is accessible to all—for it has a large circulation, having apparently been carefully written out later—but I decided to send you the comments which seem to me to be necessary, for you will through your proficiency in mathematics in general and in geometry in particular form an expert judgment on what I am about to say, and you will lend a kindly ear to my disquisition for the sake of your friendship to my father and your goodwill to me.”

The idea that Apollonius preceded Euclid must evidently have been derived from the passage just quoted. It explains other things besides. Basilides must have been confused with βασιλεύς, and we have a probable explanation of the “Alexandrian king,” and of the “learned men who visited” Alexandria. It is possible also that in the “Tyrian” of Hypsicles’ preface we have the origin of the notion that Euclid was born in Tyre. These inferences argue, no doubt, very defective knowledge of Greek: but we could expect no better from those who took the *Organon* of Aristotle to be “instrumentum musicum pneumaticum,” and who explained the name of Euclid, which they variously pronounced as *Uclides* or *Icludes*, to be compounded of *Ucli* a key, and *Dis* a measure, or, as some say, geometry, so that *Uclides* is equivalent to the *key of geometry*!

Lastly the alternative version, given in brackets above, which says that Euclid made the *Elements* out of commentaries which he wrote on two books of Apollonius on conics and prolegomena added to the doctrine of the five solids, seems to have arisen, through a like confusion, out of a later passage¹ in Hypsicles’ Book XIV.: “And this is expounded by Aristaeus in the book entitled ‘Comparison of the five figures,’ and by Apollonius in the second edition of his comparison of the dodecahedron with the icosahedron.” The “doctrine of the five solids” in the Arabic must be the “Comparison of the five figures” in the passage of Hypsicles, for nowhere else have we any information about a work bearing this title, nor can the Arabians have had. The reference to the *two books* of Apollonius on *conics* will then be the result of mixing up the fact that Apollonius wrote a book on conics with the *second edition* of the other work mentioned by Hypsicles. We do not find elsewhere in Arabian authors any mention of a commentary by Euclid on Apollonius and Aristaeus: so that the story in the passage quoted is really no more than a variation of the fable that the *Elements* were the work of Apollonius.

¹ Heiberg’s Euclid, vol. v. p. 6.

CHAPTER II.

EUCLID'S OTHER WORKS.

IN giving a list of the Euclidean treatises other than the *Elements*, I shall be brief: for fuller accounts of them, or speculations with regard to them, reference should be made to the standard histories of mathematics¹.

I will take first the works which are mentioned by Greek authors.

1. The *Pseudaria*.

I mention this first because Proclus refers to it in the general remarks in praise of the *Elements* which he gives immediately after the mention of Euclid in his summary. He says²: "But, inasmuch as many things, while appearing to rest on truth and to follow from scientific principles, really tend to lead one astray from the principles and deceive the more superficial minds, he has handed down methods for the discriminative understanding of these things as well, by the use of which methods we shall be able to give beginners in this study practice in the discovery of paralogisms, and to avoid being misled. This treatise, by which he puts this machinery in our hands, he entitled (the book) of *Pseudaria*, enumerating in order their various kinds, exercising our intelligence in each case by theorems of all sorts, setting the true side by side with the false, and combining the refutation of error with practical illustration. This book then is by way of cathartic and exercise, while the *Elements* contain the irrefragable and complete guide to the actual scientific investigation of the subjects of geometry."

The book is considered to be irreparably lost. We may conclude however from the connexion of it with the *Elements*, and the reference to its usefulness for beginners that it did not go outside the domain of elementary geometry³.

¹ See, for example, Loria, *Le scienze esatte nell' antica Grecia*, 1914, pp. 245—268; T. L. Heath, *History of Greek Mathematics*, 1921, I. pp. 421—446. Cf. Heiberg, *Litterar-geschichtliche Studien über Euklid*, pp. 36—153; *Euclidis opera omnia*, ed. Heiberg and Menge, Vols. VI.—VIII.

² Proclus, p. 70, 1—18.

³ Heiberg points out that Alexander Aphrodisiensis appears to allude to the work in his commentary on Aristotle's *Sophistici Elenchi* (fol. 25 b): "Not only those (ἐλεγχοί) which do not start from the principles of the science under which the problem is classed...but also those which do start from the proper principles of the science but in some respect admit a paralogism, e.g. the *Pseudographemata* of Euclid." Tannery (*Bull. des sciences math. et astr.* 2^e Série, VI., 1882, 1^{ère} Partie, p. 147) conjectures that it may be from this treatise that the same commentator got his information about the quadratures of the circle by Antiphon and

2. The *Data*.

The *Data* (δεδομένα) are included by Pappus in the *Treasury of Analysis* (τόπος ἀναλυόμενος), and he describes their contents¹. They are still concerned with elementary geometry, though forming part of the introduction to higher analysis. Their form is that of propositions proving that, if certain things in a figure are given (in magnitude, in species, etc.), something else is given. The subject-matter is much the same as that of the planimetric books of the *Elements*, to which the *Data* are often supplementary. We shall see this later when we come to compare the propositions in the *Elements* which give us the means of solving the general quadratic equation with the corresponding propositions of the *Data* which give the solution. The *Data* may in fact be regarded as elementary exercises in analysis.

It is not necessary to go more closely into the contents, as we have the full Greek text and the commentary by Marinus newly edited by Menge and therefore easily accessible².

3. The book *On divisions (of figures)*.

This work (περὶ διαιρέσεων βιβλίον) is mentioned by Proclus³. In one place he is speaking of the conception or definition (λόγος) of *figure*, and of the divisibility of a figure into others differing from it in kind; and he adds: "For the circle is divisible into parts unlike in definition or notion (ἀνόμοια τῷ λόγῳ), and so is each of the rectilinear figures; this is in fact the business of the writer of the *Elements* in his *Divisions*, where he divides given figures, in one case into like figures, and in another into unlike⁴." "Like" and "unlike" here mean, not "similar" and "dissimilar" in the technical sense, but "like" or "unlike in definition or notion" (λόγῳ): thus to divide a triangle into triangles would be to divide it into "like" figures, to divide a triangle into a triangle and a quadrilateral would be to divide it into "unlike" figures.

The treatise is lost in Greek but has been discovered in the Arabic. First John Dee discovered a treatise *De divisionibus* by one Muhammad Bagdadinus⁵ and handed over a copy of it (in Latin) in 1563 to Commandinus, who published it, in Dee's name and his own, in 1570⁶. Dee did not himself translate the tract from the Arabic; he

Bryson, to say nothing of the lunules of Hippocrates. I think however that there is an objection to this theory so far as regards Bryson; for Alexander distinctly says that Bryson's quadrature did *not* start from the proper principles of geometry, but from some principles more general.

¹ Pappus, VII. p. 638.

² Vol. VI. in the Teubner edition of *Euclidis opera omnia* by Heiberg and Menge. A translation of the *Data* is also included in Simson's *Euclid* (though naturally his text left much to be desired).

³ Proclus, p. 69, 4.

⁴ *ibid.* 144, 22—26.

⁵ Steinschneider places him in the 10th c. H. Suter (*Bibliotheca Mathematica*, IV, 3, 1903, pp. 24, 27) identifies him with Abū (Bekr) Muḥ. b. 'Abdalbāqī al-Baḡdādī, Qāḍī (Judge) of Māristān (circa 1070–1141), to whom he also attributes the *Liber judei* (? judicis) *super decimum Euclidis* translated by Gherard of Cremona.

⁶ *De superficierum divisionibus liber Machometo Bagdadino adscriptus, nunc primum Ioannis Dee Londinensis et Federici Commandini Urbinate opera in lucem editus*, Pisauri, 1570, afterwards included in Gregory's *Euclid* (Oxford, 1703).

found it in Latin in a MS. which was then in his own possession but was about 20 years afterwards stolen or destroyed in an attack by a mob on his house at Mortlake¹. Dee, in his preface addressed to Commandinus, says nothing of his having *translated* the book, but only remarks that the very illegible MS. had caused him much trouble and (in a later passage) speaks of "the actual, very ancient, copy from which I *wrote out*..." (in ipso unde descripsi vetustissimo exemplari). The Latin translation of this tract from the Arabic was probably made by Gherard of Cremona (1114–1187), among the list of whose numerous translations a "liber divisionum" occurs. The Arabic original cannot have been a direct translation from Euclid, and probably was not even a direct adaptation of it; it contains mistakes and unmathematical expressions, and moreover does not contain the propositions about the division of a circle alluded to by Proclus. Hence it can scarcely have contained more than a fragment of Euclid's work.

But Woepcke found in a MS. at Paris a treatise in Arabic on the division of figures, which he translated and published in 1851². It is expressly attributed to Euclid in the MS. and corresponds to the description of it by Proclus. Generally speaking, the divisions are divisions into figures of the same kind as the original figures, e.g. of triangles into triangles; but there are also divisions into "unlike" figures, e.g. that of a triangle by a straight line parallel to the base. The missing propositions about the division of a circle are also here: "to divide into two equal parts a given figure bounded by an arc of a circle and two straight lines including a given angle" and "to draw in a given circle two parallel straight lines cutting off a certain part of the circle." Unfortunately the proofs are given of only four propositions (including the two last mentioned) out of 36, because the Arabic translator found them too easy and omitted them. To illustrate the character of the problems dealt with I need only take one more example: "To cut off a certain fraction from a (parallel-) trapezium by a straight line which passes through a given point lying inside or outside the trapezium but so that a straight line can be drawn through it cutting both the parallel sides of the trapezium." The genuineness of the treatise edited by Woepcke is attested by the facts that the four proofs which remain are elegant and depend on propositions in the *Elements*, and that there is a lemma with a true Greek ring: "to apply to a straight line a rectangle equal to the rectangle contained by AB , AC and deficient by a square." Moreover the treatise is no fragment, but finishes with the words "end of the treatise," and is a well-ordered and compact whole. Hence we may safely conclude that Woepcke's is not only Euclid's own work but the whole of it. A restoration of the work, with proofs, was attempted by Offerdinger³, who however does not give Woepcke's props. 30, 31, 34, 35, 36. We have now a satisfactory restoration, with ample notes

¹ R. C. Archibald, *Euclid's Book on the Division of Figures with a restoration based on Woepcke's text and on the Practica geometriae of Leonardo Pisano*, Cambridge, 1915, pp. 4–9.

² *Journal Asiatique*, 1851, p. 233 sqq.

³ L. F. Offerdinger, *Beiträge zur Wiederherstellung der Schrift des Euklides über die Theilung der Figuren*, Ulm, 1853.

and an introduction, by R. C. Archibald, who used for the purpose Woepcke's text and a section of Leonardo of Pisa's *Practica geometriae* (1220)¹.

4. The *Porisms*.

It is not possible to give in this place any account of the controversies about the contents and significance of the three lost books of *Porisms*, or of the important attempts by Robert Simson and Chasles to restore the work. These may be said to form a whole literature, references to which will be found most abundantly given by Heiberg and Loria, the former of whom has treated the subject from the philological point of view, most exhaustively, while the latter, founding himself generally on Heiberg, has added useful details, from the mathematical side, relating to the attempted restorations, etc.² It must suffice here to give an extract from the only original source of information about the nature and contents of the *Porisms*, namely Pappus³. In his general preface about the books composing the *Treasury of Analysis* (τόπος ἀναλυόμενος) he says:

"After the Tangencies (of Apollonius) come, in three books, the *Porisms* of Euclid, [in the view of many] a collection most ingeniously devised for the analysis of the more weighty problems, [and] although nature presents an unlimited number of such porisms⁴, [they have added nothing to what was written originally by Euclid, except that some before my time have shown their want of taste by adding to a few (of the propositions) second proofs, each (proposition) admitting of a definite number of demonstrations, as we have shown, and Euclid having given one for each, namely that which is the most lucid. These porisms embody a theory subtle, natural, necessary, and of considerable generality, which is fascinating to those who can see and produce results].

"Now all the varieties of porisms belong, neither to theorems nor problems, but to a species occupying a sort of intermediate position [so that their enunciations can be formed like those of either theorems or problems], the result being that, of the great number of geometers, some regarded them as of the class of theorems, and others of problems, looking only to the form of the proposition. But that the ancients knew better the difference between these three things is clear from the definitions. For they said that a theorem is that which is proposed with a view to the demonstration of the very thing proposed, a problem that which is thrown out with a view to the construction of the very thing proposed, and a porism that which is proposed with a view to the producing of the very thing proposed. [But this definition of the porism was changed by the more recent writers who could not produce everything, but used these elements

¹ There is a remarkable similarity between the propositions of Woepcke's text and those of Leonardo, suggesting that Leonardo may have had before him a translation (perhaps by Gherard of Cremona) of the Arabic tract.

² Heiberg, *Euklid-Studien*, pp. 56—79, and Loria, *op. cit.*, pp. 253—265.

³ Pappus, ed. Hultsch, vii. pp. 648—660. I put in square brackets the words bracketed by Hultsch.

⁴ I adopt Heiberg's reading of a comma here instead of a full stop.

and proved only the fact that that which is sought really exists, but did not produce it¹ and were accordingly confuted by the definition and the whole doctrine. They based their definition on an incidental characteristic, thus: A porism is that which falls short of a locus-theorem in respect of its hypothesis². Of this kind of porisms loci are a species, and they abound in the Treasury of Analysis; but this species has been collected, named and handed down separately from the porisms, because it is more widely diffused than the other species]. But it has further become characteristic of porisms that, owing to their complication, the enunciations are put in a contracted form, much being by usage left to be understood; so that many geometers understand them only in a partial way and are ignorant of the more essential features of their contents.

"[Now to comprehend a number of propositions in one enunciation is by no means easy in these porisms, because Euclid himself has not in fact given many of each species, but chosen, for examples, one or a few out of a great multitude³. But at the beginning of the first book he has given some propositions, to the number of ten, of one species, namely that more fruitful species consisting of loci.] Consequently, finding that these admitted of being comprehended in one enunciation, we have set it out thus:

If, in a system of four straight lines⁴ which cut each other two and two, three points on one straight line be given while the rest except one lie on different straight lines given in position, the remaining point also will lie on a straight line given in position⁵.

¹ Heiberg points out that Props. 5—9 of Archimedes' treatise *On Spirals* are porisms in this sense. To take Prop. 5 as an example, DBF is a tangent to a circle with centre K . It is then possible, says Archimedes, to draw a straight line KHF , meeting the circumference in H and the tangent in F , such that

$$FH : HK < (\text{arc } BH) : c,$$

where c is the circumference of any circle. To prove this he assumes the following construction. E being any straight line greater than c , he says: let KG be parallel to DF , "and let the line GH equal to E be placed *verging* to the point B ."

Archimedes must of course have known how to effect this

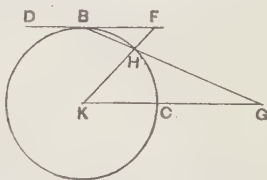
construction, which requires conics. But that it is *possible* requires very little argument, for if we draw any straight line BHG meeting the circle in H and KG in G , it is obvious that as G moves away from C , HG becomes greater and greater and may be made as great as we please. The "later writers" would no doubt have contented themselves with this consideration without actually *constructing* HG .

² As Heiberg says, this translation is made certain by a preceding passage of Pappus (p. 648, 1—3) where he compares two enunciations, the latter of which "falls short of the former in *hypothesis* but goes beyond it in *requirement*." E.g. the first enunciation requiring us, given three circles, to draw a circle touching all three, the second may require us, given only two circles (one less datum), to draw a circle touching them and of a *given size* (an extra requirement).

³ I translate Heiberg's reading with a full stop here followed by $\pi\rho\acute{o}s\ \acute{\alpha}\rho\chi\eta\tilde{\nu}\ \delta\acute{\epsilon}\ \delta\mu\omega\varsigma\ [\pi\rho\acute{o}s\ \acute{\alpha}\rho\chi\eta\tilde{\nu}\ (\delta\epsilon\delta\omicron\mu\epsilon\nu\omicron\nu)]$ Hultsch] τοῦ πρώτου βιβλίου....

⁴ The four straight lines are described in the text as (the sides) $\delta\upsilon\pi\tau\iota\omicron\nu\ \eta\ \pi\alpha\rho\upsilon\pi\tau\iota\omicron\nu$, i.e. sides of two sorts of quadrilaterals which Simson tries to explain (see p. 120 of the *Index Graecitatis* of Hultsch's edition of Pappus).

⁵ In other words (Chasles, p. 23; Loria, p. 256), if a triangle be so deformed that each of its sides turns about one of three points in a straight line, and two of its vertices lie on two straight lines given in position, the third vertex will also lie on a straight line.



"This has only been enunciated of four straight lines, of which not more than two pass through the same point, but it is not known (to most people) that it is true of any assigned number of straight lines if enunciated thus :

If any number of straight lines cut one another, not more than two (passing) through the same point, and all the points (of intersection situated) on one of them be given, and if each of those which are on another (of them) lie on a straight line given in position—

or still more generally thus :

if any number of straight lines cut one another, not more than two (passing) through the same point, and all the points (of intersection situated) on one of them be given, while of the other points of intersection in multitude equal to a triangular number a number corresponding to the side of this triangular number lie respectively on straight lines given in position, provided that of these latter points no three are at the angular points of a triangle (*sc.* having for sides three of the given straight lines)—each of the remaining points will lie on a straight line given in position¹.

"It is probable that the writer of the Elements was not unaware of this but that he only set out the principle; and he seems, in the case of all the porisms, to have laid down the principles and the seed only [of many important things], the kinds of which should be distinguished according to the differences, not of their hypotheses, but of the results and the things sought. [All the hypotheses are different from one another because they are entirely special, but each of the results and things sought, being one and the same, follow from many different hypotheses.]

"We must then in the first book distinguish the following kinds of things sought :

"At the beginning of the book² is this proposition :

I. *'If from two given points straight lines be drawn meeting on a straight line given in position, and one cut off from a straight line given in position (a segment measured) to a given point on it, the other will also cut off from another (straight line a segment) having to the first a given ratio.'*

"Following on this (we have to prove)

II. that such and such a point lies on a straight line given in position;

III. that the ratio of such and such a pair of straight lines is given ;"

etc. etc. (up to XXIX.).

"The three books of the porisms contain 38 lemmas; of the theorems themselves there are 171."

¹ Loria (p. 256, n. 3) gives the meaning of this as follows, pointing out that Simson was the discoverer of it : "If a complete n -lateral be deformed so that its sides respectively turn about n points on a straight line, and $(n-1)$ of its $n(n-1)/2$ vertices move on as many straight lines, the other $(n-1)(n-2)/2$ of its vertices likewise move on as many straight lines; but it is necessary that it should be impossible to form with the $(n-1)$ vertices any triangle having for sides the sides of the polygon."

² Reading, with Heiberg, τοῦ βέλλου [τοῦ ζ' Hultsch].

Pappus further gives lemmas to the *Porisms* (pp. 866—918, ed. Hultsch).

With Pappus' account of Porisms must be compared the passages of Proclus on the same subject. Proclus distinguishes two senses in which the word *πόρισμα* is used. The first is that of *corollary* where something appears as an incidental result of a proposition, obtained without trouble or special seeking, a sort of bonus which the investigation has presented us with¹. The other sense is that of Euclid's *Porisms*². In this sense³ "*porism* is the name given to things which are sought, but need some finding and are neither pure bringing into existence nor simple theoretic argument. For (to prove) that the angles at the base of isosceles triangles are equal is a matter of theoretic argument, and it is with reference to things existing that such knowledge is (obtained). But to bisect an angle, to construct a triangle, to cut off, or to place—all these things demand the making of something; and to find the centre of a given circle, or to find the greatest common measure of two given commensurable magnitudes, or the like, is in some sort between theorems and problems. For in these cases there is no bringing into existence of the things sought, but finding of them, nor is the procedure purely theoretic. For it is necessary to bring that which is sought into view and exhibit it to the eye. Such are the porisms which Euclid wrote, and arranged in three books of Porisms."

Proclus' definition thus agrees well enough with the first, "older," definition of Pappus. A porism occupies a place between a theorem and a problem: it deals with something already *existing*, as a theorem does, but has to *find* it (e.g. the centre of a circle), and, as a certain operation is therefore necessary, it partakes to that extent of the nature of a problem, which requires us to construct or produce something not previously existing. Thus, besides III. 1 of the *Elements* and X. 3, 4 mentioned by Proclus, the following propositions are real porisms: III. 25, VI. 11—13, VII. 33, 34, 36, 39, VIII. 2, 4, X. 10, XIII. 18. Similarly in Archimedes *On the Sphere and Cylinder* I. 2—6 might be called porisms.

The enunciation given by Pappus as comprehending ten of Euclid's propositions may not reproduce the *form* of Euclid's enunciations; but, comparing the result to be proved, that certain points lie on straight lines given in position, with the *class* indicated by II. above, where the question is of such and such a point lying on a straight line given in position, and with other classes, e.g. (V.) that such and such a line is given in position, (VI.) that such and such a line verges to a given point, (XXVII.) that there exists a given point such that straight lines drawn from it to such and such (circles) will contain a triangle given in species, we may conclude that a usual form of a porism was "to prove that it is possible to find a point with such and such a property"

¹ Proclus, pp. 212, 14; 301, 22.

² *ibid.* p. 212, 12. "The term porism is used of certain problems, like the *Porisms* written by Euclid."

³ *ibid.* pp. 301, 25 sqq.

or "a straight line on which lie all the points satisfying given conditions" etc.

Simson defined a porism thus: "Porisma est propositio in qua proponitur demonstrare rem aliquam, vel plures datas esse, cui, vel quibus, ut et cuilibet ex rebus innumeris, non quidem datis, sed quae ad ea quae data sunt eandem habent relationem, convenire ostendendum est affectionem quandam communem in propositione descriptam¹."

From the above it is easy to understand Pappus' statement that *loci* constitute a large class of porisms. A *locus* is well defined by Simson thus: "Locus est propositio in qua propositum est datam esse demonstrare, vel invenire lineam aut superficiem cuius quodlibet punctum, vel superficiem in qua quaelibet linea data lege descripta, communem quandam habet proprietatem in propositione descriptam." Heiberg cites an excellent instance of a *locus* which is a *porism*, namely the following proposition quoted by Eutocius² from the *Plane Loci* of Apollonius:

"Given two points in a plane, and a ratio between unequal straight lines, it is possible to draw, in the plane, a circle such that the straight lines drawn from the given points to meet on the circumference of the circle have (to one another) a ratio the same as the given ratio."

A difficult point, however, arises on the passage of Pappus, which says that a porism is "that which, in respect of its hypothesis, falls short of a locus-theorem" (τοπικὸν θεωρήματος). Heiberg explains it by comparing the porism from Apollonius' *Plane Loci* just given with Pappus' enunciation of the same thing, to the effect that, if from two given points two straight lines be drawn meeting in a point, and these straight lines have to one another a given ratio, the point will lie on either a straight line or a circumference of a circle given in position. Heiberg observes that in this latter enunciation something is taken into the hypothesis which was not in the hypothesis of the enunciation of the porism, viz. "that the ratio of the straight lines is the same." I confess this does not seem to me satisfactory: for there is no real difference between the enunciations, and the supposed difference in hypothesis is very like playing with words. Chasles says: "*Ce qui constitue le porisme est ce qui manque à l'hypothèse d'un théorème local* (en d'autres termes, le porisme est inférieur, par l'hypothèse, au théorème local; c'est-à-dire que quand quelques parties d'une proposition locale n'ont pas dans l'énoncé la détermination qui leur est propre, cette proposition cesse d'être regardée comme un théorème et devient un porisme)." But the subject still seems to require further elucidation.

While there is so much that is obscure, it seems certain (1) that the *Porisms* were distinctly part of higher geometry and not of elementary

¹ This was thus expressed by Chasles: "Le porisme est une proposition dans laquelle on demande de démontrer qu'une chose ou plusieurs choses sont *données*, qui, ainsi que l'une quelconque d'une infinité d'autres choses non données, mais dont chacune est avec des choses données dans une même relation, ont une certaine propriété commune, décrite dans la proposition."

² Commentary on Apollonius' *Conics* (vol. II. p. 180, ed. Heiberg).

geometry, (2) that they contained propositions belonging to the modern theory of transversals and to projective geometry. It should be remembered too that it was in the course of his researches on this subject that Chasles was led to the idea of *anharmonic ratios*.

Lastly, allusion should be made to the theory of Zeuthen¹ on the subject of the porisms. He observes that the only porism of which Pappus gives the complete enunciation, "If from two given points straight lines be drawn meeting on a straight line given in position, and one cut off from a straight line given in position (a segment measured) towards a given point on it, the other will also cut off from another (straight line a segment) bearing to the first a given ratio," is also true if there be substituted for the first given straight line a conic regarded as the "locus with respect to four lines," and that this extended porism can be used for completing Apollonius' exposition of that locus. Zeuthen concludes that the *Porisms* were in part by-products of the theory of conics and in part auxiliary means for the study of conics, and that Euclid called them by the same name as that applied to corollaries because they were corollaries with respect to conics. But there appears to be no evidence to confirm this conjecture.

5. The *Surface-loci* (τόποι πρὸς ἐπιφανείᾳ).

The two books on this subject are mentioned by Pappus as part of the *Treasury of Analysis*². As the other works in the list which were on plane subjects dealt only with straight lines, circles, and conic sections, it is *a priori* likely that among the loci in this treatise (loci which are surfaces) were included such loci as were cones, cylinders and spheres. Beyond this all is conjecture based on two lemmas given by Pappus in connexion with the treatise.

(1) The first of these lemmas³ and the figure attached to it are not satisfactory as they stand, but a possible restoration is indicated by Tannery⁴. If the latter is right, it suggests that one of the loci contained all the points on the elliptical parallel sections of a cylinder and was therefore an oblique circular cylinder. Other assumptions with regard to the conditions to which the lines in the figure may be subject would suggest that other loci dealt with were cones regarded as containing all points on particular elliptical parallel sections of the cones⁵.

(2) In the second lemma Pappus states and gives a complete proof of the focus-and-directrix property of a conic, viz. that *the locus of a point whose distance from a given point is in a given ratio to its distance from a fixed line is a conic section, which is an ellipse, a parabola or a hyperbola according as the given ratio is less than, equal to, or greater than unity*⁶. Two conjectures are possible as to the application of this theorem in Euclid's *Surface-loci*. (a) It may have been used to prove that the locus of a point whose distance from a given straight

¹ *Die Lehre von den Kegelschnitten im Altertum*, chapter VIII.

² Pappus, VII. p. 636.

³ *ibid.* VII. p. 1004.

⁴ *Bulletin des sciences math. et astron.*, 2^e Série, VI. 149.

⁵ Further particulars will be found in *The Works of Archimedes*, pp. lxii–lxiv, and in Zeuthen, *Die Lehre von den Kegelschnitten*, p. 425 sqq.

⁶ Pappus, VII. pp. 1006–1014, and Hultsch's Appendix, pp. 1270–3.

line is in a given ratio to its distance from a given plane is a certain cone. (b) It may have been used to prove that the locus of a point whose distance from a given point is in a given ratio to its distance from a given plane is the surface formed by the revolution of a conic about its major or conjugate axis¹. Thus Chasles may have been correct in his conjecture that the *Surface-loci* dealt with surfaces of revolution of the second degree and sections of the same².

6. The *Conics*.

Pappus says of this lost work: "The four books of Euclid's *Conics* were completed by Apollonius, who added four more and gave us eight books of *Conics*³." It is probable that Euclid's work was lost even by Pappus' time, for he goes on to speak of "Aristaeus, who wrote the *still extant* five books of *Solid Loci* connected with the conics." Speaking of the relation of Euclid's work to that of Aristaeus on conics regarded as loci, Pappus says in a later passage (bracketed however by Hultsch) that Euclid, regarding Aristaeus as deserving credit for the discoveries he had already made in conics, did not (try to) anticipate him or construct anew the same system. We may no doubt conclude that the book by Aristaeus on solid loci preceded Euclid's on conics and was, at least in point of originality, more important. Though both treatises dealt with the same subject-matter, the object and the point of view were different; had they been the same, Euclid could scarcely have refrained, as Pappus says he did, from attempting to improve upon the earlier treatise. No doubt Euclid wrote on the general theory of conics as Apollonius did, but confined himself to those properties which were necessary for the analysis of the *Solid Loci* of Aristaeus. The *Conics* of Euclid were evidently superseded by the treatise of Apollonius.

As regards the contents of Euclid's *Conics*, the most important source of our information is Archimedes, who frequently refers to propositions in conics as well known and not needing proof, adding in three cases that they are proved in the "elements of conics" or in "the conics," which expressions must clearly refer to the works of Aristaeus and Euclid⁴.

Euclid still used the old names for the conics (sections of a right-angled, acute-angled, or obtuse-angled cone), but he was aware that an ellipse could be obtained by cutting a cone in any manner by a plane not parallel to the base (assuming the section to lie wholly between the apex of the cone and its base) and also by cutting a cylinder. This is expressly stated in a passage from the *Phaenomena* of Euclid about to be mentioned⁵.

7. The *Phaenomena*.

This is an astronomical work and is still extant. A much inter-

¹ For further details see *The Works of Archimedes*, pp. lxiv, lxv, and Zeuthen, *l. c.*

² *Aperçu historique*, pp. 273—4.

³ Pappus, vii. p. 672.

⁴ For details of these propositions see my *Apollonius of Perga*, pp. xxxv, xxxvi.

⁵ *Phaenomena*, ed. Menge, p. 6: "If a cone or a cylinder be cut by a plane not parallel to the base, the section is a section of an acute-angled cone, which is like a shield (*θυρεός*)."

polated version appears in Gregory's Euclid. An earlier and better recension is however contained in the Ms. Vindobonensis philos. Gr. 103, though the end of the treatise, from the middle of prop. 16 to the last (18), is missing. The book, now edited by Menge¹, consists of propositions in *spheric* geometry. Euclid based it on Autolycus' work *περὶ κινουμένης σφαίρας*, but also, evidently, on an earlier textbook of *Sphaerica* of exclusively mathematical content. It has been conjectured that the latter textbook may have been due to Eudoxus².

8. The *Optics*.

This book needs no description, as it has been edited by Heiberg recently³, both in its genuine form and in the recension by Theon. The *Catoptrica* published by Heiberg in the same volume is not genuine, and Heiberg suspects that in its present form it may be Theon's. It is not even certain that Euclid wrote *Catoptrica* at all, as Proclus may easily have had Theon's work before him and inadvertently assigned it to Euclid⁴.

9. Besides the above-mentioned works, Euclid is said to have written the *Elements of Music*⁵ (*αἱ κατὰ μουσικὴν στοιχειώσεις*). Two treatises are attributed to Euclid in our MSS. of the *Musici*, the *κατατομὴ κανόνος*, *Sectio canonis* (the theory of the intervals), and the *εἰσαγωγή ἁρμονικὴ* (introduction to harmony)⁶. The first, resting on the Pythagorean theory of music, is mathematical, and the style and diction as well as the form of the propositions mostly agree with what we find in the *Elements*. Jan thought it genuine, especially as almost the whole of the treatise (except the preface) is quoted *in extenso*, and Euclid is twice mentioned by name, in the commentary on Ptolemy's *Harmonica* published by Wallis and attributed by him to Porphyry. Tannery was of the opposite opinion⁷. The latest editor, Menge, suggests that it may be a redaction by a less competent hand from the genuine Euclidean *Elements of Music*. The second treatise is not Euclid's, but was written by Cleonides, a pupil of Aristoxenus⁸.

Lastly, it is worth while to give the Arabians' list of Euclid's works. I take this from Suter's translation of the list of philosophers and mathematicians in the *Fihrist*, the oldest authority of the kind that we possess⁹. "To the writings of Euclid belong further [in addition to the *Elements*]: the book of *Phaenomena*; the book of

¹ *Euclidis opera omnia*, vol. VIII., 1916, pp. 2—156.

² Heiberg, *Euklid-Studien*, p. 46; Hultsch, *Autolycus*, p. xii; A. A. Björnbo, *Studien über Menelaos' Sphärik* (*Abhandlungen zur Geschichte der mathematischen Wissenschaften*, XIV. 1902), p. 56 sqq.

³ *Euclidis opera omnia*, vol. VII. (1895).

⁴ Heiberg, *Euclid's Optics*, etc. p. 1.

⁵ Proclus, p. 69, 3.

⁶ Both treatises edited by Jan in *Musici Scriptores Graeci*, 1895, pp. 113—166, 167—207, and by Menge in *Euclidis opera omnia*, vol. VIII., 1916, pp. 157—183, 185—223.

⁷ *Comptes rendus de l'Acad. des inscriptions et belles-lettres*, Paris, 1904, pp. 439—445. Cf. *Bibliotheca Mathematica*, VI, 1905—6, p. 225, note 1.

⁸ Heiberg, *Euklid-Studien*, pp. 52—55; Jan, *Musici Scriptores Graeci*, pp. 169—174.

⁹ H. Suter, *Das Mathematiker-Verzeichniss im Fihrist in Abhandlungen zur Geschichte der Mathematik*, VI., 1892, pp. 1—87 (see especially p. 17). Cf. Casiri, I. 339, 340, and Gartz, *De interpretibus et explanatoribus Euclidis Arabicis*, 1823, pp. 4, 5.

Given Magnitudes [*Data*]; the book of Tones, known under the name of Music, not genuine; the book of Division, emended by Thābit; the book of Utilisations or Applications [*Porisms*], not genuine; the book of the Canon; the book of the Heavy and Light; the book of Synthesis, not genuine; and the book of Analysis, not genuine."

It is to be observed that the Arabs already regarded the book of Tones (by which must be meant the *εἰσαγωγή ἁρμονικὴ*) as spurious. The book of Division is evidently the book on *Divisions (of figures)*. The next book is described by Casiri as "liber de utilitate suppositus." Suter gives reason for believing the *Porisms* to be meant¹, but does not apparently offer any explanation of why the work is supposed to be spurious. The book of the Canon is clearly the *κατατομὴ κανόνης*. The book on "the Heavy and Light" is apparently the tract *De levi et ponderoso*, included in the Basel Latin translation of 1537, and in Gregory's edition. The fragment, however, cannot safely be attributed to Euclid, for (1) we have nowhere any mention of his having written on mechanics, (2) it contains the notion of specific gravity in a form so clear that it could hardly be attributed to anyone earlier than Archimedes². Suter thinks³ that the works on Analysis and Synthesis (said to be spurious in the extract) may be further developments of the *Data* or *Porisms*, or may be the interpolated proofs of *Eucl.* XIII. 1—5, divided into *analysis* and *synthesis*, as to which see the notes on those propositions.

¹ Suter, *op. cit.* pp. 49, 50. Wenrich translated the word as "utilia." Suter says that the nearest meaning of the Arabic word as of "porism" is *use, gain* (Nutzen, Gewinn), while a further meaning is explanation, observation, addition: a gain arising out of what has preceded (cf. Proclus' definition of the porism in the sense of a corollary).

² Heiberg, *Euklid-Studien*, pp. 9, 10.

³ Suter, *op. cit.* p. 50.

CHAPTER III.

GREEK COMMENTATORS ON THE *ELEMENTS* OTHER THAN PROCLUS.

THAT there was no lack of commentaries on the *Elements* before the time of Proclus is evident from the terms in which Proclus refers to them; and he leaves us in equally little doubt as to the value which, in his opinion, the generality of them possessed. Thus he says in one place (at the end of his second prologue)¹:

"Before making a beginning with the investigation of details, I warn those who may read me not to expect from me the things which have been dinned into our ears *ad nauseam* (διατεθρύλῃται) by those who have preceded me, viz. lemmas, cases, and so forth. For I am surfeited with these things and shall give little attention to them. But I shall direct my remarks principally to the points which require deeper study and contribute to the sum of philosophy, therein emulating the Pythagoreans who even had this common phrase for what I mean 'a figure and a platform, but not a figure and sixpence'.²"

In another place³ he says: "Let us now turn to the elucidation of the things proved by the writer of the *Elements*, selecting the more subtle of the comments made on them by the ancient writers, while cutting down their interminable diffuseness, giving the things which are more systematic and follow scientific methods, attaching more importance to the working-out of the real subject-matter than to the variety of cases and lemmas to which we see recent writers devoting themselves for the most part."

At the end of his commentary on *Eucl. I.* Proclus remarks⁴ that the commentaries then in vogue were full of all sorts of confusion, and contained no account of *causes*, no dialectical discrimination, and no philosophic thought.

These passages and two others in which Proclus refers to "the commentators"⁵ suggest that these commentators were numerous. He does not however give many names; and no doubt the only important commentaries were those of Heron, Porphyry, and Pappus.

¹ Proclus, p. 84, 8.

² i.e. we reach a certain height, use the platform so attained as a base on which to build another stage, then use that as a base and so on.

³ Proclus, p. 200, 10.

⁴ *ibid.* p. 432, 15.

⁵ *ibid.* p. 289, 11; p. 328, 16.

I. Heron.

Proclus alludes to Heron twice as Heron *mechanicus*¹, in another place² he associates him with Ctesibius, and in the three other passages³ where Heron is mentioned there is no reason to doubt that the same person is meant, namely Heron of Alexandria. The date of Heron is still a vexed question. In the early stages of the controversy much was made of the supposed relation of Heron to Ctesibius. The best MS. of Heron's *Belopoeica* has the heading "Ἡρωνος Κτησιβίου βελοποιϊκά, and an anonymous Byzantine writer of the tenth century, evidently basing himself on this title, speaks of Ctesibius as Heron's *καθηγητής*, "master" or "teacher." We know of two men of the name of Ctesibius. One was a barber who lived in the time of Ptolemy Euergetes II, i.e. Ptolemy VII, called Physcon (died 117 B.C.), and who is said to have made an improved water-organ⁴. The other was a mechanician mentioned by Athenaeus as having made an elegant drinking-horn in the time of Ptolemy Philadelphus (285-247 B.C.)⁵. Martin⁶ took the Ctesibius in question to be the former and accordingly placed Heron at the beginning of the first century B.C., say 126-50 B.C. But Philo of Byzantium⁷, who repeatedly mentions Ctesibius by name, says that the first mechanicians had the advantage of being under kings who loved fame and supported the arts. Hence our Ctesibius is more likely to have been the earlier Ctesibius who was contemporary with Ptolemy II Philadelphus.

But, whatever be the date of Ctesibius, we cannot safely conclude that Heron was his immediate pupil. The title "Heron's (edition of) Ctesibius's *Belopoeica*" does not, in fact, justify any inference as to the interval of time between the two works.

We now have better evidence for a *terminus post quem*. The *Metrica* of Heron, besides quoting Archimedes and Apollonius, twice refers to "the books about straight lines (chords) in a circle" (ἐν τοῖς περὶ τῶν ἐν κύκλῳ εὐθειῶν). Now we know of no work giving a Table of Chords earlier than that of Hipparchus. We get, therefore, at once, 150 B.C. or thereabouts as the *terminus post quem*. But, again, Heron's *Mechanica* quotes a definition of "centre of gravity" as given by "Posidonius, a Stoic": and, even if this Posidonius lived before Archimedes, as the context seems to imply, it is certain that another work of Heron's, the *Definitions*, owes something to Posidonius of Apamea or Rhodes, Cicero's teacher (135-51 B.C.). This brings Heron's date down to the end of the first century B.C., at least.

We have next to consider the relation, if any, between Heron and Vitruvius. In his *De Architectura*, brought out apparently in 14 B.C., Vitruvius quotes twelve authorities on *machinationes* including Archytas

¹ Proclus, p. 305, 24; p. 346, 13.

² *ibid.* p. 41, 10.

³ *ibid.* p. 196, 16; p. 323, 7; p. 429, 13.

⁴ Athenaeus, *Deipno-Soph.* iv., c. 75, p. 174 b-c.

⁵ *ibid.* xi., c. 97, p. 497 b-c.

⁶ Martin, *Recherches sur la vie et les ouvrages d'Héron d'Alexandrie*, Paris, 1854, p. 27.

⁷ Philo, *Mechan. Synt.*, p. 50, 38, ed. Schöne.

(second), Archimedes (third), Ctesibius (fourth) and Philo of Byzantium (sixth), but does not mention Heron. Nor is it possible to establish inter-dependence between Vitruvius and Heron; the differences between them seem on the whole more numerous and important than the resemblances (e.g. Vitruvius uses 3 as the value of π , while Heron always uses the Archimedean value $3\frac{1}{7}$). The inference is that Heron can hardly have written earlier than the first century A.D.

The most recent theory of Heron's date makes him later than Claudius Ptolemy the astronomer (100–178 A.D.). The arguments are mainly these. (1) Ptolemy claims as a discovery of his own a method of measuring the distance between two places (as an arc of a great circle on the earth's surface) in the case where the places are neither on the same meridian nor on the same parallel circle. Heron, in his *Dioptra*, speaks of this method as of a thing generally known to experts. (2) The dioptra described in Heron's work is a fine and accurate instrument, much better than anything Ptolemy had at his disposal. (3) Ptolemy, in his work *Περὶ ῥοπᾶν*, asserted that water with water round it has no weight and that the diver, however deep he dives, does not feel the weight of the water above him. Heron, strangely enough, accepts as true what Ptolemy says of the diver, but is dissatisfied with the explanation given by "some," namely that it is because water is uniformly heavy—this seems to be equivalent to Ptolemy's dictum that water in water has no weight—and he essays a different explanation based on Archimedes. (4) It is suggested that the Dionysius to whom Heron dedicated his *Definitions* is a certain Dionysius who was *praefectus urbi* in 301 A.D.

On the other hand Heron was earlier than Pappus, who was writing under Diocletian (284–305 A.D.), for Pappus alludes to and draws upon the works of Heron. The net result, then, of the most recent research is to place Heron in the third century A.D. and perhaps little earlier than Pappus. Heiberg¹ accepts this conclusion, which may therefore, perhaps, be said to hold the field for the present².

That Heron wrote a systematic commentary on the *Elements* might be inferred from Proclus, but it is rendered quite certain by references to the commentary in Arabian writers, and particularly in an-Nairizi's commentary on the first ten Books of the *Elements*. The *Fihrist* says, under Euclid, that "Heron wrote a commentary on this book [the *Elements*], endeavouring to solve its difficulties³"; and under Heron, "He wrote: the book of explanation of the obscurities in Euclid⁴...." An-Nairizi's commentary quotes Heron by name very frequently, and often in such a way as to leave no doubt that the author had Heron's work actually before him. Thus the extracts are given in the first person, introduced by "Heron says" ("Dixit Yrinus"

¹ *Heronis Alexandrini opera*, vol. v. (Teubner, 1914), p. ix.

² Fuller details of the various arguments will be found in my *History of Greek Mathematics*, 1921, vol. II., pp. 298–306.

³ *Das Mathematiker-Verzeichniss im Fihrist* (tr. Suter), p. 16.

⁴ *ibid.* p. 22.

or "Heron"); and in other places we are told that Heron "says nothing," or "is not found to have said anything," on such and such a proposition. The commentary of an-Nairizī is in part edited by Besthorn and Heiberg from a Leiden MS. of the translation of the *Elements* by al-Hajjāj with the commentary attached¹. But this MS. only contains six Books, and several pages in the first Book, which contain the comments of Simplicius on the first twenty-two definitions of the first Book, are missing. Fortunately the commentary of an-Nairizī has been discovered in a more complete form, in a Latin translation by Gherardus Cremonensis of the twelfth century, which contains the missing comments by Simplicius and an-Nairizī's comments on the first ten Books. This valuable work has recently been edited by Curtze².

Thus from the three sources, Proclus, and the two versions of an-Nairizī, which supplement one another, we are able to form a very good idea of the character of Heron's commentary. In some cases observations given by Proclus without the name of their author are seen from an-Nairizī to be Heron's; in a few cases notes attributed by Proclus to Heron are found in an-Nairizī without Heron's name; and, curiously enough, one alternative proof (of I. 25) given as Heron's by Proclus is introduced by the Arab with the remark that he has not been able to discover who is the author.

Speaking generally, the comments of Heron do not seem to have contained much that can be called important. We find

(1) A few general notes, e.g. that Heron would not admit more than three axioms.

(2) Distinctions of a number of particular cases of Euclid's propositions according as the figure is drawn in one way or in another.

Of this class are the different cases of I. 35, 36, III. 7, 8 (where the chords to be compared are drawn on *different* sides of the diameter instead of on the same side), III. 12 (which is not Euclid's, but Heron's own, adding the case of external contact to that of internal contact in III. 11), VI. 19 (where the triangle in which an additional line is drawn is taken to be the *smaller* of the two), VII. 19 (where he gives the particular case of *three* numbers in continued proportion, instead of four proportionals).

(3) Alternative proofs. Of these there should be mentioned (*a*) the proofs of II. 1—10 "without a figure," being simply the algebraic forms of proof, easy but uninteresting, which are so popular nowadays, the proof of III. 25 (placed after III. 30 and starting from the *arc* instead of the chord), III. 10 (proved by III. 9), III. 13 (a proof preceded by a lemma to the effect that a straight line cannot meet a circle in more than two points). Another class of alternative proof is

¹ *Codex Leidensis* 399, 1. *Euclidis Elementa ex interpretatione al-Hadschschadschii cum commentariis al-Nairizii*. Five parts carrying the work to the end of Book IV. were issued in 1893, 1897, 1900, 1905 and 1910 respectively.

² *Anarithi in decem libros priores elementorum Euclidis commentarii ex interpretatione Gherardi Cremonensis...edit* Maximilianus Curtze (Teubner, Leipzig, 1899).

(b) that which is intended to meet a particular *objection* (ἐνστάσις) which had been or might be raised to Euclid's construction. Thus in certain cases he avoids *producing* a particular straight line, where Euclid produces it, in order to meet the objection of any one who should deny our right to assume that there is *any space available*¹. Of this class are Heron's proofs of I. 11, I. 20, and his note on I. 16. Similarly on I. 48 he supposes the right-angled triangle which is constructed to be constructed on the *same* side of the common side as the given triangle is. A third class (c) is that which avoids *reductio ad absurdum*. Thus, instead of indirect proofs, Heron gives direct proofs of I. 19 (for which he requires, and gives, a preliminary lemma), and of I. 25.

(4) Heron supplies certain *converses* of Euclid's propositions, e.g. converses of II. 12, 13, VIII. 27.

(5) A few additions to, and extensions of, Euclid's propositions are also found. Some are unimportant, e.g. the construction of isosceles and scalene triangles in a note on I. 1, the construction of *two* tangents in III. 17, the remark that VII. 3 about finding the greatest common measure of three numbers can be applied to as many numbers as we please (as Euclid tacitly assumes in VII. 31). The most important extension is that of III. 20 to the case where the angle at the circumference is greater than a right angle, and the direct deduction from this extension of the result of III. 22. Interesting also are the notes on I. 37 (on I. 24 in Proclus), where Heron proves that two triangles with two sides of one equal to two sides of the other and with the included angles supplementary are equal, and compares the areas where the sum of the two included angles (one being supposed greater than the other) is less or greater than two right angles, and on I. 47, where there is a proof (depending on preliminary lemmas) of the fact that, in the figure of the proposition, the straight lines *AL*, *BK*, *CF* meet in a point. After IV. 16 there is a proof that, in a regular polygon with an even number of sides, the bisector of one angle also bisects its opposite, and an enunciation of the corresponding proposition for a regular polygon with an odd number of sides.

Van Pesch² gives reason for attributing to Heron certain other notes found in Proclus, viz. that they are designed to meet the same sort of points as Heron had in view in other notes undoubtedly written by him. These are (a) alternative proofs of I. 5, I. 17, and I. 32, which avoid the *producing* of certain straight lines, (b) an alternative proof of I. 9 avoiding the construction of the equilateral triangle on the side of *BC* opposite to *A*; (c) partial converses of I. 35—38, starting from the equality of the areas and the fact of the parallelograms or triangles being in the same parallels, and proving that the bases are the same or equal, may also be Heron's. Van Pesch further supposes that it was in Heron's commentary that the proof by Menelaus of I. 25 and the proof by Philo of I. 8 were given.

¹ Cf. Proclus, 275, γ ἐὶ δὲ λέγει τις τόπον μὴ εἶδέναι..., 289, 18 λέγει οὖν τις ὅτι οὐκ ἔστι τόπος....

² *De Procli fontibus*, Lugduni-Batavorum, 1900.

The last reference to Heron made by an-Nairizī occurs in the note on VIII. 27, so that the commentary of the former must at least have reached that point.

II. Porphyry.

The Porphyry here mentioned is of course the Neo-Platonist who lived about 232–304 A.D. Whether he really wrote a systematic commentary on the *Elements* is uncertain. The passages in Proclus which seem to make this probable are two in which he mentions him (1) as having demonstrated the necessity of the words “not on the same side” in the enunciation of I. 14¹, and (2) as having pointed out the necessity of understanding correctly the enunciation of I. 26, since, if the particular injunctions as to the sides of the triangles to be taken as equal are not regarded, the student may easily fall into error². These passages, showing that Porphyry carefully analysed Euclid’s *enunciations* in these cases, certainly suggest that his remarks were part of a systematic commentary. Further, the list of mathematicians in the *Fihrist* gives Porphyry as having written “a book on the Elements.” It is true that Wenrich takes this book to have been a work by Porphyry mentioned by Suidas and Proclus (*Theolog. Platon.*), *περὶ ἀρχῶν libri II.*³

There is nothing of importance in the notes attributed to Porphyry by Proclus.

(1) Three alternative proofs of I. 20, which avoid *producing* a side of the triangle, are assigned to Heron and Porphyry without saying which belonged to which. If the first of the three was Heron’s, I agree with van Pesch that it is more probable that the two others were both Porphyry’s than that the second was Heron’s and only the third Porphyry’s. For they are similar in character, and the third uses a result obtained in the second⁴.

(2) Porphyry gave an alternative proof of I. 18 to meet a childish objection which is supposed to require the part of AC equal to AB to be cut off from CA and not from AC .

Proclus gives a precisely similar alternative proof of I. 6 to meet a similar supposed objection; and it may well be that, though Proclus mentions no name, this proof was also Porphyry’s, as van Pesch suggests⁵.

Two other references to Porphyry found in Proclus cannot have anything to do with commentaries on the *Elements*. In the first a work called the *Συμμικτά* is quoted, while in the second a philosophical question is raised.

III. Pappus.

The references to Pappus in Proclus are not numerous; but we have other evidence that he wrote a commentary on the *Elements*. Thus a scholiast on the definitions of the *Data* uses the phrase “as

¹ Proclus, pp. 297, 1–298, 10.

² *ibid.* p. 352, 13, 14 and the pages preceding.

³ *Fihrist* (tr. Suter), p. 9, 10 and p. 45 (note 5).

⁴ Van Pesch, *De Procli fontibus*, pp. 129, 130. Heiberg assigned them as above in his *Euklid-Studien* (p. 160), but seems to have changed his view later. (See Besthorn-Heiberg, *Codex Leidensis*, p. 93, note 2.)

⁵ Van Pesch, *op. cit.* pp. 130–1.

Pappus says at the beginning of his (commentary) on the 10th (book) of Euclid¹. Again in the *Fihrist* we are told that Pappus wrote a commentary to the tenth book of Euclid in two parts². Fragments of this still survive in a MS. described by Woepcke³, Paris. No. 952. 2 (supplément arabe de la Bibliothèque impériale), which contains a translation by Abū 'Uthmān (beginning of 10th century) of a Greek commentary on Book X. It is in two books, and there can now be no doubt that the author of the Greek commentary was Pappus⁴. Again Eutocius, in his note on Archimedes, *On the Sphere and Cylinder* I. 13, says that Pappus explained in his commentary on the *Elements* how to inscribe in a circle a polygon similar to a polygon inscribed in another circle; and this would presumably come in his commentary on Book XII, just as the problem is solved in the second scholium on Eucl. XII. I. Thus Pappus' commentary on the *Elements* must have been pretty complete, an additional confirmation of this supposition being forthcoming in the reference of Marinus (a pupil and follower of Proclus) in his preface to the *Data* to "the commentaries of Pappus on the book⁵."

The actual references to Pappus in Proclus are as follows:

(1) On the Postulate (4) that all right angles are equal, Pappus is quoted as saying that the converse, viz. that all angles equal to a right angle are right, is not true⁶, since the angle included between the arcs of two semicircles which are equal, and have their diameters at right angles and terminating at one point, is equal to a right angle, but is not a right angle.

(2) On the axioms Pappus is quoted as saying that, in addition to Euclid's axioms, others are on record as well (*συναναγράφεσθαι*) about unequals added to equals and equals added to unequals⁷; these, says Proclus, follow from the Euclidean axioms, while others given by Pappus are involved by the definitions, namely those which assert that "all parts of the plane and of the straight line coincide with one another," that "a point divides a straight line, a line a surface, and a surface a solid," and that "the infinite is (obtained) in magnitudes both by addition and diminution⁸."

¹ Euclid's *Data*, ed. Menge, p. 262.

² *Fihrist* (tr. Suter), p. 22.

³ *Mémoires présentés à l'Académie des sciences*, 1856, XIV. pp. 658—719.

⁴ Woepcke read the name of the author, in the title of the first book, as *B. los* (the dot representing a missing vowel). He quotes also from other MSS. (e.g. of the *Ta'rikh al-Hukamā* and of the *Fihrist*) where he reads the name of the commentator as *B. lis*, *B. n. s* or *B. l. s*. Woepcke takes this author to be Valens, and thinks it possible that he may be the same as the astrologer Vettius Valens. This Heiberg (*Euklid-Studien*, pp. 169, 170) proves to be impossible, because, while one of the MSS. quoted by Woepcke says that "*B. n. s*, le *Roūmi*" (late-Greek) was later than Claudius Ptolemy and the *Fihrist* says "*B. l. s*, le *Roūmi*" wrote a commentary on Ptolemy's *Planisphaerium*, Vettius Valens seems to have lived under Hadrian, and must therefore have been an elder contemporary of Ptolemy. But Suter shows (*Fihrist*, p. 22 and p. 54, note 92) that *Banos* is only distinguished from *Babos* by the position of a certain dot, and *Balos* may also easily have arisen from an original *Babos* (there is no P in Arabic), so that Pappus must be the person meant. This is further confirmed by the fact that the *Fihrist* gives this author and Valens as the subjects of two separate paragraphs, attributing to the latter astrological works only.

⁵ Heiberg, *Euklid-Studien*, p. 173; Euclid's *Data*, ed. Menge, pp. 256, lii.

⁶ Proclus, pp. 189, 190.

⁷ *ibid.* p. 197, 6—10.

⁸ *ibid.* p. 198, 3—15.

(3) Pappus gave a pretty proof of I. 5. This proof has, I think, been wrongly understood; on this point see my note on the proposition.

(4) On I. 47 Proclus says¹: "As the proof of the writer of the Elements is manifest, I think that it is not necessary to add anything further, but that what has been said is sufficient, since indeed those who have added more, like Heron and Pappus, were obliged to make use of what is proved in the sixth book, without attaining any important result." We shall see what Heron's addition consisted of; what Pappus may have added we do not know, unless it was something on the lines of his extension of I. 47 found in the *Synagoge* (IV. p. 176, ed. Hultsch).

We may fairly conclude, with van Pesch², that Pappus is drawn upon in various other passages of Proclus where he quotes no authority, but where the subject-matter reminds us of other notes expressly assigned to Pappus or of what we otherwise know to have been favourite questions with him. Thus:

1. We are reminded of the curvilinear angle which is equal to but not a right angle by the note on I. 32 to the effect that the converse (that a figure with its interior angles together equal to two right angles is a triangle) is not true unless we confine ourselves to rectilinear figures. This statement is supported by reference to a figure formed by four semicircles whose diameters form a square, and one of which is turned inwards while the others are turned outwards. The figure forms two angles "equal to" right angles in the sense described by Pappus on *Pest.* 4, while the other curvilinear angles are not considered to be angles at all, and are left out in summing the internal angles. Similarly the allusions in the notes on I. 4, 23 to curvilinear angles of which certain moon-shaped angles (*μηννοειδείς*) are shown to be "equal to" rectilinear angles savour of Pappus.

2. On I. 9 Proclus says³ that "Others, starting from the Archimedean spirals, divided any given rectilinear angle in any given ratio." We cannot but compare this with Pappus IV. p. 286, where the spiral is so used; hence this note, including remarks immediately preceding about the conchoid and the quadratrix, which were used for the same purpose, may very well be due to Pappus.

3. The subject of isoperimetric figures was a favourite one with Pappus, who wrote a recension of Zenodorus' treatise on the subject⁴. Now on I. 35 Proclus speaks⁵ about the paradox of parallelograms having equal area (between the same parallels) though the two sides between the parallels may be of any length, adding that of parallelograms with equal perimeter the rectangle is greatest if the base be given, and the square greatest if the base be not given etc. He returns to the subject on I. 37 about triangles⁶. Compare⁷ also his note on I. 4. These notes may have been taken from Pappus.

¹ Proclus, p. 429, 9—15.

² Van Pesch, *De Procli fontibus*, p. 134 sqq.

³ Proclus, p. 272, 10.

⁴ Pappus, v. pp. 304—350; for Zenodorus' own treatise see Hultsch's Appendix, pp. 1189—1211.

⁵ Proclus, pp. 396—8.

⁶ *ibid.* pp. 403—4.

⁷ *ibid.* pp. 236—7.

4. Again, on I. 21, Proclus remarks on the paradox that straight lines may be drawn from the base to a point within a triangle which are (1) together greater than the two sides, and (2) include a less angle, provided that the straight lines may be drawn from points in the base other than its extremities. The subject of straight lines satisfying condition (1) was treated at length, with reference to a variety of cases, by Pappus¹, after a collection of "paradoxes" by Erycinus, of whom nothing more is known. Proclus gives Pappus' first case, and adds a rather useless proof of the possibility of drawing straight lines satisfying condition (2) *alone*, adding that "the proposition stated has been proved by me without using the parallels of the commentators". By "the commentators" Pappus is doubtless meant.

5. Lastly, the "four-sided triangle," called by Zenodorus the "hollow-angled,"² is mentioned in the notes on I. Def. 24—29 and I. 21. As Pappus wrote on Zenodorus' work in which the term occurred³, Pappus may be responsible for these notes.

IV. Simplicius.

According to the *Fihrist*⁴, Simplicius the Greek wrote "a commentary to the beginning of Euclid's book, which forms an introduction to geometry." And in fact this commentary on the definitions, postulates and axioms (including the postulate known as the Parallel-Axiom) is preserved in the Arabic commentary of an-Nairizi⁵. On two subjects this commentary of Simplicius quotes a certain "Aganis," the first subject being the definition of an angle, and the second the definition of parallels and the parallel-postulate. Simplicius gives word for word, in a long passage placed by an-Nairizi after I. 29, an attempt by "Aganis" to prove the parallel-postulate. It starts from a definition of parallels which agrees with Geminus' view of them as given by Proclus⁶, and is closely connected with the definition given by Posidonius⁷. Hence it has been assumed that "Aganis" is none other than Geminus, and the historical importance of the commentary of Simplicius has been judged accordingly. But it has been recently shown by Tannery that the identification of "Aganis" with Geminus is practically impossible⁸. In the translation of Besthorn-Heiberg Aganis is called by Simplicius in one place "philosophus Aganis," in another "magister noster Aganis," in Gherard's version he is "socius Aganis" and "socius noster Aganis." These expressions seem to leave no doubt that Aganis was a contemporary and friend, if not master, of Simplicius; and it is impossible to suppose that Simplicius (fl. about 500 A.D.) could have used them of a man who lived four and

¹ Pappus, III. pp. 104—130.

³ Proclus, p. 165, 24; cf. pp. 328, 329.

⁵ *Fihrist* (tr. Suter), p. 21.

² Proclus, p. 328, 15.

⁴ See Pappus, ed. Hultsch, pp. 1154, 1206.

⁶ An-Nairizi, ed. Besthorn-Heiberg, pp. 9—41, 119—133, ed. Curtze, pp. 1—37, 65—73. The *Codex Leidensis*, from which Besthorn and Heiberg's edition is taken, has unfortunately lost some leaves, so that there is a gap from Def. 1 to Def. 23 (parallels). The loss is, however, made good by Curtze's edition of the translation by Gherard of Cremona.

⁷ Proclus, p. 177, 21.

⁸ *ibid.* p. 176, 7.

⁹ *Bibliotheca Mathematica*, 113, 1900, pp. 9—11.

a half centuries before his time. A phrase in Simplicius' word-for-word quotation from Aganis leads to the same conclusion. He speaks of people who objected "even in ancient times" (*iam antiquitus*) to the use by geometers of this postulate. This would not have been an appropriate phrase had Geminus been the writer. I do not think that this difficulty can be got over by Suter's suggestion¹ that the passages in question may have been taken out of *Heron's* commentary, and that an-Nairīzī may have forgotten to name the author; it seems clear that Simplicius is the person who described "Aganis." Hence we are driven to suppose that Aganis was not Geminus, but some unknown contemporary of Simplicius². Considerable interest will however continue to attach to the comments of Simplicius so fortunately preserved.

Proclus tells us that one Aegaeas (? Aenaeas) of Hierapolis wrote an epitome of the *Elements*³; but we know nothing more of him or of it.

¹ *Zeitschrift für Math. u. Physik*, XLIV., hist.-litt. Abth. p. 61.

² The above argument seems to me quite insuperable. The other arguments of Tannery do not, however, carry conviction to my mind. I do not follow the reasoning based on Aganis' definition of an angle. It appears to me a pure assumption that Geminus would have seen that Posidonius' definition of parallels was not admissible. Nor does it seem to me to count for much that Proclus, while telling us that Geminus held that the postulate ought to be proved and warned the unwary against hastily concluding that two straight lines approaching one another must necessarily meet (cf. a curve and its asymptote), gives no hint that Geminus did try to prove the postulate. It may well be that Proclus omitted Geminus' "proof" (if he wrote one) because he preferred Ptolemy's attempt which he gives (pp. 365—7).

³ Proclus, p. 361, 21.

CHAPTER IV.

PROCLUS AND HIS SOURCES¹.

IT is well known that the commentary of Proclus on Eucl. Book I. is one of the two main sources of information as to the history of Greek geometry which we possess, the other being the *Collection* of Pappus. They are the more precious because the original works of the forerunners of Euclid, Archimedes and Apollonius are lost, having probably been discarded and forgotten almost immediately after the appearance of the masterpieces of that great trio.

Proclus himself lived 410–485 A.D., so that there had already passed a sufficient amount of time for the tradition relating to the pre-Euclidean geometers to become obscure and defective. In this connexion a passage is quoted from Simplicius² who, in his account of the quadrature of certain lunes by Hippocrates of Chios, while mentioning two authorities for his statements, Alexander Aphrodisiensis (about 220 A.D.) and Eudemus, says in one place³, “As regards Hippocrates of Chios we must pay more attention to Eudemus, *since he was nearer the times*, being a pupil of Aristotle.”

The importance therefore of a critical examination of Proclus' commentary with a view to determining from what original sources he drew need not be further emphasised.

Proclus received his early training in Alexandria, where Olympiodorus was his instructor in the works of Aristotle, and mathematics was taught him by one Heron⁴ (of course a different Heron from the “*mechanicus* Hero” of whom we have already spoken). He afterwards went to Athens where he was imbued by Plutarch, and by Syrianus, with the Neo-Platonic philosophy, to which he then devoted

¹ My task in this chapter is made easy by the appearance, in the nick of time, of the dissertation *De Procli fontibus* by J. G. van Pesch (Lugduni-Batavorum, Apud L. van Nifterik, MDCCC). The chapters dealing directly with the subject show a thorough acquaintance on the part of the author with all the literature bearing on it; he covers the whole field and he exercises a sound and sober judgment in forming his conclusions. The same cannot always be said of his only predecessor in the same inquiry, Tannery (in *La Géométrie grecque*, 1887), who often robs his speculations of much of their value through his proneness to run away with an idea; he does so in this case, basing most of his conclusions on an arbitrary and unwarranted assumption as to the significance of the words *oi περί τινα* (e.g. “*Ἡρώνα, Ποσειδώνιον* etc.) as used in Proclus.

² Simplicius on Aristotle's *Physics*, ed. Diels, pp. 54–69.

³ *ibid.* p. 68, 32.

⁴ Cf. Martin, *Recherches sur la vie et les ouvrages d'Héron d'Alexandrie*, pp. 240–2.

heart and soul, becoming one of its most prominent exponents. He speaks everywhere with the highest respect of his masters, and was in turn regarded with extravagant veneration by his contemporaries, as we learn from Marinus his pupil and biographer. On the death of Syrianus he was put at the head of the Neo-Platonic school. He was a man of untiring industry, as is shown by the number of books which he wrote, including a large number of commentaries, mostly on the dialogues of Plato. He was an acute dialectician, and pre-eminent among his contemporaries in the range of his learning¹; he was a competent mathematician; he was even a poet. At the same time he was a believer in all sorts of myths and mysteries and a devout worshipper of divinities both Greek and Oriental.

Though he was a competent mathematician, he was evidently much more a philosopher than a mathematician². This is shown even in his commentary on Eucl. I., where, not only in the Prologues (especially the first), but also in the notes themselves, he seizes any opportunity for a philosophical digression. He says himself that he attaches most importance to "the things which require deeper study and contribute to the sum of philosophy³"; alternative proofs, *cases*, and the like (though he gives many) have no attraction for him; and, in particular, he attaches no value to the addition of Heron to I. 47⁴, which is of considerable mathematical interest. Though he esteemed mathematics highly, it was only as a handmaid to philosophy. He quotes Plato's opinion to the effect that "mathematics, as making use of hypotheses, falls short of the non-hypothetical and perfect science⁵"... "Let us then not say that Plato excludes mathematics from the sciences, but that he declares it to be secondary to the one supreme science⁶." And again, while "mathematical science must be considered desirable in itself, though not with reference to the needs of daily life," "if it is necessary to refer the benefit arising from it to something else, we must connect that benefit with intellectual knowledge (*νοερὰν γνῶσιν*), to which it leads the way and is a propaedeutic, clearing the eye of the soul and taking away the impediments which the senses place in the way of the knowledge of universals (*τῶν ὁλῶν*)⁷."

We know that in the Neo-Platonic school the younger pupils learnt mathematics; and it is clear that Proclus taught this subject, and that this was the origin of the commentary. Many passages show him as a master speaking to scholars. Thus "we have illustrated

¹ Zeller calls him "Der Gelehrte, dem kein Feld damaligen Wissens verschlossen ist."

² Van Pesch observes that in his commentaries on the *Timaeus* (pp. 671—2) he speaks as no real mathematician could have spoken. In the passage referred to the question is whether the sun occupies a middle place among the planets. Proclus rejects the view of Hipparchus and Ptolemy because "*ὁ θεουργός*" (sc. the Chaldean, says Zeller) thinks otherwise, "whom it is not lawful to disbelieve." Martin says rather neatly, "Pour Proclus, les Éléments d'Euclide ont l'heureuse chance de n'être contredits ni par les Oracles chaldaïques, ni par les spéculations des pythagoriciens anciens et nouveaux....."

³ Proclus, p. 84, 13.

⁴ *ibid.* p. 429, 12.

⁵ *ibid.* p. 31, 20.

⁶ *ibid.* p. 32, 2.

⁷ *ibid.* p. 27, 27 to 28, 7; cf. also p. 21, 25, pp. 46, 47.

and made plain all these things in the case of the first problem, but it is necessary that *my hearers* should make the same inquiry as regards the others as well¹, and "I do not indicate these things as a merely incidental matter but as preparing *us* beforehand for the doctrine of the *Timaeus*?" Further, the pupils whom he was addressing were *beginners* in mathematics; for in one place he says that he omits "for the present" to speak of the discoveries of those who employed the curves of Nicomedes and Hippias for trisecting an angle, and of those who used the Archimedean spiral for dividing an angle in any given ratio, because these things would be too difficult for beginners (*δυσθεωρήτους τοῖς εἰσαγόμενοις*)². Again, if his pupils had not been beginners, it would not have been necessary for Proclus to explain what is meant by saying that sides subtend certain angles³, the difference between *adjacent* and *vertical* angles⁴ etc., or to exhort them, as he often does, to work out other particular cases for themselves, for practice (*γυμνασίας ἕνεκα*)⁵.

The commentary seems then to have been founded on Proclus' lectures to beginners in mathematics. But there are signs that it was revised and re-edited for a larger public; thus he gives notice in one place⁷ "to those who shall come upon" his work (*τοῖς ἐντευξόμενοις*). There are also passages which could not have been understood by the beginners to whom he lectured, e.g. passages about the cylindrical helix⁸, conchoids and cissoids⁹. These passages may have been added in the revised edition, or, as van Pesch conjectures, the explanations given in the lectures may have been much fuller and more comprehensible to beginners, and they may have been shortened on revision.

In his comments on the propositions of Euclid, Proclus generally proceeds in this way: first he gives explanations regarding Euclid's proofs, secondly he gives a few different cases, mainly for the sake of practice, and thirdly he addresses himself to refuting objections raised by cavillers to particular propositions. The latter class of note he deems necessary because of "sophistical cavils" and the attitude of the people who rejoiced in finding paralogisms and in causing annoyance to scientific men¹⁰. His commentary does not seem to have been written for the purpose of correcting or improving Euclid. For there are very few passages of mathematical content in which Proclus can be supposed to be propounding anything of his own; nearly all are taken from the works of others, mostly earlier commentators, so that, for the purpose of improving on or correcting Euclid, there was no need for his commentary at all. Indeed only in one place does he definitely bring forward anything of his own to get over a difficulty which he finds in Euclid¹¹; this is where he tries to

¹ Proclus, p. 210, 18.

² *ibid.* p. 272, 12.

³ *ibid.* p. 298, 14.

⁴ *ibid.* p. 84, 9.

⁵ *ibid.* p. 113.

⁶ *ibid.* pp. 368—373.

⁷ *ibid.* p. 384, 2.

⁸ *ibid.* p. 238, 12.

⁹ Cf. p. 224, 15 (on 1. 2).

¹⁰ *ibid.* p. 105.

¹¹ *ibid.* p. 375, 8.

prove the parallel-postulate, after first giving Ptolemy's attempt and then pointing out objections to it. On the other hand, there are a number of passages in which he extols Euclid; thrice¹ also he supports Euclid against Apollonius where the latter had given proofs which he considered better than Euclid's (I. 10, 11, and 23).

Allusion must be made to the debated question whether Proclus continued his commentaries beyond Book I. His intention to do so is clear from the following passages. Just after the words above quoted about the trisection etc. of an angle by means of certain curves he says, "For we may perhaps more appropriately examine these things on the third book, where the writer of the Elements bisects a given circumference²." Again, after saying that of all parallelograms which have the same perimeter the square is the greatest "and the rhomboid least of all," he adds: "But this we will prove in another place; for it is more appropriate to the (discussion of the) hypotheses of the second book³." Lastly, when alluding (on I. 45) to the squaring of the circle, and to Archimedes' proposition that any circle is equal to the right-angled triangle in which the perpendicular is equal to the radius of the circle and the base to its perimeter, he adds, "But of this elsewhere⁴"; this may imply an intention to treat of the subject on Eucl. XII., though Heiberg doubts it⁵. But it is clear that, at the time when the commentary on Book I. was written, Proclus had not yet begun to write on the other Books and was uncertain whether he would be able to do so: for at the end he says⁶, "For my part, if I should be able to discuss the other books⁷ in the same manner, I should give thanks to the gods; but, if other cares should draw me away, I beg those who are attracted by this subject to complete the exposition of the other books as well, following the same method, and addressing themselves throughout to the deeper and better defined questions involved" (*τὸ πραγματειῶδες πανταχοῦ καὶ εὐδιαίρετον μεταδιώκοντας*).

There is in fact no satisfactory evidence that Proclus did actually write any more commentaries than that on Book I.⁸ The contrary view receives support from two facts pointed out by Heiberg, viz. (1) that the scholiast's copy of Proclus was not so much better than our

¹ Proclus, p. 280, 9; p. 282, 20; pp. 335, 336.

² *ibid.* p. 398, 18.

³ Heiberg, *Euklid-Studien*, p. 165, note.

⁴ *ibid.* p. 272, 14.

⁵ *ibid.* p. 423, 6.

⁶ Proclus, p. 432, 9.

⁷ The words in the Greek are: *εἰ μὲν δυναθῆμεν καὶ τοῖς λοιποῖς τὸν αὐτὸν τρόπον ἐξελεῖν*. For *ἐξελεῖν* Heiberg would read *ἐπεξελεῖν*.

⁸ True, a Vatican MS. has a collection of scholia on Books I. (extracts from the extant commentary of Proclus), II., V., VI., X. headed *Εἰς τὰ Εὐκλείδου στοιχεῖα προλαμβανόμενα ἐκ τῶν Πρόκλου σποράδην καὶ κατ' ἐπιτομήν*. Heiberg holds that this title itself suggests that the authorship of Proclus was limited to the scholia on Book I.; for *προλαμβανόμενα ἐκ τῶν Πρόκλου* suits extracts from Proclus' *prologues*, but hardly scholia to later Books. Again, a certain scholium (Heiberg in *Hermes* XXXVIII., 1903, p. 341, No. 17) purports to quote words from the end of "a scholium of Proclus" on X. 9. The words quoted are from the scholium X. No. 62, one of the Scholia Vaticana. But none of the other, older, sources connect Proclus' name with X. No. 62; it is probable therefore that a Byzantine, who had in his MS. of Euclid the collection of Schol. Vat. and knew that those on Book I. came from Proclus, himself attached Proclus' name to the others.

MSS. as to suggest that the scholiast had further commentaries of Proclus which have vanished for us¹; (2) that there is no trace in the scholia of the notes which Proclus promised in the passages quoted above.

Coming now to the question of the sources of Proclus, we may say that everything goes to show that his commentary is a compilation, though a compilation "in the better sense" of the term². He does not even give us to understand that we shall find in it much of his own; "let us," he says, "now turn to the exposition of the theorems proved by Euclid, selecting the more subtle of the comments made on them by the ancient writers, and cutting down their interminable diffuseness...³": not a word about anything of his own. At the same time, he seems to imply that he will not necessarily on each occasion quote the source of each extract from an earlier commentary; and, in fact, while he quotes the name of his authority in many places, especially where the subject is important, in many others, where it is equally certain that he is not giving anything of his own, he mentions no authority. Thus he quotes Heron by name six times; but we now know, from the commentary of an-Nairizî, that a number of other passages, where he mentions no name, are taken from Heron, and among them the not unimportant addition of an alternative proof to I. 19. Hence we can by no means conclude that, where no authority is mentioned, Proclus is giving notes of his own. The presumption is generally the other way; and it is often possible to arrive at a conclusion, either that a particular note is not Proclus' own, or that it is definitely attributable to someone else, by applying the ordinary principles of criticism. Thus, where the note shows an unmistakable affinity to another which Proclus definitely attributes to some commentator by name, especially when both contain some peculiar and distinctive idea, we cannot have much doubt in assigning both to the same commentator⁴. Again, van Pesch finds a criterion in the *form* of a note, where the explanation is so condensed as to be only just intelligible; the note is that in which a converse of I. 32 is proved⁵, the proposition namely that a rectilineal figure which has all its interior angles together equal to two right angles is a triangle.

It is not safe to attribute a passage to Proclus himself because he uses the first person in such expressions as "I say" or "I will prove"—for he was in the habit of putting into his own words the substance of notes borrowed from others—nor because, in speaking of an

¹ While one class of scholia (Schol. Vat.) have some better readings than our MSS. of Proclus have, and partly fill up the gaps at I. 36, 37 and I. 41—43, the other class (Schol. Vind.) derive from an inferior Proclus MS. which also had the same lacunae.

² Knoche, *Untersuchungen über des Proklus Diadochus Commentar zu Euklid's Elementen* (1862), p. 11.

³ Proclus, p. 200, 10—13.

⁴ Instances of the application of this criterion will be found in the discussion of Proclus' indebtedness to the commentaries of Heron, Porphyry and Pappus.

⁵ Van Pesch attributes this converse and proof to Pappus, arguing from the fact that the proof is followed by a passage which, on comparison with Pappus' note on the postulate that all right angles are equal, he feels justified in assigning to Pappus. I doubt if the evidence is sufficient.

objection raised to a particular proposition, he uses such expressions as "perhaps someone may object" (ἴσως δ' ἂν τινες ἐνσταλῖεν...): for sometimes other words in the same passage indicate that the objection had actually been taken by someone¹. Speaking generally, we shall not be justified in concluding that Proclus is stating something new of his own unless he indicates this himself in express terms.

As regards the form of Proclus' references to others by name, van Pesch notes that he very seldom mentions the particular *work* from which he is borrowing. If we leave out of account the references to Plato's dialogues, there are only the following references to books: the *Bacchae* of Philolaus², the *Symmikta* of Porphyry³, Archimedes *On the Sphere and Cylinder*⁴, Apollonius *On the cochlias*⁵, a book by Eudemus on *The Angle*⁶, a whole book of Posidonius directed against Zeno of the Epicurean sect⁷, Carpus' *Astronomy*⁸, Eudemus' *History of Geometry*⁹, and a tract by Ptolemy on the parallel-postulate¹⁰.

Again, Proclus does not always indicate that he is quoting something at second-hand. He often does so, e.g. he quotes Heron as the authority for a statement about Philippus, Eudemus as attributing a certain theorem to Oenopides etc.; but he says on I. 12 that "Oenopides first investigated this problem, thinking it useful for astronomy" when he cannot have had Oenopides' work before him.

It has been said above that Proclus was in the habit of stating in his own words the substance of the things which he borrowed. We are prepared for this when we find him stating that he will select the best things from ancient commentaries and "cut short their interminable diffuseness," that he will "briefly describe" (συντομως ἱστορήσαι) the other proofs of I. 20 given by Heron and Porphyry and also the proofs of I. 25 by Menelaus and Heron. But the best evidence is of course to be found in the passages where he quotes works still extant, e.g. those of Plato, Aristotle and Plotinus. Examination of these passages shows great divergences from the original; even where he purports to quote textually, using the expressions "Plato says," or "Plotinus says," he by no means quotes word for word¹¹. In fact, he seems to have had a positive distaste for quoting textually from other works. He cannot conquer this even when quoting from Euclid; he says in his note on I. 22, "we will follow the words of the geometer" but fails, nevertheless, to reproduce the text of Euclid unchanged¹².

We now come to the sources themselves from which Proclus drew

¹ Van Pesch illustrates this by an objection refuted in the note on I. 9, p. 273, 11 sqq. After using the above expression to introduce the objection, Proclus uses further on (p. 273, 25) the term "they say" (φασιν).

² Proclus, p. 22, 15.

⁴ *ibid.* p. 71, 18.

⁶ *ibid.* p. 125, 8.

⁸ *ibid.* p. 241, 19.

¹⁰ *ibid.* p. 362, 15.

³ *ibid.* p. 56, 25.

⁵ *ibid.* p. 105, 5.

⁷ *ibid.* p. 200, 2.

⁹ *ibid.* p. 352, 15.

¹¹ See the passages referred to by van Pesch (p. 70). The most glaring case is a passage (p. 21, 19) where he quotes Plotinus, using the expression "Plotinus says....." Comparison with Plotinus, *Ennead.* I. 3, 3, shows that *very few* words are those of Plotinus himself; the rest represent Plotinus' views in Proclus' own language.

¹² Proclus, p. 330, 19 sqq

in writing his commentary. Three have already been disposed of, viz. Heron, Porphyry and Pappus, who had all written commentaries on the *Elements*¹. We go on to

Eudemus, the pupil of Aristotle, who, among other works, wrote a history of arithmetic, a history of astronomy, and a history of geometry. The importance of the last mentioned work is attested by the frequent use made of it by ancient writers. That there was no other history of geometry written after the time of Eudemus seems to be proved by the remark of Proclus in the course of his famous summary: "Those who compiled histories bring the development of this science up to this point. *Not much younger than these is Euclid*..." The loss of Eudemus' history is one of the gravest which fate has inflicted upon us, for it cannot be doubted that Eudemus had before him a number of the actual works of earlier geometers, which, as before observed, seem to have vanished completely when they were superseded by the treatises of Euclid, Archimedes and Apollonius. As it is, we have to be thankful for the fragments from Eudemus which such writers as Proclus have preserved to us.

I agree with van Pesch² that there is no sufficient reason for doubting that the work of Eudemus was accessible to Proclus at first hand. For the later writers Simplicius and Eutocius refer to it in terms such as leave no room for doubt that *they* had it before them. I have already quoted a passage from Simplicius' account of the lunes of Hippocrates to the effect that Eudemus must be considered the best authority since he lived nearer the times³. In the same place Simplicius says⁴, "I will set out what Eudemus says word for word (*κατὰ λέξιν λεγόμενα*), adding only a little explanation in the shape of reference to Euclid's *Elements* owing to the memorandum-like style of Eudemus (*διὰ τὸν ὑπομνηματικὸν τρόπον τοῦ Εὐδήμου*) who sets out his explanations in the abbreviated form usual with ancient writers. Now in the second book of the history of geometry he writes as follows⁵." It is not possible to suppose that Simplicius would have written in this way about the style of Eudemus if he had merely been copying certain passages second-hand out of some other author and had not the original work itself to refer to. In like manner, Eutocius speaks of the paralogisms handed down in connexion with the attempts of Hippocrates and Antiphon to square the circle⁶, "with which I imagine that those are accurately acquainted who have examined (*ἐπεσκεμμένους*) the geometrical history of Eudemus and know the *Ceria Aristotelica*." How could the contemporaries of Eutocius have *examined* the work of Eudemus unless it was still extant in his time?

The passages in which Proclus quotes Eudemus by name as his authority are as follows:

(1) On I. 26 he says that Eudemus in his history of geometry

¹ See pp. 20 to 27 above.

² Proclus, p. 68, 4—7.

³ See above, p. 29.

⁴ Archimedes, ed. Heiberg, vol. III. p. 228.

⁵ *De Procli fontibus*, pp. 73—75.

⁶ Simplicius, *loc. cit.*, ed. Diels, p. 60, 27.

referred this theorem to Thales, inasmuch as it was necessary to Thales' method of ascertaining the distance of ships from the shore¹.

(2) Eudemus attributed to Thales the discovery of Eucl. I. 15², and

(3) to Oenopides the problem of I. 23³.

(4) Eudemus referred the discovery of the theorem in I. 32 to the Pythagoreans, and gave their proof of it, which Proclus reproduces⁴.

(5) On I. 44 Proclus tells us⁵ that Eudemus says that "these things are ancient, being discoveries of the Pythagorean muse, the application (*παραβολή*) of areas, their exceeding (*ὑπερβολή*) and their falling short (*ἔλλειψις*)." The next words about the appropriation of these terms (parabola, hyperbola and ellipse) by later writers (i.e. Apollonius) to denote the conic sections are of course not due to Eudemus.

Coming now to notes where Eudemus is not named by Proclus, we may fairly conjecture, with van Pesch, that Eudemus was really the authority for the statements (1) that Thales first proved that a circle is bisected by its diameter⁶ (though the proof by *reductio ad absurdum* which follows in Proclus cannot be attributed to Thales⁷), (2) that "Plato made over to Leodamas the analytical method, by means of which *it is recorded* (*ιστόρηται*) that the latter too made many discoveries in geometry⁸," (3) that the theorem of I. 5 was due to Thales, and that for equal angles he used the more archaic expression "similar" angles⁹, (4) that Oenopides first investigated the problem of I. 12, and that he called the perpendicular the *gnomonic* line (*κατὰ γνώμονα*)¹⁰, (5) that the theorem that only three sorts of polygons can fill up the space round a point, viz. the equilateral triangle, the square and the regular hexagon, was Pythagorean¹¹. Eudemus may also be the authority for Proclus' description of the two methods, referred to Plato and Pythagoras respectively, of forming right-angled triangles in whole numbers¹².

We cannot attribute to Eudemus the beginning of the note on I. 47 where Proclus says that "if we listen to those who like to recount ancient history, we may find some of them referring this theorem to Pythagoras and saying that he sacrificed an ox in honour of his discovery¹³." As such a sacrifice was contrary to the Pythagorean tenets, and Eudemus could not have been unaware of this, the story cannot rest on his authority. Moreover Proclus speaks as though he were not certain of the correctness of the tradition; indeed,

¹ Proclus, p. 352, 14—18.

³ *ibid.* p. 333, 5.

⁵ *ibid.* p. 419, 15—18.

⁷ Cantor (*Gesch. d. Math.* 13, p. 221) points out the connexion between the *reductio ad absurdum* and the analytical method said to have been discovered by Plato. Proclus gives the proof by *reductio ad absurdum* to meet an imaginary critic who desires a *mathematical* proof; possibly Thales may have been satisfied with the argument in the same sentence which mentions Thales, "the cause of the bisection being the unswerving course of the straight line through the centre."

⁸ Proclus, p. 211, 19—23.

¹⁰ *ibid.* p. 283, 7—10.

¹² *ibid.* pp. 428, 7—429, 9.

² *ibid.* p. 299, 3.

⁴ *ibid.* p. 379, 1—16.

⁶ *ibid.* p. 157, 10, 11.

⁹ *ibid.* p. 250, 20.

¹¹ *ibid.* pp. 304, 11—305, 3.

¹³ *ibid.* p. 426, 6—9.

so far as the story of the sacrifice is concerned, the same thing is told of Thales in connexion with his discovery that the angle in a semi-circle is a right angle¹, and Plutarch is not certain whether the ox was sacrificed on the discovery of I. 47 or of the problem about application of areas². Plutarch's doubt suggests that he knew of no evidence for the story beyond the vague allusion in the distich of Apollodorus "Logisticus" (the "calculator") cited by Diogenes Laertius also³; and Proclus may have had in mind this couplet with the passages of Plutarch.

We come now to the question of the famous historical summary given by Proclus⁴. No one appears to maintain that Eudemus is the author of even the early part of this summary in the form in which Proclus gives it. It is, as is well known, divided into two distinct parts, between which comes the remark, "Those who compiled histories⁵ bring the development of this science up to this point. Not much younger than these is Euclid, who put together the *Elements*, collecting many of the theorems of Eudoxus, perfecting many others by Theaetetus, and bringing to irrefragable demonstration the things which had only been somewhat loosely proved by his predecessors." Since Euclid was later than Eudemus, it is impossible that Eudemus can have written this. Yet the style of the summary after this point does not show any such change from that of the former portion as to suggest different authorship. The author of the earlier portion recurs frequently to the question of the origin of the *elements* of geometry in a way in which no one would be likely to do who was not later than Euclid; and it must be the same hand which in the second portion connects Euclid's *Elements* with the work of Eudoxus and Theaetetus⁶.

If then the summary is the work of one author, and that author not Eudemus, who is it likely to have been? Tannery answers that it is Geminus⁷; but I think, with van Pesch, that he has failed to show why it should be Geminus rather than another. And certainly the extracts which we have from Geminus' work suggest that the sort of topics which it dealt with was quite different; they seem rather to have been general questions of the *content* of mathematics, and even Tannery admits that historical details could only have come incidentally into the work⁸.

Could the author have been Proclus himself? Circumstances

¹ Diogenes Laertius, I. 24, p. 6, ed. Cobet.

² Plutarch, *non posse suaviter vivi secundum Epicurum*, II; *Symp.* VIII, 2.

³ Diog. Laert. VIII. 12, p. 207, ed. Cobet:

Ἡνίκα Πυθαγόρης τὸ περικλεές εὗρετο γράμμα,
κεῖν' ἐφ' ὅτῳ κλεινὴν ἤγαγε βουβυστήν.

See on this subject Tannery, *La Géométrie grecque*, p. 105.

⁴ Proclus, pp. 64—70.

⁵ The plural is well explained by Tannery, *La Géométrie grecque*, pp. 73, 74. No doubt the author of the summary tried to supplement Eudemus by means of any other histories which threw light on the subject. Thus e.g. the allusion (p. 64, 21) to the Nile recalls Herodotus. Cf. the expression in Proclus, p. 64, 19, *παρὰ τῶν πολλῶν ιστόρηται*.

⁶ Tannery, *La Géométrie grecque*, p. 75.

⁷ *ibid.* pp. 66—75.

⁸ *ibid.* p. 19.

which seem to suggest this possibility are (1) that, as already stated, the question of the origin of the *Elements* is kept prominent, (2) that there is no mention of Democritus, whom Eudemus would not be likely to have ignored, while a follower of Plato would be likely enough to do him the injustice, following the example of Plato who was an opponent of Democritus, never once mentions him, and is said to have wished to burn all his writings¹, and (3) the allusion at the beginning to the "inspired Aristotle" (ὁ δαιμονιος Ἀριστοτέλης)², though this may easily have been inserted by Proclus in a quotation made by him from someone else. On the other hand there are considerations which suggest that Proclus himself was *not* the writer. (1) The style of the whole passage is not such as to point to him as the author. (2) If he wrote it, it is hardly conceivable that he would have passed over in silence the discovery of the analytical method, the invention of Plato to which he attached so much importance³.

There is nothing improbable in the conjecture that Proclus quoted the summary from a compendium of Eudemus' history made by some later writer: but as yet the question has not been definitely settled. All that is certain is that the early part of the summary must have been made up from scattered notices found in the great work of Eudemus.

Proclus refers to another work of Eudemus besides the history, viz. a book on *The Angle* (βιβλίου περὶ γωνίας)⁴. Tannery assumes that this must have been part of the history, and uses this assumption to confirm his idea that the history was arranged according to *subjects*, not according to chronological order⁵. The phraseology of Proclus however unmistakably suggests a separate work; and that the history was *chronologically* arranged seems to be clearly indicated by the remark of Simplicius that Eudemus "also counted Hippocrates among the more ancient writers" (ἐν τοῖς παλαιότεροις)⁶.

The passage of Simplicius about the lunes of Hippocrates throws considerable light on the style of Eudemus' history. Eudemus wrote in a memorandum-like or summary manner (τὸν ὑπομνηματικὸν τρόπον τοῦ Εὐδήμου)⁷ when reproducing what he found in the ancient writers; sometimes it is clear that he left out altogether proofs or constructions of things by no means easy⁸.

Geminus.

The discussions about the date and birthplace of Geminus form a whole literature, as to which I must refer the reader to Manitius and Tittel⁹. Though the name looks like a Latin name (Geminus), Mani-

¹ Diog. Laertius, IX. 40, p. 237, ed. Cobet.

² Proclus, p. 64, 8.

³ Proclus, p. 211, 19 sqq.; the passage is quoted above, p. 36.

⁴ *ibid.* p. 125, 8.

⁵ Tannery, *La Géométrie grecque*, p. 26.

⁶ Simplicius, ed. Diels, p. 69, 23.

⁷ *ibid.* p. 60, 29.

⁸ Cf. Simplicius, p. 63, 19 sqq.; p. 64, 25 sqq.; also Usener's note "de supplendis Hippocratis quas omisit Eudemus constructionibus" added to Diels' preface, pp. xxiii—xxvi.

⁹ Manitius, *Gemini elementa astronomiae* (Teubner, 1898), pp. 237—252; Tittel, art. "Geminus" in Pauly-Wissowa's *Real-Encyclopädie der classischen Altertumswissenschaft*, vol. VII., 1910.

tius concluded that, since it appears as Γεμῖνος in all Greek MSS. and as Γεμείνος in some inscriptions, it is Greek and possibly formed from γεμ as Ἐργῖνος is from ἐργ and Ἀλεξῖνος from ἀλεξ (cf. also Ἰκτῖνος, Κρατῖνος). Tittel is equally positive that it is Geminus and suggests that Γεμῖνος is due to a false analogy with Ἀλεξῖνος etc. and Γεμείνος wrongly formed on the model of Ἀντωνεῖνος, Ἀγριππείνα. Geminus, a Stoic philosopher, born probably in the island of Rhodes, was the author of a comprehensive work on the classification of mathematics, and also wrote, about 73–67 B.C., a not less comprehensive commentary on the meteorological textbook of his teacher Posidonius of Rhodes.

It is the former work in which we are specially interested here. Though Proclus made great use of it, he does not mention its title, unless we may suppose that, in the passage (p. 177, 24) where, after quoting from Geminus a classification of lines which never meet, he says, "these remarks I have selected from the φιλοκαλία of Geminus," φιλοκαλία is a title or an alternative title. Pappus however quotes a work of Geminus "on the classification of the mathematics" (ἐν τῷ περὶ τῆς τῶν μαθημάτων τάξεως)¹, while Eutocius quotes from "the sixth book of the doctrine of the mathematics" (ἐν τῷ ἕκτῳ τῆς τῶν μαθημάτων θεωρίας)². Tannery³ pointed out that the former title corresponds well enough to the long extract⁴ which Proclus gives in his first prologue, and also to the fragments contained in the *Anonymi variae collectiones* published by Hultsch at the end of his edition of Heron⁵; but it does not suit most of the other passages borrowed by Proclus. The correct title was therefore probably that given by Eutocius, *The Doctrine, or Theory, of the Mathematics*; and Pappus probably refers to one particular portion of the work, say the first Book. If the sixth Book treated of conics, as we may conclude from Eutocius, there must have been more Books to follow, because Proclus has preserved us details about higher curves, which must have come later. If again Geminus finished his work and wrote with the same fulness about the other branches of mathematics as he did about geometry, there must have been a considerable number of Books altogether. At all events it seems to have been designed to give a complete view of the whole science of mathematics, and in fact to be a sort of encyclopaedia of the subject.

I shall now indicate first the certain, and secondly the probable, obligations of Proclus to Geminus, in which task I have only to follow van Pesch, who has embodied the results of Tittel's similar inquiry also⁶. I shall only omit the passages as regards which a case for attributing them to Geminus does not seem to me to have been made out.

First come the following passages which must be attributed to Geminus, because Proclus mentions his name:

(1) (In the first prologue of Proclus⁷) on the division of mathe-

¹ Pappus, ed. Hultsch, p. 1026, 9.

² Apollonius, ed. Heiberg, vol. II. p. 170.

³ Tannery, *La Géométrie grecque*, pp. 18, 19.

⁴ Proclus, pp. 38, 1–42, 8.

⁵ Heron, ed. Hultsch, pp. 246, 16–249, 12.

⁶ Van Pesch, *De Procli fontibus*, pp. 97–113. The dissertation of Tittel is entitled *De Geminii Stoici studii mathematicis* (1895).

⁷ Proclus, pp. 38, 1–42, 8, except the allusion in p. 41, 8–10, to Ctesibius and Heron and

mathematical sciences into arithmetic, geometry, mechanics, astronomy, optics, geodesy, canonic (science of musical harmony), and logistic (apparently arithmetical problems);

(2) (in the note on the definition of a straight line) on the classification of lines (including curves) as simple (straight or circular) and mixed, composite and incomposite, uniform (*ὁμοιομερεῖς*) and non-uniform (*ἀνομοιομερεῖς*), lines "about solids" and lines produced by cutting solids, including conic and spiric sections¹;

(3) (in the note on the definition of a plane surface) on similar distinctions extended to surfaces and solids²;

(4) (in the note on the definition of parallels) on lines which *do not meet* (*ἀσύμπτωτοι*) but which are not on that account parallel, e.g. a curve and its asymptote, showing that the property of *not meeting* does not make lines parallel—a favourite observation of Geminus—and, incidentally, on *bounded* lines or those which *enclose a figure* and those which *do not*³;

(5) (in the same note) the definition of parallels given by Posidonius⁴;

(6) on the distinction between postulates and axioms, the futility of trying to prove axioms, as Apollonius tried to prove Axiom I, and the equal incorrectness of assuming what really requires proof, "as Euclid did in the fourth postulate [equality of right angles] and in the fifth postulate [the parallel-postulate]"⁵;

(7) on Postulates I, 2, 3, which Geminus makes depend on the idea of a straight line being described by the motion of a point⁶;

(8) (in the note on Postulate 5) on the inadmissibility in geometry of an argument which is merely plausible, and the danger in this particular case owing to the existence of lines which do converge *ad infinitum* and yet never meet⁷;

(9) (in the note on I. 1) on the subject-matter of geometry, theorems, problems and *διορισμοί* (conditions of possibility) for problems⁸;

(10) (in the note on I. 5) on a generalisation of I. 5 by Geminus through the substitution for the rectilinear base of "one uniform line (curve)," by means of which he proved that the only "uniform lines"

their pneumatic devices (*θανατοποικίη*), as regards which Proclus' authority may be Pappus (VIII. p. 1024, 24—27) who uses very similar expressions. Heron, even if not later than Geminus, could hardly have been included in a historical work by him. Perhaps Geminus may have referred to Ctesibius only, and Proclus may have inserted "and Heron" himself.

¹ Proclus, pp. 103, 21—107, 10; pp. 111, 1—113, 3.

² *ibid.* pp. 117, 14—120, 12, where perhaps in the passage pp. 117, 22—118, 23 we may have Geminus' own words.

³ *ibid.* pp. 176, 18—177, 25; perhaps also p. 175. The note ends with the words "These things too we have selected from Geminus' *Φιλοκαλία* for the elucidation of the matters in question." Tannery (p. 27) takes these words coming at the end of the commentary on the definitions as referring to the whole of the portion of the commentary dealing with the definitions. Van Pesch properly regards them as only applying to the note on *parallels*. This seems to me clear from the use of the word *τοῦ* (*τῶσαῦτα καὶ*).

⁴ Proclus, p. 176, 5—17.

⁵ *ibid.* pp. 178—182, 4; pp. 183, 14—184, 10; cf. p. 188, 3—11.

⁶ *ibid.* p. 185, 6—25.

⁷ *ibid.* p. 192, 5—29.

⁸ *ibid.* pp. 200, 21—202, 25.

(alike in all their parts) are a straight line, a circle, and a cylindrical helix¹;

(11) (in the note on I. 10) on the question whether a line is made up of indivisible parts (*ἀμερῆ*), as affecting the problem of bisecting a given straight line²;

(12) (in the note on I. 35) on *topical*, or *locus*-theorems³, where the illustration of the equal parallelograms described between a hyperbola and its asymptotes may also be due to Geminus⁴.

Other passages which may fairly be attributed to Geminus, though his name is not mentioned, are the following:

(1) in the prologue, where there is the same allusion as in the passage (8) above to a remark of Aristotle that it is equally absurd to expect scientific proofs from a rhetorician and to accept mere plausibilities from a geometer⁵;

(2) a passage in the prologue about the subject-matter, methods, and bases of geometry, the latter including axioms and postulates⁶;

(3) another on the definition and nature of *elements*⁷;

(4) a remark on the Stoic use of the term axiom for every simple statement (*ἀπόφανσις ἀπλή*)⁸;

(5) another discussion on theorems and problems⁹, in the middle of which however there are some sentences by Proclus himself¹⁰.

(6) another passage, in connexion with Def. 3, on lines including or not including a figure (with which cf. part of the passage (4) above)¹¹;

(7) a classification of different sorts of angles according as they are contained by simple or mixed lines (or curves)¹²;

(8) a similar classification of figures¹³, and of plane figures¹⁴;

(9) Posidonius' definition of a *figure*¹⁵;

(10) a classification of triangles into seven kinds¹⁶;

(11) a note distinguishing lines (or curves) producible indefinitely or not so producible, whether forming a figure or not forming a figure (like the "single-turn spiral")¹⁷;

(12) passages distinguishing different sorts of problems¹⁸, different sorts of theorems¹⁹, and two sorts of converses (complete and partial)²⁰;

(13) the definition of the term "porism" as used in the title of Euclid's *Porisms*, as distinct from the other meaning of "corollary"²¹;

(14) a note on the Epicurean objection to I. 20 as being obvious even to an ass²²;

(15) a passage on the properties of parallels, with allusions to

¹ Proclus, p. 251, 2—11.

² *ibid.* pp. 394, 11—395, 2 and p. 395, 13—21.

³ *ibid.* pp. 33, 21—34, 1.

⁴ *ibid.* pp. 72, 3—75, 4.

⁵ *ibid.* pp. 77, 7—78, 13, and 79, 3—81, 4.

⁶ *ibid.* pp. 102, 22—103, 18.

⁷ *ibid.* pp. 159, 12—160, 9.

⁸ *ibid.* p. 143, 5—11.

⁹ *ibid.* p. 187, 19—27.

¹⁰ *ibid.* pp. 244, 14—246, 12.

¹¹ *ibid.* pp. 301, 21—302, 13.

¹² *ibid.* pp. 277, 25—279, 11.

¹³ *ibid.* p. 395, 8—12.

¹⁴ *ibid.* pp. 57, 9—58, 3.

¹⁵ *ibid.* p. 77, 3—6.

¹⁶ *ibid.* pp. 78, 13—79, 2.

¹⁷ *ibid.* pp. 126, 7—127, 16.

¹⁸ *ibid.* pp. 162, 27—164, 6.

¹⁹ *ibid.* p. 168, 4—12.

²⁰ *ibid.* pp. 220, 7—222, 14; also p. 330, 6—9.

²¹ *ibid.* p. 252, 5—254, 20.

²² *ibid.* pp. 322, 4—323, 3.

Apollonius' *Conics*, and the curves invented by Nicomedes, Hippias and Perseus¹;

(16) a passage on the parallel-postulate regarded as the converse of I. 17².

Of the authors to whom Proclus was indebted in a less degree the most important is **Apollonius of Perga**. Two passages allude to his *Conics*³, one to a work on irrationals⁴, and two to a treatise *On the cochlias* (apparently the cylindrical helix) by Apollonius⁵. But more important for our purpose are six references to Apollonius in connexion with elementary geometry.

(1) He appears as the author of an attempt to explain the idea of a line (possessing length but no breadth) by reference to daily experience, e.g. when we tell someone to measure, merely, the length of a road or of a wall⁶; and doubtless the similar passage showing how we may in like manner get a notion of a surface (without depth) is his also⁷.

(2) He gave a new general definition of an angle⁸.

(3) He tried to prove certain axioms⁹, and Proclus gives his attempt to prove Axiom I, word for word¹⁰.

Proclus further quotes:

(4) Apollonius' solution of the problem in Eucl. I. 10, avoiding Euclid's use of I. 9¹¹,

(5) his solution of the problem in I. 11, differing only slightly from Euclid's¹², and

(6) his solution of the problem in I. 23¹³.

Heiberg¹⁴ conjectures that Apollonius departed from Euclid's method in these propositions because he objected to solving problems of a more general, by means of problems of a more particular, character. Proclus however considers all three solutions inferior to Euclid's; and his remarks on Apollonius' handling of these elementary matters generally suggest that he was nettled by criticisms of Euclid in the work containing the things which he quotes from Apollonius, just as we conclude that Pappus was offended by the remarks of Apollonius about Euclid's incomplete treatment of the "three- and four-line locus¹⁵." If this was the case, Proclus can hardly have got his information about these things at second-hand; and there seems to be no reason to doubt that he had the actual work of Apollonius before him. This work may have been the treatise mentioned by Marinus in the words "Apollonius in his general treatise" (*Ἀπολλώνιος ἐν τῇ καθόλου πραγματείᾳ*)¹⁶. If the notice in the *Fihrist*¹⁷ stating, on the authority of Thābit b. Qurra, that

¹ Proclus, pp. 355, 20—356, 16.

² *ibid.* p. 71, 19; p. 356, 8, 6.

³ *ibid.* pp. 105, 5, 6, 14, 15.

⁷ *ibid.* p. 114, 20—25.

⁹ *ibid.* p. 183, 13, 14.

¹¹ *ibid.* pp. 279, 16—280, 4.

¹³ *ibid.* pp. 335, 16—336, 5.

¹⁵ See above, pp. 2, 3.

¹⁷ *Fihrist*, tr. Suter, p. 19.

² *ibid.* p. 364, 9—12; pp. 364, 20—365, 4.

⁴ *ibid.* p. 74, 23, 24.

⁶ *ibid.* p. 100, 5—19.

⁸ *ibid.* p. 123, 15—19 (cf. p. 124, 17, p. 125, 17).

¹⁰ *ibid.* pp. 194, 25—195, 5.

¹² *ibid.* p. 282, 8—19.

¹⁴ *Philologus*, vol. XLIII, p. 489.

¹⁶ *Marinus in Euclidis Data*, ed. Menge, p. 234, 16.

Apollonius wrote a tract on the parallel-postulate be correct, it may have been included in the same work. We may conclude generally that, in it, Apollonius tried to remodel the beginnings of geometry, reducing the number of axioms, appealing, in his definitions of lines, surfaces etc., more to experience than to abstract reason, and substituting for certain proofs others of a more general character.

The probabilities are that, in quoting from the tract of **Ptolemy** in which he tried to prove the parallel-postulate, Proclus had the actual work before him. For, after an allusion to it as "a certain book¹" he gives two long extracts², and at the beginning of the second indicates the title of the tract, "in the (book) about the meeting of straight lines produced from (angles) less than two right angles," as he has very rarely done in other cases.

Certain things from **Posidonius** are evidently quoted at second-hand, the authority being Geminus (e.g. the definitions of *figure* and *parallels*); but besides these we have quotations from a separate work which he wrote to controvert Zeno of Sidon, an Epicurean who had sought to destroy the whole of geometry³. We are told that Zeno had argued that, even if we admit the fundamental principles (*ἀρχαί*) of geometry, the deductions from them cannot be proved without the admission of something else as well, which has not been included in the said principles⁴. On I. 1 Proclus gives at some length the arguments of Zeno and the reply of Posidonius as regards this proposition⁵. In this case Zeno's "something else" which he considers to be assumed is the fact that two straight lines cannot have a common segment, and then, as regards the "proof" of it by means of the bisection of a circle by its diameter, he objects that it has been assumed that two *circumferences* (arcs) of circles cannot have a common part. Lastly, he makes up, for the purpose of attacking it, another supposed "proof" of the fact that two straight lines cannot have a common part. Proclus appears, more than once, to be quoting the actual words of Zeno and Posidonius; in particular, two expressions used by Posidonius about "the acrid Epicurean" (*τὸν δριμύν Ἐπικούρειον*)⁶ and his "misrepresentations" (*Ποσειδωνίος φησι τὸν Ζήνωνα συκοφαντεῖν*)⁷. It is not necessary to suppose that Proclus had the original work of Zeno before him, because Zeno's arguments may easily have been got from Posidonius' reply; but he would appear to have quoted direct from the latter at all events.

The work of **Carpus** *mechanicus* (a treatise on astronomy) quoted from by Proclus⁸ must have been accessible to him at first-hand, because a portion of the extract from it about the relation of theorems and problems⁹ is reproduced word for word. Moreover, if he were not using the book itself, Proclus would hardly be in a position to question whether the introduction of the subject of theorems and problems

¹ Proclus, p. 191, 23.

³ *ibid.* p. 200, 1-3.

⁵ *ibid.* pp. 214, 18-215, 13; pp. 216, 10-218, 11.

⁶ *ibid.* p. 216, 21.

⁸ *ibid.* pp. 241, 19-243, 11.

² *ibid.* pp. 362, 14-363, 18; pp. 365, 7-367, 27.

⁴ *ibid.* pp. 199, 11-200, 1.

⁷ *ibid.* p. 218, 1.

⁹ *ibid.* pp. 242, 22-243, 11.

was opportune in the place where it was found (εἰ μὲν κατὰ καιρὸν ἡ μὴ, παρείσθω πρὸς τὸ παρόν)¹.

It is of course evident that Proclus had before him the original works of Plato, Aristotle, Archimedes and Plotinus, as well as the *Συμμικτά* of Porphyry and the works of his master Syrianus (ὁ ἡμέτερος καθηγεμών)², from whom he quotes in his note on the definition of an angle. Tannery also points out that he must have had before him a group of works representing the Pythagorean tradition on its mystic, as distinct from its mathematical, side, from Philolaus downwards, and comprising the more or less apocryphal *ἱερὸς λόγος* of Pythagoras, the Oracles (*λόγια*), and Orphic verses³.

Besides quotations from writers whom we can identify with more or less certainty, there are many other passages which are doubtless quoted from other commentators whose names we do not know. A list of such passages is given by van Pesch⁴, and there is no need to cite them here.

Van Pesch also gives at the end of his work⁵ a convenient list of the books which, as the result of his investigation, he deems to have been accessible to and directly used by Proclus. The list is worth giving here, on the same ground of convenience. It is as follows:

Eudemus: *history of geometry*.

Geminus: *the theory of the mathematical sciences*.

Heron: *commentary on the Elements of Euclid*.

Porphyry: " " "

Pappus: " " "

Apollonius of Perga: a work relating to elementary geometry.

Ptolemy: *on the parallel-postulate*.

Posidonius: a book controverting Zeno of Sidon.

Carpus: *astronomy*.

Syrianus: a discussion on the *angle*.

Pythagorean philosophical tradition.

Plato's works.

Aristotle's works.

Archimedes' works.

Plotinus: *Enneades*.

Lastly we come to the question what passages, if any, in the commentary of Proclus represent his own contributions to the subject. As we have seen, the *onus probandi* must be held to rest upon him who shall maintain that a particular note is original on the part of Proclus. Hence it is not enough that it should be impossible to point to another writer as the probable source of a note; we must have a *positive* reason for attributing it to Proclus. The criterion must therefore be found either (1) in the general terms in which Proclus points out the deficiencies in previous commentaries and indicates the respects in which his own will differ from them, or (2) in specific expressions used by him in introducing particular notes which may

¹ Proclus, p. 241, 21, 22.

² Tannery, *La Géométrie grecque*, pp. 25, 26.

⁴ Van Pesch, *De Procli fontibus*, p. 139.

⁵ *ibid.* p. 123, 19.

⁵ *ibid.* p. 155.

indicate that he is giving his own views. Besides indicating that he paid more attention than his predecessors to questions requiring deeper study (τὸ πραγματειώδες) and "pursued clear distinctions" (τὸ εὐδιαίρετον μεταδιώκοντας)¹—by which he appears to imply that his predecessors had confused the different departments of their commentaries, viz. lemmas, cases, and objections (ἐνστάσεις)²—Proclus complains that the earlier commentators had failed to indicate the ultimate grounds or *causes* of propositions³. Although it is from Geminus that he borrowed a passage maintaining that it is one of the proper functions of geometry to inquire into causes (τὴν αἰτίαν καὶ τὸ διὰ τί)⁴, yet it is not likely that Geminus dealt with Euclid's propositions one by one; and consequently, when we find Proclus, on I. 8, 16, 17, 18, 32, and 47⁵, endeavouring to explain *causes*, we have good reason to suppose that the explanations are his own.

Again, his remarks on certain things which he quotes from Pappus can scarcely be due to anyone else, since Pappus is the latest of the commentators whose works he appears to have used. Under this head come

(1) his objections to certain new axioms introduced by Pappus⁶,

(2) his conjecture as to how Pappus came to think of his alternative proof of I. 5⁷,

(3) an addition to Pappus' remarks about the curvilinear angle which is equal to a right angle without being one⁸.

The defence of Geminus against Carpus, who combated his view of theorems and problems, is also probably due to Proclus⁹, as well as an observation on I. 38 to the effect that I. 35—38 are really comprehended in VI. 1 as particular cases¹⁰.

Lastly, we can have no hesitation in attributing to Proclus himself (1) the criticism of Ptolemy's attempt to prove the parallel-postulate¹¹, and (2) the other attempted proof given in the same note¹² (on I. 29) and assuming as an axiom that "if from one point two straight lines forming an angle be produced *ad infinitum* the distance between them when so produced *ad infinitum* exceeds any finite magnitude (i.e. length)," an assumption which purports to be the equivalent of a statement in Aristotle¹³. It is introduced by words in which the writer appears to claim originality for his proof: "To him who desires to see this proved (κατασκευαζόμενον) let it be said by us (λεγέσθω παρ' ἡμῶν)" etc.¹⁴ Moreover, Philoponus, in a note on Aristotle's *Anal. post.* I. 10, says that "the geometer (Euclid) assumes this as an axiom, but it wants a great deal of proof, insomuch that both Ptolemy and Proclus wrote a whole book upon it¹⁵."

¹ Proclus, p. 84, 13, p. 432, 14, 15.

² Cf. *ibid.* p. 289, 11—15; p. 432, 15—17.

³ *ibid.* p. 432, 17.

⁴ *ibid.* p. 202, 9—25.

⁵ See Proclus, p. 270, 5—24 (I. 8); pp. 309, 3—310, 8 (I. 16); pp. 310, 19—311, 23 (I. 17); pp. 316, 14—318, 2 (I. 18); p. 384, 13—21 (I. 32); pp. 426, 22—427, 8 (I. 47).

⁶ Proclus, p. 198, 5—15.

⁷ *ibid.* p. 250, 12—19.

⁸ *ibid.* p. 190, 9—23.

⁹ *ibid.* p. 243, 12—29.

¹⁰ *ibid.* pp. 405, 6—406, 9.

¹¹ *ibid.* p. 368, 1—23.

¹² *ibid.* pp. 371, 11—373, 2.

¹³ Aristotle, *de caelo*, I. 5 (271 b 28—30).

¹⁴ Proclus, p. 371, 10.

¹⁵ Berlin Aristotle, vol. IV, p. 214 a 9—12.

CHAPTER V.

THE TEXT¹.

IT is well known that the title of Simson's edition of Euclid (first brought out in Latin and English in 1756) claims that, in it, "the errors by which Theon, or others, have long ago vitiated these books are corrected, and some of Euclid's demonstrations are restored"; and readers of Simson's notes are familiar with the phrases used, where anything in the text does not seem to him satisfactory, to the effect that the demonstration has been spoiled, or things have been interpolated or omitted, by Theon "or some other unskilful editor." Now most of the MSS. of the Greek text prove by their titles that they proceed from the recension of the *Elements* by Theon; they purport to be either "from the edition of Theon" (ἐκ τῆς Θέωνος ἐκδόσεως) or "from the lectures of Theon" (ἀπὸ συνουσιῶν τοῦ Θέωνος). This was Theon of Alexandria (4th c. A.D.) who also wrote a commentary on Ptolemy, in which there occurs a passage of the greatest importance in this connexion²: "But that sectors in equal circles are to one another as the angles on which they stand *has been proved by me in my edition of the Elements at the end of the sixth book.*" Thus Theon himself says that he edited the *Elements* and also that the second part of VI. 33, found in nearly all the MSS., is his addition.

This passage is the key to the whole question of Theon's changes in the text of Euclid; for, when Peyrard found in the Vatican the MS. 190 which contained neither the words from the titles of the other MSS. quoted above nor the interpolated second part of VI. 33, he was justified in concluding, as he did, that in the Vatican MS. we have an edition more ancient than Theon's. It is also clear that the copyist of P, or rather of its archetype, had before him the two recensions and systematically gave the preference to the earlier one; for at XIII. 6 in P the first hand has added a note in the margin: "This theorem is not given in most copies of the *new edition*, but is found in those of the old." Thus we are more fortunate than Simson, since our judgment of Theon's recension can be formed on the basis, not of mere conjecture, but of the documentary evidence afforded by a comparison of the Vatican MS. just mentioned with what we may conveniently call, after Heiberg, the Theonine MSS.

¹ The material for the whole of this chapter is taken from Heiberg's edition of the *Elements*, introduction to vol. v., and from the same scholar's *Litterargeschichtliche Studien über Euklid*, p. 174 sqq. and *Paralipomena zu Euklid in Hermes*, XXXVIII., 1903.

² I. p. 201 ed. Halma = p. 50 ed. Basel.

The MSS. used for Heiberg's edition of the *Elements* are the following :

- (1) P = Vatican MS. numbered 190, 4to, in two volumes (doubtless one originally) ; 10th c.

This is the MS. which Peyrard was able to use ; it was sent from Rome to Paris for his use and bears the stamp of the Paris Imperial Library on the last page. It is well and carefully written. There are corrections some of which are by the original hand, but generally in paler ink, others, still pretty old, by several different hands, or by one hand with different ink in different places (P m. 2), and others again by the latest hand (P m. rec.). It contains, first, the *Elements* I.—XIII. with scholia, then Marinus' commentary on the *Data* (without the name of the author), followed by the *Data* itself and scholia, then the *Elements* XIV., XV. (so called), and lastly three books and a part of a fourth of a commentary by Theon εἰς τοὺς προχείρους κανόνας Πτολεμαίου.

The other MSS. are "Theonine."

- (2) F = MS. XXVIII, 3, in the Laurentian Library at Florence, 4to ; 10th c.

This MS. is written in a beautiful and scholarly hand and contains the *Elements* I.—XV., the *Optics* and the *Phaenomena*, but is not well preserved. Not only is the original writing renewed in many places, where it had become faint, by a later hand of the 16th c., but the same hand has filled certain smaller lacunae by gumming on to torn pages new pieces of parchment, and has replaced bodily certain portions of the MS., which had doubtless become illegible, by fresh leaves. The larger gaps so made good extend from Eucl. VII. 12 to IX. 15, and from XII. 3 to the end ; so that, besides the conclusion of the *Elements*, the *Optics* and *Phaenomena* are also in the later hand, and we cannot even tell what in addition to the *Elements* I.—XIII. the original MS. contained. Heiberg denotes the later hand by ϕ and observes that, while in restoring words which had become faint and filling up minor lacunae the writer used no other MS., yet in the two larger restorations he used the Laurentian MS. XXVIII, 6, belonging to the 13th—14th c. The latter MS. (which Heiberg denotes by f) was copied from the Viennese MS. (V) to be described below.

- (3) B = Bodleian MS., D'Orville X. 1 inf. 2, 30, 4to ; A.D. 888.

This MS. contains the *Elements* I.—XV. with many scholia. Leaves 15—118 contain I. 14 (from about the middle of the proposition) to the end of Book VI., and leaves 123—387 (wrongly numbered 397) Books VII.—XV. in one and the same elegant hand (9th c.). The leaves preceding leaf 15 seem to have been lost at some time, leaves 6 to 14 (containing *Elem.* I. to the place in I. 14 above referred to) being carelessly written by a later hand on thick and common parchment (13th c.). On leaves 2 to 4 and 122 are certain notes in the hand of Arethas, who also wrote a two-line epigram on leaf 5, the greater part of the scholia in uncial letters, a few notes and corrections, and two sentences on the last leaf, the first of which states that the MS. was written by one Stephen clericus in the year of the world 6397

(= 888 A.D.), while the second records Arethas' own acquisition of it. Arethas lived from, say, 865 to 939 A.D. He was Archbishop of Caesarea and wrote a commentary on the Apocalypse. The portions of his library which survive are of the greatest interest to palaeography on account of his exact notes of dates, names of copyists, prices of parchment etc. It is to him also that we owe the famous Plato MS. from Patmos (Cod. Clarkianus) which was written for him in November 895¹.

(4) V = Viennese MS. Philos. Gr. No. 103; probably 12th c.

This MS. contains 292 leaves, Eucl. *Elements* I.—XV. occupying leaves 1 to 254, after which come the *Optics* (to leaf 271), the *Phaenomena* (mutilated at the end) from leaf 272 to leaf 282, and lastly scholia, on leaves 283 to 292, also imperfect at the end. The different material used for different parts and the varieties of handwriting make it necessary for Heiberg to discuss this MS. at some length². The handwriting on leaves 1 to 183 (Book I. to the middle of x. 105) and on leaves 203 to 234 (from XI. 31, towards the end of the proposition, to XIII. 7, a few lines down) is the same; between leaves 184 and 202 there are two varieties of handwriting, that of leaves 184 to 189 and that of leaves 200 (verso) to 202 being the same. Leaf 235 begins in the same handwriting, changes first gradually into that of leaves 184 to 189 and then (verso) into a third more rapid cursive writing which is the same as that of the greater part of the scholia, and also as that of leaves 243 and 282, although, as these leaves are of different material, the look of the writing and of the ink seems altered. There are corrections both by the first and a second hand, and scholia by many hands. On the whole, in spite of the apparent diversity of handwriting in the MS., it is probable that the whole of it was written at about the same time, and it may (allowing for changes of material, ink etc.) even have been written by the same man. It is at least certain that, when the Laurentian MS. XXVIII, 6 was copied from it, the whole MS. was in the condition in which it is now, except as regards the later scholia and leaves 283 to 292 which are not in the Laurentian MS., that MS. coming to an end where the *Phaenomena* breaks off abruptly in V. Hence Heiberg attributes the whole MS. to the 12th c.

But it was apparently in two volumes originally, the first consisting of leaves 1 to 183; and it is certain that it was not all copied at the same time or from one and the same original. For leaves 184 to 202 were evidently copied from two MSS. different both from one another and from that from which the rest was copied. Leaves 184 to the middle of leaf 189. (recto) must have been copied from a MS. similar to P, as is proved by similarity of readings, though not from P itself. The rest, up to leaf 202, were copied from the Bologna MS. (b) to be mentioned below. It seems clear that the content of leaves 184 to 202 was supplied from other MSS. because there was a lacuna in the original from which the rest of V was copied.

¹ See Pauly-Wissowa, *Real-Encyclopädie der class. Altertumswissenschaft*, vol. II., 1896, p. 675.

² Heiberg, vol. v. pp. xxix—xxxiii.

Heiberg sums up his conclusions thus. The copyist of V first copied leaves 1 to 183 from an original in which two *quaterniones* were missing (covering from the middle of Eucl. X. 105 to near the end of XI. 31). Noticing the lacuna he put aside one *quaternio* of the parchment used up to that point. Then he copied onwards from the end of the lacuna in the original to the end of the *Phaenomena*. After this he looked about him for another MS. from which to fill up the lacuna; finding one, he copied from it as far as the middle of leaf 189 (recto). Then, noticing that the MS. from which he was copying was of a different class, he had recourse to yet another MS. from which he copied up to leaf 202. At the same time, finding that the lacuna was longer than he had reckoned for, he had to use twelve more leaves of a different parchment in addition to the *quaternio* which he had put aside. The whole MS. at first formed two volumes (the first containing leaves 1 to 183 and the second leaves 184 to 282); then, after the last leaf had perished, the two volumes were made into one to which two more *quaterniones* were also added. A few leaves of the latter of these two have since perished.

(5) b = MS. numbered 18—19 in the Communal Library at Bologna, in two volumes, 4to; 11th c.

This MS. has scholia in the margin written both by the first hand and by two or three later hands; some are written by the latest hand, Theodorus Cabasilas (a descendant apparently of Nicolaus Cabasilas, 14th c.) who owned the MS. at one time. It contains (a) in 14 *quaterniones* the definitions and the enunciations (without proofs) of the *Elements* I.—XIII. and of the *Data*, (b) in the remainder of the volumes the *Proem to Geometry* (published among the *Variae Collectiones* in Hultsch's edition of Heron, pp. 252, 24 to 274, 14) followed by the *Elements* I.—XIII. (part of XIII. 18 to the end being missing), and then by part of the *Data* (from the last three words of the enunciation of Prop. 38 to the end of the penultimate clause in Prop. 87, ed. Menge). From XI. 36 inclusive to the end of XII. this MS. appears to represent an entirely different recension. Heiberg is compelled to give this portion of b separately in an appendix. He conjectures that it is due to a Byzantine mathematician who thought Euclid's proofs too long and tiresome and consequently contented himself with indicating the course followed¹. At the same time this Byzantine must have had an excellent MS. before him, probably of the ante-Theonine variety of which the Vatican MS. 190 (P) is the sole representative.

(6) p = Paris MS. 2466, 4to; 12th c.

This manuscript is written in two hands, the finer hand occupying leaves 1 to 53 (recto), and a more careless hand leaves 53 (verso) to 64, which are of the same parchment as the earlier leaves, and leaves 65 to 239, which are of a thinner and rougher parchment showing traces of writing of the 8th—9th c. (a Greek version of the Old Testament). The MS. contains the *Elements* I.—XIII. and some scholia after Books XI., XII. and XIII.

¹ *Zeitschrift für Math. u. Physik*, xxix., hist.-litt. Abtheilung, p. 13.

(7) q = Paris MS. 2344, folio ; 12th c.

It is written by one hand but includes scholia by many hands. On leaves 1 to 16 (recto) are scholia with the same title as that found by Wachsmuth in a Vatican MS. and relied upon by him to prove that Proclus continued his commentaries beyond Book I.¹ Leaves 17 to 357 contain the *Elements* I.—XIII. (except that there is a lacuna from the middle of VIII. 25 to the *ἐκθεσις* of IX. 14); before Books VII. and X. there are some leaves filled with scholia only, and leaves 358 to 366 contain nothing but scholia.

(8) Heiberg also used a palimpsest in the British Museum (Add. 17211). Five pages are of the 7th—8th c. and are contained (leaves 49—53) in the second volume of the Syrian MS. Brit. Mus. 687 of the 9th c.; half of leaf 50 has perished. The leaves contain various fragments from Book X. enumerated by Heiberg, Vol. III., p. v, and nearly the whole of XIII. 14.

Since his edition of the *Elements* was published, Heiberg has collected further material bearing on the history of the text². Besides giving the results of further or new examination of MSS., he has collected the fresh evidence contained in an-Nairizi's commentary, and particularly in the quotations from Heron's commentary given in it (often word for word), which enable us in several cases to trace differences between our text and the text as Heron had it, and to identify some interpolations which actually found their way into the text from Heron's commentary itself; and lastly he has dealt with some valuable fragments of ancient papyri which have recently come to light, and which are especially important in that the evidence drawn from them necessitates some modification in the views expressed in the preface to Vol. v. as to the nature of the changes made in Theon's recension, and in the principles laid down for differentiating between Theon's recension and the original text, on the basis of a comparison between P and the Theonine MSS. alone.

The fragments of ancient papyri referred to are the following.

1. *Papyrus Herculanensis* No. 1061³.

This fragment quotes Def. 15 of Book I. in Greek, and *omits* the words *ἡ καλεῖται περιφέρεια*, "which is called the circumference," found in all our MSS., and the further addition *πρὸς τὴν τοῦ κύκλου περιφέρειαν* also found in practically all the MSS. Thus Heiberg's assumption that both expressions are interpolations is now confirmed by this oldest of all sources.

2. *The Oxyrhynchus Papyri* I. p. 58, No. XXIX. of the 3rd or 4th c.

This fragment contains the enunciation of Eucl. II. 5 (with figure, apparently without letters, immediately following, and not, as usual in our MSS., at the end of the proof) and before it the part of a word *περιεχομε* belonging to II. 4 (with room for *-νφ ὀρθογωνίφ. ὅπερ ἔδει*

¹ [ἐλς τ]ὰ τοῦ Εὐκλείδου στοιχεῖα προλαμβανόμενα ἐκ τῶν Πρόκλου σποράδην καὶ κατ' ἐπιτομήν. Cf. p. 32, note 8, above.

² Heiberg, *Paralipomena zu Euklid* in *Hermes*, XXXVIII., 1903, pp. 46—74, 161—201, 321—356.

³ Described by Heiberg in *Oversigt over det kongl. danske Videnskabernes Selskabs Forhandlinger*, 1900, p. 161.

δείξαι and a stroke to mark the end), showing that the fragment *had not* the Porism which appears in all the Theonine MSS. and (in a later hand) in P, and thereby confirming Heiberg's assumption that the Porism was due to Theon.

3. A fragment in *Fayum towns and their papyri*, p. 96, No. IX. of 2nd or 3rd c.

This contains I. 39 and I. 41 following one another and almost complete, showing that I. 40 was wanting, whereas it is found in all the MSS. and is recognised by Proclus. Moreover the text of the beginning of I. 39 is better than ours, since it has no double διορισμός but omits the first ("I say that they are also in the same parallels") and has "*and*" instead of "*for* let *AD* be joined" in the next sentence. It is clear that I. 40 was interpolated by someone who thought there ought to be a proposition following I. 39 and related to it as I. 38 is related to I. 37 and I. 36 to I. 35, although Euclid nowhere uses I. 40, and therefore was not likely to include it. The same interpolator failed to realise that the words "let *AD* be joined" were part of the *ἐκθεσις* or *setting-out*, and took them for the *κατασκευή* or "construction" which generally follows the διορισμός or "particular statement" of the conclusion to be proved, and consequently thought it necessary to insert a διορισμός *before* the words.

The conclusions drawn by Heiberg from a consideration of particular readings in this papyrus along with those of our MSS. will be referred to below.

We now come to the principles which Heiberg followed, when preparing his edition, in differentiating the original text from the Theonine recension by means of a comparison of the readings of P and of the Theonine MSS. The rules which he gives are subject to a certain number of exceptions (mostly in cases where one MS. or the other shows readings due to copyists' errors), but in general they may be relied upon to give conclusive results.

The possible alternatives which the comparison of P with the Theonine MSS. may give in particular passages are as follows:

I. There may be *agreement* in three different degrees.

(1) P and *all* the Theonine MSS. may agree.

In this case the reading common to all, even if it is corrupt or interpolated, is more ancient than Theon, i.e. than the 4th c.

(2) P may agree with *some* (only) of the Theonine MSS.

In this case Heiberg considered that the latter give the true reading of Theon's recension, and the other Theonine MSS. have departed from it.

(3) P and *one* only of the Theonine MSS. may agree.

In this case too Heiberg assumed that the *one* Theonine MS. which agrees with P gives the true Theonine reading, and that this rule even supplies a sort of measure of the quality and faithfulness of the Theonine MSS. Now none of them agrees *alone* with P in preserving the true reading so often as F. Hence F must be held to have preserved Theon's recension more faithfully than the other Theonine MSS.; and it would follow that in those portions where F fails us P must

carry rather more weight even though it may differ from the Theonine MSS. BVpq. (Heiberg gives many examples in proof of this, as of his main rules generally, for which reference must be made to his *Prolegomena* in Vol. v.) The specially close relation of F and P is also illustrated by passages in which they have the same *errors*; the explanation of these common errors (where not due to accident) is found by Heiberg in the supposition that they existed, but were not noticed by Theon, in the original copy in which he made his changes.

Although however F is by far the best of the Theonine MSS., there are a considerable number of passages where one of the others (B, V, p or q) *alone* with P gives the genuine reading of Theon's recension.

As the result of the discovery of the papyrus fragment containing I. 39, 41, the principles above enunciated under (2) and (3) are found by Heiberg to require some qualification. For there is in some cases a remarkable agreement between the papyrus and the Theonine MSS. (some or all) as against P. This shows that Theon took more trouble to follow older MSS., and made fewer arbitrary changes of his own, than has hitherto been supposed. Next, when the papyrus agrees with some of the Theonine MSS. against P, it must now be held that these MSS. (and not, as formerly supposed, those which agree with P) give the true reading of Theon. If it were otherwise, the agreement between the papyrus and the Theonine MSS. would be accidental: but it happens too often for this. It is clear also that there must have been contamination between the two recensions; otherwise, whence could the Theonine MSS. which agree with P and not with the papyrus have got their readings? The influence of the P class on the Theonine F is especially marked.

II. There may be *disagreement* between P and all the Theonine MSS.

The following possibilities arise.

(1) The Theonine MSS. differ also among themselves.

In this case Heiberg considered that P nearly always has the true reading, and the Theonine MSS. have suffered interpolation in different ways after Theon's time.

(2) The Theonine MSS. all combine against P.

In this case the explanation was assumed by Heiberg to be one or other of the following.

(a) The common reading is due to an error which cannot be imputed to Theon (though it may have escaped him when putting together the archetype of his edition); such error may either have arisen accidentally in all alike, or (more frequently) may be referred to a common archetype of all the MSS.

(β) There may be an accidental error in P; e.g. something has dropped out of P in a good many places, generally through *ὁμοιοτέλετον*

(γ) There may be words interpolated in P.

(δ) Lastly, *we may have in the Theonine MSS. a change made by Theon himself.*

(The discovery of the ancient papyrus showing readings agreeing

with some, or with all, of the Theonine MSS. against P now makes it necessary to be very cautious in applying these criteria.)

It is of course the last class (δ) of changes which we have to investigate in order to get a proper idea of Theon's recension.

Heiberg first observes, as regards these, that we shall find that Theon, in editing the *Elements*, altered hardly anything without some reason, often inadequate according to our ideas, but still some reason which seemed to him sufficient. Hence, in cases of very slight differences where both the Theonine MSS. and P have readings good and probable in themselves, Heiberg is not prepared to put the differences down to Theon. In those passages where we cannot see the least reason why Theon, if he had the reading of P before him, should have altered it, Heiberg would not at once assume the superiority of P unless there was such a consistency in the differences as would indicate that they were due not to accident but to design. In the absence of such indications, he thinks that the ordinary principles of criticism should be followed and that proper weight should be attached to the *antiquity* of the sources. And it cannot be denied that the sources of the Theonine version are the more ancient. For not only is the British Museum palimpsest (L), which is intimately connected with the rest of our MSS., at least two centuries older than P, but the other Theonine MSS. are so nearly allied that they must be held to have had a common archetype intermediate between them and the actual edition of Theon; and, since they themselves are as old as, or older than P, their archetype must have been much older. Heiberg gives (pp. xlv, xlvii) a list of passages where, for this reason, he has followed the Theonine MSS. in preference to P.

It has been mentioned above that the copyist of P or rather of its archetype wished to give an ancient recension. Therefore (apart from clerical errors and interpolations) the first hand in P may be relied upon as giving a genuine reading even where a correction by the first hand has been made *at the same time*. But in many places the first hand has made corrections *afterwards*; on these occasions he must have used new sources, e.g. when inserting the scholia to the first Book which P *alone* has, and in a number of passages he has made additions from Theonine MSS.

We cannot make out any "family tree" for the different Theonine MSS. Although they all proceeded from a common archetype later than the edition of Theon itself, they cannot have been copied one from the other; for, if they had been, how could it have come about that in one place or other each of them agrees *alone* with P in preserving the genuine reading? Moreover the great variety in their agreements and disagreements indicates that they have all diverged to about the same extent from their archetype. As we have seen that P contains corrections from the Theonine family, so they show corrections from P or other MSS. of the same family. Thus V has part of the lacuna in the MS. from which it was copied filled up from a MS. similar to P, and has corrections apparently derived from the same; the copyist, however, in correcting V, also used another MS. to which

he alludes in the additions to IX. 19 and 30 (and also on X. 23 Por.): "in the book of the Ephesian (this) is not found." Who this Ephesian of the 12th c. was, we do not know.

We now come to the alterations made by Theon in his edition of the *Elements*. I shall indicate *classes* into which these alterations may be divided but without details (except in cases where they affect the *mathematical content* as distinct from form or language pure and simple)¹.

I. *Alterations made by Theon where he found, or thought he found, mistakes in the original.*

1. Real blots in the original which Theon saw and tried to remove.

(a) Euclid has a porism (corollary) to VI. 19, the enunciation of which speaks of similar and similarly described *figures* though the proposition itself refers only to triangles, and therefore the porism should have come after VI. 20. Theon substitutes *triangle* for *figure* and proves the more general porism after VI. 20.

(b) In IX. 19 there is a statement which is obviously incorrect. Theon saw this and altered the proof by reducing four alternatives to two, with the result that it fails to correspond to the enunciation even with Theon's substitution of "if" for "when" in the enunciation.

(c) Theon omits a porism to IX. 11, although it is necessary for the proof of the succeeding proposition, apparently because, owing to an error in the text (*κατὰ τὸν* corrected by Heiberg into *ἐπὶ τὸ*), he could not get out of it the right sense.

(d) I should also put into this category a case which Heiberg classifies among those in which Theon merely fancied that he found mistakes, viz. the porism to V. 7 stating that, if four magnitudes are proportional, they are proportional inversely. Theon puts this after V. 4 with a proof, which however has no necessary connexion with V. 4 but is obvious from the definition of proportion.

(e) I should also put under this head XI. 1, where Euclid's argument to prove that two straight lines cannot have a common segment is altered.

2. Passages which seemed to Theon to contain blots, and which he therefore set himself to correct, though more careful consideration would have shown that Euclid's words are right or at least may be excused and offer no difficulty to an intelligent reader. Under this head come:

(a) an alteration in III. 24.

(b) a perfectly unnecessary alteration, in VI. 14, of "equiangular parallelograms" into "parallelograms having one angle equal to one angle," where Theon followed the false analogy of VI. 15.

(c) an omission of words in V. 26, owing to his having been misled by a wrong figure.

(d) an alteration of the order of XI. Deff. 27, 28.

(e) the substitution of "parallelepipedal solid" for "cube" in XI.

¹ Exhaustive details under all the different heads are given by Heiberg (Vol. v. pp. lii—lxxv).

38, because Theon observed, correctly enough, that it was true of the parallelepipedal solid in general as well as of the cube, but failed to give weight to the fact that Euclid must have given the particular case of the cube for the simple reason that that was all he wanted for use in XIII. 17.

(f) the substitution of the letter Φ for Ω (V for Z in my figure) because he saw that the perpendicular from K to $B\Phi$ would fall on Φ itself, so that Φ, Ω coincide. But, if the substitution is made, it should be *proved* that Φ, Ω coincide. Euclid can hardly have failed to notice the fact, but it may be that he deliberately ignored it as unnecessary for his purpose, because he did not want to lengthen his proposition by giving the proof.

II. *Emendations intended to improve the form or diction of Euclid.*

Some of these emendations of Theon affect passages of appreciable length. Heiberg notes about ten such passages; the longest is in Eucl. XII. 4 where a whole page of Heiberg's text is affected and Theon's version is put in the Appendix. The kind of alteration may be illustrated by that in IX. 15 where Euclid uses successively the propositions VII. 24, 25, quoting the enunciation of the former but not of the latter; Theon does exactly the reverse. In a few of the cases here quoted by Heiberg, Theon shortened the original somewhat.

But, as a rule, the emendations affect only a few words in each sentence. Sometimes they are considerable enough to alter the conformation of the sentence, sometimes they are trifling alterations "more magistellorum ineptorum" and unworthy of Theon. Generally speaking, they were prompted by a desire to change anything which was out of the common in expression or in form, in order to reduce the language to one and the same standard or norm. Thus Theon changed the order of words, substituted one word for another where the latter was used in a sense unusual with Euclid (e.g. *ἐπειδήπερ*, "since," for *ὅτι* in the sense of "because"), or one expression for another in like circumstances (e.g. where, finding "that which was enjoined would be done" in a *theorem*, VII. 31, and deeming the phrase more appropriate to a *problem*, he substituted for it "that which is sought would be manifest"; probably also and for similar reasons he made certain variations between the two expressions usual at the end of propositions *ὅπερ ἔδει δεῖξαι* and *ὅπερ ἔδει ποιῆσαι*, *quod erat demonstrandum* and *quod erat faciendum*). Sometimes his alterations show carelessness in the use of technical terms, as when he uses *ἀπτεσθαι* (to *meet*) for *ἐφάπτεσθαι* (to *touch*) although the ancients carefully distinguished the two words. The desire of keeping to a standard phraseology also led Theon to omit or add words in a number of cases, and also, sometimes, to change the lettering of figures.

But Theon seems, in editing the *Elements*, to have bestowed the most attention upon

III. *Additions designed to supplement or explain Euclid.*

First, he did not hesitate to interpolate whole propositions where he thought there was room or use for them. We have already

mentioned the addition to VI. 33 of the second part relating to *sectors*, for which Theon himself takes credit in his commentary on Ptolemy. Again, he interpolated the proposition commonly known as VII. 22 (*ex aequo in proportionone perturbata* for numbers, corresponding to V. 23), and perhaps also VII. 20, a particular case of VII. 19 as VI. 17 is of VI. 16. He added a second case to VI. 27, a porism to II. 4, a second porism to III. 16, and a lemma after X. 12; perhaps also the porism to V. 19 and the first porism to VI. 20. He also inserted alternative proofs here and there, e.g. in II. 4 (where the alternative differs little from the original) and in VII. 31; perhaps also in X. 1, 6, and 9.

Secondly, he sometimes repeats an argument where Euclid had said "For the same reason," adds specific references to points, straight lines etc. in the figures in order to exclude the possibility of mistake arising from Euclid's reference to them in general terms, or inserts words to make the meaning of Euclid more plain, e.g. *componendo* and *alternately*, where Euclid had left them out. Sometimes he thought to increase by his additions the mathematical precision of Euclid's language in enunciations or elsewhere, sometimes to make smoother and clearer things which Euclid had expressed with unusual brevity and harshness or carelessness, in reliance on the intelligence of his readers.

Thirdly, he supplied intermediate steps where Euclid's argument seemed too rapid and not easy enough to follow. The form of these additions varies; they are sometimes placed as a definite intermediate step with "therefore" or "so that," sometimes they are additions to the statement of premisses, sometimes phrases introduced by "since," "for" and the like, after the inference.

Lastly, there is a very large class of additions of a word, or one or two words, for the sake of clearness or consistency. Heiberg gives a number of examples of the addition of such nouns as "triangle," "square," "rectangle," "magnitude," "number," "point," "side," "circle," "straight line," "area" and the like, of adjectives such as "remaining," "right," "whole," "proportional," and of other parts of speech, even down to words like "is" (*ἐστί*) which is added 600 times, *δὲ, ἄρα, μέν, γάρ, καί* and the like.

IV. *Omissions by Theon.*

Heiberg remarks that, Theon's object having been, as above shown, to amplify and explain Euclid, we should not naturally have expected to find him doing much in the contrary process of compression, and it is only owing to the recurrence of a certain sort of omissions so frequently (especially in the first Books) as to exclude the hypothesis of their being all due to chance that we are bound to credit him with alterations making for greater brevity. We have seen, it is true, that he made omissions as well as additions for the purpose of reducing the language to a certain standard form. But there are also a good number of cases where in the enunciation of propositions, and in the *exposition* (the re-statement of them with reference to the figure), he has left out words because, apparently, he regarded Euclid's language as being *too* careful and precise.

Again, he is apparently responsible for the frequent omission of the words *ὅπερ ἔδει δεῖξαι* (or *ποιῆσαι*), Q.E.D. (or F.), at the end of propositions. This is often the case at the end of porisms, where, in omitting the words, Theon seems to have deliberately departed from Euclid's practice. The MS. P seems to show clearly that, where Euclid put a porism at the end of a proposition, he omitted the Q.E.D. at the end of the proposition but inserted it at the end of the porism, as if he regarded the latter as being actually a part of the proposition itself. As in the Theonine MSS. the Q.E.D. is generally omitted, the omission would seem to have been due to Theon. Sometimes in these cases the Q.E.D. is interpolated at the end of the proposition.

Heiberg summed up the discussion of Theon's edition by the remark that Theon evidently took no pains to discover and restore from MSS. the actual words which Euclid had written, but aimed much more at removing difficulties that might be felt by learners in studying the book. His edition is therefore not to be compared with the editions of the Alexandrine grammarians, but rather with the work done by Eutocius in editing Apollonius and with an interpolated recension of some of the works of Archimedes by a certain Byzantine, Theon occupying a position midway between these two editors, being superior to the latter in mathematical knowledge but behind Eutocius in industry (these views now require to be somewhat modified, as above stated). But however little Theon's object may be approved by those of us who would rather know the *ipsissima verba* of Euclid, there is no doubt that his work was approved by his pupils at Alexandria for whom it was written; and his edition was almost exclusively used by later Greeks, with the result that the more ancient text is only preserved to us in one MS.

As the result of the above investigation, we may feel satisfied that, where P and the Theonine MSS. agree, they give us (except in a few accidental instances) Euclid as he was read by the Greeks of the 4th c. But even at that time the text had been passed from hand to hand through more than six centuries, so that it is certain that it had already suffered changes, due partly to the fault of copyists and partly to the interpolations of mathematicians. Some errors of copyists escaped Theon and were corrected in some MSS. by later hands. Others appear in all our MSS. and, as they cannot have arisen accidentally in all, we must put them down to a common source more ancient than Theon. A somewhat serious instance is to be found in III. 8; and the use of *ἀπτέσθω* for *ἐφαπτέσθω* in the sense of "touch" may also be mentioned, the proper distinction between the words having been ignored as it was by Theon also. But there are a number of imperfections in the ante-Theonine text which it would be unsafe to put down to the errors of copyists, those namely where the good MSS. agree and it is not possible to see any motive that a copyist could have had for altering a correct reading. In these cases it is possible that the imperfections are due to a certain degree of carelessness on the part of Euclid himself; for it

is not possible "Euclidem ab omni naevo vindicare," to use the words of Saccheri¹, and consequently Simson is not right in attributing to Theon and other editors all the things in Euclid to which mathematical objection can be taken. Thus, when Euclid speaks of "the ratio compounded of the sides" for "the ratio compounded of the *ratios of the sides*," there is no reason for doubting that Euclid himself is responsible for the more slipshod expression. Again, in the Books XI.—XIII. relating to solid geometry there are blots neither few nor altogether unimportant which can only be attributed to Euclid himself²; and there is the less reason for hesitation in so attributing them because solid geometry was then being treated in a thoroughly systematic manner for the first time. Sometimes the *conclusion* (συμπέρασμα) of a proposition does not correspond exactly to the enunciation, often it is cut short with the words καὶ τὰ ἑξῆς "and the rest" (especially from Book X. onwards), and very often in Books VIII., IX. it is omitted. Where all the MSS. agree, there is no ground for hesitating to attribute the abbreviation or omission to Euclid; though, of course, where one or more MSS. have the longer form, it must be retained because this is one of the cases where a copyist has a temptation to abbreviate.

Where the true reading is preserved in one of the Theonine MSS. alone, Heiberg attributes the wrong reading to a mistake which arose before Theon's time, and the right reading of the single MS. to a successful correction.

We now come to the most important question of the *Interpolations introduced before Theon's time*.

I. Alternative proofs or additional cases.

It is not in itself probable that Euclid would have given two proofs of the same proposition; and the doubt as to the genuineness of the alternatives is increased when we consider the character of some of them and the way in which they are introduced. First of all, we have those of VI. 20 and XII. 17 introduced by "we shall prove this otherwise *more readily* (προχειρότερον)" or that of X. 90 "it is possible to prove *more shortly* (συντομώτερον)." Now it is impossible to suppose that Euclid would have given one proof as that definitely accepted by him and then added another with the express comment that the latter has certain advantages over the former. Had he considered the two proofs and come to this conclusion, he would have inserted the latter in the received text instead of the former. These alternative proofs must therefore have been interpolated. The same argument applies to alternatives introduced with the words "or even thus" (ἢ καὶ οὕτως), "or even otherwise" (ἢ καὶ ἄλλως). Under this head come the alternatives for the last portions of III. 7, 8; and Heiberg also compares the alternatives for parts of III. 31 (that the angle in a semicircle is a right angle) and XIII. 18, and the alternative proof of the lemma after X. 32. The alternatives to X. 105 and 106,

¹ *Euclides ab omni naevo vindicatus*, Mediolani, 1733.

² Cf. especially the assumption, without proof or definition, of the criterion for *equal* solid angles, and the incomplete proof of XII. 17.

again, are condemned by the place in which they occur, namely after an alternative proof to X. 115. The above alternatives being all admitted to be spurious, suspicion must necessarily attach to the few others which are in themselves unobjectionable. Heiberg instances the alternative proofs to III. 9, III. 10, VI. 30, VI. 31 and XI. 22, observing that it is quite comprehensible that any of these might have occurred to a teacher or editor and seemed to him, rightly or wrongly, to be better than the corresponding proofs in Euclid. Curiously enough, Simson adopted the alternatives to III. 9, 10 in preference to the genuine proofs. Since Heiberg's preface was written, his suspicion has been amply confirmed as regards III. 10 by the commentary of an-Nairizī (ed. Curtze) which shows not only that this alternative is Heron's, but also that the substantive proposition III. 12 in Euclid is also Heron's, having been given by him to supplement III. 11 which must originally have been enunciated of circles "touching one another" simply, i.e. so as to include the case of external as well as internal contact, though the proof covered the case of internal contact only. "Euclid, in the 11th proposition," says Heron, "supposed two circles touching one another internally and wrote the proposition on this case, proving what it was required to prove in it. *But I will show how it is to be proved if the contact be external*¹." This additional proposition of Heron's is by way of adding another *case*, which brings us to that class of interpolation. It was the practice of Euclid and the ancients to give only one case (generally the most difficult one) and to leave the others to be investigated by the reader for himself. One interpolation of a second case (VI. 27) is due, as we have seen, to Theon. The two extra cases of XI. 23 were manifestly interpolated before Theon's time, for the preliminary distinction of three cases, "(the centre) will either be within the triangle *LMN*, or on one of the sides, or outside. First let it be within," is a spurious addition (B and V only). Similarly an unnecessary case is interpolated in III. 11.

II. Lemmas:

Heiberg has unhesitatingly placed in his Appendix to Vol. III. certain lemmas interpolated either by Theon (on X. 13) or later writers (on X. 27, 29, 31, 32, 33, 34, where V only has the lemmas). But we are here concerned with the lemmas found in all the MSS., which however are, for different reasons, necessarily suspected. We will deal with the Book X. lemmas last.

(1) There is an *a priori* ground of objection to those lemmas which come *after* the propositions to which they relate and prove properties used in those propositions; for, if genuine, they would be a sign of faulty arrangement such as would not be likely in a systematic work so carefully ordered as the *Elements*. The lemma to VI. 22 is one of this class, and there is the further objection to it that in VI. 28 Euclid makes an assumption which would equally require a lemma though none is found. The lemma after XII. 4 is open to the further objections that certain altitudes are used but are not drawn in the

¹ An-Nairizī, ed. Curtze, p. 121.

figure (which is not in the manner of Euclid), and that a peculiar expression "parallelepipedal solids *described on* (*ἀναγραφόμενα ἀπὸ*) *prisms*" betrays a hand other than Euclid's. There is an objection on the score of language to the lemma after XIII. 2. The lemmas on XI. 23, XIII. 13, XIII. 18, besides coming after the propositions to which they relate, are not very necessary in themselves and, as regards the lemma to XIII. 13, it is to be noticed that the writer of a gloss in the proposition could not have had it, and the words "as will be proved afterwards" in the text are rightly suspected owing to differences between the MS. readings. The lemma to XII. 2 also, to which Simson raised objection, comes *after* the proposition; but, if it is rejected, the words "as was proved before" used in XII. 5 and 18, and referring to this lemma, must be struck out.

(2) Reasons of substance are fatal to the lemma before X. 60, which is really assumed in X. 44 and therefore should have appeared there if anywhere, and to the lemma on X. 20, which tries to prove what is already stated in X. Def. 4.

We now come to the remaining lemmas in Book X., eleven in number, which come *before* the propositions to which they relate and remove difficulties in the way of their demonstration. That before X. 42 introduces a set of propositions with the words "that the said irrational straight lines are uniquely divided ... we will prove after premising the following lemma," and it is not possible to suppose that these words are due to an interpolator; nor are there any objections to the lemmas before X. 14, 17, 22, 33, 54, except perhaps that they are rather easy. The lemma before X. 10 and X. 10 itself should probably be removed from the *Elements*; for X. 10 really uses the following proposition X. 11, which is moreover numbered 10 by the first hand in P, and the words in X. 10 referring to the lemma "for we learnt (how to do this)" betray the interpolator. Heiberg gives reason also for rejecting the lemmas before X. 19 and 24 with the words "in any of the aforesaid ways" (omitted in the Theonine MSS.) in the enunciations of X. 19, 24 and in the *exposition* of X. 20. Lastly, the lemmas before X. 29 may be genuine, though there is an addition to the second of them which is spurious.

Heiberg includes under this heading of interpolated lemmas two which purport to be substantive propositions, XI. 38 and XIII. 6. These must be rejected as spurious for reasons which will be found in detail in my notes on XI. 37 and XIII. 6 respectively. The latter proposition is only quoted once (in XIII. 17); probably the words quoting it (with *γραμμῇ* instead of *εὐθείᾳ*) are themselves interpolated, and Euclid thought the fact stated a sufficiently obvious inference from XIII. 1.

III. Porisms (or corollaries).

Most of the porisms in the text are both genuine and necessary; but some are shown by differences in the MSS. not to be so, e.g. those to I. 15 (though Proclus has it), III. 31 and VI. 20 (Por. 2). Sometimes parts of porisms are interpolated. Such are the last few lines in the porisms to IV. 5, VI. 8; the latter addition is proved later by

means of VI. 4, 8, so that the writer of these proofs could not have had the addition to VI. 8 Por. before him. Lastly, interpolators have added a sort of proof to some porisms, as though they were not quite obvious enough; but to add a demonstration is inconsistent with the idea of a porism, which, according to Proclus, is a by-product of a proposition appearing without our seeking it.

IV. Scholia.

Several interpolated scholia betray themselves by their wording, e.g. those given by Heiberg in the Appendix to Book X. and containing the words *καλεῖ, ἐκάλεσε* ("he calls" or "called"); these scholia were apparently written as marginal notes before Theon's time, and, being adopted as such by Theon, found their way into the text in P and some of the Theonine MSS. The same thing no doubt accounts for the interpolated analyses and syntheses to XIII. 1—5, as to which see my note on XIII. 1.

V. Interpolations in Book X.

First comes the proposition "*Let it be proposed to us* to show that in square figures the diameter is incommensurable in length with the side," which, with a scholium after it, ends the tenth Book. The form of the enunciation is suspicious enough and the proposition, the proof of which is indicated by Aristotle and perhaps was Pythagorean, is perfectly unnecessary when X. 9 has preceded. The scholium ends with remarks about commensurable and incommensurable *solids*, which are of course out of place before the Books on solids. The scholiast on Book X. alludes to this particular scholium as being due to "Theon and some others." But it is doubtless much more ancient, and may, as Heiberg conjectures, have been the beginning of Apollonius' more advanced treatise on incommensurables. Not only is everything in Book X. after X. 115 interpolated, but Heiberg doubts the genuineness even of X. 112—115, on the ground that X. 111 rounds off the theory of incommensurables as we want it in the Books on solid geometry, while X. 112—115 are not really connected with what precedes, nor wanted for the later Books, but seem to form the starting-point of a new and more elaborate theory of irrationals.

VI. Other minor interpolations are found of the same character as those above attributed to Theon. First there are two places (XI. 35 and XI. 26) where, after "similarly we shall prove" and "for the same reason," an actual proof is nevertheless given. Clearly the proofs are interpolated; and there are other similar interpolations. There are also interpolations of intermediate steps in proofs, unnecessary explanations and so on, as to which I need not enter into details.

Lastly, following Heiberg's order, I come to

VII. Interpolated definitions, axioms etc.

Apart from VI. Def. 5 (which may have been interpolated by Theon although it is found written in the margin of P by the first hand), the definition of a segment of a circle in Book I. is interpolated, as is clear from the fact that it occurs in a more appropriate place in Book III. and Proclus omits it. VI. Def. 2 (reciprocal figures) is rightly condemned by Simson—perhaps it was taken from Heron—and

Heiberg would reject VII. Def. 10, as to which see my note on that definition. Lastly the double definition of a solid angle (XI. Def. 11) constitutes a difficulty. The use of the word ἐπιφάνεια suggests that the first definition may have been older than Euclid, and he may have quoted it from older *elements*, especially as his own definition which follows only includes solid angles contained by *planes*, whereas the other includes other sorts (cf. the words γραμμῶν, γραμμαῖς) which are also distinguished by Heron (Def. 22). If the first definition had come last, it could have been rejected without hesitation: but it is not so easy to reject the first part up to and including "otherwise" (ἄλλως). No difficulty need be felt about the definitions of "oblong," "rhombus," and "rhomboid," which are not actually used in the *Elements*; they were no doubt taken from earlier *elements* and given for the sake of completeness.

As regards the axioms or, as they are called in the text, *common notions* (κοινὰ ἔννοιαι), it is to be observed that Proclus says¹ that Apollonius tried to prove "the axioms," and he gives Apollonius' attempt to prove Axiom 1. This shows at all events that Apollonius had *some* of the axioms now appearing in the text. But how could Apollonius have taken a controversial line against Euclid on the subject of axioms if these axioms had not been Euclid's to his knowledge? And, if they had been interpolated between Euclid's time and his own, how could Apollonius, living so comparatively short a time after Euclid, have been ignorant of the fact? Therefore *some* of the axioms are Euclid's (whether he called them *common notions*, or *axioms*, as is perhaps more likely since Proclus calls them axioms): and we need not hesitate to accept as genuine the first three discussed by Proclus, viz. (1) things equal to the same equal to one another, (2) if equals be added to equals, wholes equal, (3) if equals be subtracted from equals, remainders equal. The other two mentioned by Proclus (whole greater than part, and congruent figures equal) are more doubtful, since they are omitted by Heron, Martianus Capella, and others. The axiom that "two lines cannot enclose a space" is however clearly an interpolation due to the fact that I. 4 appeared to require it. The others about equals added to unequals, doubles of the same thing, and halves of the same thing are also interpolated; they are connected with other interpolations, and Proclus clearly used some source which did not contain them.

Euclid evidently limited his formal axioms to those which seemed to him most essential and of the widest application; for he not unfrequently assumes other things as axiomatic, e.g. in VII. 28 that, if a number measures two numbers, it measures their difference.

The differences of reading appearing in Proclus suggest the question of the comparative purity of the sources used by Proclus, Heron and others, and of our text. The omission of the definition of a segment in Book I, and of the old gloss "which is called the circumference" in I. Def. 15 (also omitted by Heron, Taurus, Sextus

¹ Proclus, pp. 194, 10 sqq.

Empiricus and others) indicates that Proclus had better sources than we have; and Heiberg gives other cases where Proclus omits words which are in all our MSS. and where Proclus' reading should perhaps be preferred. But, except in these instances (where Proclus may have drawn from some ancient source such as one of the older commentaries), Proclus' MS. does not seem to have been among the best. Often it agrees with our worst MSS., sometimes it agrees with F where F alone has a certain reading in the text, so that (e.g. in I. 15 Por.) the common reading of Proclus and F must be rejected, thrice only does it agree with P alone, sometimes it agrees with P and some Theonine MSS., and once it agrees with the Theonine MSS. against P and other sources.

Of the other external sources, those which are older than Theon generally agree with our best MSS., e.g. Heron, allowing for the difference in the plan of his definitions and the somewhat free adaptation to his purpose of the Euclidean definitions in Books X., XI.

Heiberg concludes that the *Elements* were most spoiled by interpolations about the 3rd c., for Sextus Empiricus had a correct text, while Iamblichus had an interpolated one; but doubtless the purer text continued for a long time in circulation, as we conclude from the fact that our MSS. are free from interpolations already found in Iamblichus' MS.

CHAPTER VI.

THE SCHOLIA.

HEIBERG has collected scholia, to the number of about 1500, in Vol. v. of his edition of Euclid, and has also discussed and classified them in a separate short treatise, in which he added a few others¹.

These scholia cannot be regarded as doing much to facilitate the reading of the *Elements*. As a rule, they contain only such observations as any intelligent reader could make for himself. Among the few exceptions are XI. Nos. 33, 35 (where XI. 22, 23 are extended to solid angles formed by any number of plane angles), XII. No. 85 (where an assumption tacitly made by Euclid in XII. 17 is proved), IX. Nos. 28, 29 (where the scholiast has pointed out the error in the text of IX. 19).

Nor are they very rich in historical information; they cannot be compared in this respect with Proclus' commentary on Book I. or with those of Eutocius on Archimedes and Apollonius. But even under this head they contain some things of interest, e.g. II. No. 11 explaining that the gnomon was invented by geometers for the sake of brevity, and that its name was suggested by an incidental characteristic, namely that "from it the whole is known (*γνωρίζεται*), either of the whole area or of the remainder, when it (the *γνώμων*) is either placed round or taken away"; II. No. 13, also on the gnomon; IV. No. 2 stating that Book IV. was the discovery of the Pythagoreans; V. No. 1 attributing the content of Book V. to Eudoxus; X. No. 1 with its allusion to the discovery of incommensurability by the Pythagoreans and to Apollonius' work on irrationals; X. No. 62 definitely attributing X. 9 to Theaetetus; XIII. No. 1 about the "Platonic" figures, which attributes the cube, the pyramid, and the dodecahedron to the Pythagoreans, and the octahedron and icosahedron to Theaetetus.

Sometimes the scholia are useful in connexion with the settlement of the text, (1) directly, e.g. III. No. 16 on the interpolation of the word "within" (*ἐντός*) in the enunciation of III. 6, and X. No. 1 alluding to the discussion by "Theon and some others" of irrational "surfaces" and "solids," as well as "lines," from which we may

¹ Heiberg, *Om Scholierne til Euklids Elementer*, Kjøbenhavn, 1888. The tract is written in Danish, but, fortunately for those who do not read Danish easily, the author has appended (pp. 70—78) a résumé in French.

conclude that the scholium at the end of Book X. is not genuine; (2) indirectly in that they sometimes throw light on the connexion of certain MSS.

Lastly, they have their historical importance as enabling us to judge of the state of mathematical science at the times when they were written.

Before passing to the classification of the scholia, Heiberg remarks that we must separate from them a number of additions in the nature of scholia which are found in the text of our MSS. but which can, in one way or another, be proved to be spurious. As they are found both in P and in the Theonine MSS., they must have been in the MSS. anterior to Theon (4th c.). But they are, in great part, only found in the margin of P and the Theonine MSS.; in V they are half in the text and half in the margin. This can hardly be explained except on the supposition that these additions were originally (in the MSS. before Theon) in the margin, and that Theon kept them there in his edition, but that they afterwards found their way gradually into the text of P as well as of the Theonine MSS., or were omitted altogether, while particular MSS. have in certain places preserved the old arrangement. Of such spurious additions Heiberg enumerates the following: the axiom about equals subtracted from unequals, the last lines of the porism to VI. 8, second porisms to V. 19 and to VI. 20, the porism to III. 31, VI. Def. 5, various additions in Book X., the analyses and syntheses of XIII. 1—5, and the proposition XIII. 6.

The two first classes of scholia distinguished by Heiberg are denoted by the convenient abbreviations "Schol. Vat." and "Schol. Vind."

I. Schol. Vat.

It is first necessary to set out the letters by which Heiberg denotes certain collections of scholia.

P = Scholia in P written by the first hand.

B = Scholia in B by a hand of the same date as the MS. itself, generally that of Arethas.

F = Scholia in F by the first hand.

Vat. = Scholia of the Vatican MS. 204 of the 10th c., which has these scholia on leaves 198—205 (the end is missing) as an independent collection. It does not contain the text of the *Elements*.

V^c = Scholia found on leaves 283—292 of V and written in the same hand as that part of the MS. itself which begins at leaf 235.

Vat. 192 = a Vatican MS. of the 14th c. which contains, after (1) the *Elements* I.—XIII. (without scholia), (2) the *Data* with scholia, (3) Marinus on the *Data*, the Schol. Vat. as an independent collection and in their entirety, beginning with I. No. 88 and ending with XIII. No. 44.

The Schol. Vat., the most ancient and important collection of scholia, comprise those which are found in PBF Vat. and, from VII. 12 to IX. 15, in PB Vat. only, since in that portion of the *Elements* F was restored by a later hand without scholia; they also include I.

No. 88 which only happens to be erased in F, and IX. Nos. 28, 29 which may be left out because F. here has a different text. In F and Vat. the collection ends with Book x.; but it must also include Schol. PB of Books XI.—XIII., since these are found along with Schol. Vat. to Books I.—X. in several MSS. (of which Vat. 192¹ is one) as a separate collection. The Schol. Vat. to Books x.—XIII. are also found in the collection V^c (where, curiously enough, XIII. Nos. 43, 44 are at the beginning). The Schol. Vat. accordingly include Schol. PBV^c Vat. 192, and doubtless also those which are found in two of these sources. The total number of scholia classified by Heiberg as Schol. Vat. is 138.

As regards the contents of Schol. Vat. Heiberg has the following observations. The thirteen scholia to Book I. are extracts made from Proclus by a writer thoroughly conversant with the subject, and cleverly recast (with some additions). Their author does not seem to have had the two lacunae which our text of Proclus has (at the end of the note on I. 36 and the beginning of the next note, and at the beginning of the note on I. 43), for the scholia I. Nos. 125 and 137 seem to fill the gaps appropriately, at least in part. In some passages he had better readings than our MSS. have. The rest of Schol. Vat. (on Books II.—XIII.) are essentially of the same character as those on Book I., containing prolegomena, remarks on the object of the propositions, critical remarks on the text, converses, lemmas; they are, in general, exact and true to tradition. The reason of the resemblance between them and Proclus appears to be due to the fact that they have their origin in the commentary of Pappus, of which we know that Proclus also made use. In support of the view that Pappus is the source, Heiberg places some of the Schol. Vat. to Book x. side by side with passages from the commentary of Pappus in the Arabic translation discovered by Woepcke¹; he also refers to the striking confirmation afforded by the fact that XII. No. 2 contains the solution of the problem of inscribing in a given circle a polygon similar to a polygon inscribed in another circle, which problem Eutocius says² that Pappus gave in his commentary on the *Elements*.

But, on the other hand, Schol. Vat. contain some things which cannot have come from Pappus, e.g. the allusion in x. No. 1 to Theon and irrational surfaces and solids, Theon being later than Pappus; III. No. 10 about *porisms* is more like Proclus' treatment of the subject than Pappus', though one expression recalls that of Pappus about *forming* (σχηματίζεσθαι) the enunciations of porisms like those of either theorems or problems.

The Schol. Vat. give us important indications as regards the text of the *Elements* as Pappus had it. In particular, they show that he could not have had in his text certain of the lemmas in Book x. For example, three of these are identical with what we find in Schol.

¹ *Om Scholierne til Euklids Elementer*, pp. 11, 12 : cf. *Euklid-Studien*, pp. 170, 171; Woepcke, *Mémoires présent. à l'Acad. des Sciences*, 1856, XIV. p. 658 sqq.

² Archimedes, ed. Heiberg, III. p. 28, 19—22.

Vat. (the lemma to x. 17 = Schol. x. No. 106, and the lemmas to x. 54, 60 come in Schol. x. No. 328); and it is not possible to suppose that these lemmas, if they were already in the text, would also be given as scholia. Of these three lemmas, that before x. 60 has already been condemned for other reasons; the other two, unobjectionable in themselves, must be rejected on the ground now stated. There were four others against which Heiberg found nothing to urge when writing his prolegomena to Vol. v., viz. the lemmas before x. 42, x. 14, x. 22 and x. 33. Of these, the lemma to x. 22 is not reconcilable with Schol. x. No. 161, which takes up the assumption in the text of Eucl. x. 22 as if no lemma had gone before. The lemma to x. 42, which, on account of the words introducing it (see p. 60 above), Heiberg at first hesitated to regard as an interpolation, is identical with Schol. x. No. 270. It is true that in Schol. x. No. 269 we find the words "this lemma has been proved before (*ἐν τοῖς ἔμπροσθεν*), but it shall also be proved now for convenience' sake (*τοῦ ἐτοίμου ἕνεκα*)," and it is possible to suppose that "before" may mean in Euclid's text *before* x. 42; but a proof in that place would surely have been as "convenient" as could be desired, and it is therefore more probable that the proof had been given by *Pappus* in some earlier place. (It may be added that the lemma to x. 14, which is identical with the lemma to XI. 23, condemned on other grounds, is for that reason open to suspicion.)

Heiberg's conclusion is that *all* the lemmas are spurious, and that most or all of them have found their way into the text from *Pappus'* commentary, though at a time anterior to Theon's edition, since they are found in all our MSS. This enables us to fix a date for these interpolations, namely the first half of the 4th c.

Of course *Pappus* had not in his text the interpolations which, from the fact of their appearing only in some of our MSS., are seen to be later than those above-mentioned. Such are the lemmas which are found in the text of V only after x. 29 and x. 31 respectively and are given in Heiberg's Appendix to Book x. (numbered 10 and 11). On the other hand it appears from *Woepcke's tract*¹ that *Pappus* already had x. 115 in his text: though it does not follow from this that the proposition is genuine but only that interpolations began very early.

Theon interpolated a proposition (or lemma) between x. 12 and x. 13 (No. 5 in Heiberg's Appendix). Schol. Vat. has the same thing (x. No. 125): The writer of the scholia therefore did not find this lemma in the text. Schol. Vat. IX. Nos. 28, 29 show that neither did he find in his text the alterations which Theon made in Eucl. IX. 19; the scholia in fact only agree with the text of P, not with Theon's. This suggests that Schol. Vat. were written for use with a MS. of the ante-Theonine recension such as P is. This probability is further confirmed by a certain independence which P shows in several places when compared with the Theonine MSS. Not only has P better readings in some passages, but more substantial divergences; and,

¹ *Woepcke, op. cit. p. 702.*

in particular, the absence in P of three notes of a historical character which are added, wholly or partly from Proclus, in the Theonine MSS. attests an independent and more primitive point of view in P.

In view of the distinctive character of P, it is possible that some of the scholia found in it in the first hand, but not in the other sources of Schol. Vat., also belong to that collection; and several circumstances confirm this. Schol. XIII. No. 45, found in P only, which relates to a passage in Eucl. XIII. 13, shows that certain words in the text, though older than Theon, are interpolated; and, as the scholium is itself older than Theon, is headed "*third lemma*," and follows a "*second lemma*" relating to a passage in the text immediately preceding, which "*second lemma*" belongs to Schol. Vat. and is taken from Pappus, the "*third*" in all probability came from Pappus also. The same is true of Schol. XII. No. 72 and XIII. No. 69, which are respectively identical with the propositions *vulgo* XI. 38 (Heiberg, App. to Book XI, No. 3) and XIII. 6; for both of these interpolations are older than Theon. Moreover most of the scholia which P in the first hand alone has are of the same character as Schol. Vat. Thus VII. No. 7 and XIII. No. 1 introducing Books VII. and XIII. respectively are of the same historical character as several of Schol. Vat.; that VII. No. 7 appears in the *text* of P at the beginning of Book VII. constitutes no difficulty. There are a number of *converses*, remarks on the relation of propositions to one another, explanations such as XII. No. 89 in which it is remarked that Φ , Ω in Euclid's figure to XII. 17 (Z , V in my figure) are really the same point but that this makes no difference in the proof. Two other Schol. P on XII. 17 are connected by their headings with XII. No. 72 mentioned above. XI. No. 10 (P) is only another form of XI. No. 11 (B); and B often, alone with P, has preserved Schol. Vat. On the whole Heiberg considers some 40 scholia found in P alone to belong to Schol. Vat.

The history of Schol. Vat. appears to have been, in its main outlines, the following. They were put together after 500 A.D., since they contain extracts from Proclus, to which we ought not to assign a date too near to that of Proclus' work itself; and they must at least be earlier than the latter half of the 9th c., in which B was written. As there must evidently have been several intermediate links between the archetype and B, we must assign them rather to the first half of the period between the two dates, and it is not improbable that they were a new product of the great development of mathematical studies at the end of the 6th c. (Isidorus of Miletus). The author extracted what he found of interest in the commentary of Proclus on Book I. and in that of Pappus on the rest of the work, and put these extracts in the margin of a MS. of the class of P. As there are no scholia to I. 1—22, the first leaves of the archetype or of one of the earliest copies must have been lost at an early date, and it was from that mutilated copy that partly P and partly a MS. of the Theonine class were taken, the scholia being put in the margin in both. Then the collection spread through the Theonine MSS., gradually losing some

scholia which could not be read or understood, or which were accidentally or deliberately omitted. Next it was extracted from one of these MSS. and made into a separate work which has been preserved, in part, in its entirety (Vat. 192 etc.) and, in part, divided into sections, so that the scholia to Books X.—XIII. were detached (V^c). It had the same fate in the MSS. which kept the original arrangement (in the margin), and in consequence there are some MSS. where the scholia to the stereometric Books are missing, those Books having come to be less read in the period of decadence. It is from one of these MSS. that the collection was extracted as a separate work such as we find it in Vat. (10th c.).

II. The second great division of the scholia is **Schol. Vind.**

This title is taken from the Viennese MS. (V), and the letters used by Heiberg to indicate the sources here in question are as follows.

V^a = scholia in V written by the same hand that copied the MS. itself from fol. 235 onward.

q = scholia of the Paris MS. 2344 (q) written by the first hand.

l = scholia of the Florence MS. Laurent. XXVIII, 2 written in the 13th—14th c., mostly in the first hand, but partly in two later hands.

V^b = scholia in V written by the same hand as the first part (leaves 1—183) of the MS. itself; V^b wrote his scholia after V^a .

q^1 = scholia of the Paris MS. (q) found here and there in another hand of early date.

Schol. Vind. include scholia found in V^aq . l is nearly related to q ; and in fact the three MSS. which, so far as Euclid's text is concerned, show no direct interdependence, are, as regards their scholia, derived from one original. Heiberg proves this by reference to the readings of the three in two passages (found in Schol. I. No. 109 and X. No. 39 respectively). The common source must have contained, besides the scholia found in the three MSS. V^aq^1 , those also which are contained in two of them, for it is more unlikely that two of the three should contain common interpolations than that a particular scholium should drop out of one of them. Besides V^a and q , the scholia V^b and q^1 must equally be referred to Schol. Vind., since the greater part of their scholia are found in l . There is a lacuna in q from Eucl. VIII. 25 to IX. 14, so that for this portion of the *Elements* Schol. Vind. are represented by l only. Heiberg gives about 450 numbers in all as belonging to this collection.

Schol. Vind. did not all come from one source; this is shown by differences of substance, e.g. between X. Nos. 36 and 39, and by differences of time of writing: e.g. VI. No. 52 refers at the beginning to No. 55 with the words "as the scholium has it" and is therefore later than that scholium; X. No. 247 is also later than X. No. 246.

The scholia to Book I. are here also extracts from Proclus, but more copious and more verbatim than in Schol. Vat. The author has not always understood Proclus; and he had a text as bad as that of our MSS., with the same lacunae. The scholia to the other

Books are partly drawn (1) from Schol. Vat., the MSS. representing Schol. Vind. and Schol. Vat. in these cases showing nearly all possible combinations; but there is no certain trace in Schol. Vind. of the scholia peculiar to P. The author used a copy of Schol. Vat. in the form in which they were attached to the Theonine text; thus Schol. Vind. correspond to BF Vat., where these diverge from P, and especially closely to B. Besides Schol. Vat., the editors of Schol. Vind. used (2) other old collections of scholia of which we find traces in B and F; Schol. Vind. have also some scholia common with b. The scholia which Schol. Vind. have in common with BF come from two different sources, and were apparently afterwards introduced into the other MSS.; one result of this is that several scholia are reproduced twice.

But, besides the scholia derived from these sources, Schol. Vind. contain a large number of others of late date, characterised by incorrect language or by triviality of content (there are many examples in numbers, citations of propositions used, absurd ἀπορίαι, and the like). Unlike Schol. Vat., these scholia often quote words from Euclid as a heading (in one case a heading is inserted in Schol. Vind. where a scholium without the heading is quoted from Schol. Vat., see v. No. 14). The explanations given often presuppose very little knowledge on the part of the reader and frequently contain obscurities and gross errors.

Schol. Vind. were collected for use with a MS. of the Theonine class; this follows from the fact that they contain a note on the proposition *vulgo* VII. 22 interpolated by Theon (given in Heiberg's App. to Vol. II. p. 430). Since the scholium to VII. 39 given in V and p in the text after the title of Book VIII. quotes the proposition as VII. 39, it follows that this scholium must have been written before the interpolation of the two propositions *vulgo* VII. 20, 22; Schol. Vind. contain (VII. No. 80) the first sentence of it, but without the heading referring to VII. 39. Schol. VII. No. 97 quotes VII. 33 as VII. 34, so that the proposition *vulgo* VII. 22 may have stood in the scholiast's text but not the later interpolation *vulgo* VII. 20 (later because only found in B in the margin by the first hand). Of course the scholiast had also the interpolations earlier than Theon.

For the date of the collection we have a lower limit in the date (12th c.) of MSS. in which the scholia appear. That it was not much earlier than the 12th c. is indicated (1) by the poverty of its contents, (2) by the quality of the MS. of Proclus which was used in the compilation of it (the Munich MS. used by Friedlein with which the scholiast's excerpts are essentially in agreement belongs to the 11th—12th c.), (3) by the fact that Schol. Vind. appear only in MSS. of the 12th c. and no trace of them is found in our MSS. belonging to the 9th—10th c. in which Schol. Vat. are found. The collection may therefore probably be assigned to the 11th c. Perhaps it may be in part due to Psellus who lived towards the end of that century: for in a Florence MS. (Magliabecch. XI, 53 of the 15th c.) containing a mathematical compendium intended for use in the reading of Aristotle

the scholia I. Nos. 40 and 49 appear with the name of Psellus attached.

Schol. Vind. are not found without the admixture of foreign elements in any of our three sources. In l there are only very few such in the first hand. In q there are several new scholia in the first hand, for the most part due to the copyist himself. The collection of scholia on Book x. in q (Heiberg's q^c) is also in the first hand; it is not original, and it may perhaps be due to Psellus (Maglb. has some definitions of Book x. with a heading "scholia of ... Michael Psellus on the definitions of Euclid's 10th *Element*" and Schol. x. No. 9), whose name must have been attached to it in the common source of Maglb. and q; to a great extent it consists of extracts from Schol. Vind. taken from the same source as VI. The scholia q¹ (in an ancient hand in q), confined to Book II., partly belong to Schol. Vind. and partly correspond to b¹ (Bologna MS.). q^a and q^b are in one hand (Theodorus Antiochita), the nearest to the first hand of q; they are doubtless due to an early possessor of the MS. of whom we know nothing more.

V^a has, besides Schol. Vind., a number of scholia which also appear in other MSS., one in BFb, some others in P, and some in v (Codex Vat. 1038, 13th c.); these scholia were taken from a source in which many abbreviations were used, as they were often misunderstood by V^a. Other scholia in V^a which are not found in the older sources—some appearing in V^a alone—are also not original, as is proved by mistakes or corruptions which they contain; some others may be due to the copyist himself.

V^b seldom has scholia common with the other older sources; for the most part they either appear in V^b alone or only in the later sources as v or F² (later scholia in F), some being original, others not. In Book x. V^b has three series of numerical examples, (1) with Greek numerals, (2) alternatives added later, also mostly with Greek numerals, (3) with Arabic numerals. The last class were probably the work of the copyist himself. These examples (cf. p. 74 below) show the facility with which the Byzantines made calculations at the date of the MS. (12th c.). They prove also that the use of the Arabic numerals (in the East-Arabian form) was thoroughly established in the 12th c.; they were actually known to the Byzantines a century earlier, since they appear, in the first hand, in an Escorial MS. of the 11th c.

Of collections in other hands in V distinguished by Heiberg (see preface to Vol. v.), V¹ has very few scholia which are found in other sources, the greater part being original; V², V³ are the work of the copyist himself; V⁴ are so in part only, and contain several scholia from Schol. Vat. and other sources. V³ and V⁴ are later than 13th—14th c., since they are not found in f (cod. Laurent. xxviii, 6) which was copied from V and contains, besides V^a V^b, the greater part of V¹ and VI. No. 20 of V² (in the text).

In P there are, besides P³ (a quite late hand, probably one of the old *Scriptores Graeci* at the Vatican), two late hands (P²), one of which has some new and independent scholia, while the other has

added the greater part of Schol. Vind., partly in the margin and partly on pieces of leaves stitched on.

Our sources for Schol. Vat. also contain other elements. In P there were introduced a certain number of extracts from Proclus, to supplement Schol. Vat. to Book I.; they are all written with a different ink from that used for the oldest part of the MS., and the text is inferior. There are additions in the other sources of Schol. Vat. (F and B) which point to a common source for FB and which are nearly all found in other MSS., and, in particular, in Schol. Vind., which also used the same source; that they are not assignable to Schol. Vat. results only from their not being found in Vat. Of other additions in F, some are peculiar to F and some common to it and b; but they are not original. F² (scholia in a later hand in F) contains three original scholia; the rest come from V. B contains, besides scholia common to it and F, b or other sources, several scholia which seem to have been put together by Arethas, who wrote at least a part of them with his own hand.

Heiberg has satisfied himself, by a closer study of b, that the scholia which he denotes by b, β and b¹ are by one hand; they are mostly to be found in other sources as well, though some are original. By the same hand (Theodorus Cabasilas, 15th c.) are also the scholia denoted by b², B², b³ and B³. These scholia come in great part from Schol. Vind., and in making these extracts Theodorus probably used one of our sources, l, mistakes in which often correspond to those of Theodorus. To one scholium is attached the name of Demetrius (who must be Demetrius Cydonius, a friend of Nicolaus Cabasilas, 14th c.); but it could not have been written by him, since it appears in B and Schol. Vind. Nor are all the scholia which bear the name of Theodorus due to Theodorus himself, though some are so.

As B³ (a late hand in B) contains several of the original scholia of b², B³ must have used b itself as his source, and, as all the scholia in B³ are in b, the latter is also the source of the scholia in B³ which are found in other MSS. B and b were therefore, in the 15th c., in the hands of the same person; this explains, too, the fact that b in a late hand has some scholia which can only come from B. We arrive then at the conclusion that Theodorus Cabasilas, in the 15th c., owned both the MSS. B and b, and that he transferred to B scholia which he had before written in b, either independently or after other sources, and inversely transferred some scholia from B to b. Further, B² are earlier than Theodorus Cabasilas, who certainly himself wrote B³ as well as b² and b³.

An author's name is also attached to the scholia VI. No. 6 and x. No. 223, which are attributed to Maximus Planudes (end of 13th c.) along with scholia on I. 31, x. 14 and x. 18 found in l in a quite late hand and published on pp. 46, 47 of Heiberg's dissertation. These seem to have been taken from lectures of Planudes on the *Elements* by a pupil who used l as his copy.

There are also in l two other Byzantine scholia, written by a late hand, and bearing the names Ioannes and Pediasimus respectively;

these must in like manner have been written by a pupil after lectures of Ioannes Pediasimus (first half of 14th c.), and this pupil must also have used I.

Before these scholia were edited by Heiberg, very few of them had been published in the original Greek. The Basel *editio princeps* has a few (v. No. 1, VI. Nos. 3, 4 and some in Book x.) which are taken, some from the Paris MS. (Paris. Gr. 2343) used by Grynaeus, others probably from the Venice MS. (Marc. 301) also used by him; one published by Heiberg, not in his edition of Euclid but in his paper on the scholia, may also be from Venet. 301, but appears also in Paris. Gr. 2342. The scholia in the Basel edition passed into the Oxford edition in the text, and were also given by August in the Appendix to his Vol. II.

Several specimens of the two series of scholia (Vat. and Vind.) were published by C. Wachsmuth (*Rhein. Mus.* XVIII. p. 132 sqq.) and by Knoche (*Untersuchungen über die neu aufgefundenen Scholien des Proklus*, Herford, 1865).

The scholia published in Latin were much more numerous. G. Valla (*De expetendis et fugiendis rebus*, 1501) reproduced apparently some 200 of the scholia included in Heiberg's edition. Several of these he obtained from two Modena MSS. which at one time were in his possession (Mutin. III B, 4 and II E, 9, both of the 15th c.); but he must have used another source as well, containing extracts from other series of scholia, notably Schol. Vind. with which he has some 87 scholia in common. He has also several that are new.

Commandinus included in his translation under the title "Scholia antiqua" the greater part of the Schol. Vat. which he certainly obtained from a MS. of the class of Vat. 192; on the whole he adhered closely to the Greek text. Besides these scholia Commandinus has the scholia and lemmas which he found in the Basel *editio princeps*, and also three other scholia not belonging to Schol. Vat., as well as one new scholium (to XII. 13) not included in Heiberg's edition, which are distinguished by different type and were doubtless taken from the Greek MS. used by him along with the Basel edition.

In Conrad Dasypodius' *Lexicon mathematicum* published in 1573 there is (on fol. 42—44) "Graecum scholion in definitiones Euclidis libri quinti elementorum appendicis loco propter pagellas vacantes annexum." This contains four scholia, and part of two others, published in Heiberg's edition, with some variations of readings, and with some new matter added (for which see pp. 64—6 of Heiberg's pamphlet). The source of these scholia is revealed to us by another work of Dasypodius, *Isaaci Monachi Scholia in Euclidis elementorum geometriae sex priores libros per C. Dasypodium in latinum sermonem translata et in lucem edita* (1579). This work contains, besides excerpts from Proclus on Book I. (in part closely related to Schol. Vind.), some 30 scholia included in Heiberg's edition, several new scholia, and the above-mentioned scholia to the definitions of Book v. published in Greek in 1573. After the scholia follow "Isaaci Monachi

prolegomena in Euclidis Elementorum geometriae libros" (two definitions of geometry) and "Varia miscellanea ad geometriae cognitionem necessaria ab Isaaco Monacho collecta" (mostly the same as pp. 252, 24—272, 27 in the *Variae Collectiones* included in Hultsch's Heron); lastly, a note of Dasypodius to the reader says that these scholia were taken "ex clarissimi viri Joannis Sambuci antiquo codice manu propria Isaaci Monachi scripto." Isaak Monachus is doubtless Isaak Argyrus, 14th c.; and Dasypodius used a MS. in which, besides the passage in Hultsch's *Variae Collectiones*, there were a number of scholia marked in the margin with the name of Isaak (cf. those in b under the name of Theodorus Cabasilas). Whether the new scholia are original cannot be decided until they are published in Greek; but it is not improbable that they are at all events independent arrangements of older scholia. All but five of the others, and all but one of the Greek scholia to Book v., are taken from Schol. Vat.; three of the excepted ones are from Schol. Vind., and the other three seem to come from F (where some words of them are illegible, but can be supplied by means of Mut. III B, 4, which has these three scholia and generally shows a certain likeness to Isaak's scholia).

Dasypodius also published in 1564 the arithmetical commentary of Barlaam the monk (14th c.) on Eucl. Book II., which finds a place in Appendix IV. to the Scholia in Heiberg's edition.

Hultsch has some remarks on the origin of the scholia¹. He observes that the scholia to Book I. contain a considerable portion of Geminus' commentary on the definitions and are specially valuable because they contain extracts from Geminus *only*, whereas Proclus, though drawing mainly upon him, quotes from others as well. On the postulates and axioms the scholia give more than is found in Proclus. Hultsch conjectures that the scholium on Book v., No. 3, attributing the discovery of the theorems to Eudoxus but their arrangement to Euclid, represents the tradition going back to Geminus, and that the scholium XIII., No. 1, has the same origin.

A word should be added about the numerical illustrations of Euclid's propositions in the scholia to Book x. They contain a large number of calculations with sexagesimal fractions²; the fractions go as far as *fourth-sixtieths* ($1/60^4$). Numbers expressed in these fractions are handled with skill and include some results of surprising accuracy³.

¹ Art. "Eukleides" in Pauly-Wissowa's *Real-Encyclopädie*.

² Hultsch has written upon these in *Bibliotheca Mathematica*, v₃, 1904, pp. 225—233.

³ Thus $\sqrt{(27)}$ is given (allowing for a slight correction by means of the context) as $5\ 11'\ 46''\ 10'''$, which gives for $\sqrt{3}$ the value $1\ 43'\ 55''\ 23'''$, being the same value as that given by Hipparchus in his Table of Chords, and correct to the seventh decimal place. Similarly $\sqrt{8}$ is given as $2\ 49'\ 42''\ 20''' 10''''$, which is equivalent to $\sqrt{2} = 1.41421335$. Hultsch gives instances of the various operations, addition, subtraction, etc., carried out in these fractions, and shows how the extraction of the square root was effected. Cf. T. L. Heath, *History of Greek Mathematics*, I., pp. 59—63.

CHAPTER VII.

EUCLID IN ARABIA.

WE are told by Hāji Khalfa¹ that the Caliph al-Mansūr (754-775) sent a mission to the Byzantine Emperor as the result of which he obtained from him a copy of Euclid among other Greek books, and again that the Caliph al-Ma'mūn (813-833) obtained manuscripts of Euclid, among others, from the Byzantines. The version of the *Elements* by al-Ḥajjāj b. Yūsuf b. Maṭar is, if not the very first, at least one of the first books translated from the Greek into Arabic². According to the *Fihrist*³ it was translated by al-Ḥajjāj twice; the first translation was known as "Hārūni" ("for Hārūn"), the second bore the name "Ma'mūni" ("for al-Ma'mūn") and was the more trustworthy. Six Books of the second of these versions survive in a Leiden MS. (Codex Leidensis 399, 1) now in part published by Besthorn and Heiberg⁴. In the preface to this MS. it is stated that, in the reign of Hārūn ar-Rashīd (786-809), al-Ḥajjāj was commanded by Yaḥyā b. Khālīd b. Barmak to translate the book into Arabic. Then, when al-Ma'mūn became Caliph, as he was devoted to learning, al-Ḥajjāj saw that he would secure the favour of al-Ma'mūn "if he illustrated and expounded this book and reduced it to smaller dimensions. He accordingly left out the superfluities, filled up the gaps, corrected or removed the errors, until he had gone through the book and reduced it, when corrected and explained, to smaller dimensions, as in this copy, but without altering the substance, for the use of men endowed with ability and devoted to learning, the earlier edition being left in the hands of readers."

The *Fihrist* goes on to say that the work was next translated by Ishāq b. Hunain, and that this translation was improved by Thābit b. Qurra. This Abū Ya'qub Ishāq b. Hunain b. Ishāq al-'Ibādī (d. 910) was the son of the most famous of Arabic translators, Hunain b. Ishāq al-'Ibādī (809-873), a Christian and physician to the Caliph al-Mutawakkil (847-861). There seems to be no doubt that Ishāq, who

¹ *Lexicon bibliogr. et encyclop.* ed. Flügel, III. pp. 91, 92.

² Klamroth, *Zeitschrift der Deutschen Morgenländischen Gesellschaft*, xxxv. p. 303.

³ *Fihrist* (tr. Suter), p. 16.

⁴ *Codex Leidensis* 399, 1. *Euclidis Elementa ex interpretatione al-Ḥajjādjadschadschii cum commentariis al-Narizii*, Hauniae, part I. i. 1893, part I. ii. 1897, p. i. ii. i. 1900, part II. ii. 1905, part III. i. 1910.

must have known Greek as well as his father, made his translation direct from the Greek. The revision must apparently have been the subject of an arrangement between Ishāq and Thābit, as the latter died in 901 or nine years before Ishāq. Thābit undoubtedly consulted Greek MSS. for the purposes of his revision. This is expressly stated in a marginal note to a Hebrew version of the *Elements*, made from Ishāq's, attributed to one of two scholars belonging to the same family, viz. either to Moses b. Tibbon (about 1244–1274) or to Jakob b. Machir (who died soon after 1306)¹. Moreover Thābit observes, on the proposition which he gives as IX. 31, that he had not found this proposition and the one before it in the Greek but only in the Arabic; from which statement Klamroth draws two conclusions, (1) that the Arabs had already begun to interest themselves in the authenticity of the text and (2) that Thābit did not alter the numbers of the propositions in Ishāq's translation². The *Fihrist* also says that Yuḥannā al-Qass (i.e. "the Priest") had seen in the Greek copy in his possession the proposition in Book I. which Thābit took credit for, and that this was confirmed by Naẓīf, the physician, to whom Yuhannā had shown it. This proposition may have been wanting in Ishāq, and Thābit may have added it, but without claiming it as his own discovery³. As a fact, I. 45 is missing in the translation by al-Ḥajjāj.

The original version of Ishāq *without* the improvements by Thābit has probably not survived any more than the first of the two versions by al-Ḥajjāj; the divergences between the MSS. are apparently due to the voluntary or involuntary changes of copyists, the former class varying according to the degree of mathematical knowledge possessed by the copyists and the extent to which they were influenced by considerations of practical utility for teaching purposes⁴. Two MSS. of the Ishāq-Thābit version exist in the Bodleian Library (No. 279 belonging to the year 1238, and No. 280 written in 1260–1)⁵; Books I.—XIII. are in the Ishāq-Thābit version, the non-Euclidean Books XIV., XV. in the translation of Qusṭā b. Lūqā al-Ba'labakkī (d. about 912). The first of these MSS. (No. 279) is that (O) used by Klamroth for the purpose of his paper on the Arabian Euclid. The other MS. used by Klamroth is (K) Kjøbenhavn LXXXI, undated but probably of the 13th c., containing Books v.—xv., Books v.—x. being in the Ishāq-Thābit version, Books XI.—XIII. purporting to be in al-Ḥajjāj's translation, and Books XIV., XV. in the version of Qusṭā b. Lūqā. In not a few propositions K and O show not the slightest difference, and, even where the proofs show considerable differences, they are generally such that, by a careful comparison, it is possible to reconstruct the common archetype, so that it is fairly clear that we have in these cases, not two recensions of one translation, but arbitrarily altered and

¹ Steinschneider, *Zeitschrift für Math. u. Physik*, xxxi., hist.-litt. Abtheilung, pp. 85, 86, 99.

² Klamroth, p. 279.

³ Steinschneider, p. 88.

⁴ Klamroth, p. 306.

⁵ These MSS. are described by Nicoll and Pusey, *Catalogus cod. mss. orient. bibl. Bodleianae*, pt. II. 1835 (pp. 257–262).

shortened copies of one and the same recension¹. The Bodleian MS. No. 280 contains a preface, translated by Nicoll, which cannot be by Thābit himself because it mentions Avicenna (980–1037) and other later authors. The MS. was written at Marāḡa in the year 1260–1 and has in the margin readings and emendations from the edition of Naṣīraddīn aṭ-Ṭūsī (shortly to be mentioned) who was living at Marāḡa at the time. Is it possible that aṭ-Ṭūsī himself is the author of the preface? Be this as it may, the preface is interesting because it throws light on the liberties which the Arabians allowed themselves to take with the text. After the observation that the book (in spite of the labours of many editors) is not free from errors, obscurities, redundancies, omissions etc., and is without certain definitions necessary for the proofs, it goes on to say that the man has not yet been found who could make it perfect, and next proceeds to explain (1) that Avicenna “cut out postulates and many definitions” and attempted to clear up difficult and obscure passages, (2) that Abū’l Wafā al-Būzjānī (939–997) “introduced unnecessary additions and left out many things of great importance and entirely necessary,” inasmuch as he was too long in various places in Book VI. and too short in Book X. where he left out entirely the proofs of the *apotomae*, while he made an unsuccessful attempt to emend XII. 14, (3) that Abū Ja’far al-Khāzin (d. between 961 and 971) arranged the postulates excellently but “disturbed the number and order of the propositions, reduced several propositions to one” etc. Next the preface describes the editor’s own claims² and then ends with the sentences, “But we have kept to the order of the books and propositions in the work itself (i.e. Euclid’s) except in the twelfth and thirteenth books. For we have dealt in Book XIII. with the (solid) bodies and in Book XII. with the surfaces by themselves.”

After Thābit the *Fihrist* mentions Abū ‘Uthmān ad-Dimashqī as having translated some Books of the *Elements* including Book X. (It is Abū ‘Uthmān’s translation of Pappus’ commentary on Book X. which Woepcke discovered at Paris.) The *Fihrist* adds also that “Nazīf the physician told me that he had seen the tenth Book of Euclid in Greek, that it had 40 propositions more than the version in common circulation which had 109 propositions, and that he had determined to translate it into Arabic.”

But the third form of the Arabian Euclid actually accessible to us is the edition of Abū Ja’far Muḥ. b. Muḥ. b. al-Hasan Naṣīraddīn aṭ-Ṭūsī (whom we shall call aṭ-Ṭūsī for short), born at Tūs (in Khurāsān) in 1201 (d. 1274). This edition appeared in two forms, a larger and a smaller. The larger is said to survive in Florence only (Pal. 272 and 313, the latter MS. containing only six Books); this was published at Rome in 1594, and, remarkably enough, some copies of

¹ Klamroth, pp. 306–8.

² Steinschneider, p. 98. Heiberg has quoted the whole of this preface in the *Zeitschrift für Math. u. Physik*, XXIX., hist.-litt. Abth. p. 16.

³ This seems to include a rearrangement of the contents of Books XIV., XV. added to the *Elements*.

this edition are to be found with 12 and some with 13 Books, some with a Latin title and some without¹. But the book was printed in Arabic, so that Kästner remarks that he will say as much about it as can be said about a book which one cannot read². The shorter form, which however, in most MSS., is in 15 Books, survives at Berlin, Munich, Oxford, British Museum (974, 1334³, 1335), Paris (2465, 2466), India Office, and Constantinople; it was printed at Constantinople in 1801, and the first six Books at Calcutta in 1824⁴.

At-Tūsī's work is however not a *translation* of Euclid's text, but a re-written Euclid based on the older Arabic translations. In this respect it seems to be like the Latin version of the *Elements* by Campanus (Campano), which was first published by Erhard Ratdolt at Venice in 1482 (the first printed edition of Euclid⁵). Campanus (13th c.) was a mathematician, and it is likely enough that he allowed himself the same liberty as at-Tūsī in reproducing Euclid. Whatever may be the relation between Campanus' version and that of Athelhard of Bath (about 1120), and whether, as Curtze thinks⁶, they both used one and the same Latin version of 10th—11th c., or whether Campanus used Athelhard's version in the same way as at-Tūsī used those of his predecessors⁷, it is certain that both versions came from an Arabian source, as is evident from the occurrence of Arabic words in them⁸. Campanus' version is not of much service for the purpose of forming a judgment on the relative authenticity of the Greek and Arabian tradition; but it sometimes preserves traces of the purer source, as when it omits Theon's addition to VI. 33⁹. A curious circumstance is that, while Campanus' version agrees with at-Tūsī's in the number of the propositions in all the genuine Euclidean Books except V. and IX., it agrees with Athelhard's in having 34 propositions in Book V. (as against 25 in other versions), which confirms the view that the two are not independent, and also leads, as Klamroth says, to this dilemma: either the additions to Book V. are Athelhard's own, or he used an Arabian Euclid which is not known to us¹⁰. Heiberg also notes that Campanus' Books XIV., XV. show a certain agreement with the preface to the Thābit-Ishāq version, in which the author claims to have (1) given a method of inscribing spheres in the five regular solids, (2) carried further the solution of the problem how

¹ Suter, *Die Mathematiker und Astronomen der Araber*, p. 151. The Latin title is *Euclidis elementorum geometricorum libri tredecim. Ex traditione doctissimi Nasiridini Tusini nunc primum arabice impressi. Romae in typographia Medicea MDXCIV. Cum licentia superiorum.*

² Kästner, *Geschichte der Mathematik*, I. p. 367.

³ Suter has a note that this MS. is very old, having been copied from the original in the author's lifetime.

⁴ Suter, p. 151.

⁵ Described by Kästner, *Geschichte der Mathematik*, I. pp. 289—299, and by Weissenborn, *Die Übersetzungen des Euklid durch Campano und Zamberti*, Halle a. S., 1882, pp. 1—7. See also *infra*, Chapter VIII, p. 97.

⁶ Sonderabdruck des *Jahresberichtes über die Fortschritte der klassischen Alterthums-wissenschaft vom Okt. 1879—1882*, Berlin, 1884.

⁷ Klamroth, p. 271.

⁸ Curtze, *op. cit.* p. 20; Heiberg, *Euklid-Studien*, p. 178.

⁹ Heiberg's Euclid, vol. v. p. ci.

¹⁰ Klamroth, pp. 273—4.

to inscribe any one of the solids in any other and (3) noted the cases where this could not be done¹.

With a view to arriving at what may be called a common measure of the Arabian tradition, it is necessary to compare, in the first place, the numbers of propositions in the various Books. Hāji Khalfa says that al-Hajjāj's translation contained 468 propositions, and Thābit's 478; this is stated on the authority of aṭ-Ṭūsī, whose own edition contained 468². The fact that Thābit's version had 478 propositions is confirmed by an index in the Bodleian MS. 279 (called O by Klamroth). A register at the beginning of the Codex Leidensis 399, 1 which gives Ishāq's numbers (although the translation is that of al-Hajjāj) apparently makes the total 479 propositions (the number in Book XIV. being apparently 11, instead of the 10 of O³). I subjoin a table of relative numbers taken from Klamroth, to which I have added the corresponding numbers in August's and Heiberg's editions of the Greek text

	The Arabian Euclid			The Greek Euclid		
Books	Ishāq	aṭ-Ṭūsī	Campanus	Gregory	August	Heiberg
I	48	48	48	48	48	48
II	14	14	14	14	14	14
III	36	36	36	37	37	37
IV	16	16	16	16	16	16
V	25	25	34	25	25	25
VI	33	32	32	33	33	33
VII	39	39	39	41	41	39
VIII	27	25	25	27	27	27
IX	38	36	39	36	36	36
X	109	107	107	117	116	115
XI	41	41	41	40	40	39
XII	15	15	15	18	18	18
XIII	21	18	18	18	18	18
	462	452	464	470	469	465
[XIV	10	10	18	7		?
XV	6	6	13	10		
	478	468	495	487		?]]

The numbers in the case of Heiberg include all propositions which he has printed in the text; they include therefore XIII. 6 and III. 12 now to be regarded as spurious, and X. 112—115 which he brackets as doubtful. He does not number the propositions in Books XIV., XV., but I conclude that the numbers in P reach at least 9 in XIV., and 9 in XV.

¹ Heiberg, *Zeitschrift für Math. u. Physik*, XXIX., hist.-litt. Abtheilung, p. 21.

² Klamroth, p. 274; Steinschneider, *Zeitschrift für Math. u. Physik*, XXXI., hist.-litt.

Abth. p. 98.

³ Besthorn-Heiberg read "11?" as the number, Klamroth had read it as 21 (p. 273).

The *Fihrist* confirms the number 109 for Book X., from which Klamroth concludes that Ishāq's version was considered as by far the most authoritative.

In the text of O, Book IV. consists of 17 propositions and Book XIV. of 12, differing in this respect from its own table of contents; IV. 15, 16 in O are really two proofs of the same proposition.

In al-Ḥajjāj's version Book I. consists of 47 propositions only, I. 45 being omitted. It has also one proposition fewer in Book III., the Heronic proposition III. 12 being no doubt omitted.

In speaking of particular propositions, I shall use Heiberg's numbering, except where otherwise stated.

The difference of 10 propositions between Thābit-Ishāq and at-Ṭūsī is accounted for thus:

(1) The three propositions VI. 12 and X. 28, 29 which both Ishāq and the Greek text have are omitted in at-Ṭūsī.

(2) Ishāq divides each of the propositions XIII. 1—3 into two, making six instead of three in at-Ṭūsī and in the Greek.

(3) Ishāq has four propositions (numbered by him VIII. 24, 25, IX. 30, 31) which are neither in the Greek Euclid nor in at-Ṭūsī.

Apart from the above differences al-Ḥajjāj (so far as we know), Ishāq and at-Ṭūsī agree; but their Euclid shows many differences from our Greek text. These differences we will classify as follows¹.

I. *Propositions.*

The Arabian Euclid omits VII. 20, 22 of Gregory's and August's editions (Heiberg, App. to Vol. II. pp. 428—32); VIII. 16, 17; X. 7, 8, 13, 16, 24, 112, 113, 114, besides a lemma *vulgo* X. 13, the proposition X. 117 of Gregory's edition, and the scholium at the end of the Book (see for these Heiberg's Appendix to Vol. III. pp. 382, 408—416); XI. 38 in Gregory and August (Heiberg, App. to Vol. IV. p. 354); XII. 6, 13, 14; (also all but the first third of Book XV.).

The Arabian Euclid makes III. 11, 12 into one proposition, and divides some propositions (X. 31, 32; XI. 31, 34; XIII. 1—3) into two each.

The order is also changed in the Arabic to the following extent. V. 12, 13 are interchanged and the order in Books VI., VII., IX.—XIII. is:

VI. 1—8, 13, 11, 12, 9, 10, 14—17, 19, 20, 18, 21, 22, 24, 26, 23, 25, 27—30, 32, 31, 33.

VII. 1—20, 22, 21, 23—28, 31, 32, 29, 30, 33—39.

IX. 1—13, 20, 14—19, 21—25, 27, 26, 28—36, with two new propositions coming before prop. 30.

X. 1—6, 9—12, 15, 14, 17—23, 26—28, 25, 29—30, 31, 32, 33—III, 115.

XI. 1—30, 31, 32, 34, 33, 35—39.

XII. 1—5, 7, 9, 8, 10, 12, 11, 15, 16—18.

XIII. 1—3, 5, 4, 6, 7, 12, 9, 10, 8, 11, 13, 15, 14, 16—18.

¹ See Klamroth, pp. 275—6, 280, 282—4, 314—15, 326; Heiberg, vol. v. pp. xcvi, xcvi.

2. *Definitions.*

The Arabic omits the following definitions: IV. Deff. 3—7, VII. Def. 9 (or 10), XI. Deff. 5—7, 15, 17, 23, 25—28; but it has the spurious definitions VI. Deff. 2, 5, and those of *proportion* and *ordered proportion* in Book V. (Deff. 8, 19 August), and wrongly interchanges V. Deff. 11, 12 and also VI. Deff. 3, 4.

The order of the definitions is also different in Book VII. where, after Def. 11, the order is 12, 14, 13, 15, 16, 19, 20, 17, 18, 21, 22, 23, and in Book XI. where the order is 1, 2, 3, 4, 8, 10, 9, 13, 14, 16, 12, 21, 22, 18, 19, 20, 11, 24.

3. *Lemmas and porisms.*

All are omitted in the Arabic except the porisms to VI. 8, VIII. 2, X. 3; but there are slight additions here and there, not found in the Greek, e.g. in VIII. 14, 15 (in K).

4. *Alternative proofs.*

These are all omitted in the Arabic, except that in X. 105, 106 they are substituted for the genuine proofs; but one or two alternative proofs are peculiar to the Arabic (VI. 32 and VIII. 4, 6).

The analyses and syntheses to XIII. 1—5 are also omitted in the Arabic.

Klamroth is inclined, on a consideration of all these differences, to give preference to the Arabian tradition over the Greek (1) "on historical grounds," subject to the proviso that no Greek MS. as ancient as the 8th c. is found to contradict his conclusions, which are based generally (2) on the improbability that the Arabs would have *omitted* so much if they had found it in their Greek MSS., it being clear from the *Fihrist* that the Arabs had already shown an anxiety for a pure text, and that the old translators were subjected in this matter to the check of public criticism. Against the "historical grounds," Heiberg is able to bring a considerable amount of evidence¹. First of all there is the British Museum palimpsest (L) of the 7th or the beginning of the 8th c. This has fragments of propositions in Book X. which are omitted in the Arabic; the numbering of one proposition, which agrees with the numbering in other Greek MS., is not comprehensible on the assumption that eight preceding propositions were omitted in it, as they are in the Arabic; and lastly, the readings in L are tolerably like those of our MSS., and surprisingly like those of B. It is also to be noted that, although P dates from the 10th c. only, it contains, according to all appearance, an ante-Theonine recension.

Moreover there is positive evidence against certain omissions by the Arabians. Aṭ-Ṭūsī omits VI. 12, but it is scarcely possible that, if Eutocius had not had it, he would have quoted VI. 23 by that number². This quotation of VI. 23 by Eutocius also tells against Ishāq who has the proposition as VI. 25. Again, Simplicius quotes VI. 10 by that number, whereas it is VI. 13 in Ishāq; and Pappus quotes, by number, XIII. 2 (Ishāq 3, 4), XIII. 4 (Ishāq 8), XIII. 16 (Ishāq 19).

¹ Heiberg in *Zeitschrift für Math. u. Physik*, xxix., hist.-litt. Abth. p. 3 sqq.

² Apollonius, ed. Heiberg, vol. II. p. 218, 3—5.

On the other hand the contraction of III. 11, 12 into one proposition in the Arabic tells in favour of the Arabic.

Further, the omission of certain porisms in the Arabic cannot be supported; for Pappus quotes the porism to XIII. 17¹, Proclus those to II. 4, III. 1, VII. 2², and Simplicius that to IV. 15.

Lastly, some propositions omitted in the Arabic are required in later propositions. Thus x. 13 is used in x. 18, 22, 23, 26 etc.; x. 17 is wanted in x. 18, 26, 36; XII. 6, 13 are required for XII. 11 and XII. 15 respectively.

It must also be remembered that some of the things which were properly omitted by the Arabians are omitted or marked as doubtful in Greek MSS. also, especially in P, and others are rightly suspected for other reasons (e.g. a number of alternative proofs, lemmas, and porisms, as well as the analyses and syntheses of XIII. 1—5). On the other hand, the Arabic has certain interpolations peculiar to our inferior MSS. (cf. the definition VI. Def. 2 and those of *proportion* and *ordered proportion*).

Heiberg comes to the general conclusion that, not only is the Arabic tradition not to be preferred offhand to that of the Greek MSS., but it must be regarded as inferior in authority. It is a question how far the differences shown in the Arabic are due to the use of Greek MSS. differing from those which have been most used as the basis of our text, and how far to the arbitrary changes made by the Arabians themselves. Changes of order and arbitrary omissions could not surprise us, in view of the preface above quoted from the Oxford MS. of Thābit-Ishāq, with its allusion to the many important and necessary things left out by Abū 'l Wafā and to the author's own rearrangement of Books XII., XIII. But there is evidence of differences due to the use by the Arabs of other Greek MSS. Heiberg³ is able to show considerable resemblances between the Arabic text and the Bologna MS. b in that part of the MS. where it diverges so remarkably from our other MSS. (see the short description of it above, p. 49); in illustration he gives a comparison of the proofs of XII. 7 in b and in the Arabic respectively, and points to the omission in both of the proposition given in Gregory's edition as XI. 38, and to a remarkable agreement between them as regards the order of the propositions of Book XII. As above stated, the remarkable divergence of b only affects Books XI. (at end) and XII.; and Book XIII. in b shows none of the transpositions and other peculiarities of the Arabic. There are many differences between b and the Arabic, especially in the definitions of Book XI., as well as in Book XIII. It is therefore a question whether the Arabians made arbitrary changes, or the Arabic form is the more ancient, and b has been altered through contact with other MSS. Heiberg points out that the Arabians must be alone responsible for their definition of a prism, which only covers a prism with a triangular base. This could not have been Euclid's own, for the word *prism* already has the wider meaning in Archimedes, and

¹ Pappus, v. p. 436, 5.

² Proclus, pp. 303—4.

³ *Zeitschrift für Math. u. Physik*, xxix., hist.-litt. Abth. p. 6 sqq.

Euclid himself speaks of prisms with parallelograms and polygons as bases (XI. 39; XII. 10). Moreover, a Greek would not have been likely to leave out the definitions of the "Platonic" regular solids.

Heiberg considers that the Arabian translator had before him a MS. which was related to b, but diverged still further from the rest of our MSS. He does not think that there is evidence of the existence of a redaction of Books I.—X. similar to that of Books XI., XII. in b; for Klamroth observes that it is the Books on solid geometry (XI.—XIII.) which are more remarkable than the others for omissions and shorter proofs, and it is a noteworthy coincidence that it is just in these Books that we have a divergent text in b.

An advantage in the Arabic version is the omission of VII. Def. 10, although, as Iamblichus had it, it may have been deliberately omitted by the Arabic translator. Another advantage is the omission of the analyses and syntheses of XIII. 1—5; but again these may have been omitted purposely, as were evidently a number of porisms which are really necessary.

One or two remarks may be added about the Arabic versions as compared with one another. Al-Ḥajjāj's object seems to have been less to give a faithful reflection of the original than to write a useful and convenient mathematical text-book. One characteristic of it is the careful references to earlier propositions when their results are used. Such specific quotations of earlier propositions are rare in Euclid; but in al-Ḥajjāj we find not only such phrases as "by prop. so and so," "which was proved" or "which we showed how to do in prop. so and so," but also still longer phrases. Sometimes he *repeats* a construction, as in I. 44 where, instead of constructing "the parallelogram $BEFG$ equal to the triangle C in the angle EBG which is equal to the angle D " and placing it in a certain position, he produces AB to G , making BG equal to half DE (the base of the triangle CDE in his figure), and on GB so constructs the parallelogram $BHKG$ by I. 42 that it is equal to the triangle CDE , and its angle GBH is equal to the given angle.

Secondly, al-Ḥajjāj, in the arithmetical books, in the theory of proportion, in the applications of the Pythagorean I. 47, and generally where possible, illustrates the proofs by numerical examples. It is true, observes Klamroth, that these examples are not apparently separated from the commentary of an-Nairizi, and might not therefore have been due to al-Ḥajjāj himself; but the marginal notes to the Hebrew translation in Munich MS. 36 show that these additions were in the copy of al-Ḥajjāj used by the translator, for they expressly give these proofs in numbers as variants taken from al-Ḥajjāj¹.

These characteristics, together with al-Ḥajjāj's freer formulation of the propositions and expansion of the proofs, constitute an intelligible reason why Ishāq should have undertaken a fresh translation from the Greek. Klamroth calls Ishāq's version a model of a good translation of a mathematical text; the introductory and transitional

¹ Klamroth, p. 310; Steinschneider, pp. 85—6.

phrases are stereotyped and few in number, the technical terms are simply and consistently rendered, and the less formal expressions connect themselves as closely with the Greek as is consistent with intelligibility and the character of the Arabic language. Only in isolated cases does the formulation of definitions and enunciations differ to any considerable extent from the original. In general, his object seems to have been to get rid of difficulties and unevennesses in the Greek text by neat devices, while at the same time giving a faithful reproduction of it¹.

There are curious points of contact between the versions of al-Ḥajjāj and Thābit-Ishāq. For example, the definitions and enunciations of propositions are often word for word the same. Presumably this is owing to the fact that Ishāq found these definitions and enunciations already established in the schools in his time, where they would no doubt be learnt by heart, and refrained from translating them afresh, merely adopting the older version with some changes². Secondly, there is remarkable agreement between the Arabic versions as regards the figures, which show considerable variations from the figures of the Greek text, especially as regards the letters; this is also probably to be explained in the same way, all the later translators having most likely borrowed al-Ḥajjāj's adaptation of the Greek figures³. Lastly, it is remarkable that the version of Books XI.—XIII. in the Kjøbenhavn MS. (K), purporting to be by al-Ḥajjāj, is almost exactly the same as the Thābit-Ishāq version of the same Books in O. Klamroth conjectures that Ishāq may not have translated the Books on solid geometry at all, and that Thābit took them from al-Ḥajjāj, only making some changes in order to fit them to the translation of Ishāq⁴.

From the facts (1) that at-Ṭūsī's edition had the same number of propositions (468) as al-Ḥajjāj's version, while Thābit-Ishāq's had 478, and (2) that at-Ṭūsī has the same careful references to earlier propositions, Klamroth concludes that at-Ṭūsī deliberately preferred al-Ḥajjāj's version to that of Ishāq⁵. Heiberg, however, points out (1) that at-Ṭūsī left out VI. 12 which, if we may judge by Klamroth's silence, al-Ḥajjāj had, and (2) al-Ḥajjāj's version had one proposition less in Books I. and III. than at-Ṭūsī has. Besides, in a passage quoted by Hāji Khalfa⁶ from at-Ṭūsī, the latter says that "he separated the things which, in the approved editions, were taken from the archetype from the things which had been added thereto," indicating that he had compiled his edition from *both* the earlier translations⁷.

There were a large number of Arabian commentaries on, or reproductions of, the *Elements* or portions thereof, which will be

¹ Klamroth, p. 290, illustrates Ishāq's method by his way of distinguishing *ἐφαρμόζειν* (to be congruent with) and *ἐφαρμόζεσθαι* (to be applied to), the confusion of which by translators was animadverted on by Savile. Ishāq avoided the confusion by using two entirely different words.

² Klamroth, pp. 310—1.

³ *ibid.* p. 287.

⁴ *ibid.* pp. 304—5.

⁵ *ibid.* p. 274.

⁶ Hāji Khalfa, I. p. 383.

⁷ Heiberg, *Zeitschrift für Math. u. Physik*, xxix., hist.-litt. Abth. pp. 2, 3.

found fully noticed by Steinschneider¹. I shall mention here the commentators etc. referred to in the *Fihrist*, with a few others.

1. Abū 'l 'Abbās al-Faḍl b. Ḥatim an-Nairīzī (born at Nairīz, died about 922) has already been mentioned². His commentary survives, as regards Books I.—VI., in the Codex Leidensis 399, I, now edited, as to four Books, by Besthorn and Heiberg, and as regards Books I.—X. in the Latin translation made by Gherard of Cremona in the 12th c. and now published by Curtze from a Cracow MS.³ Its importance lies mainly in the quotations from Heron and Simplicius.

2. Ahmad b. 'Umar al-Karābīsī (date uncertain, probably 9th—10th c.), "who was among the most distinguished geometers and arithmeticians⁴."

3. Al-'Abbās b. Sa'īd al-Jauharī (fl. 830) was one of the astronomical observers under al-Ma'mūn, but devoted himself mostly to geometry. He wrote a commentary to the whole of the *Elements*, from the beginning to the end; also the "Book of the propositions which he added to the first book of Euclid⁵."

4. Muḥ. b. 'Īsā Abū 'Abdallāh al-Māhānī (d. between 874 and 884) wrote, according to the *Fihrist*, (1) a commentary on Eucl. Book v., (2) "On proportion," (3) "On the 26 propositions of the first Book of Euclid which are proved without *reductio ad absurdum*⁶." The work "On proportion" survives and is probably identical with, or part of, the commentary on Book v.⁷ He also wrote, what is not mentioned by the *Fihrist*, a commentary on Eucl. Book x., a fragment of which survives in a Paris MS.⁸

5. Abū Ja'far al-Khāzin (i.e. "the treasurer" or "librarian"), one of the first mathematicians and astronomers of his time, was born in Khurāsān and died between the years 961 and 971. The *Fihrist* speaks of him as having written a commentary on the whole of the *Elements*⁹, but only the commentary on the beginning of Book x. survives (in Leiden, Berlin and Paris); therefore either the notes on the rest of the Books have perished, or the *Fihrist* is in error¹⁰. The latter would seem more probable, for, at the end of his commentary, al-Khāzin remarks that the rest had already been commented on by Sulaimān b. 'Uṣma (Leiden MS.)¹¹ or 'Oqba (Suter), to be mentioned below. Al-Khāzin's method is criticised unfavourably in the preface to the Oxford MS. quoted by Nicoll (see p. 77 above).

6. Abū 'l Wafā al-Būzjānī (940–997), one of the greatest Arabian mathematicians, wrote a commentary on the *Elements*, but

¹ Steinschneider, *Zeitschrift für Math. u. Physik*, xxxi., hist.-litt. Abth. pp. 86 sqq.

² Steinschneider, p. 86, *Fihrist* (tr. Suter), pp. 16, 67; Suter, *Die Mathematiker und Astronomen der Araber* (1900), p. 45.

³ *Supplementum ad Euclidis opera omnia*, ed. Heiberg and Menge, Leipzig, 1899.

⁴ *Fihrist*, pp. 16, 38; Steinschneider, p. 87; Suter, p. 65.

⁵ *Fihrist*, pp. 16, 25; Steinschneider, p. 88; Suter, p. 12.

⁶ *Fihrist*, pp. 16, 25, 58.

⁷ Suter, p. 26, note, quotes the Paris MS. 2467, 16° containing the work "on proportion" as the authority for this conjecture.

⁸ MS. 2457, 39° (cf. Woepcke in *Mém. prés. à l'acad. des sciences*, x.v., 1856, p. 669).

⁹ *Fihrist*, p. 17.

¹⁰ Suter, p. 58, note b.

¹¹ Steinschneider, p. 89.

did not complete it¹. His method is also unfavourably regarded in the same preface to the Oxford MS. 280. According to Ḥājī Khalfā, he also wrote a book on geometrical constructions, in thirteen chapters. Apparently a book answering to this description was compiled by a gifted pupil from lectures by Abū 'l Wafā, and a Paris MS. (Anc. fonds 169) contains a Persian translation of this work, not that of Abū 'l Wafā himself. An analysis of the work was given by Woepcke², and some particulars will be found in Cantor³. Abū 'l Wafā also wrote a commentary on Diophantus, as well as a separate "book of proofs to the propositions which Diophantus used in his book and to what he (Abū 'l Wafā) employed in his commentary⁴."

7. Ibn Rāhawaihi al-Arjānī also commented on Eucl. Book x.⁵

8. 'Alī b. Aḥmad Abū 'l-Qāsim al-Anṭākī (d. 987) wrote a commentary on the whole book⁶; part of it seems to survive (from the 5th Book onwards) at Oxford (Catal. MSS. orient. II. 281)⁷.

9. Sind b. 'Alī Abū 't-Taiyib was a Jew who went over to Islam in the time of al-Ma'mūn, and was received among his astronomical observers, whose head he became⁸ (about 830); he died after 864. He wrote a commentary on the whole of the *Elements*; "Abū 'Alī saw nine books of it, and a part of the tenth⁹." His book "On the Apotomae and the Medials," mentioned by the *Fihrist*, may be the same as, or part of, his commentary on Book x.

10. Abū Yūsuf Ya'qūb b. Muḥ. ar-Rāzī "wrote a commentary on Book x., and that an excellent one, at the instance of Ibn al-'Amīd¹⁰."

11. The *Fihrist* next mentions al-Kindī (Abū Yūsuf Ya'qūb b. Ishāq b. as-Ṣabbāḥ al-Kindī, d. about 873), as the author (1) of a work "on the objects of Euclid's book," in which occurs the statement that the *Elements* were originally written by Apollonius, the carpenter (see above, p. 5 and note), (2) of a book "on the improvement of Euclid's work," and (3) of another "on the improvement of the 14th and 15th Books of Euclid." "He was the most distinguished man of his time, and stood alone in the knowledge of the old sciences collectively; he was called 'the philosopher of the Arabians'; his writings treat of the most different branches of knowledge, as logic, philosophy, geometry, calculation, arithmetic, music, astronomy and others¹¹." Among the other geometrical works of al-Kindī mentioned by the *Fihrist*¹² are treatises on the closer investigation of the results of Archimedes concerning the measure of the diameter of a circle in terms of its circumference, on the construction of the figure of the two mean proportionals, on the approximate determination of the chords

¹ *Fihrist*, p. 17.

² Woepcke, *Journal Asiatique*, Sér. v. T. v. pp. 218—256 and 309—359.

³ *Gesch. d. Math.* vol. I₃, pp. 743—6.

⁴ *Fihrist*, p. 39; Suter, p. 71.

⁵ *Fihrist*, p. 17.

⁶ *Fihrist*, p. 17, 29; Suter, pp. 13, 14.

⁷ *Fihrist*, p. 17; Suter, p. 66.

⁸ *Fihrist*, p. 17, 10—15.

⁹ *Fihrist*, p. 17; Suter, p. 64.

¹⁰ *Fihrist*, p. 17; Suter, p. 66.

¹¹ *Fihrist*, p. 17, 10—15.

¹² The mere catalogue of al-Kindī's works on the various branches of science takes up four octavo pages (11—15) of Suter's translation of the *Fihrist*.

of the circle, on the approximate determination of the chord (side) of the nonagon, on the division of triangles and quadrilaterals and constructions for that purpose, on the manner of construction of a circle which is equal to the surface of a given cylinder, on the division of the circle, in three chapters etc.

12. The physician **Naẓīf** b. Yumn (or Yaman) al-Qass ("the priest") is mentioned by the *Fihrist* as having seen a Greek copy of Eucl. Book x. which had 40 more propositions than that which was in general circulation (containing 109), and having determined to translate it into Arabic¹. Fragments of such a translation exist at Paris, Nos. 18 and 34 of the MS. 2457 (952, 2 Suppl. Arab. in Woepcke's tract); No. 18 contains "additions to some propositions of the 10th Book, existing in the Greek language²." Naẓīf must have died about 990³.

13. **Yūḥannā** b. Yūsuf b. al-Ḥārith b. al-Bīṭriq **al-Qass** (d. about 980) lectured on the *Elements* and other geometrical books, made translations from the Greek, and wrote a tract on the "proof" of the case of two straight lines both meeting a third and making with it, on one side, two angles together less than two right angles⁴. Nothing of his appears to survive, except that a tract "on rational and irrational magnitudes," No. 48 in the Paris MS. just mentioned, is attributed to him.

14. **Abū Muḥ. al-Ḥasan** b. 'Ubaidallāh b. Sulaimān b. **Wahb** (d. 901) was a geometer of distinction, who wrote works under the two distinct titles "A commentary on the difficult parts of the work of Euclid" and "The Book on Proportion⁵." Suter thinks that another reading is possible in the case of the second title, and that it may refer to the Euclidean work "on the divisions (of figures)⁶."

15. **Qusṭā** b. **Lūqā** al-Ba'labakkī (d. about 912), a physician, philosopher, astronomer, mathematician and translator, wrote "on the difficult passages of Euclid's book" and "on the solution of arithmetical problems from the third book of Euclid⁷"; also an "introduction to geometry," in the form of question and answer⁸.

16. **Thābit** b. **Qurra** (826-901), besides translating some parts of Archimedes and Books v.—vii. of the *Conics* of Apollonius, and revising Ishāq's translation of Euclid's *Elements*, also revised the translation of the *Data* by the same Ishāq and the book *On divisions of figures* translated by an anonymous writer. We are told also that he wrote the following works: (1) On the Premises (Axioms, Postulates etc.) of Euclid, (2) On the Propositions of Euclid, (3) On the propositions and questions which arise when two straight lines are cut by a third (or on the "proof" of Euclid's famous postulate). The last tract is extant in the MS. discovered by Woepcke (Paris 2457, 32^o). He is also credited with "an excellent work" in the shape of an "Introduction to the Book of Euclid," a treatise on

¹ *Fihrist*, pp. 16, 17.

² Woepcke, *Mém. prés. à l'acad. des sciences*, xiv. pp. 666, 668

³ Suter, p. 68.

⁴ *Fihrist*, p. 38; Suter, p. 60.

⁵ *Fihrist*, p. 26, and Suter's note, p. 60.

⁶ Suter, p. 211, note 23.

⁷ *Fihrist*, p. 43.

⁸ *Fihrist*, p. 43; Suter, p. 41.

Geometry dedicated to Ismā'il b. Bulbul, a Compendium of Geometry, and a large number of other works for the titles of which reference may be made to Suter, who also gives particulars as to which are extant¹.

17. Abū Sa'id Sinān b. Thābit b. Qurra, the son of the translator of Euclid, followed in his father's footsteps as geometer, astronomer and physician. He wrote an "improvement of the book of on the Elements of Geometry, in which he made various additions to the original." It is natural to conjecture that *Euclid* is the name missing in this description (by Ibn abī Uṣaibi'a); Casiri has the name Aqāṭon². The latest editor of the *Ta'rikh al-Hukamā*, however, makes the name to be Iḥlāton (= Plato), and he refers to the statement by the *Fihrist* and Ibn al-Qiftī attributing to Plato a work on the Elements of Geometry translated by Qusṭā. It is just possible, therefore, that at the time of Qusṭā the Arabs were acquainted with a book on the Elements of Geometry translated from the Greek, which they attributed to Plato³. Sinān died in 943.

18. Abū Sahl Wijan (or Waijan) b. Rustam al-Kūhī (fl. 988), born at Kūh in Ṭabaristān, a distinguished geometer and astronomer, wrote, according to the *Fihrist*, a "Book of the Elements" after that of Euclid⁴; the 1st and 2nd Books survive at Cairo, and a part of the 3rd Book at Berlin (5922)⁵. He wrote also a number of other geometrical works: Additions to the 2nd Book of Archimedes on the Sphere and Cylinder (extant at Paris, at Leiden, and in the India Office), On the finding of the side of a heptagon in a circle (India Office and Cairo), On two mean proportionals (India Office), which last may be only a part of the Additions to Archimedes' On the Sphere and Cylinder, etc.

19. Abū Naṣr Muḥ. b. Muḥ. b. Ṭarkhān b. Uzlağ al-Fārābī (870-950) wrote a commentary on the difficulties of the introductory matter to Books I. and V.⁶ This appears to survive in the Hebrew translation which is, with probability, attributed to Moses b. Tibbon⁷.

20. Abū 'Alī al-Ḥasan b. al-Hasan b. al-Haitham (about 965-1039), known by the name Ibn al-Haitham or Abū 'Alī al-Baṣrī, was a man of great powers and knowledge, and no one of his time approached him in the field of mathematical science. He wrote several works on Euclid the titles of which, as translated by Woepcke from Uṣaibi'a, are as follows⁸:

1. Commentary and abridgment of the *Elements*.
2. Collection of the Elements of Geometry and Arithmetic, drawn from the treatises of Euclid and Apollonius.
3. Collection of the Elements of the Calculus deduced from the principles laid down by Euclid in his *Elements*.

¹ Suter, pp. 34-8.

² *Fihrist* (ed. Suter), p. 59, note 132; Suter, p. 52, note b.

³ See Suter in *Bibliotheca Mathematica*, IV₃, 1903-4, pp. 296-7, review of Julius Lippert's *Ibn al-Qiftī. Ta'rikh al-ḥukamā*, Leipzig, 1903.

⁴ *Fihrist*, p. 40.

⁶ Suter, p. 55.

⁵ Suter, p. 75.

⁸ Steinschneider, pp. 92-3.

⁷ Steinschneider, p. 92.

4. Treatise on "measure" after the manner of Euclid's *Elements*.

5. Memoir on the solution of the difficulties in Book I.

6. Memoir for the solution of a doubt about Euclid, relative to Book V.

7. Memoir on the solution of a doubt about the stereometric portion.

8. Memoir on the solution of a doubt about Book XII.

9. Memoir on the division of the two magnitudes mentioned in X. I (the theorem of exhaustion).

10. Commentary on the definitions in the work of Euclid (where Steinschneider thinks that some more general expression should be substituted for "definitions").

The last-named work (which Suter calls a commentary on the *Postulates* of Euclid) survives in an Oxford MS. (Catal. MSS. orient. I. 908) and in Algiers (1446, 1°).

A Leiden MS. (966) contains his Commentary "on the difficult places" up to Book V. We do not know whether in this commentary, which the author intended to form, with the commentary on the *Musādarāt*, a sort of complete commentary, he had collected the separate memoirs on certain doubts and difficult passages mentioned in the above list.

A commentary on Book V. and following Books found in a Bodleian MS. (Catal. II. p. 262) with the title "Commentary on Euclid and solution of his difficulties" is attributed to b. Haitham; this might be a continuation of the Leiden MS.

The memoir on X. I appears to survive at St Petersburg, MS. de l'Institut des langues orient. 192, 5° (Rosen, Catal. p. 125).

21. Ibn Sīnā, known as *Avicenna* (980–1037), wrote a Compendium of Euclid, preserved in a Leiden MS. No. 1445, and forming the geometrical portion of an encyclopaedic work embracing Logic, Mathematics, Physics and Metaphysics¹.

22. Aḥmad b. al-Ḥusain al-Ahwāzī al-Kātib wrote a commentary on Book X., a fragment of which (some 10 pages) is to be found at Leiden (970), Berlin (5923) and Paris (2467, 18°)².

23. Naṣīraddīn aṭ-Ṭūsī (1201–1274) who, as we have seen, brought out a Euclid in two forms, wrote:

1. A treatise on the postulates of Euclid (Paris, 2467, 5°).

2. A treatise on the 5th postulate, perhaps only a part of the foregoing (Berlin, 5942, Paris, 2467, 6°).

3. Principles of Geometry taken from Euclid, perhaps identical with No. 1 above (Florence, Pal. 298).

4. 105 problems out of the *Elements* (Cairo). He also edited the *Data* (Berlin, Florence, Oxford etc.)³.

24. Muḥ. b. Ashraf Shamsaddīn as-Samarqandī (fl. 1276) wrote "Fundamental Propositions, being elucidations of 35 selected proposi-

¹ Steinschneider, p. 92; Suter, p. 89.

² Suter, p. 57.

³ Suter, pp. 150–1.

tions of the first Books of Euclid," which are extant at Gotha (1496 and 1497), Oxford (Catal. I. 967, 2^o), and Brit. Mus.¹

25. Mūsā b. Muḥ. b. Maḥmūd, known as Qāḍīzāde ar-Rūmī (i.e. the son of the judge from Asia Minor), who died between 1436 and 1446, wrote a commentary on the "Fundamental Propositions" just mentioned, of which many MSS. are extant². It contained biographical statements about Euclid alluded to above (p. 5, note).

26. Abū Dā'ūd Sulaimān b. 'Uqba, a contemporary of al-Khāzin (see above, No. 5), wrote a commentary on the second half of Book X., which is, at least partly, extant at Leiden (974) under the title "On the binomials and apotomae found in the 10th Book of Euclid³."

27. The Codex Leidensis 399, 1 containing al-Ḥajjāj's translation of Books I.—VI. is said to contain glosses to it by Sa'īd b. Mas'ūd b. al-Qass, apparently identical with Abū Naṣr Ġars al-Na'ma, son of the physician Mas'ūd b. al-Qass al-Baġdādī, who lived in the time of the last Caliph al-Musta'ṣim (d. 1258)⁴.

28. Abū Muḥammad b. Abdalbāqī al-Baġdādī al-Faraḍī (d. 1141, at the age of over 70 years) is stated in the *Ta'riḫ al-Ḥukamā* to have written an excellent commentary on Book x. of the *Elements*, in which he gave numerical examples of the propositions⁵. This is published in Curtze's edition of an-Nairīzī where it occupies pages 252—386⁶.

29. Yaḥyā b. Muḥ. b. 'Abdān b. 'Abdalwāḥid, known by the name of Ibn al-Lubūdī (1210—1268), wrote a Compendium of Euclid, and a short presentation of the postulates⁷.

30. Abū 'Abdallāh Muḥ. b. Mu'ādh al-Jayyānī wrote a commentary on Eucl. Book v. which survives at Algiers (1446, 3^o)⁸.

31. Abū Naṣr Mansūr b. 'Alī b. 'Irāq wrote, at the instance of Muḥ. b. Ahmad Abū 'r-Raiḥān al-Birūnī (973—1048), a tract "on a doubtful (difficult) passage in Eucl. Book XIII." (Berlin, 5925)⁹.

¹ Suter, p. 157.

² *ibid.* p. 175.

³ *ibid.* p. 56.

⁴ *ibid.* pp. 153—4, 227.

⁵ Gartz, p. 14; Steinschneider, pp. 94—5.

⁶ Suter in *Bibliotheca Mathematica*, IV₃, 1903, pp. 25, 295; Suter has also an article on its contents, *Bibliotheca Mathematica*, VII₃, 1906—7, pp. 234—251.

⁷ Steinschneider, p. 94; Suter, p. 146.

⁸ Suter, *Nachträge und Berichtigungen*, in *Abhandlungen zur Gesch. der math. Wissenschaften*, XIV., 1902, p. 170.

⁹ Suter, p. 81, and *Nachträge*, p. 172.

CHAPTER VIII.

PRINCIPAL TRANSLATIONS AND EDITIONS OF THE ELEMENTS.

CICERO is the first Latin author to mention Euclid¹; but it is not likely that in Cicero's time Euclid had been translated into Latin or was studied to any considerable extent by the Romans; for, as Cicero says in another place², while geometry was held in high honour among the Greeks, so that nothing was more brilliant than their mathematicians, the Romans limited its scope by having regard only to its utility for measurements and calculations. How very little theoretical geometry satisfied the Roman *agrimensores* is evidenced by the work of Balbus *de mensuris*³, where some of the definitions of Eucl. Book I. are given. Again, the extracts from the *Elements* found in the fragment attributed to Censorinus (fl. 238 A.D.)⁴ are confined to the definitions, postulates, and common notions. But by degrees the *Elements* passed even among the Romans into the curriculum of a liberal education; for Martianus Capella speaks of the effect of the enunciation of the proposition "how to construct an equilateral triangle on a given straight line" among a company of philosophers, who, recognising the first proposition of the *Elements*, straightway break out into encomiums on Euclid⁵. But the *Elements* were then (c. 470 A.D.) doubtless read in Greek; for what Martianus Capella gives⁶ was drawn from a Greek source, as is shown by the occurrence of Greek words and by the wrong translation of I. def. 1 ("punctum vero est cuius pars *nihil* est"). Martianus may, it is true, have quoted, not from Euclid himself, but from Heron or some other ancient source.

But it is clear from a certain palimpsest at Verona that some scholar had already attempted to translate the *Elements* into Latin. This palimpsest⁷ has part of the "Moral reflections on the Book of Job" by Pope Gregory the Great written in a hand of the 9th c. above certain fragments which in the opinion of the best judges date from the 4th c. Among these are fragments of Vergil and of Livy, as well as a geometrical fragment which purports to be taken from the 14th and 15th Books of Euclid. As a matter of fact it is from Books XII. and XIII. and is of the nature of a free rendering, or rather a new

¹ *De oratore* III. 132.

² *Tusc.* I. 5.

³ *Gromatici veteres*, I. 97 sq. (ed. F. Blume, K. Lachmann and A. Rudorff. Berlin, 1848, 1852).

⁴ Censorinus, ed. Hultsch, pp. 60—3.

⁵ Martianus Capella, VI. 724.

⁶ *ibid.* VI. 708 sq.

⁷ Cf. Cantor, I₃, p. 565.

arrangement, of Euclid with the propositions in different order¹. The MS. was evidently the translator's own copy, because some words are struck out and replaced by synonyms. We do not know whether the translator completed the translation of the whole, or in what relation his version stood to our other sources.

Magnus Aurelius Cassiodorus (b. about 475 A.D.) in the geometrical part of his encyclopaedia *De artibus ac disciplinis liberalium literarum* says that geometry was represented among the Greeks by Euclid, Apollonius, Archimedes, and others, "of whom Euclid was given us translated into the Latin language by the same great man Boethius"; also in his collection of letters² is a letter from Theodoric to Boethius containing the words, "for in your translations... Nicomachus the arithmetician, and Euclid the geometer, are heard in the Ausonian tongue." The so-called Geometry of Boethius which has come down to us by no means constitutes a translation of Euclid. The MSS. variously give five, four, three or two Books, but they represent only two distinct compilations, one normally in five Books and the other in two. Even the latter, which was edited by Friedlein, is not genuine³, but appears to have been put together in the 11th c., from various sources. It begins with the definitions of Eucl. I., and in these are traces of perfectly correct readings which are not found even in the MSS. of the 10th c., but which can be traced in Proclus and other ancient sources; then come the Postulates (five only), the Axioms (three only), and after these some definitions of Eucl. II., III., IV. Next come the enunciations of Eucl. I., of ten propositions of Book II., and of some from Books III., IV., but always without proofs; there follows an extraordinary passage which indicates that the author will now give something of his own in elucidation of Euclid, though what follows is a literal translation of the proofs of Eucl. I. 1—3. This latter passage, although it affords a strong argument against the genuineness of this part of the work, shows that the Pseudoboethius had a Latin translation of Euclid from which he extracted the three propositions.

Curtze has reproduced, in the preface to his edition of the translation by Gherard of Cremona of an-Nairizi's Arabic commentary on Euclid, some interesting fragments of a translation of Euclid taken from a Munich MS. of the 10th c. They are on two leaves used for the cover of the MS. (Bibliothecae Regiae Universitatis Monacensis 2° 757) and consist of portions of Eucl. I. 37, 38 and II. 8, translated literally word for word from the Greek text. The translator seems to have been an Italian (cf. the words "capitolo nono" used for the ninth prop. of Book II.) who knew very little Greek and had moreover little mathematical knowledge. For example, he translates the capital letters denoting points in figures as if they were numerals: thus $\tau\alpha$ ABΓ,

¹ The fragment was deciphered by W. Studemund, who communicated his results to Cantor.

² Cassiodorus, *Variae*, I. 45, p. 40, 12 ed. Mommsen.

³ See especially Weissenborn in *Abhandlungen zur Gesch. d. Math.* II. p. 185 sq.; Heiberg in *Philologus*, XLIII. p. 507 sq.; Cantor, I₃, p. 580 sq.

ΔEZ is translated "que primo secundo et tertio quarto quinto et septimo," T becomes "tricentissimo" and so on. The Greek MS. which he used was evidently written in uncials, for $\Delta EZ\Theta$ becomes in one place "quod autem septimo nono," showing that he mistook ΔE for the particle $\delta\epsilon$, and $\kappa\alpha\iota\ \acute{o}\ \Sigma TU$ is rendered "sicut tricentissimo et quadringentissimo," showing that the letters must have been written $KAIOCTU$.

The date of the Englishman Athelhard (*Æthelhard*) is approximately fixed by some remarks in his work *Perdifficiles Quaestiones Naturales* which, on the ground of the personal allusions they contain, must be assigned to the first thirty years of the 12th c.¹ He wrote a number of philosophical works. Little is known about his life. He is said to have studied at Tours and Laon, and to have lectured at the latter school. He travelled to Spain, Greece, Asia Minor and Egypt, and acquired a knowledge of Arabic, which enabled him to translate from the Arabic into Latin, among other works, the *Elements* of Euclid. The date of this translation must be put at about 1120. MSS. purporting to contain Athelhard's version are extant in the British Museum (Harleian No. 5404 and others), Oxford (Trin. Coll. 47 and Ball. Coll. 257 of 12th c.), Nürnberg (Johannes Regiomontanus' copy) and Erfurt.

Among the very numerous works of Gherard of Cremona (1114—1187) are mentioned translations of "15 Books of Euclid" and of the *Data*². Till recently this translation of the *Elements* was supposed to be lost; but Axel Anthon Björnbo has succeeded (1904) in discovering a translation from the Arabic which is different from the two others known to us (those by Athelhard and Campanus respectively), and which he, on grounds apparently convincing, holds to be Gherard's. Already in 1901 Björnbo had found Books X.—XV. of this translation in a MS. at Rome (Codex Reginensis lat. 1268 of 14th c.)³; but three years later he had traced three MSS. containing the whole of the same translation at Paris (Cod. Paris. 7216, 15th c.), Boulogne-sur-Mer (Cod. Bononiens. 196, 14th c.), and Bruges (Cod. Brugens. 521, 14th c.), and another at Oxford (Cod. Digby 174, end of 12th c.) containing a fragment, XI. 2 to XIV.⁴ The occurrence of Greek words in this translation such as *rombus*, *romboides* (where Athelhard keeps the Arabic terms), *ambligonius*, *orthogonius*, *gnomo*, *pyramis* etc., show that the translation is independent of Athelhard's. Gherard appears to have had before him an old translation of Euclid from the Greek which Athelhard also often followed, especially in his terminology, using it however in a very different manner. Again, there are some Arabic terms, e.g. *meguar* for *axis of rotation*, which Athelhard did not use, but which is found in almost all the translations that are with certainty attributed to Gherard of Cremona; there occurs also the

¹ Cantor, *Gesch. d. Math.* I, p. 906.

² Boncompagni, *Della vita e delle opere di Gherardo Cremonese*, Rome, 1851, p. 5.

³ Described in an appendix to *Studien über Menelaos' Sphärik* (Abhandlungen zur Geschichte der mathematischen Wissenschaften, XIV., 1902).

⁴ See *Bibliotheca Mathematica*, VI, 3, 1905-6, pp. 242-8.

expression "superficies equidistantium laterum et rectorum angulorum," found also in Gherard's translation of an-Nairizī, where Athelhard says "parallelogrammum rectangulum." The translation is much clearer than Athelhard's: it is neither abbreviated nor "edited" as Athelhard's appears to have been; it is a word-for-word translation of an Arabic MS. containing a revised and critical edition of Thābit's version. It contains several notes quoted from Thābit himself (*Thebit dixit*), e.g. about alternative proofs etc. which Thābit found "in another Greek MS.," and is therefore a further testimony to Thābit's critical treatment of the text after Greek MSS. The new editor also added critical remarks of his own, e.g. on other proofs which he found in other Arabic versions, but not in the Greek: whence it is clear that he compared the Thābit version before him with other versions as carefully as Thābit collated the Greek MSS. Lastly, the new editor speaks of "Thebit qui transtulit hunc librum in arabicam linguam" and of "translatio Thebit," which may tend to confirm the statement of al-Qiftī who credited Thābit with an independent translation, and not (as the *Fihrist* does) with a mere improvement of the version of Ishāq b. Hunain.

Gherard's translation of the Arabic commentary of an-Nairizī on the first ten Books of the *Elements* was discovered by Maximilian Curtze in a MS. at Cracow and published as a supplementary volume to Heiberg and Menge's Euclid¹: it will often be referred to in this work.

Next in chronological order comes Johannes Campanus (Campano) of Novara. He is mentioned by Roger Bacon (1214-1294) as a prominent mathematician of his time², and this indication of his date is confirmed by the fact that he was chaplain to Pope Urban IV, who was Pope from 1261 to 1281³. His most important achievement was his edition of the *Elements* including the two Books XIV. and XV. which are not Euclid's. The sources of Athelhard's and Campanus' translations, and the relation between them, have been the subject of much discussion, which does not seem to have led as yet to any definite conclusion. Cantor (II, p. 91) gives references⁴ and some particulars. It appears that there is a MS. at Munich (Cod. lat. Mon. 13021) written by Sigboto in the 12th c. at Prüfning near Regensburg, and denoted by Curtze by the letter R, which contains the enunciations of part of Euclid. The Munich MSS. of Athelhard and Campanus' translations have many enunciations textually identical with those in R, so that the source of all three must, for these enunciations, have

¹ *Anarithi in decem libros priores Elementorum Euclidis Commentarii ex interpretatione Gherardi Cremonensis in codice Cracoviensi 569 servata* edidit Maximilianus Curtze, Leipzig (Teubner), 1899.

² Cantor, II, p. 88.

³ Tiraboschi, *Storia della letteratura italiana*, IV. 145-160.

⁴ H. Weissenborn in *Zeitschrift für Math. u. Physik*, xxv., Supplement, pp. 143-166, and in his monograph, *Die Übersetzungen des Euklid durch Campano und Zamberti* (1882); Max. Curtze in *Philologische Rundschau* (1881), I. pp. 943-950, and in *Jahresbericht über die Fortschritte der klassischen Alterthumswissenschaft*, XL. (1884, III.) pp. 19-22; Heiberg in *Zeitschrift für Math. u. Physik*, xxxv., hist.-litt. Abth., pp. 48-58 and pp. 81-6.

been the same; in others Athelhard and Campanus diverge completely from R, which in these places follows the Greek text and is therefore genuine and authoritative. In the 32nd definition occurs the word "elinuam," the Arabic term for "rhombus," and throughout the translation are a number of Arabic figures. But R was not translated from the Arabic, as is shown by (among other things) its close resemblance to the translation from Euclid given on pp. 377 sqq. of the *Gromatici Veteres* and to the so-called geometry of Boethius. The explanation of the Arabic figures and the word "elinuam" in Def. 32 appears to be that R was a late copy of an earlier original with corruptions introduced in many places; thus in Def. 32 a part of the text was completely lost and was supplied by some intelligent copyist who inserted the word "elinuam," which was known to him, and also the Arabic figures. Thus Athelhard certainly was not the first to translate Euclid into Latin; there must have been in existence before the 11th c. a Latin translation which was the common source of R, the passage in the *Gromatici*, and "Boethius." As in the two latter there occur the *proofs* as well as the enunciations of I. 1—3, it is possible that this translation originally contained the proofs also. Athelhard must have had before him this translation of the *enunciations*, as well as the Arabic source from which he obtained his proofs. That some sort of translation, or at least fragments of one, were available before Athelhard's time even in England is indicated by some old English verses¹:

"The clerk Euclide on this wyse hit fonde
Thys craft of gemetry yn Egypte londe
Yn Egypte he tawghte hyt ful wyde,
In dyvers londe on every syde.
Mony erys afterwarde y understonde
Yer that the craft com ynto thys londe.
Thys craft com into England, as y yow say,
Yn tyme of good kyng Adelstone's day,"

which would put the introduction of Euclid into England as far back as 924–940 A.D.

We now come to the relation between Athelhard and Campanus. That their translations were not independent, as Weissenborn would have us believe, is clear from the fact that in all MSS. and editions, apart from orthographical differences and such small differences as are bound to arise when MSS. are copied by persons with some knowledge of the subject-matter, the definitions, postulates, axioms, and the 364 enunciations are word for word identical in Athelhard and Campanus; and this is the case not only where both have the same text as R but where they diverge from it. Hence it would seem that Campanus used Athelhard's translation and only developed the proofs by means of another redaction of the Arabian Euclid. It is true that the difference between the proofs of the propositions in the two translations is considerable; Athelhard's are short and com-

¹ Quoted by Halliwell in *Rara Mathematica* (p. 56 note) from MS. Bib. Reg. Mus. Brit. 17 A. 1. f. 2^b—3.

pressed, Campanus' clearer and more complete, following the Greek text more closely, though still at some distance. Further, the arrangement in the two is different; in Athelhard the proofs regularly precede the enunciations, Campanus follows the usual order. It is a question how far the differences in the proofs, and certain additions in each, are due to the two translators themselves or go back to Arabic originals. The latter supposition seems to Curtze and Cantor the more probable one. Curtze's general view of the relation of Campanus to Athelhard is to the effect that Athelhard's translation was gradually altered, from the form in which it appears in the two Erfurt MSS. described by Weissenborn, by successive copyists and commentators *who had Arabic originals before them*, until it took the form which Campanus gave it and in which it was published. In support of this view Curtze refers to Regiomontanus' copy of the Athelhard-Campanus translation. In Regiomontanus' own preface the title is given, and this attributes the translation to Athelhard; but, while this copy agrees almost exactly with Athelhard in Book I., yet, in places where Campanus is more lengthy, it has similar additions, and in the later Books, especially from Book III. onwards, agrees absolutely with Campanus; Regiomontanus, too, himself implies that, though the translation was Athelhard's, Campanus had revised it; for he has notes in the margin such as the following, "Campani est hec," "dubito an demonstret hic Campanus" etc.

We come now to the printed editions of the whole or of portions of the *Elements*. This is not the place for a complete bibliography, such as Riccardi has attempted in his valuable memoir issued in five parts between 1887 and 1893, which makes a large book in itself¹. I shall confine myself to saying something of the most noteworthy translations and editions. It will be convenient to give first the Latin translations which preceded the publication of the *editio princeps* of the Greek text in 1533, next the most important editions of the Greek text itself, and after them the most important translations arranged according to date of first appearance and languages, first the Latin translations after 1533, then the Italian, German, French and English translations in order.

It may be added here that the first allusion, in the West, to the Greek text as still extant is found in Boccaccio's commentary on the *Divina Commedia* of Dante². Next Johannes Regiomontanus, who intended to publish the *Elements* after the version of Campanus, but with the latter's mistakes corrected, saw in Italy (doubtless when staying with his friend Bessarion) some Greek MSS. and noticed how far they differed from the Latin version (see a letter of his written in the year 1471 to Christian Roder of Hamburg)³.

¹ *Saggio di una Bibliografia Euclidea*, memoria del Prof. Pietro Riccardi (Bologna, 1887, 1888, 1890, 1893).

² I. p. 404.

³ Published in C. T. de Murr's *Memorabilia Bibliothecarum Norimbergensium*, Part I. p. 190 sqq.

I. LATIN TRANSLATIONS PRIOR TO 1533.

1482. In this year appeared the first printed edition of Euclid, which was also the first printed mathematical book of any importance. This was printed at Venice by Erhard Ratdolt and contained Campanus' translation¹. Ratdolt belonged to a family of artists at Augsburg, where he was born about 1443. Having learnt the trade of printing at home, he went in 1475 to Venice, and founded there a famous printing house which he managed for 11 years, after which he returned to Augsburg and continued to print important books until 1516. He is said to have died in 1528. Kästner² gives a short description of this first edition of Euclid and quotes the dedication to Prince Mocenigo of Venice which occupies the page opposite to the first page of text. The book has a margin of $2\frac{1}{2}$ inches, and in this margin are placed the figures of the propositions. Ratdolt says in his dedication that at that time, although books by ancient and modern authors were printed every day in Venice, little or nothing mathematical had appeared: a fact which he puts down to the difficulty involved by the figures, which no one had up to that time succeeded in printing. He adds that after much labour he had discovered a method by which figures could be produced as easily as letters³. Experts are in doubt as to the nature of Ratdolt's discovery. Was it a method of making figures up out of separate parts of figures, straight or curved lines, put together as letters are put together to make words? In a life of Joh. Gottlob Immanuel Breitkopf, a contemporary of Kästner's own, this member of the great house of Breitkopf is credited with this particular discovery. Experts in that same house expressed the opinion that Ratdolt's figures were woodcuts, while the letters denoting points in the figures were like the other letters in the text; yet it was with carved wooden blocks that printing began. If Ratdolt was the first to print geometrical figures, it was not long before an emulator arose; for in the very same year Mattheus Cordonis of Windischgrätz employed woodcut mathematical figures in printing Oresme's *De latitudinibus*⁴. How eagerly the opportunity of spreading geometrical knowledge was seized upon is proved by the number of editions which followed in the next few years. Even the year 1482 saw two forms of the book, though they only differ in the first sheet. Another edition came out in 1486 (*Ulmae, apud Io. Regerum*) and another in 1491 (*Vincentiae per*

¹ Curtze (*An-Nairiz*, p. xiii) reproduces the heading of the first page of the text as follows (there is no title-page): *Preclarissimū opus elemento*²⁴. *Euclidis megarensis vna cū cōmentis Campani pspicacissimi in artē geometriā incipit felicit'*, after which the definitions begin at once. Other copies have the shorter heading: *Preclarissimus liber elementorum Euclidis perspicacissimi: in artem Geometrie incipit quam foelicissime*. At the end stands the following: *¶ Opus elementorū euclidis megarenfis in geometriā artē Jn id quoq3 Campani pspicacissimi Cōmentationes finiūt. Erhardus ratdolt Augustensis impressor solertissimus. venetijs impressit. Anno salutis . M.cccc.lxxxij . Octauis . Caleñ . Juñ . Lector . Vale.*

² Kästner, *Geschichte der Mathematik*, I. p. 289 sqq. See also Weissenborn, *Die Übersetzungen des Euklid durch Campano und Zamberti*, pp. 1-7.

³ "Mea industria non sine maximo labore effecti vi qua facilitate litterarum elementa imprimuntur ea etiam geometrice figure conficerentur."

⁴ Curtze in *Zeitschrift für Math. u. Physik*, xx., hist.-litt. Abth. p. 58.

Leonardum de Basilea et Gulielmum de Papia), but without the dedication to Mocenigo who had died in the meantime (1485). If Campanus added anything of his own, his additions are at all events not distinguished by any difference of type or otherwise; the enunciations are in large type, and the rest is printed continuously in smaller type. There are no superscriptions to particular passages such as *Euclides ex Campano*, *Campanus*, *Campani additio*, or *Campani annotatio*, which are found for the first time in the Paris edition of 1516 giving both Campanus' version and that of Zamberti (presently to be mentioned).

1501. G. Valla included in his encyclopaedic work *De expetendis et fugiendis rebus* published in this year at Venice (*in aedibus Aldi Romani*) a number of propositions with proofs and scholia translated from a Greek MS. which was once in his possession (cod. Mutin. III B, 4 of the 15th c.).

1505. In this year Bartolomeo Zamberti (Zambertus) brought out at Venice the first translation, from the Greek text, of the whole of the *Elements*. From the title¹, as well as from his prefaces to the *Catoptrica* and *Data*, with their allusions to previous translators "who take some things out of authors, omit some, and change some," or "to that most barbarous translator" who filled a volume purporting to be Euclid's "with extraordinary scarecrows, nightmares and phantasies," one object of Zamberti's translation is clear. His animus against Campanus appears also in a number of notes, e.g. when he condemns the terms "helmuain" and "helmuariphe" used by Campanus as barbarous, un-Latin etc., and when he is roused to wrath by Campanus' unfortunate mistranslation of v. Def. 5. He does not seem to have had the penetration to see that Campanus was translating from an Arabic, and not from a Greek, text. Zamberti tells us that he spent seven years over his translation of the thirteen Books of the *Elements*. As he seems to have been born in 1473, and the *Elements* were printed as early as 1500, though the complete work (including the *Phaenomena*, *Optica*, *Catoptrica*, *Data* etc.) has the date 1505 at the end, he must have translated Euclid before the age of 30. Heiberg has not been able to identify the MS. of the *Elements* which Zamberti used; but it is clear that it belonged to the worse class of MSS., since it contains most of the interpolations of the Theonine variety. Zamberti, as his title shows, attributed the *proofs* to Theon.

1509. As a counterblast to Zamberti, Luca Paciolo brought out an edition of Euclid, apparently at the expense of Ratdolt, at Venice (*per Paganinum de Paganinis*), in which he set himself to vindicate Campanus. The title-page of this now very rare edition² begins thus: "The works of Euclid of Megara, a most acute philosopher and without

¹ The title begins thus: "Euclidis megaresis philosophi platonicj mathematicarum disciplinarum Janitoris: Habent in hoc volumine quicunque ad mathematicam substantiam aspirant: elementorum libros xiiij cum expositione Theonis insignis mathematici. quibus multa quae deerant ex lectione graeca sumpta addita sunt nec non plurima peruersa et praepostere: voluta in Campani interpretatione: ordinata digesta et castigata sunt etc." For a description of the book see Weissenborn, p. 12 sqq.

² See Weissenborn, p. 30 sqq.

question the chief of all mathematicians, translated by Campanus their *most faithful interpreter*." It proceeds to say that the translation had been, through the fault of copyists, so spoiled and deformed that it could scarcely be recognised as Euclid. Luca Paciuolo accordingly has polished and emended it with the most critical judgment, has corrected 129 figures wrongly drawn and added others, besides supplying short explanations of difficult passages. It is added that Scipio Vegius of Milan, distinguished for his knowledge "*of both languages*" (i.e. of course Latin and Greek), as well as in medicine and the more sublime studies, had helped to make the edition more perfect. Though Zamberti is not once mentioned, this latter remark must have reference to Zamberti's statement that his translation was from the Greek text; and no doubt Zamberti is aimed at in the wish of Paciuolo's "that others too would seek to acquire knowledge instead of merely showing off, or that they would not try to make a market of the things of which they are ignorant, as it were (selling) smoke¹." Weissenborn observes that, while there are many trivialities in Paciuolo's notes, they contain some useful and practical hints and explanations of terms, besides some new proofs which of course are not difficult if one takes the liberty, as Paciuolo does, of diverging from Euclid's order and assuming for the proof of a proposition results not arrived at till later. Two not inapt terms are used in this edition to describe the figures of III. 7, 8, the former of which is called the *goose's foot* (*pes anseris*), the second the *peacock's tail* (*cauda pavonis*). Paciuolo as the *castigator* of Campanus' translation, as he calls himself, failed to correct the mistranslation of v. Def. 5². Before the fifth Book he inserted a discourse which he gave at Venice on the 15th August, 1508, in S. Bartholomew's Church, before a select audience of 500, as an introduction to his elucidation of that Book.

1516. The first of the editions giving Campanus' and Zamberti's translations in conjunction was brought out at Paris (*in officina Henrici Stephani e regione scholae Decretorum*). The idea that only the enunciations were Euclid's, and that Campanus was the author of the proofs in his translation, while Theon was the author of the proofs in the Greek text, reappears in the title of this edition; and the enunciations of the added Books XIV., XV. are also attributed to Euclid, Hypsicles being credited with the proofs³. The date is not on the title-page nor at the

¹ "Atque utinam et alii cognoscere vellent non ostentare aut ea quae nesciunt veluti fumum venditare non conarentur."

² Campanus' translation in Raddolt's edition is as follows: "Quantitates quae dicuntur continuam habere proportionalitatem, sunt, quarum eque multiplicia aut equa sunt aut eque sibi sine interruptione addunt aut minuunt" (!), to which Campanus adds the note: "Continuè proportionalia sunt quorum omnia multiplicia equalia sunt continuè proportionalia. Sed noluit ipsam diffinitionem proponere sub hac forma, quia tunc diffiniret idem per idem, aperte (? a parte) tamen rei est istud cum sua diffinitione convertibile."

³ "Euclidis Megarensis Geometricorum Elementorum Libri xv. Campani Galli transalpini in eosdem commentariorum libri xv. Theonis Alexandrini Bartholomaeo Zamberto Veneto interprete, in tredecim priores, commentationum libri XIII. Hypsicles Alexandrini in duos posteriores, eodem Bartholomaeo Zamberto Veneto interprete, commentariorum libri II." On the last page (261) is a similar statement of content, but with the difference that the expression "ex Campani...deinde Theonis...et Hypsicles...traditionibus." For description see Weissenborn, p. 56 sqq.

end, but the letter of dedication to François Bricconnet by Jacques Lefèvre is dated the day after the Epiphany, 1516. The figures are in the margin. The arrangement of the propositions is as follows: first the enunciation with the heading *Euclides ex Campano*, then the proof with the note *Campanus*, and after that, as *Campani additio*, any passage found in the edition of Campanus' translation but not in the Greek text; then follows the text of the enunciation translated from the Greek with the heading *Euclides ex Zamberto*, and lastly the proof headed *Theo ex Zamberto*. There are separate figures for the two proofs. This edition was reissued with few changes in 1537 and 1546 at Basel (*apud Iohannem Hervagium*), but with the addition of the *Phaenomena*, *Optica*, *Catoptrica* etc. For the edition of 1537 the Paris edition of 1516 was collated with "a Greek copy" (as the preface says) by Christian Herlin, professor of mathematical studies at Strassburg, who however seems to have done no more than correct one or two passages by the help of the Basel *editio princeps* (1533), and add the Greek word in cases where Zamberti's translation of it seemed unsuitable or inaccurate.

We now come to

II. EDITIONS OF THE GREEK TEXT.

1533 is the date of the *editio princeps*, the title-page of which reads as follows:

ΕΤΚΛΕΙΔΟΥ ΣΤΟΙΧΕΙΩΝ ΒΙΒΛΗ ΙΕΒ

ΕΚ ΤΩΝ ΘΕΩΝΟΣ ΣΥΝΟΥΣΙΩΝ.

Εἰς τοῦ αὐτοῦ τὸ πρῶτον, ἐξηγημάτων Πρόκλου βιβλ. δ.

Adiecta praefatiuncula in qua de disciplinis

Mathematicis nonnihil.

BASILEAE APVD IOAN. HERVAGIVM ANNO

M.D.XXXIII. MENSE SEPTEMBRI.

The editor was Simon Grynaeus the elder (d. 1541), who, after working at Vienna and Ofen, Heidelberg and Tübingen, taught last of all at Basel, where theology was his main subject. His "praefatiuncula" is addressed to an Englishman, Cuthbert Tonstall (1474–1559), who, having studied first at Oxford, then at Cambridge, where he became Doctor of Laws, and afterwards at Padua, where in addition he learnt mathematics—mostly from the works of Regiomontanus and Paciolo—wrote a book on arithmetic¹ as "a farewell to the sciences," and then, entering politics, became Bishop of London and member of the Privy Council, and afterwards (1530) Bishop of Durham. Grynaeus tells us that he used two MSS. of the text of the *Elements*, entrusted to friends of his, one at Venice by "Lazarus Bayfius" (Lazare de Barf, then the ambassador of the King of France at Venice), the other at Paris by "Ioann. Rvellius" (Jean Ruel, a French doctor and a Greek scholar), while the commentaries of Proclus were put at

¹ *De arte supputandi libri quatuor.*

the disposal of Grynaeus himself by "Ioann. Claymundus" at Oxford. Heiberg has been able to identify the two MSS. used for the text; they are (1) cod. Venetus Marcianus 301 and (2) cod. Paris. gr. 2343 of the 16th c., containing Books I.—XV., with some scholia which are embodied in the text. When Grynaeus notes in the margin the readings from "the other copy," this "other copy" is as a rule the Paris MS., though sometimes the reading of the Paris MS. is taken into the text and the "other copy" of the margin is the Venice MS. Besides these two MSS. Grynaeus consulted Zamberti, as is shown by a number of marginal notes referring to "Zampertus" or to "latinum exemplar" in certain propositions of Books IX.—XI. When it is considered that the two MSS. used by Grynaeus are among the worst, it is obvious how entirely unauthoritative is the text of the *editio princeps*. Yet it remained the source and foundation of later editions of the Greek text for a long period, the editions which followed being designed, not for the purpose of giving, from other MSS., a text more nearly representing what Euclid himself wrote, but of supplying a handy compendium to students at a moderate price.

1536. Orontius Finaeus (Oronce Fine) published at Paris (*apud Simonem Colinaeum*) "demonstrations on the first six books of Euclid's elements of geometry," "in which the Greek text of Euclid himself is inserted in its proper places, with the Latin translation of Barth. Zamberti of Venice," which seems to imply that only the enunciations were given in Greek. The preface, from which Kästner quotes¹, says that the University of Paris at that time required, from all who aspired to the laurels of philosophy, a most solemn oath that they had attended lectures on the said first six Books. Other editions of Fine's work followed in 1544 and 1551.

1545. The *enunciations* of the fifteen Books were published in Greek, with an Italian translation by Angelo Caiani, at Rome (*apud Antonium Bladum Asulanum*). The translator claims to have corrected the books and "purged them of six hundred things which did not seem to savour of the almost divine genius and the perspicuity of Euclid²."

1549. Joachim Camerarius published the enunciations of the first six Books in Greek and Latin (Leipzig). The book, with preface, purports to be brought out by Rhaeticus (1514–1576), a pupil of Copernicus. Another edition with proofs of the propositions of the first three Books was published by Moritz Steinmetz in 1577 (Leipzig); a note by the printer attributes the preface to Camerarius himself.

1550. Ioan. Scheubel published at Basel (also *per Ioan. Hervagium*) the first six Books in Greek and Latin "together with true and appropriate proofs of the propositions, without the use of letters" (i.e. letters denoting points in the figures), the various straight lines and angles being described in words³.

1557 (also 1558). Stephanus Gracilis published another edition (repeated 1573, 1578, 1598) of the enunciations (alone) of Books I.—XV.

¹ Kästner, I. p. 260.

² Heiberg, vol. v. p. cvii.

³ Kästner, I. p. 359.

in Greek and Latin at Paris (*apud Gulielmum Cavellat*). He remarks in the preface that for want of time he had changed scarcely anything in Books I.—VI., but in the remaining Books he had emended what seemed obscure or inelegant in the Latin translation, while he had adopted in its entirety the translation of Book X. by Pierre Mondoré (*Petrus Montaurcus*), published separately at Paris in 1551. Gracilis also added a few “scholia.”

1564. In this year Conrad Dasypodius (*Rauchfuss*), the inventor and maker of the clock in Strassburg cathedral, similar to the present one, which did duty from 1571 to 1789, edited (*Strassburg, Chr. Mylius*) (1) Book I. of the *Elements* in Greek and Latin with scholia, (2) Book II. in Greek and Latin with Barlaam’s arithmetical version of Book II., and (3) the *enunciations* of the remaining Books III.—XIII. Book I. was reissued with “*vocabula quaedam geometrica*” of Heron, the enunciations of all the Books of the *Elements*, and the other works of Euclid, all in Greek and Latin. In the preface to (1) he says that it had been for twenty-six years the rule of his school that all who were promoted from the classes to public lectures should learn the first Book, and that he brought it out, because there were then no longer any copies to be had, and in order to prevent a good and fruitful regulation of his school from falling through. In the preface to the edition of 1571 he says that the first Book was generally taught in all gymnasia and that it was prescribed in his school for the first class. In the preface to (3) he tells us that he published the enunciations of Books III.—XIII. in order not to leave his work unfinished, but that, as it would be irksome to carry about the whole work of Euclid in extenso, he thought it would be more convenient to students of geometry to learn the *Elements* if they were compressed into a smaller book.

1620. Henry Briggs (of Briggs’ logarithms) published the first six Books in Greek with a Latin translation after Commandinus, “corrected in many places” (*London, G. Jones*).

1703 is the date of the Oxford edition by David Gregory which, until the issue of Heiberg and Menge’s edition, was still the only edition of the complete works of Euclid¹. In the Latin translation attached to the Greek text Gregory says that he followed Commandinus in the main, but corrected numberless passages in it by means of the books in the Bodleian Library which belonged to Edward Bernard (1638–1696), formerly Savilian Professor of Astronomy, who had conceived the plan of publishing the complete works of the ancient mathematicians in fourteen volumes, of which the first was to contain Euclid’s *Elements* I.—XV. As regards the Greek text, Gregory tells us that he consulted, as far as was necessary, not a few MSS. of the better sort, bequeathed by the great Savile to the University, as well as the corrections made by Savile in his own hand in the margin of the Basel edition. He had the help of John Hudson, Bodley’s Librarian, who

¹ ΕΥΚΛΕΙΔΟΥ ΤΑ ΣΩΖΟΜΕΝΑ. Euclidis quae supersunt omnia. Ex recensione Davidis Gregorii M.D. Astronomiae Professoris Saviliani et R.S.S. Oxoniae, e Theatro Sheldoniano, An. Dom. MDCCIII.

punctuated the Basel text before it went to the printer, compared the Latin version with the Greek throughout, especially in the *Elements* and *Data*, and, *where they differed or where he suspected the Greek text*, consulted the Greek MSS. and put their readings in the margin if they agreed with the Latin and, if they did not agree, affixed an asterisk in order that Gregory might judge which reading was geometrically preferable. Hence it is clear that no Greek MS., but the Basel edition, was the foundation of Gregory's text, and that Greek MSS. were only referred to in the special passages to which Hudson called attention.

1814-1818. A most important step towards a good Greek text was taken by F. Peyrard, who published at Paris, between these years, in three volumes, the *Elements* and *Data* in Greek, Latin and French¹. At the time (1808) when Napoleon was having valuable MSS. selected from Italian libraries and sent to Paris, Peyrard managed to get two ancient Vatican MSS. (190 and 1038) sent to Paris for his use (Vat. 204 was also at Paris at the time, but all three were restored to their owners in 1814). Peyrard noticed the excellence of Cod. Vat. 190, adopted many of its readings, and gave in an appendix a conspectus of these readings and those of Gregory's edition; he also noted here and there readings from Vat. 1038 and various Paris MSS. He therefore pointed the way towards a better text, but committed the error of correcting the Basel text instead of rejecting it altogether and starting afresh.

1824-1825. A most valuable edition of Books I.—VI. is that of J. G. Camerer (and C. F. Hauber) in two volumes published at Berlin². The Greek text is based on Peyrard, although the Basel and Oxford editions were also used. There is a Latin translation and a collection of notes far more complete than any other I have seen and well nigh inexhaustible. There is no editor or commentator of any mark who is not quoted from; to show the variety of important authorities drawn upon by Camerer, I need only mention the following names: Proclus, Pappus, Tartaglia, Commandinus, Clavius, Peletier, Barrow, Borelli, Wallis, Tacquet, Austin, Simson, Playfair. No words of praise would be too warm for this veritable encyclopaedia of information.

1825. J. G. C. Neide edited, from Peyrard, the text of Books I.—VI., XI. and XII. (*Halis Saxoniae*).

1826-9. The last edition of the Greek text before Heiberg's is that of E. F. August, who followed the Vatican MS. more closely than Peyrard did, and consulted at all events the Viennese MS. Gr. 103 (Heiberg's V). August's edition (Berlin, 1826-9) contains Books I.—XIII.

¹ *Euclidis quae supersunt. Les Œuvres d'Euclide, en Grec, en Latin et en Français d'après un manuscrit très-ancien, qui était resté inconnu jusqu'à nos jours.* Par F. Peyrard. Ouvrage approuvé par l'Institut de France (Paris, chez M. Patris).

² *Euclidis elementorum libri sex priores graece et latine commentariis e scriptis veterum ac recentiorum mathematicorum et Pfeidereri maxime illustrati* (Berolini, sumptibus G. Reimeri). Tom. I. 1824; tom. II. 1825.

III. LATIN VERSIONS OR COMMENTARIES AFTER 1533.

1545. Petrus Ramus (Pierre de la Ramée, 1515–1572) is credited with a translation of Euclid which appeared in 1545 and again in 1549 at Paris¹. Ramus, who was more rhetorician and logician than geometer, also published in his *Scholae mathematicae* (1559, Frankfurt; 1569, Basel) what amounts to a series of lectures on Euclid's *Elements*, in which he criticises Euclid's arrangement of his propositions, the definitions, postulates and axioms, all from the point of view of logic.

1557. Demonstrations to the geometrical Elements of Euclid, six Books, by Peletarius (Jacques Peletier). The second edition (1610) contained the same with the addition of the "Greek text of Euclid"; but only the *enunciations* of the propositions, as well as the definitions etc., are given in Greek (with a Latin translation), the rest is in Latin only. He has some acute observations, for instance about the "angle" of contact.

1559. Johannes Buteo, or Borrel (1492–1572), published in an appendix to his book *De quadratura circuli* some notes "on the errors of Campanus, Zambertus, Orontius, Peletarius, Pena, interpreters of Euclid." Buteo in these notes proved, by reasoned argument based on original authorities, that Euclid himself and not Theon was the author of the proofs of the propositions.

1566. Franciscus Flussates Candalla (François de Foix, Comte de Candale, 1502–1594), "restored" the fifteen Books, following, as he says, the terminology of Zamberti's translation from the Greek, but drawing, for his proofs, on both Campanus and Theon (i.e. Zamberti) except where mistakes in them made emendation necessary. Other editions followed in 1578, 1602, 1695 (in Dutch).

1572. The most important Latin translation is that of Commandinus (1509–1575) of Urbino, since it was the foundation of most translations which followed it up to the time of Peyrard, including that of Simson and therefore of those editions, numerous in England, which give Euclid "chiefly after the text of Simson." Simson's first (Latin) edition (1756) has "ex versione Latina Federici Commandini" on the title-page. Commandinus not only followed the original Greek more closely than his predecessors but added to his translation some ancient scholia as well as good notes of his own. The title of his work is

Euclidis elementorum libri XV, una cum scholiis antiquis. A Federico Commandino Urbinatense nuper in latinum conversi, commentariisque quibusdam illustrati (Pisauri, apud Camillum Francischinum).

He remarks in his preface that Orontius Finaeus had only edited six Books without reference to any Greek MS., that Peletarius had followed Campanus' version from the Arabic rather than the Greek text, and that Candalla had diverged too far from Euclid, having rejected as inelegant the proofs given in the Greek text and substituted faulty proofs of his own. Commandinus appears to have

¹ Described by Boncompagni, *Bullettino*, II. p. 389.

used, in addition to the Basel *editio princeps*, some Greek MS., so far not identified; he also extracted his "scholia antiqua" from a MS. of the class of Vat. 192 containing the scholia distinguished by Heiberg as "Schol. Vat." New editions of Commandinus' translation followed in 1575 (in Italian), 1619, 1749 (in English, by Keill and Stone), 1756 (Books I.—VI., XI., XII. in Latin and English, by Simson), 1763 (Keill). Besides these there were many editions of parts of the whole work, e.g. the first six Books.

1574. The first edition of the Latin version by Clavius (Christoph Klau [?], born at Bamberg 1537, died 1612) appeared in 1574, and new editions of it in 1589, 1591, 1603, 1607, 1612. It is not a translation, as Clavius himself states in the preface, but it contains a vast amount of notes collected from previous commentators and editors, as well as some good criticisms and elucidations of his own. Among other things, Clavius finally disposed of the error by which Euclid had been identified with Euclid of Megara. He speaks of the differences between Campanus who followed the Arabic tradition and the "commentaries of Theon," by which he appears to mean the Euclidean proofs as handed down by Theon; he complains of predecessors who have either only given the first six Books, or have rejected the ancient proofs and substituted worse proofs of their own, but makes an exception as regards Commandinus, "a geometer not of the common sort, who has lately restored Euclid, in a Latin translation, to his original brilliancy." Clavius, as already stated, did not give a translation of the *Elements* but rewrote the proofs, compressing them or adding to them, where he thought that he could make them clearer. Altogether his book is a most useful work.

1621. Henry Savile's lectures (*Praelectiones tresdecim in principium Elementorum Euclidis Oxoniae habitae* MDC.XX., Oxonii 1621), though they do not extend beyond I. 8, are valuable because they grapple with the difficulties connected with the preliminary matter, the definitions etc., and the tacit assumptions contained in the first propositions.

1654. André Tacquet's *Elementa geometriae planae et solidae* containing apparently the eight geometrical Books arranged for general use in schools. It came out in a large number of editions up to the end of the eighteenth century.

1655. Barrow's *Euclidis Elementorum Libri xv breviter demonstrati* is a book of the same kind. In the preface (to the edition of 1659) he says that he would not have written it but for the fact that Tacquet gave only eight Books of Euclid. He compressed the work into a very small compass (less than 400 small pages, in the edition of 1659, for the whole of the fifteen Books and the *Data*) by abbreviating the proofs and using a large quantity of symbols (which, he says, are generally Oughtred's). There were several editions up to 1732 (those of 1660 and 1732 and one or two others are in English).

¹ *Euclidis elementorum libri xv. Accessit xvi. de solidorum regularium comparatione. Omnes perspicuis demonstrationibus, accuratisque scholiis illustrati. Auctore Christophoro Clavio* (Romae, apud Vincentium Accoltum), 2 vols.

1658. Giovanni Alfonso Borelli (1608–1679) published *Euclides restitutus*, on apparently similar lines, which went through three more editions (one in Italian, 1663).

1660. Claude François Milliet Dechaies' eight geometrical Books of Euclid's *Elements* made easy. Dechaies' versions of the *Elements* had great vogue, appearing in French, Italian and English as well as Latin. Riccardi enumerates over twenty editions.

1733. Saccheri's *Euclides ab omni naevo vindicatus sive conatus geometricus quo stabiliuntur prima ipsa geometriae principia* is important for his elaborate attempt to prove the parallel-postulate, forming an important stage in the history of the development of non-Euclidean geometry.

1756. Simson's first edition, in Latin and in English. The Latin title is

Euclidis elementorum libri priores sex, item undecimus et duodecimus, ex versione latina Federici Commandini; sublati iis quibus olim libri hi a Theone, aliisque, vitii sunt, et quibusdam Euclidis demonstrationibus restitutis. A Roberto Simson M.D. Glasgae, in aedibus Academicis excudebant Robertus et Andreas Foulis, Academiae typographi.

1802. *Euclidis elementorum libri priores XII ex Commandini et Gregorii versionibus latinis. In usum juventutis Academicæ...* by Samuel Horsley, Bishop of Rochester. (Oxford, Clarendon Press.)

IV. ITALIAN VERSIONS OR COMMENTARIES.

1543. Tartaglia's version, a second edition of which was published in 1565¹, and a third in 1585. It does not appear that he used any Greek text, for in the edition of 1565 he mentions as available only "the first translation by Campano," "the second made by Bartolomeo Zamberto Veneto who is still alive," "the editions of Paris or Germany in which they have included both the aforesaid translations," and "our own translation into the vulgar (tongue)."

1575. Commandinus' translation turned into Italian and revised by him.

1613. The first six Books "reduced to practice" by Pietro Antonio Cataldi, re-issued in 1620, and followed by Books VII.—IX. (1621) and Book X. (1625).

1663. Borelli's Latin translation turned into Italian by Domenico Magni.

1680. *Euclide restituito* by Vitale Giordano.

1690. Vincenzo Viviani's *Elementi piani e solidi di Euclide* (Book V. in 1674).

¹ The title-page of the edition of 1565 is as follows: *Euclide Megarense philosopho, solo introduttore delle scientie mathematiche, diligentemente rassettato, et alla integrità ridotto, per il degno professore di tal scientie Nicolo Tartalea Brisciano. secondo le due tradottioni. con una ampla esposizione dello istesso traduttore di nuouo aggiunta. talmente chiara, che ogni mediocre ingegno, senza la notitia, ouer suffragio di alcun' altra scientia con facilità serà capace a poterlo intendere. In Venetia, Appresso Curtio Troiano, 1565.*

1731. *Elementi geometrici piani e solidi di Euclide* by Guido Grandi. No translation, but an abbreviated version, of which new editions followed one another up to 1806.

1749. Italian translation of Dechales with Ozanam's corrections and additions, re-issued 1785, 1797.

1752. Leonardo Ximenes (the first six Books). Fifth edition, 1819.

1818. Vincenzo Flauti's *Corso di geometria elementare e sublime* (4 vols.) contains (Vol. I.) the first six Books, with additions and a dissertation on Postulate 5, and (Vol. II.) Books XI., XII. Flauti also published the first six Books in 1827 and the *Elements of geometry of Euclid* in 1843 and 1854.

V. GERMAN.

1558. The arithmetical Books VII.—IX. by Scheubel¹ (cf. the edition of the first six Books, with enunciations in Greek and Latin, mentioned above, under date 1550).

1562. The version of the first six Books by Wilhelm Holtzmann (Xylander)². This work has its interest as the first edition in German, but otherwise it is not of importance. Xylander tells us that it was written for practical people such as artists, goldsmiths, builders etc., and that, as the simple amateur is of course content to know facts, without knowing how to prove them, he has often left out the proofs altogether. He has indeed taken the greatest possible liberties with Euclid, and has not grappled with any of the theoretical difficulties, such as that of the theory of parallels.

1651. Heinrich Hoffmann's *Teutscher Euclides* (2nd edition 1653), not a translation.

1694. Ant. Ernst Burkh. v. Pirckenstein's *Teutsch Redender Euclides* (eight geometrical Books), "for generals, engineers etc." "proved in a new and quite easy manner." Other editions 1699, 1744.

1697. Samuel Reyher's *In teutscher Sprache vorgestellter Euclides* (six Books), "made easy, with symbols algebraical or derived from the newest art of solution."

1714. *Euclidis xv Bücher teutsch*, "treated in a special and brief manner, yet completely," by Chr. Schessler (another edition in 1729).

1773. The first six Books translated from the Greek for the use of schools by J. F. Lorenz. The first attempt to reproduce Euclid in German word for word.

1781. Books XI., XII. by Lorenz (supplementary to the preceding). Also *Euklid's Elemente fünfzehn Bücher* translated from

¹ Das sibend acht und neunt buch des hochberümbten Mathematici Euclidis Megarensis... durch Magistram Johann Scheybl, der löblichen universitet zu Tübingen, des Euclidis und Arithmetici Ordinarien, auss dem latein ins teutsch gebracht....

² Die sechs erste Bücher Euclidis vom anfang oder grund der Geometrij... Auss Griechischer sprach in die Teutsch gebracht eigentlich erklärt... Demassen vormals .n Teutscher sprach nie gesehen worden... Durch Wilhelm Holtzman genant Xylander von Augsburg. Getruckht zu Basel.

the Greek by Lorenz (second edition 1798; editions of 1809, 1818, 1824 by Mollweide, of 1840 by Dippe). The edition of 1824, and I presume those before it, are shortened by the use of symbols and the compression of the enunciation and "setting-out" into one.

1807. Books I.—VI., XI., XII. "newly translated from the Greek," by J. K. F. Hauff.

1828. The same Books by Joh. Jos. Ign. Hoffmann "as guide to instruction in elementary geometry," followed in 1832 by observations on the text by the same editor.

1833. *Die Geometrie des Euklid und das Wesen derselben* by E. S. Unger; also 1838, 1851.

1901. Max Simon, *Euclid und die sechs planimetrischen Bücher*.

VI. FRENCH.

1564–1566. Nine Books translated by Pierre Forcadel, a pupil and friend of P. de la Ramée.

1604. The first nine Books translated and annotated by Jean Errard de Bar-le-Duc; second edition, 1605.

1615. Denis Henrion's translation of the 15 Books (seven editions up to 1676).

1639. The first six Books "demonstrated by symbols, by a method very brief and intelligible," by Pierre Hérigone, mentioned by Barrow as the only editor who, before him, had used symbols for the exposition of Euclid.

1672. Eight Books "rendus plus faciles" by Claude François Milliet Dechales, who also brought out *Les élémens d'Euclide expliqués d'une manière nouvelle et très facile*, which appeared in many editions, 1672, 1677, 1683 etc. (from 1709 onwards revised by Ozanam), and was translated into Italian (1749 etc.) and English (by William Halifax, 1685).

1804. In this year, and therefore before his edition of the Greek text, F. Peyrard published the *Elements* literally translated into French. A second edition appeared in 1809 with the addition of the fifth Book. As this second edition contains Books I.—VI. XI., XII. and X. I, it would appear that the first edition contained Books I.—IV., VI., XI., XII. Peyrard used for this translation the Oxford Greek text and Simson.

VII. DUTCH.

1606. Jan Pieterszoon Dou (six Books). There were many later editions. Kästner, in mentioning one of 1702, says that Dou explains in his preface that he used Xylander's translation, but, having afterwards obtained the French translation of the six Books by Errard de Bar-le-Duc (see above), the proofs in which sometimes pleased him more than those of the German edition, he made his Dutch version by the help of both.

1617. Frans van Schooten, "The Propositions of the Books of Euclid's Elements"; the fifteen Books in this version "enlarged" by Jakob van Leest in 1662.

1695. C. J. Vooght, fifteen Books complete, with Candalla's "16th."

1702. Hendrik Coets, six Books (also in Latin, 1692); several editions up to 1752. Apparently not a translation, but an edition for school use.

1763. Pybo Steenstra, Books I.—VI., XI., XII., likewise an abbreviated version, several times reissued until 1825.

VIII. ENGLISH.

1570 saw the first and the most important translation, that of Sir Henry Billingsley. The title-page is as follows:

THE ELEMENTS
OF GEOMETRIE
of the most auncient Philosopher
EVCLIDE
of Megara

Faithfully (now first) translated into the Englishe tounge, by H. Billingsley, Citizen of London. Whereunto are annexed certaine Scholies, Annotations, and Inuentions, of the best Mathematicians, both of time past, and in this our age.

With a very fruitfull Preface by M. I. Dee, specifying the chiefe Mathematicall Sciēces, what they are, and whereunto commodious: where, also, are disclosed certaine new Secrets Mathematicall and Mechanicall, vntill these our daies, greatly missed.

Imprinted at London by *John Daye*.

The Preface by the translator, after a sentence observing that without the diligent study of Euclides Elementes it is impossible to attain unto the perfect knowledge of Geometry, proceeds thus. "Wherefore considering the want and lacke of such good authors hitherto in our Englishe tounge, lamenting also the negligence, and lacke of zeale to their country in those of our nation, to whom God hath geuen both knowledge and also abilitie to translate into our tounge, and to publishe abroad such good authors and bookes (the chiefe instrumentes of all learninges): seing moreouer that many good wittes both of gentlemen and of others of all degrees, much desirous and studious of these artes, and seeking for them as much as they can, sparing no paines, and yet frustrate of their intent, by no meanes attaining to that which they seeke: I haue for their sakes, with some charge and great trauaile, faithfully translated into our vulgare tōge, and set abroad in Print, this booke of Euclide. Whereunto I haue added easie and plaine declarations and examples by figures, of the definitions. In which booke also ye shall in due place finde manifolde additions, Scholies, Annotations, and Inuentions: which I haue gathered out of many of the most famous and chiefe Mathematiciēs, both of old time, and in our age: as by diligent reading it in course, ye shall well perceauē...."

It is truly a monumental work, consisting of 464 leaves, and therefore 928 pages, of folio size, excluding the lengthy preface by Dee. The notes certainly include all the most important that had ever been

written, from those of the Greek commentators, Proclus and the others whom he quotes, down to those of Dee himself on the last books. Besides the fifteen Books, Billingsley included the "sixteenth" added by Candalla. The print and appearance of the book are worthy of its contents; and, in order that it may be understood how no pains were spared to represent everything in the clearest and most perfect form, I need only mention that the figures of the propositions in Book XI. are nearly all duplicated, one being the figure of Euclid, the other an arrangement of pieces of paper (triangular, rectangular etc.) pasted at the edges on to the page of the book so that the pieces can be turned up and made to show the real form of the solid figures represented.

Billingsley was admitted Lady Margaret Scholar of St John's College, Cambridge, in 1551, and he is also said to have studied at Oxford, but he did not take a degree at either University. He was afterwards apprenticed to a London haberdasher and rapidly became a wealthy merchant. Sheriff of London in 1584, he was elected Lord Mayor on 31st December, 1596, on the death, during his year of office, of Sir Thomas Skinner. From 1589 he was one of the Queen's four "customers," or farmers of customs, of the port of London. In 1591 he founded three scholarships at St John's College for poor students, and gave to the College for their maintenance two messuages and tenements in Tower Street and in Mark Lane, Allhallows, Barking. He died in 1606.

1651. *Elements of Geometry. The first VI Books: In a compendious form contracted and demonstrated* by Captain Thomas Rudd, with the mathematicall preface of John Dee (London).

1660. The first English edition of Barrow's Euclid (published in Latin in 1655), appeared in London. It contained "the whole fifteen books compendiously demonstrated"; several editions followed, in 1705, 1722, 1732, 1751.

1661. *Euclid's Elements of Geometry, with a supplement of divers Propositions and Corollaries. To which is added a Treatise of regular Solids by Campane and Flussat; likewise Euclid's Data and Marinus his Preface. Also a Treatise of the Divisions of Superficies, ascribed to Machomet Bagdedine, but published by Commandine at the request of J. Dee of London.* Published by care and industry of John Leeke and Geo. Serle, students in the Math. (London). According to Potts this was a second edition of Billingsley's translation.

1685. William Halifax's version of Dechales' "Elements of Euclid explained in a new but most easy method" (London and Oxford).

1705. *The English Euclide; being the first six Elements of Geometry, translated out of the Greek, with annotations and usefull supplements* by Edmund Scarburgh (Oxford). A noteworthy and useful edition.

1708. Books I.—VI., XI., XII., translated from Commandinus' Latin version by Dr John Keill, Savilian Professor of Astronomy at Oxford.

Keill complains in his preface of the omissions by such editors as Tacquet and Dechales of many necessary propositions (e.g. VI. 27—29), and of their substitution of proofs of their own for Euclid's. He praises Barrow's version on the whole, though objecting to the "algebraical"

form of proof adopted in Book II., and to the excessive use of notes and symbols, which (he considers) make the proofs *too* short and thereby obscure; his edition was therefore intended to hit a proper mean between Barrow's excessive brevity and Clavius' prolixity.

Keill's translation was revised by Samuel Cunn and several times reissued. 1749 saw the eighth edition, 1772 the eleventh, and 1782 the twelfth.

1714. W. Whiston's English version (abridged) of *The Elements of Euclid with select theorems out of Archimedes by the learned Andr. Tacquet*.

1756. Simson's first English edition appeared in the same year as his Latin version under the title:

The Elements of Euclid, viz. the first six Books together with the eleventh and twelfth. In this Edition the Errors by which Theon or others have long ago vitiated these Books are corrected and some of Euclid's Demonstrations are restored. By Robert Simson (Glasgow).

As above stated, the Latin edition, by its title, purports to be "ex versione latina Federici Commandini," but to the Latin edition, as well as to the English editions, are appended

Notes Critical and Geometrical; containing an Account of those things in which this Edition differs from the Greek text; and the Reasons of the Alterations which have been made. As also Observations on some of the Propositions.

Simson says in the Preface to some editions (e.g. the tenth, of 1799) that "the translation is much amended by the friendly assistance of a learned gentleman."

Simson's version and his notes are so well known as not to need any further description. The book went through some thirty successive editions. The first five appear to have been dated 1756, 1762, 1767, 1772 and 1775 respectively; the tenth 1799, the thirteenth 1806, the twenty-third 1830, the twenty-fourth 1834, the twenty-sixth 1844. The *Data* "in like manner corrected" was added for the first time in the edition of 1762 (the first octavo edition).

1781, 1788. In these years respectively appeared the two volumes containing the complete translation of the whole thirteen Books by James Williamson, the last English translation which reproduced Euclid word for word. The title is

The Elements of Euclid, with Dissertations intended to assist and encourage a critical examination of these Elements, as the most effectual means of establishing a juster taste upon mathematical subjects than that which at present prevails. By James Williamson.

In the first volume (Oxford, 1781) he is described as "M.A. Fellow of Hertford College," and in the second (London, printed by T. Spilsbury, 1788) as "B.D." simply. Books v., vi. with the Conclusion in the first volume are pagged separately from the rest.

1781. *An examination of the first six Books of Euclid's Elements*, by William Austin (London).

1795. John Playfair's first edition, containing "the first six Books of Euclid with two Books on the Geometry of Solids." The book

reached a fifth edition in 1819, an eighth in 1831, a ninth in 1836, and a tenth in 1846.

1826. Riccardi notes under this date *Euclid's Elements of Geometry containing the whole twelve Books translated into English, from the edition of Peyrard*, by George Phillips. The editor, who was President of Queens' College, Cambridge, 1857–1892, was born in 1804 and matriculated at Queens' in 1826, so that he must have published the book as an undergraduate.

1828. A very valuable edition of the first six Books is that of Dionysius Lardner, with commentary and geometrical exercises, to which he added, in place of Books XI., XII., a Treatise on Solid Geometry mostly based on Legendre. Lardner compresses the propositions by combining the enunciation and the setting-out, and he gives a vast number of riders and additional propositions in smaller print. The book had reached a ninth edition by 1846, and an eleventh by 1855. Among other things, Lardner gives an Appendix "on the theory of parallel lines," in which he gives a short history of the attempts to get over the difficulty of the parallel-postulate, down to that of Legendre.

1833. T. Perronet Thompson's *Geometry without axioms, or the first Book of Euclid's Elements with alterations and notes; and an intercalary book in which the straight line and plane are derived from properties of the sphere, with an appendix containing notices of methods proposed for getting over the difficulty in the twelfth axiom of Euclid*.

Thompson (1783–1869) was 7th wrangler 1802, midshipman 1803, Fellow of Queens' College, Cambridge, 1804, and afterwards general and politician. The book went through several editions, but, having been well translated into French by Van Tenac, is said to have received more recognition in France than at home.

1845. Robert Potts' first edition (and one of the best) entitled:

Euclid's Elements of Geometry chiefly from the text of Dr Simson with explanatory notes...to which is prefixed an introduction containing a brief outline of the History of Geometry. Designed for the use of the higher forms in Public Schools and students in the Universities (Cambridge University Press, and London, John W. Parker), to which was added (1847) *An Appendix to the larger edition of Euclid's Elements of Geometry, containing additional notes on the Elements, a short tract on transversals, and hints for the solution of the problems etc.*

1862. Todhunter's edition.

The later English editions I will not attempt to enumerate; their name is legion and their object mostly that of adapting Euclid for school use, with all possible gradations of departure from his text and order.

IX. SPANISH.

1576. The first six Books translated into Spanish by Rodrigo Camorano.

1637. The first six Books translated, with notes, by L. Carduchi.

1689. Books I.—VI., XI., XII., translated and explained by Jacob Knesa.

X. RUSSIAN.

1739. Ivan Astaroff (translation from Latin).
 1789. Pr. Suvoroff and Yos. Nikitin (translation from Greek).
 1880. Vachtchenko-Zakhartchenko.
 (1817. A translation into Polish by Jo. Czecha.)

XI. SWEDISH.

1744. Mårten Strömer, the first six Books ; second edition 1748.
 The third edition (1753) contained Books XI.—XII. as well ; new
 editions continued to appear till 1884.
 1836. H. Falk, the first six Books.
 1844, 1845, 1859. P. R. Bråkenhjelm, Books I.—VI., XI., XII.
 1850. F. A. Å. Lundgren.
 1850. H. A. Witt and M. E. Areskong, Books I.—VI., XI., XII.

XII. DANISH.

1745. Ernest Gottlieb Ziegenbalg.
 1803. H. C. Linderup, Books I.—VI.

XIII. MODERN GREEK.

1820. Benjamin of Lesbos.

I should add a reference to certain editions which have appeared in recent years.

A Danish translation (*Euklid's Elementer* oversat af Thyra Eibe) was completed in 1912 ; Books I.—II. were published (with an Introduction by Zeuthen) in 1897, Books III.—IV. in 1900, Books V.—VI. in 1904, Books VII.—XIII. in 1912.

The Italians, whose great services to elementary geometry are more than once emphasised in this work, have lately shown a noteworthy disposition to make the *ipsissima verba* of Euclid once more the object of study. Giovanni Vacca has edited the text of Book I. (*Il primo libro degli Elementi*. Testo greco, versione italiana, introduzione e note, Firenze 1916.) Federigo Enriques has begun the publication of a complete Italian translation (*Gli Elementi d' Euclide e la critica antica e moderna*) ; Books I.—IV. appeared in 1925 (Alberto Stock, Roma).

An edition of Book I. by the present writer was published in 1918 (*Euclid in Greek, Book I., with Introduction and Notes*, Camb. Univ. Press).

CHAPTER IX.

§ 1. ON THE NATURE OF *ELEMENTS*.

IT would not be easy to find a more lucid explanation of the terms *element* and *elementary*, and of the distinction between them, than is found in Proclus¹, who is doubtless, here as so often, quoting from Geminus. There are, says Proclus, in the whole of geometry certain leading theorems, bearing to those which follow the relation of a principle, all-pervading, and furnishing proofs of many properties. Such theorems are called by the name of *elements*; and their function may be compared to that of the letters of the alphabet in relation to language, letters being indeed called by the same name in Greek (*στοιχεῖα*).

The term *elementary*, on the other hand, has a wider application: it is applicable to things "which extend to greater multiplicity, and, though possessing simplicity and elegance, have no longer the same dignity as the *elements*, because their investigation is not of general use in the whole of the science, e.g. the proposition that in triangles the perpendiculars from the angles to the transverse sides meet in a point."

"Again, the term *element* is used in two senses, as Menaechmus says. For that which is the means of obtaining is an element of that which is obtained, as the first proposition in Euclid is of the second, and the fourth of the fifth. In this sense many things may even be said to be elements of each other, for they are obtained from one another. Thus from the fact that the exterior angles of rectilineal figures are (together) equal to four right angles we deduce the number of right angles equal to the internal angles (taken together)², and *vice versa*. Such an element is like a *lemma*. But the term *element* is otherwise used of that into which, being more simple, the composite is divided; and in this sense we can no longer say that everything is an element of everything, but only that things which are more of the nature of principles are elements of those which stand to them in the relation of results, as postulates are elements of theorems. It is

¹ Proclus, *Comm. on Eucl. I.*, ed. Friedlein, pp. 72 sqq.

² τὸ πλῆθος τῶν ἐντὸς ὀρθαῖς ἴσων. If the text is right, we must apparently take it as "the number of the angles equal to right angles that there are inside," i.e. that are made up by the internal angles.

according to this signification of the term *element* that the elements found in Euclid were compiled, being partly those of plane geometry, and partly those of stereometry. In like manner many writers have drawn up elementary treatises in arithmetic and astronomy.

"Now it is difficult, in each science, both to select and arrange in due order the elements from which all the rest proceeds, and into which all the rest is resolved. And of those who have made the attempt some were able to put together more and some less; some used shorter proofs, some extended their investigation to an indefinite length; some avoided the method of *reductio ad absurdum*, some avoided *proportion*; some contrived preliminary steps directed against those who reject the principles; and, in a word, many different methods have been invented by various writers of elements.

"It is essential that such a treatise should be rid of everything superfluous (for this is an obstacle to the acquisition of knowledge); it should select everything that embraces the subject and brings it to a point (for this is of supreme service to science); it must have great regard at once to clearness and conciseness (for their opposites trouble our understanding); it must aim at the embracing of theorems in general terms (for the piecemeal division of instruction into the more partial makes knowledge difficult to grasp). In all these ways Euclid's system of elements will be found to be superior to the rest; for its utility avails towards the investigation of the primordial figures¹, its clearness and organic perfection are secured by the progression from the more simple to the more complex and by the foundation of the investigation upon common notions, while generality of demonstration is secured by the progression through the theorems which are primary and of the nature of principles to the things sought. As for the things which seem to be wanting, they are partly to be discovered by the same methods, like the construction of the scalene and isosceles (triangle), partly alien to the character of a selection of elements as introducing hopeless and boundless complexity, like the subject of *unordered irrationals* which Apollonius worked out at length², and partly developed from things handed down (in the elements) as causes, like the many species of angles and of lines. These things then have been omitted in Euclid, though they have received full discussion in other works; but the knowledge of them is derived from the simple (elements)."

Proclus, speaking apparently on his own behalf, in another place distinguishes two objects aimed at in Euclid's *Elements*. The first has reference to the *matter* of the investigation, and here, like a good Platonist, he takes the whole subject of geometry to be concerned with the "cosmic figures," the five regular solids, which in Book XIII.

¹ τῶν ἀρχικῶν σχημάτων, by which Proclus probably means the regular polyhedra (Tannery, p. 143 n.).

² We have no more than the most obscure indications of the character of this work in an Arabic MS. analysed by Woepcke, *Essai d'une restitution de travaux perdus d'Apollonius sur les quantités irrationnelles d'après des indications tirées d'un manuscrit arabe* in *Mémoires présentés à l'académie des sciences*, XIV. 658—720, Paris, 1856. Cf. Cantor, *Gesch. d. Math.* I, pp. 348—9: details are also given in my notes to Book X.

are constructed, inscribed in a sphere and compared with one another. The second object is relative to the learner; and, from this standpoint, the elements may be described as "a means of perfecting the learner's understanding with reference to the whole of geometry. For, starting from these (elements), we shall be able to acquire knowledge of the other parts of this science as well, while without them it is impossible for us to get a grasp of so complex a subject, and knowledge of the rest is unattainable. As it is, the theorems which are most of the nature of principles, most simple, and most akin to the first hypotheses are here collected, in their appropriate order; and the proofs of all other propositions use these theorems as thoroughly well known, and start from them. Thus Archimedes in the books on the sphere and cylinder, Apollonius, and all other geometers, clearly use the theorems proved in this very treatise as constituting admitted principles¹."

Aristotle too speaks of *elements* of geometry in the same sense. Thus: "in geometry it is well to be thoroughly versed in the elements²"; "in general the first of the elements are, given the definitions, e.g. of a straight line and of a circle, most easy to prove, although of course there are not many data that can be used to establish each of them because there are not many middle terms³"; "among geometrical propositions we call those 'elements' the proofs of which are contained in the proofs of all or most of such propositions⁴"; "(as in the case of bodies), so in like manner we speak of the elements of geometrical propositions and, generally, of demonstrations; for the demonstrations which come first and are contained in a variety of other demonstrations are called elements of those demonstrations... the term element is applied by analogy to that which, being one and small, is useful for many purposes⁵."

§ 2. *ELEMENTS* ANTERIOR TO EUCLID'S.

The early part of the famous summary of Proclus was no doubt drawn, at least indirectly, from the history of geometry by Eudemus; this is generally inferred from the remark, made just after the mention of Philippus of Medma, a disciple of Plato, that "those who have written histories bring the development of this science up to this point." We have therefore the best authority for the list of writers of *elements* given in the summary. Hippocrates of Chios (fl. in second half of 5th c.) is the first; then Leon, who also discovered *diorismi*, put together a more careful collection, the propositions proved in it being more numerous as well as more serviceable⁶. Leon was a little older than Eudoxus (about 408-355 B.C.) and a little younger than Plato (428/7-347/6 B.C.), but did not belong to the latter's school. The

¹ Proclus, pp. 70, 19-71, 21.

² *Topics* VIII. 14, 163 b 23.

³ *Topics* VIII. 3, 158 b 35.

⁴ *Metaph.* 998 a 25.

⁵ *Metaph.* 1014 a 35-b 5.

⁶ Proclus, p. 66, 20 ὥστε τὸν Λέοντα καὶ τὰ στοιχεῖα συνθεῖναι τῷ τε πλήθει καὶ τῇ χρείᾳ τῶν δεικνυμένων ἐπιμελέστερον.

geometrical text-book of the Academy was written by Theudius of Magnesia, who, with Amyclas of Heraclea, Menaechmus the pupil of Eudoxus, Menaechmus' brother Dinostratus and Athenaeus of Cyzicus consorted together in the Academy and carried on their investigations in common. Theudius "put together the elements admirably, making many partial (or limited) propositions more general¹." Eudemus mentions no text-book after that of Theudius, only adding that Hermotimus of Colophon "discovered many of the elements²." Theudius then must be taken to be the immediate precursor of Euclid, and no doubt Euclid made full use of Theudius as well as of the discoveries of Hermotimus and all other available material. Naturally it is not in Euclid's *Elements* that we can find much light upon the state of the subject when he took it up; but we have another source of information in Aristotle. Fortunately for the historian of mathematics, Aristotle was fond of mathematical illustrations; he refers to a considerable number of geometrical propositions, definitions etc., in a way which shows that his pupils must have had at hand some text-book where they could find the things he mentions; and this text-book must have been that of Theudius. Heiberg has made a most valuable collection of mathematical extracts from Aristotle³, from which much is to be gathered as to the changes which Euclid made in the methods of his predecessors; and these passages, as well as others not included in Heiberg's selection, will often be referred to in the sequel.

§ 3. FIRST PRINCIPLES: DEFINITIONS, POSTULATES, AND AXIOMS.

On no part of the subject does Aristotle give more valuable information than on that of the first principles as, doubtless, generally accepted at the time when he wrote. One long passage in the *Posterior Analytics* is particularly full and lucid, and is worth quoting *in extenso*. After laying it down that every demonstrative science starts from necessary principles⁴, he proceeds⁵:

"By first principles in each genus I mean those the truth of which it is not possible to prove. What is *denoted* by the first (terms) and those derived from them is assumed; but, as regards their *existence*, this must be assumed for the principles but proved for the rest. Thus what a unit is, what the straight (line) is, or what a triangle is (must be assumed); and the existence of the unit and of magnitude must also be assumed, but the rest must be proved. Now of the premisses used in demonstrative sciences some are peculiar to each science and others common (to all), the latter being common by analogy, for of course they are actually useful in so far as they are applied to the subject-matter included under the particular science. Instances of first

¹ Proclus, p. 67, 14 καὶ γὰρ τὰ στοιχεῖα καλῶς συνέταξεν καὶ πολλὰ τῶν μερικῶν [ὀρίκων (?) Friedlein] καθολικώτερα ἐποίησεν.

² Proclus, p. 67, 22 τῶν στοιχείων πολλὰ ἀνέυρε.

³ *Mathematisches zu Aristoteles in Abhandlungen zur Gesch. d. math. Wissenschaften*, XVIII. Heft (1904), pp. 1—49.

⁴ *Anal. post.* I. 6, 74 b 5.

⁵ *ibid.* I. 10, 76 a 31—77 a 4.

principles peculiar to a science are the assumptions that a line is of such and such a character, and similarly for the straight (line); whereas it is a common principle, for instance, that, if equals be subtracted from equals, the remainders are equal. But it is enough that each of the common principles is true so far as regards the particular genus (subject-matter); for (in geometry) the effect will be the same even if the common principle be assumed to be true, not of everything, but only of magnitudes, and, in arithmetic, of numbers.

"Now the things peculiar to the science, the existence of which must be assumed, are the things with reference to which the science investigates the essential attributes, e.g. arithmetic with reference to units, and geometry with reference to points and lines. With these things it is assumed that they exist and that they are of such and such a nature. But, with regard to their essential properties, what is assumed is only the meaning of each term employed: thus arithmetic assumes the answer to the question what is (meant by) 'odd' or 'even,' 'a square' or 'a cube,' and geometry to the question what is (meant by) 'the irrational' or 'deflection' or (the so-called) 'verging' (to a point); but that there are such things is proved by means of the common principles and of what has already been demonstrated. Similarly with astronomy. For every demonstrative science has to do with three things, (1) the things which are assumed to exist, namely the genus (subject-matter) in each case, the essential properties of which the science investigates, (2) the common axioms so-called, which are the primary source of demonstration, and (3) the properties with regard to which all that is assumed is the meaning of the respective terms used. There is, however, no reason why some sciences should not omit to speak of one or other of these things. Thus there need not be any supposition as to the existence of the genus, if it is manifest that it exists (for it is not equally clear that number exists and that cold and hot exist); and, with regard to the properties, there need be no assumption as to the meaning of terms if it is clear: just as in the common (axioms) there is no assumption as to what is the meaning of subtracting equals from equals, because it is well known. But none the less is it true that there are three things naturally distinct, the subject-matter of the proof, the things proved, and the (axioms) from which (the proof starts).

"Now that which is *per se* necessarily true, and must necessarily be thought so, is not a hypothesis nor yet a postulate. For demonstration has not to do with reasoning from outside but with the reason dwelling in the soul, just as is the case with the syllogism. It is always possible to raise objection to reasoning from outside, but to contradict the reason within us is not always possible. Now anything that the teacher assumes, though it is matter of proof, without proving it himself, is a hypothesis if the thing assumed is believed by the learner, and it is moreover a hypothesis, not absolutely, but relatively to the particular pupil; but, if the same thing is assumed when the learner either has no opinion on the subject or is of a contrary opinion, it is a postulate. This is the difference

between a hypothesis and a postulate; for a postulate is that which is rather contrary than otherwise to the opinion of the learner, or whatever is assumed and used without being proved, although matter for demonstration. Now definitions are not hypotheses, for they do not assert the existence or non-existence of anything, while hypotheses are among propositions. Definitions only require to be understood: a definition is therefore not a hypothesis, unless indeed it be asserted that any audible speech is a hypothesis. A hypothesis is that from the truth of which, if assumed, a conclusion can be established. Nor are the geometer's hypotheses false, as some have said: I mean those who say that 'you should not make use of what is false, and yet the geometer falsely calls the line which he has drawn a foot long when it is not, or straight when it is not straight.' The geometer bases no conclusion on the particular line which he has drawn being that which he has described, but (he refers to) what is *illustrated* by the figures. Further, the postulate and every hypothesis are either universal or particular statements; definitions are neither" (because the subject is of equal extent with what is predicated of it).

Every demonstrative science, says Aristotle, must start from indemonstrable principles: otherwise, the steps of demonstration would be endless. Of these indemonstrable principles some are (*a*) common to all sciences, others are (*b*) particular, or peculiar to the particular science; (*a*) the common principles are the *axioms*, most commonly illustrated by the axiom that, if equals be subtracted from equals, the remainders are equal. Coming now to (*b*) the principles peculiar to the particular science which must be assumed, we have first the *genus* or subject-matter, the *existence* of which must be assumed, viz. magnitude in the case of geometry, the unit in the case of arithmetic. Under this we must assume *definitions* of manifestations or attributes of the genus, e.g. straight lines, triangles, deflection etc. The definition in itself says nothing as to the existence of the thing defined: it only requires to be understood. But in geometry, in addition to the *genus* and the *definitions*, we have to assume the *existence* of a few *primary* things which are defined, viz. points and lines only: the existence of everything else, e.g. the various figures made up of these, as triangles, squares, tangents, and their properties, e.g. incommensurability etc., has to be proved (as it is proved by construction and demonstration). In arithmetic we assume the *existence* of the *unit*: but, as regards the rest, only the *definitions*, e.g. those of odd, even, square, cube, are assumed, and *existence* has to be *proved*. We have then clearly distinguished, among the indemonstrable principles, *axioms* and *definitions*. A *postulate* is also distinguished from a *hypothesis*, the latter being made with the assent of the learner, the former without such assent or even in opposition to his opinion (though, strangely enough, immediately after saying this, Aristotle gives a wider meaning to "postulate" which would cover "hypothesis" as well, namely whatever is assumed, though it is matter for proof, and used without being proved). Heiberg remarks that there is no trace in Aristotle of Euclid's Postulates, and that "postulate" in Aristotle has

a different meaning. He seems to base this on the alternative description of postulate, indistinguishable from a hypothesis; but, if we take the other description in which it is distinguished from a hypothesis as being an assumption of something which is a proper subject of demonstration without the assent or against the opinion of the learner, it seems to fit Euclid's Postulates fairly well, not only the first three (postulating three constructions), but eminently also the other two, that all right angles are equal, and that two straight lines meeting a third and making the internal angles on the same side of it less than two right angles will meet on that side. Aristotle's description also seems to me to suit the "postulates" with which Archimedes begins his book *On the equilibrium of planes*, namely that equal weights balance at equal distances, and that equal weights at unequal distances do not balance but that the weight at the longer distance will prevail.

Aristotle's distinction also between *hypothesis* and *definition*, and between *hypothesis* and *axiom*, is clear from the following passage: "Among immediate syllogistic principles; I call that a *thesis* which it is neither possible to prove nor essential for any one to hold who is to learn anything; but that which it is necessary for any one to hold who is to learn anything whatever is an *axiom*: for there are some principles of this kind, and that is the most usual name by which we speak of them. But, of *theses*, one kind is that which assumes one or other side of a predication, as, for instance, that something exists or does not exist, and this is a *hypothesis*; the other, which makes no such assumption, is a *definition*. For a definition is a thesis: thus the arithmetician posits (*τίθεται*) that a unit is that which is indivisible in respect of quantity; but this is not a hypothesis, since what is meant by a unit and the fact that a unit exists are different things¹."

Aristotle uses as an alternative term for axioms "common (things)," τὰ κοινά, or "common opinions" (κοινὰ δόξα), as in the following passages. "That, when equals are taken from equals, the remainders are equal is (a) common (principle) in the case of all quantities, but mathematics takes a separate department (*ἀπολαβούσα*) and directs its investigation to some portion of its proper subject-matter, as e.g. lines or angles, numbers, or any of the other quantities²." "The common (principles), e.g. that one of two contradictories must be true, that equals taken from equals etc., and the like³..." "With regard to the principles of demonstration, it is questionable whether they belong to one science or to several. By principles of demonstration I mean the *common opinions* from which all demonstration proceeds, e.g. that one of two contradictories must be true, and that it is impossible for the same thing to be and not be⁴." Similarly "every demonstrative (science) investigates, with regard to some subject-matter, the essential attributes, starting from the *common opinions*⁵." We have then here, as Heiberg says, a sufficient explanation of Euclid's term for axioms,

¹ *Anal. post.* I. 2, 72 a 14—24.

² *Anal. post.* I. 11, 77 a 30.

³ *Metaph.* 997 a 20—22.

⁴ *Metaph.* 1061 b 19—24.

⁵ *Metaph.* 996 b 26—30.

viz. *common notions* (κοινὰ ἔννοια), and there is no reason to suppose it to be a substitution for the original term due to the Stoics: cf. Proclus' remark that, according to Aristotle and the geometers, axiom and common notion are the same thing¹.

Aristotle discusses the *indemonstrable* character of the axioms in the *Metaphysics*. Since "all the demonstrative sciences use the axioms²," the question arises, to what science does their discussion belong³? The answer is that, like that of Being (ὄντῃα), it is the province of the (first) philosopher⁴. It is impossible that there should be demonstration of everything, as there would be an infinite series of demonstrations: if the axioms were the subject of a demonstrative science, there would have to be here too, as in other demonstrative sciences, a *subject-genus*, its *attributes* and corresponding *axioms*⁵; thus there would be axioms behind axioms, and so on continually. The axiom is the most firmly established of all principles⁶. It is ignorance alone that could lead any one to try to prove the axioms⁷; the supposed proof would be a *petitio principii*⁸. If it is admitted that not everything can be proved, no one can point to any principle more truly indemonstrable⁹. If any one thought he could prove them, he could at once be refuted; if he did not attempt to say anything, it would be ridiculous to argue with him: he would be no better than a vegetable¹⁰. The first condition of the possibility of any argument whatever is that words should signify something both to the speaker and to the hearer: without this there can be no reasoning with any one. And, if any one admits that words can mean anything to both hearer and speaker, he admits that something can be true without demonstration. And so on¹¹.

It was necessary to give some sketch of Aristotle's view of the first principles, if only in connexion with Proclus' account, which is as follows. As in the case of other sciences, so "the compiler of elements in geometry must give separately the principles of the science, and after that the conclusions from those principles, not giving any account of the principles but only of their consequences. No science proves its own principles, or even discourses about them: they are treated as self-evident.... Thus the first essential was to distinguish the principles from their consequences. Euclid carries out this plan practically in every book and, as a preliminary to the whole enquiry, sets out the common principles of this science. Then he divides the common principles themselves into *hypotheses*, *postulates*, and *axioms*. For all these are different from one another: an axiom, a postulate and a hypothesis are not the same thing, as the inspired Aristotle somewhere says. But, whenever that which is assumed and ranked as a principle is both known to the learner and convincing in itself, such a thing is an *axiom*, e.g. the statement that things which are equal to the same thing are also equal to one another. When, on

¹ Proclus, p. 194, 8.

² *ibid.* 996 b 26.

³ *ibid.* 1005 b 11—17.

⁴ *ibid.* 1006 a 10.

⁵ *Metaph.* 997 a 10.

⁶ *ibid.* 1005 a 21—b 11.

⁷ *ibid.* 1006 a 5.

⁸ *ibid.* 1006 a 11—15.

⁹ *ibid.* 997 a 5—8.

¹⁰ *ibid.* 1006 a 17.

¹¹ *ibid.* 1006 a 18 sqq.

the other hand, the pupil has not the notion of what is told him which carries conviction in itself, but nevertheless lays it down and assents to its being assumed, such an assumption is a *hypothesis*. Thus we do not preconceive by virtue of a common notion, and without being taught, that the circle is such and such a figure, but, when we are told so, we assent without demonstration. When again what is asserted is both unknown and assumed even without the assent of the learner, then, he says, we call this a *postulate*, e.g. that all right angles are equal. This view of a postulate is clearly implied by those who have made a special and systematic attempt to show, with regard to one of the postulates, that it cannot be assented to by any one straight off. According then to the teaching of Aristotle, an axiom, a postulate and a hypothesis are thus distinguished¹.

We observe, first, that Proclus in this passage confuses *hypotheses* and *definitions*, although Aristotle had made the distinction quite plain. The confusion may be due to his having in his mind a passage of Plato from which he evidently got the phrase about "not giving an account of" the principles. The passage is²: "I think you know that those who treat of geometries and calculations (arithmetic) and such things take for granted (*ὑποθέμενοι*) odd and even, figures, angles of three kinds, and other things akin to these in each subject, implying that they know these things, and, though using them as hypotheses, do not even condescend to give any account of them either to themselves or to others, but begin from these things and then go through everything else in order, arriving ultimately, by recognised methods, at the conclusion which they started in search of." But the hypothesis is here the assumption, e.g. 'that *there may be such a thing* as length without breadth, henceforward called a line³,' and so on, without any attempt to show that there is such a thing; it is mentioned in connexion with the distinction between Plato's 'superior' and 'inferior' intellectual method, the former of which uses successive hypotheses as stepping-stones by which it mounts upwards to the idea of Good.

We pass now to Proclus' account of the difference between *postulates* and *axioms*. He begins with the view of Geminus, according to which "they differ from one another in the same way as theorems are also distinguished from problems. For, as in theorems we propose to see and determine what follows on the premisses, while in problems we are told to find and do something, in like manner in the *axioms* such things are assumed as are manifest of themselves and easily apprehended by our untaught notions, while in the *postulates* we assume such things as are easy to find and effect (our understanding suffering no strain in their assumption), and we require no complication of machinery⁴." "Both must have the characteristic of being simple

¹ Proclus, pp. 75, 10—77, 2.

² *Republic*, vi. 510 C. Cf. Aristotle, *Nic. Eth.* 1151 a 17.

³ H. Jackson, *Journal of Philology*, vol. x. p. 144.

⁴ Proclus, pp. 178, 12—179, 8. In illustration Proclus contrasts the drawing of a straight line or a circle with the drawing of a "single-turn spiral" or of an equilateral triangle, the

and readily grasped, I mean both the postulate and the axiom; but the postulate bids us contrive and find some subject-matter ($\psi\lambda\eta$) to exhibit a property simple and easily grasped, while the axiom bids us assert some essential attribute which is self-evident to the learner, just as is the fact that fire is hot, or any of the most obvious things¹."

Again, says Proclus, "some claim that all these things are alike postulates, in the same way as some maintain that all things that are sought are problems. For Archimedes begins his first book on *In-equilibrium*² with the remark 'I postulate that equal weights at equal distances are in equilibrium,' though one would rather call this an axiom. Others call them all axioms in the same way as some regard as theorems everything that requires demonstration³."

"Others again will say that postulates are peculiar to geometrical subject-matter, while axioms are common to all investigation which is concerned with quantity and magnitude. Thus it is the geometer who knows that all right angles are equal and how to produce in a straight line any limited straight line, whereas it is a common notion that things which are equal to the same thing are also equal to one another, and it is employed by the arithmetician and any scientific person who adapts the general statement to his own subject⁴."

The third view of the distinction between a postulate and an axiom is that of Aristotle above described⁵.

The difficulties in the way of reconciling Euclid's classification of postulates and axioms with any one of the three alternative views are next dwelt upon. If we accept the first view according to which an axiom has reference to something known, and a postulate to something done, then the 4th postulate (that all right angles are equal) is not a postulate; neither is the 5th which states that, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the straight lines, if produced indefinitely, will meet on that side on which are the angles less than two right angles. On the second view, the assumption that two straight lines cannot enclose a space, "which even now," says Proclus, "some add as an axiom," and which is peculiar to the subject-matter of geometry, like the fact that all right angles are equal, is not an axiom. According to the third (Aristotelian) view, "everything which is confirmed ($\pi\sigma\tau\omicron\upsilon\tau\alpha\iota$) by a sort of demonstration

spiral requiring more complex machinery and even the equilateral triangle needing a certain method. "For the geometrical intelligence will say that by conceiving a straight line fixed at one end but, as regards the other end, moving round the fixed end, and a point moving along the straight line from the fixed end, I have described the single-turn spiral; for the end of the straight line describing a circle, and the point moving on the straight line simultaneously, when they arrive and meet at the same point, complete such a spiral. And again, if I draw equal circles, join their common point to the centres of the circles and draw a straight line from one of the centres to the other, I shall have the equilateral triangle. These things then are far from being completed by means of a single act or of a moment's thought" (p. 180, 8—21).

¹ Proclus, p. 181, 4—11.

² It is necessary to coin a word to render $\acute{\alpha}\nu\iota\sigma\omicron\rho\rho\omicron\pi\iota\omega\acute{\nu}$, which is moreover in the plural. The title of the treatise as we have it is *Equilibria of planes or centres of gravity of planes* in Book I and *Equilibria of planes* in Book II.

³ Proclus, p. 181, 16—23.

⁴ *ibid.* p. 182, 6—14.

⁵ Pp. 118, 119.

will be a postulate, and what is incapable of proof will be an axiom¹." This last statement of Proclus is loose, as regards the axiom, because it omits Aristotle's requirement that the axiom should be a self-evident truth, and one that must be admitted by any one who is to learn anything at all, and, as regards the postulate, because Aristotle calls a postulate something assumed without proof though it is "matter of demonstration" (*ἀποδεικτὸν ὄν*), but says nothing of a *quasi*-demonstration of the postulates. On the whole I think it is from Aristotle that we get the best idea of what Euclid understood by a postulate and an axiom or common notion. Thus Aristotle's account of an axiom as a principle common to all sciences, which is self-evident, though incapable of proof, agrees sufficiently with the contents of Euclid's *common notions* as reduced to five in the most recent text (not omitting the fourth, that "things which coincide are equal to one another"). As regards the *postulates*, it must be borne in mind that Aristotle says elsewhere² that, "other things being equal, that proof is the better which proceeds from the fewer postulates or hypotheses or propositions." If then we say that a geometer must lay down as principles, first certain axioms or common notions, and then an *irreducible minimum* of postulates in the Aristotelian sense concerned only with the subject-matter of geometry, we are not far from describing what Euclid in fact does. As regards the postulates we may imagine him saying: "Besides the common notions there are a few other things which I must assume without proof, but which differ from the common notions in that they are not self-evident. The learner may or may not be disposed to agree to them; but he must accept them at the outset on the superior authority of his teacher, and must be left to convince himself of their truth in the course of the investigation which follows. In the first place certain simple constructions, the drawing and producing of a straight line, and the drawing of a circle, must be assumed to be possible, and with the constructions the existence of such things as straight lines and circles; and besides this we must lay down some postulate to form the basis of the theory of parallels." It is true that the admission of the 4th postulate that all right angles are equal still presents a difficulty to which we shall have to recur.

There is of course no foundation for the idea, which has found its way into many text-books, that "the object of the postulates is to declare that the only instruments the use of which is permitted in geometry are the *rule* and *compass*."

§ 4. THEOREMS AND PROBLEMS.

"Again the deductions from the first principles," says Proclus, "are divided into **problems** and **theorems**, the former embracing the

¹ Proclus, pp. 182, 21—183, 13.

² *Anal. post.* I. 25, 86 a 33—35.

³ Cf. Lardner's Euclid: also Todhunter.

generation, division, subtraction or addition of figures, and generally the changes which are brought about in them, the latter exhibiting the essential attributes of each¹."

"Now, of the ancients, some, like Speusippus and Amphinomus, thought proper to call them all theorems, regarding the name of theorems as more appropriate than that of problems to theoretic sciences, especially as these deal with eternal objects. For there is no becoming in things eternal, so that neither could the problem have any place with them, since it promises the generation and making of what has not before existed, e.g. the construction of an equilateral triangle, or the describing of a square on a given straight line, or the placing of a straight line at a given point. Hence they say it is better to assert that all (propositions) are of the same kind, and that we regard the generation that takes place in them as referring not to actual *making* but to *knowledge*, when we treat things existing eternally as if they were subject to becoming: in other words, we may say that everything is treated by way of theorem and not by way of problem² (πάντα θεωρηματικῶς ἀλλ' οὐ προβληματικῶς λαμβάνεσθαι).

"Others on the contrary, like the mathematicians of the school of Menaechmus, thought it right to call them all problems, describing their purpose as twofold, namely in some cases to furnish (πορίσασθαι) the thing sought, in others to take a determinate object and see either what it is, or of what nature, or what is its property, or in what relations it stands to something else.

"In reality both assertions are correct. Speusippus is right because the problems of geometry are not like those of mechanics, the latter being matters of sense and exhibiting becoming and change of every sort. The school of Menaechmus are right also because the discoveries even of theorems do not arise without an issuing-forth into matter, by which I mean intelligible matter. Thus forms going out into matter and giving it shape may fairly be said to be like processes of becoming. For we say that the motion of our thought and the throwing-out of the forms in it is what produces the figures in the imagination and the conditions subsisting in them. It is in the imagination that constructions, divisions, placings, applications, additions and subtractions (take place), but everything in the mind is fixed and immune from becoming and from every sort of change³."

"Now those who distinguish the theorem from the problem say that every problem implies the possibility, not only of that which is predicated of its subject-matter, but also of its opposite, whereas every theorem implies the possibility of the thing predicated but not of its opposite as well. By the subject-matter I mean the genus which is the subject of inquiry, for example, a triangle or a square or a circle, and by the property predicated the essential attribute, as equality, section, position, and the like. When then any one

¹ Proclus, p. 77, 7—12.

² *ibid.* pp. 78, 8—79, 2.

³ *ibid.* pp. 77, 15—78, 8.

enunciates thus, *To inscribe an equilateral triangle in a circle*, he states a *problem*; for it is also possible to inscribe in it a triangle which is not equilateral. Again, if we take the enunciation *On a given limited straight line to construct an equilateral triangle*, this is a *problem*; for it is possible also to construct one which is not equilateral. But, when any one enunciates that *In isosceles triangles the angles at the base are equal*, we must say that he enunciates a *theorem*; for it is not also possible that the angles at the base of isosceles triangles should be unequal. It follows that, if any one were to use the form of a problem and say *In a semicircle to describe a right angle*, he would be set down as no geometer. For every angle in a semicircle is right¹."

"Zenodotus, who belonged to the succession of Oenopides, but was a disciple of Andron, distinguished the theorem from the problem by the fact that the theorem inquires what is the property predicated of the subject-matter in it, but the problem what is the cause of what effect (τίνος ὄντος τί ἐστίν). Hence too Posidonius defined the one (the problem) as a proposition in which it is inquired whether a thing exists or not (εἰ ἔστιν ἢ μὴ), the other (the theorem²) as a proposition in which it is inquired what (a thing) is or of what nature (τί ἐστίν ἢ ποῖόν τι); and he said that the theoretic proposition must be put in a declaratory form, e.g., *Any triangle has two sides (together) greater than the remaining side* and *In any isosceles triangle the angles at the base are equal*, but that we should state the problematic proposition as if inquiring whether it is possible to construct an equilateral triangle upon such and such a straight line. For there is a difference between inquiring absolutely and indeterminately (ἀπλῶς τε καὶ ἀορίστως) whether there exists a straight line from such and such a point at right angles to such and such a straight line and investigating which is the straight line at right angles³."

"That there is a certain difference between the problem and the theorem is clear from what has been said; and that the Elements of Euclid contain partly problems and partly theorems will be made manifest by the individual propositions, where Euclid himself adds at the end of what is proved in them, in some cases, 'that which it was required to do,' and in others, 'that which it was required to prove,' the latter expression being regarded as characteristic of theorems, in spite of the fact that, as we have said, demonstration is found in problems also. In problems, however, even the demonstration is for the purpose of (confirming) the construction: for we bring in the demonstration in order to show that what was enjoined has been done; whereas in theorems the demonstration is worthy of study for its own sake as being capable of putting before us the nature of the thing sought. And you will find that Euclid sometimes interweaves theorems with problems and employs them in turn, as in the first

¹ Proclus, pp. 79, 11—80, 5.

² In the text we have τὸ δὲ πρόβλημα answering to τὸ μὲν without substantive: πρόβλημα was obviously inserted in error.

³ Proclus, pp. 80, 15—81, 4.

book, while at other times he makes one or other preponderate. For the fourth book consists wholly of problems, and the fifth of theorems¹."

Again, in his note on Eucl. I. 4, Proclus says that Carpus, the writer on mechanics, raised the question of theorems and problems in his treatise on astronomy. Carpus, we are told, "says that the class of problems is in order prior to theorems. For the subjects, the properties of which are sought, are discovered by means of problems. Moreover in a problem the enunciation is simple and requires no skilled intelligence; it orders you plainly to do such and such a thing, *to construct an equilateral triangle*, or, *given two straight lines, to cut off from the greater (a straight line) equal to the lesser*, and what is there obscure or elaborate in these things? But the enunciation of a theorem is a matter of labour and requires much exactness and scientific judgment in order that it may not turn out to exceed or fall short of the truth; an example is found even in this proposition (I. 4), the first of the theorems. Again, in the case of problems, one general way has been discovered, that of *analysis*, by following which we can always hope to succeed; it is this method by which the more obscure problems are investigated. But, in the case of theorems, the method of setting about them is hard to get hold of since 'up to our time,' says Carpus, 'no one has been able to hand down a general method for their discovery. Hence, by reason of their easiness, the class of problems would naturally be more simple.' After these distinctions, he proceeds: 'Hence it is that in the Elements too problems precede theorems, and the Elements begin from them; the first theorem is fourth in order, not because the fifth² is proved from the problems, but because, even if it needs for its demonstration none of the propositions which precede it, it was necessary that they should be first because they are problems, while it is a theorem. In fact, in this theorem he uses the common notions exclusively, and in some sort takes the same triangle placed in different positions; the coincidence and the equality proved thereby depend entirely upon sensible and distinct apprehension. Nevertheless, though the demonstration of the first theorem is of this character, the problems properly preceded it, because in general problems are allotted the order of precedence³.'"

Proclus himself explains the position of Prop. 4 after Props. 1—3 as due to the fact that a theorem about the essential properties of triangles ought not to be introduced before we know that such a thing as a triangle can be constructed, nor a theorem about the equality of sides or straight lines until we have shown, by constructing them, that there can be two straight lines which are equal to one another⁴. It is plausible enough to argue in this way that Props. 2 and 3 at all events should precede Prop. 4. And Prop. 1 is used in

¹ Proclus, p. 81, 5—22.

² τὸ πέμπτον. This should apparently be the *fourth* because in the next words it is implied that none of the first three propositions are required in proving it.

³ Proclus, pp. 241, 19—243, 11.

⁴ *ibid.* pp. 233, 21—234, 6.

Prop. 2, and must therefore precede it. But Prop. 1 showing how to construct an *equilateral* triangle on a given base is not important, in relation to Prop. 4, as dealing with the "production of triangles" in general: for it is of no use to say, as Proclus does, that the construction of the equilateral triangle is "common to the three species (of triangles)¹," as we are not in a position to know this at such an early stage. The *existence* of triangles in general was doubtless assumed as following from the existence of straight lines and points in one plane and from the possibility of drawing a straight line from one point to another.

Proclus does not however seem to reject definitely the view of Carpus, for he goes on²: "And perhaps problems are in order before theorems, and especially for those who need to ascend from the arts which are concerned with things of sense to theoretical investigation. But in dignity theorems are prior to problems....It is then foolish to blame Geminus for saying that the theorem is more perfect than the problem. For Carpus himself gave the priority to problems in respect of *order*, and Geminus to theorems in point of more perfect *dignity*," so that there was no real inconsistency between the two.

Problems were classified according to the number of their possible solutions. Amphinómus said that those which had a unique solution (*μοναχῶς*) were called "ordered" (the word has dropped out in Proclus, but it must be *τεταγμένα*, in contrast to the third kind, *ἄτακτα*); those which had a definite number of solutions "intermediate" (*μέσσα*); and those with an infinite variety of solutions "unordered" (*ἄτακτα*)³. Proclus gives as an example of the last the problem *To divide a given straight line into three parts in continued proportion*⁴. This is the same thing as solving the equations $x+y+z=a$, $xz=y^2$. Proclus' remarks upon the problem show that it was solved, like all quadratic equations, by the method of "application of areas." The straight line a was first divided into any two parts, $(x+z)$ and y , subject to the sole limitation that $(x+z)$ must not be less than $2y$, which limitation is the *διορισμός*, or condition of possibility. Then an area was applied to $(x+z)$, or $(a-y)$, "*falling short by a square figure*" (*ἐλλείπον εἶδει τετραγώνῳ*) and equal to the square on y . This determines x and z separately in terms of a and y . For, if z be the side of the square by which the area (i.e. rectangle) "falls short," we have $\{(a-y)-z\}z=y^2$, whence $2z=(a-y) \pm \sqrt{\{(a-y)^2-4y^2\}}$. And y may be chosen arbitrarily, provided that it is not greater than $a/3$. Hence there are an infinite number of solutions. If $y=a/3$, then, as Proclus remarks, the three parts are equal.

Other distinctions between different kinds of problems are added by Proclus. The word "problem," he says, is used in several senses. In its widest sense it may mean anything "propounded" (*προτείνον-μενον*), whether for the purpose of instruction (*μαθήσεως*) or construction (*ποιήσεως*). (In this sense, therefore, it would include a theorem.)

¹ Proclus, p. 234, 21.

³ *ibid.* p. 220, 7—12.

² *ibid.* p. 243, 12—25.

⁴ *ibid.* pp. 220, 16—221, 6.

But its special sense in mathematics is that of something "propounded with a view to a theoretic construction¹."

Again you may apply the term (in this restricted sense) even to something which is *impossible*, although it is more appropriately used of what is *possible* and neither asks too much nor contains too little in the shape of data. According as a problem has one or other of these defects respectively, it is called (1) a problem *in excess* (πλεονάζον) or (2) a *deficient* problem (ἐλλειπὲς πρόβλημα). The problem *in excess* (1) is of two kinds, (a) a problem in which the properties of the figure to be found are either *inconsistent* (ἀσύμβατα) or *non-existent* (ἀνύπαρκτα), in which case the problem is called impossible, or (b) a problem in which the enunciation is merely redundant: an example of this would be a problem requiring us to construct an equilateral triangle with its vertical angle equal to two-thirds of a right angle; such a problem is possible and is called "more than a problem" (μεῖζον ἢ πρόβλημα). The *deficient* problem (2) is similarly called "less than a problem" (ἐλασσον ἢ πρόβλημα), its characteristic being that something has to be added to the enunciation in order to convert it from indeterminateness (ἀοριστία) to order (τάξις) and scientific determinateness (ὁρος ἐπιστημονικός): such would be a problem bidding you "to construct an isosceles triangle," for the varieties of isosceles triangles are unlimited. Such "problems" are not problems in the proper sense (κυρίως λεγόμενα προβλήματα), but only equivocally².

§ 5. THE FORMAL DIVISIONS OF A PROPOSITION.

"Every problem," says Proclus³, "and every theorem which is complete with all its parts perfect purports to contain in itself all of the following elements: **enunciation** (πρότασις), **setting-out** (ἐκθεσις), **definition** or **specification** (διορισμός), **construction** or **machinery** (κατασκευή), **proof** (ἀπόδειξις), **conclusion** (συμπέρασμα). Now of these the *enunciation* states what is given and what is that which is sought, the perfect *enunciation* consisting of both these parts. The *setting-out* marks off what is given, by itself, and adapts it beforehand for use in the investigation. The *definition* or *specification* states separately and makes clear what the particular thing is which is sought. The *construction* or *machinery* adds what is wanting to the datum for the purpose of finding what is sought. The *proof* draws the required inference by reasoning scientifically from acknowledged facts. The *conclusion* reverts again to the *enunciation*, confirming what has been demonstrated. These are all the parts of problems and theorems, but the most essential and those which are found in all are *enunciation*, *proof*, *conclusion*. For it is equally necessary to know beforehand what is sought, to prove this by means of the intermediate steps, and to state the proved fact as a conclusion; it is impossible to dispense with any of these three things. The remaining parts are often brought in, but are often left out as serving no purpose.

¹ Proclus, p. 221, 7—11.

² *ibid.* pp. 221, 13—222, 14.

³ *ibid.* pp. 203, 1—204, 13; 204, 23—205, 8.

Thus there is neither *setting-out* nor *definition* in the problem of constructing an isosceles triangle having each of the angles at the base double of the remaining angle, and in most theorems there is no *construction* because the *setting-out* suffices without any addition for proving the required property from the data. When then do we say that the *setting-out* is wanting? The answer is, when there is nothing *given* in the *enunciation*; for, though the *enunciation* is in general divided into what is given and what is sought, this is not always the case, but sometimes it states only what is sought, i.e. what must be known or found, as in the case of the problem just mentioned. That problem does not, in fact, state beforehand with what datum we are to construct the isosceles triangle having each of the equal angles double of the remaining angle, but (simply) that we are to find such a triangle.... When, then, the *enunciation* contains both (what is given and what is sought), in that case we find both *definition* and *setting-out*, but, whenever the datum is wanting, they too are wanting. For not only is the *setting-out* concerned with the datum, but so is the *definition* also, as, in the absence of the datum, the *definition* will be identical with the *enunciation*. In fact, what could you say in defining the object of the aforesaid problem except that it is required to find an isosceles triangle of the kind referred to? But that is what the *enunciation* stated. If then the *enunciation* does not include, on the one hand, what is given and, on the other, what is sought, there is no *setting-out* in virtue of there being no datum, and the *definition* is left out in order to avoid a mere repetition of the *enunciation*."

The constituent parts of an Euclidean proposition will be readily identified by means of the above description. As regards the *definition* or *specification* (διορισμός) it is to be observed that we have here only one of its uses. Here it means a closer definition or description of the object aimed at, by means of the concrete lines or figures set out in the ἐκθεσις instead of the general terms used in the *enunciation*; and its purpose is to rivet the attention better, as Proclus indicates in a later passage (τρόπον τινὰ προσεχέας εἶστιν αὐτίος ὁ διορισμός)¹.

The other technical use of the word to signify the limitations to which the possible solutions of a problem are subject is also described by Proclus, who speaks of διορισμοί determining "whether what is sought is impossible or possible, and how far it is practicable and in how many ways²"; and the διορισμός in this sense appears in Euclid as well as in Archimedes and Apollonius. Thus we have in Eucl. I. 22 the *enunciation* "From three straight lines which are equal to three given straight lines to construct a triangle," followed immediately by the *limiting condition* (διορισμός). "Thus two of the straight lines taken together in any manner must be greater than the remaining one." Similarly in VI. 28 the *enunciation* "To a given straight line to apply a parallelogram equal to a given rectilinear

¹ Proclus, p. 208, 21.

² *ibid.* p. 202, 3.

figure and falling short by a parallelogrammic figure similar to a given one" is at once followed by the necessary condition of possibility: "Thus the given rectilinear figure must not be greater than that described on half the line and similar to the defect."

Tannery supposed that, in giving the other description of the *διορισμός* as quoted above, Proclus, or rather his guide, was using the term incorrectly. The *διορισμός* in the better known sense of the determination of limits or conditions of possibility was, we are told, invented by Leon. Pappus uses the word in this sense only. The other use of the term might, Tannery thought, be due to a confusion occasioned by the use of the same words (*δεῖ δὴ*) in introducing the parts of a proposition corresponding to the two meanings of the word *διορισμός*¹. On the other hand it is to be observed that Eutocius distinguishes clearly between the two uses and implies that the difference was well known². The *διορισμός* in the sense of condition of possibility follows immediately on the enunciation, is even part of it; the *διορισμός* in the other sense of course comes immediately after the *setting-out*.

Proclus has a useful observation respecting the *conclusion* of a proposition³. "The conclusion they are accustomed to make double in a certain way: I mean, by proving it in the given case and then drawing a general inference, passing, that is, from the partial conclusion to the general. For, inasmuch as they do not make use of the *individuality* of the subjects taken, but only draw an angle or a straight line with a view to placing the datum before our eyes, they consider that this same fact which is established in the case of the particular figure constitutes a conclusion true of every other figure of the same kind. They pass accordingly to the general in order that we may not conceive the conclusion to be partial. And they are justified in so passing, since they use for the demonstration the particular things set out, not *quâ* particulars, but *quâ* typical of the rest. For it is not in virtue of such and such a size attaching to the angle which is set out that I effect the bisection of it, but in virtue of its being rectilinear and nothing more. Such and such size is peculiar to the angle set out, but its quality of being rectilinear is common to all rectilinear angles. Suppose, for example, that the given angle is a right angle. If then I had employed in the proof the fact of its being right, I should not have been able to pass to every species of rectilinear angle; but, if I make no use of its being right, and only consider it as rectilinear, the argument will equally apply to rectilinear angles in general."

¹ *La Géométrie grecque*, p. 149 note. Where *δεῖ δὴ* introduces the closer description of the problem we may translate, "it is then required" or "thus it is required" (to construct etc.): when it introduces the condition of possibility we may translate "thus it is necessary etc." Heiberg originally wrote *δεῖ δὲ* in the latter sense in I. 22 on the authority of Proclus and Eutocius, and against that of the MSS. Later, on the occasion of XI. 23, he observed that he should have followed the MSS. and written *δεῖ δὴ* which he found to be, after all, the right reading in Eutocius (Apollonius, ed. Heiberg, II. p. 178). *δεῖ δὴ* is also the expression used by Diophantus for introducing conditions of possibility.

² See the passage of Eutocius referred to in last note.

³ Proclus, p. 207, 4—25.

§ 6. OTHER TECHNICAL TERMS.

1. Things said to be *given*.

Proclus attaches to his description of the formal divisions of a proposition an explanation of the different senses in which the word *given* or *datum* (δεδομένον) is used in geometry. "Everything that is given is given in one or other of the following ways, *in position*, *in ratio*, *in magnitude*, or *in species*. The point is given *in position* only, but a line and the rest may be given in all the senses¹."

The illustrations which Proclus gives of the four senses in which a thing may be *given* are not altogether happy, and, as regards things which are given *in position*, *in magnitude*, and *in species*, it is best, I think, to follow the definitions given by Euclid himself in his book of *Data*. Euclid does not mention the fourth class, things given *in ratio*, nor apparently do any of the great geometers.

(1) *Given in position* really needs no definition; and, when Euclid says (*Data*, Def. 4) that "Points, lines and angles are said to be *given in position* which always occupy the same place," we are not really the wiser.

(2) *Given in magnitude* is defined thus (*Data*, Def. 1): "Areas, lines and angles are called *given in magnitude* to which we can find equals." Proclus' illustration is in this case the following: when, he says, two unequal straight lines are given from the greater of which we have to cut off a straight line equal to the lesser, the straight lines are obviously *given in magnitude*, "for greater and less, and finite and infinite are predications peculiar to magnitude." But he does not explain that part of the implication of the term is that a thing is given *in magnitude only*, and that, for example, its position is not given and is a matter of indifference.

(3) *Given in species*. Euclid's definition (*Data*, Def. 3) is: "Rectilineal figures are said to be *given in species* in which the angles are severally given and the ratios of the sides to one another are given." And this is the recognised use of the term (cf. Pappus, *passim*). Proclus uses the term in a much wider sense for which I am not aware of any authority. Thus, he says, when we speak of (bisecting) a given rectilineal angle, the angle is given *in species* by the word *rectilineal*, which prevents our attempting, by the same method, to bisect a curvilineal angle! On Eucl. I. 9, to which he here refers, he says that an angle is given *in species* when e.g. we say that it is right or acute or obtuse or rectilineal or "mixed," but that the actual angle in the proposition is given *in species* only. As a matter of fact, we should say that the actual angle in the figure of the proposition is given *in magnitude* and not *in species*, part of the implication of *given in species* being that the actual magnitude of the thing *given in species* is indifferent; an angle cannot be *given in species* in this sense at all. The confusion in Proclus' mind is shown when, after saying that a right angle is given *in species*, he describes a third of a right angle as given *in magnitude*.

¹ Proclus, p. 205, 13—15.

No better example of what is meant by *given in species*, in its proper sense, as limited to rectilinear figures, can be quoted than the given parallelogram in Eucl. VI. 28, to which the required parallelogram has to be made similar; the former parallelogram is in fact *given in species*, though its actual size, or scale, is indifferent.

(4) *Given in ratio* presumably means something which is given by means of its ratio to some other given thing. This we gather from Proclus' remark (in his note on I. 9) that an angle may be given in ratio "as when we say that it is double and treble of such and such an angle or, generally, greater and less." The term, however, appears to have no authority and to serve no purpose. Proclus may have derived it from such expressions as "in a given ratio" which are common enough.

2. Lemma.

"The term *lemma*," says Proclus¹, "is often used of any proposition which is assumed for the construction of something else: thus it is a common remark that a proof has been made out of such and such lemmas. But the special meaning of *lemma* in geometry is a proposition requiring confirmation. For when, in either construction or demonstration, we assume anything which has not been proved but requires argument, then, because we regard what has been assumed as doubtful in itself and therefore worthy of investigation, we call it a *lemma*², differing as it does from the postulate and the axiom in being matter of demonstration, whereas they are immediately taken for granted, without demonstration, for the purpose of confirming other things. Now in the discovery of lemmas the best aid is a mental aptitude for it. For we may see many who are quick at solutions and yet do not work by method; thus Cratistus in our time was able to obtain the required result from first principles, and those the fewest possible, but it was his natural gift which helped him to the discovery.

¹ Proclus, pp. 211, 1—212, 4.

² It would appear, says Tannery (p. 151 n.), that Geminus understood a lemma as being simply λαμβανόμενον, something assumed (cf. the passage of Proclus, p. 73, 4, relating to Menaechmus' view of *elements*): hence we cannot consider ourselves authorised in attributing to Geminus the more technical definition of the term here given by Proclus, according to which it is only used of propositions not proved beforehand. This view of a lemma must be considered as relatively modern. It seems to have had its origin in an imperfection of method. In the course of a demonstration it was necessary to assume a proposition which required proof, but the proof of which would, if inserted in the particular place, break the thread of the demonstration: hence it was necessary either to prove it beforehand as a preliminary proposition or to postpone it to be proved afterwards (ὡς ἐξῆς δειχθῆσεται). When, after the time of Geminus, the progress of original discovery in geometry was arrested, geometers occupied themselves with the study and elucidation of the works of the great mathematicians who had preceded them. This involved the investigation of propositions explicitly quoted or tacitly assumed in the great classical treatises; and naturally it was found that several such remained to be demonstrated, either because the authors had omitted them as being easy enough to be left to the reader himself to prove, or because books in which they were proved had been lost in the meantime. Hence arose a class of complementary or auxiliary propositions which were called *lemmas*. Thus Pappus gives in his Book VII a collection of lemmas in elucidation of the treatises of Euclid and Apollonius included in the so-called "Treasury of Analysis" (τόπος ἀναλυόμενος). When Proclus goes on to distinguish three methods of discovering lemmas, *analysis*, *division*, and *reductio ad absurdum*, he seems to imply that the principal business of contemporary geometers was the investigation of these auxiliary propositions.

Nevertheless certain methods have been handed down. The finest is the method which by means of *analysis* carries the thing sought up to an acknowledged principle, a method which Plato, as they say, communicated to Leodamas¹, and by which the latter, too, is said to have discovered many things in geometry. The second is the method of *division*², which divides into its parts the genus proposed for consideration and gives a starting-point for the demonstration by means of the elimination of the other elements in the construction of what is proposed, which method also Plato extolled as being of assistance to all sciences. The third is that by means of the *reductio ad absurdum*, which does not show what is sought directly, but refutes its opposite and discovers the truth incidentally."

3. Case.

"The *case*³ (πτῶσις)," Proclus proceeds⁴, "announces different ways of construction and alteration of positions due to the transposition of points or lines or planes or solids. And, in general, all its varieties are seen in the figure, and this is why it is called *case*, being a transposition in the construction."

4. Porism.

"The term *porism* is used also of certain problems such as the Porisms written by Euclid. But it is specially used when from what has been demonstrated some other theorem is revealed at the same time without our propounding it, which theorem has on this very account been called a *porism* (corollary) as being a sort of incidental gain arising from the scientific demonstration⁵." Cf. the note on I. 15.

¹ This passage and another from Diogenes Laërtius (III. 24, p. 74 ed. Cobet) to the effect that "He [Plato] explained (ἐξηγήσατο) to Leodamas of Thasos the method of inquiry by analysis" have been commonly understood as ascribing to Plato the *invention* of the method of analysis; but Tannery points out forcibly (pp. 112, 113) how difficult it is to explain in what Plato's discovery could have consisted if *analysis* be taken in the sense attributed to it in Pappus, where we can see no more than a series of successive, *reductions* of a problem until it is finally reduced to a known problem. On the other hand, Proclus' words about carrying up the thing sought to "an acknowledged principle" suggest that what he had in mind was the process described at the end of Book VI of the *Republic* by which the dialectician (unlike the mathematician) uses hypotheses as stepping-stones up to a principle which is not hypothetical, and then is able to descend step by step verifying every one of the hypotheses by which he ascended. This description does not of course refer to mathematical analysis, but it may have given rise to the idea that analysis was Plato's discovery, since *analysis* and *synthesis* following each other are related in the same way as the upward and the downward progression in the dialectician's intellectual method. And it may be that Plato's achievement was to observe the importance, from the point of view of logical rigour, of the confirmatory synthesis following analysis, and to regularise in this way and elevate into a completely irrefragable method the partial and uncertain analysis upon which the works of his predecessors depended.

² Here again the successive bipartitions of genera into species such as we find in the *Sophist* and *Republic* have very little to say to geometry, and the very fact that they are here mentioned side by side with analysis suggests that Proclus confused the latter with the philosophical method of *Rep.* VI.

³ Tannery rightly remarks (p. 152) that the subdivision of a theorem or problem into several cases is foreign to the really classic form; the ancients preferred, where necessary, to multiply enunciations. As, however, some omissions necessarily occurred, the writers of lemmas naturally added separate *cases*, which in some instances found their way into the text. A good example is Euclid I. 7, the second case of which, as it appears in our text-books, was interpolated. On the commentary of Proclus on this proposition Th. Taylor rightly remarks that "Euclid everywhere avoids a multitude of cases."

⁴ Proclus, p. 212, 5-11.

⁵ Tannery notes however that, so far from distinguishing his corollaries from the con-

5. Objection.

"The *objection* (ἐνστάσις) obstructs the whole course of the argument by appearing as an obstacle (or crying 'halt,' ἀπαντῶσα) either to the construction or to the demonstration. There is this difference between the *objection* and the *case*, that, whereas he who propounds the case has to prove the proposition to be true of it, he who makes the objection does not need to prove anything: on the contrary it is necessary to destroy the objection and to show that its author is saying what is false¹."

That is, in general the *objection* endeavours to make it appear that the demonstration is not true in every case; and it is then necessary to prove, in refutation of the objection, either that the supposed case is impossible, or that the demonstration is true even for that case. A good instance is afforded by Eucl. I. 7. The text-books give a second case which is not in the original text of Euclid. Proclus remarks on the proposition as given by Euclid that the objection may conceivably be raised that what Euclid declares to be impossible may after all be possible in the event of one pair of straight lines falling completely within the other pair. Proclus then refutes the objection by proving the impossibility in that case also. His proof then came to be given in the text-books as part of Euclid's proposition.

The *objection* is one of the technical terms in Aristotle's logic and its nature is explained in the *Prior Analytics*². "An *objection* is a proposition contrary to a proposition.... Objections are of two sorts, general or partial.... For when it is maintained that an attribute belongs to every (member of a class), we object either that it belongs to none (of the class) or that there is some one (member of the class) to which it does not belong."

6. Reduction.

This is again an Aristotelian term, explained in the *Prior Analytics*³. It is well described by Proclus in the following passage:

"*Reduction* (ἀπαγωγή) is a transition from one problem or theorem to another, the solution or proof of which makes that which is propounded manifest also. For example, after the doubling of the cube had been investigated, they transformed the investigation into another upon which it follows, namely the finding of the two means; and from that time forward they inquired how between two given straight lines two mean proportionals could be discovered. And they say that the first to effect the reduction of difficult constructions was Hippocrates of Chios, who also squared a lune and discovered many other things in geometry, being second to none in ingenuity as regards constructions⁴."

clusions of his propositions, Euclid inserts them before the closing words "(being) what it was required to do" or "to prove." In fact the porism-corollary is with Euclid rather a modified form of the regular conclusion than a separate proposition.

¹ Proclus, p. 212, 18—23.

² *Anal. prior.* II. 26, 69 a 37.

³ *ibid.* II. 25, 69 a 20.

⁴ Proclus, pp. 212, 24—213, 11. This passage has frequently been taken as crediting Hippocrates with the discovery of the method of geometrical reduction: cf. Taylor (Translation of Proclus, II. p. 26), Allman (p. 41 n., 59), Gow (pp. 169, 170). As Tannery remarks (p. 110), if the particular reduction of the duplication problem to that of the two means is

7. *Reductio ad absurdum.*

This is variously called by Aristotle "*reductio ad absurdum*" (ἡ εἰς τὸ ἀδύνατον ἀπαγωγή)¹, "proof *per impossibile*" (ἡ διὰ τοῦ ἀδυνάτου δείξις or ἀπόδειξις)², "proof leading to the impossible" (ἡ εἰς τὸ ἀδύνατον ἄγουσα ἀπόδειξις)³. It is part of "proof (starting) from a hypothesis" (ἐξ ὑποθέσεως). "All (syllogisms) which reach the conclusion *per impossibile* reason out a conclusion which is false, and they prove the original contention (by the method starting) from a hypothesis, when something impossible results from assuming the contradictory of the original contention, as, for example, when it is proved that the diagonal (of a square) is incommensurable because, if it be assumed commensurable, it will follow that odd (numbers) are equal to even (numbers)⁴." Or again, "proof (leading) to the impossible differs from the direct (δεικτικῆς) in that it assumes what it desires to destroy [namely the hypothesis of the falsity of the conclusion] and then reduces it to something admittedly false, whereas the direct proof starts from premisses admittedly true⁵."

Proclus has the following description of the *reductio ad absurdum*. "Proofs by *reductio ad absurdum* in every case reach a conclusion manifestly impossible, a conclusion the contradictory of which is admitted. In some cases the conclusions are found to conflict with the common notions, or the postulates, or the hypotheses (from which we started); in others they contradict propositions previously established"..."Every *reductio ad absurdum* assumes what conflicts with the desired result, then, using that as a basis, proceeds until it arrives at an admitted absurdity, and, by thus destroying the hypothesis, establishes the result originally desired. For it is necessary to understand generally that all mathematical arguments either proceed from the first principles or lead back to them, as Porphyry somewhere says. And those which proceed from the first principles are again of two kinds, for they start either from common notions and the clearness of the self-evident alone, or from results previously proved; while those which lead back to the principles are either by way of assuming the principles or by way of destroying them. Those which assume the principles are called *analyses*, and the opposite of these are *syntheses*—for it is possible to start from the said principles and to proceed in the regular order to the desired conclusion, and this process is *synthesis*—while the arguments which would destroy the principles are

the first noted in history, it is difficult to suppose that it was really the first; for Hippocrates must have found instances of it in the Pythagorean geometry. Bretschneider, I think, comes nearer the truth when he boldly (p. 99) translates: "This reduction of the aforesaid construction is said to have been first given by Hippocrates." The words are πρῶτον δὲ φασὶ τῶν ἀπορούμενων διαγραμμάτων τὴν ἀπαγωγὴν ποιήσασθαι, which must, literally, be translated as in the text above; but, when Proclus speaks vaguely of "difficult constructions," he probably means to say simply that "this first recorded instance of a reduction of a difficult construction is attributed to Hippocrates."

¹ Aristotle, *Anal. prior.* I. 7, 29 b 5; I. 44, 50 a 30.

² *ibid.* I. 21, 39 b 32; I. 29, 45 a 35.

³ *Anal. post.* I. 24, 85 a 16 etc.

⁴ *Anal. prior.* I. 23, 41 a 24.

⁵ Proclus, p. 254, 22—27.

⁴ *Anal. prior.* I. 23, 40 b 25.

⁶ *ibid.* II. 14, 62 b 29.

called *reductiones ad absurdum*. For it is the function of this method to upset something admitted as clear¹."

8. Analysis and Synthesis.

It will be seen from the note on Eucl. XIII. 1 that the MSS. of the *Elements* contain definitions of *Analysis* and *Synthesis* followed by alternative proofs of XIII. 1—5 after that method. The definitions and alternative proofs are interpolated, but they have great historical interest because of the possibility that they represent an ancient method of dealing with these propositions, anterior to Euclid. The propositions give properties of a line cut "in extreme and mean ratio," and they are preliminary to the construction and comparison of the five regular solids. Now Pappus, in the section of his *Collection* dealing with the latter subject², says that he will give the comparisons between the five figures, the pyramid, cube, octahedron, dodecahedron and icosahedron, which have equal surfaces, "not by means of the so-called *analytical* inquiry, by which some of the ancients worked out the proofs, but by the *synthetical* method³...." The conjecture of Bretschneider that the matter interpolated in Eucl. XIII. is a survival of investigations due to Eudoxus has at first sight much to commend it⁴. In the first place, we are told by Proclus that Eudoxus "greatly added to the number of the theorems which Plato originated regarding *the section*, and employed in them the method of analysis⁵." It is obvious that "*the section*" was some particular section which by the time of Plato had assumed great importance; and the one section of which this can safely be said is that which was called the "golden section," namely, the division of a straight line in extreme and mean ratio which appears in Eucl. II. 11 and is therefore most probably Pythagorean. Secondly, as Cantor points out⁶, Eudoxus was the founder of the theory of proportions in the form in which we find it in Euclid V., VI., and it was no doubt through meeting, in the course of his investigations, with proportions not expressible by whole numbers that he came to realise the necessity for a new theory of proportions which should be applicable to incommensurable as well as commensurable magnitudes. The "golden section" would furnish such a case. And it is even mentioned by Proclus in this connexion. He is explaining⁷ that it is only in arithmetic that all quantities bear "rational" ratios (*ῥητὸς λόγος*) to one another, while in geometry there are "irrational" ones (*ἄρρητος*) as well. "Theorems about sections like those in Euclid's second Book are common to both [arithmetic and geometry] *except that in which the straight line is cut in extreme and mean ratio*⁸."

¹ Proclus, p. 255, 8—26.

² Pappus, v. p. 410 sqq.

³ *ibid.* pp. 410, 27—412, 2.

⁴ Bretschneider, p. 168. See however Heiberg's recent suggestion (*Paralipomena zu Euklid in Hermes*, xxxviii., 1903) that the author was Heron. The suggestion is based on a comparison with the remarks on analysis and synthesis quoted from Heron by an-Nairizi (ed. Curtze, p. 89) at the beginning of his commentary on Eucl. Book II. On the whole, this suggestion commends itself to me more than that of Bretschneider.

⁵ Proclus, p. 67, 6.

⁶ Cantor, *Gesch. d. Math.* I, p. 241.

⁷ Proclus, p. 60, 7—9.

⁸ *ibid.* p. 60, 16—19.

The definitions of *Analysis* and *Synthesis* interpolated in Eucl. XIII. are as follows (I adopt the reading of B and V, the only intelligible one, for the second).

"**Analysis** is an assumption of that which is sought as if it were admitted < and the passage > through its consequences to something admitted (to be) true.

"**Synthesis** is an assumption of that which is admitted < and the passage > through its consequences to the finishing or attainment of what is sought."

The language is by no means clear and has, at the best, to be filled out.

Pappus has a fuller account¹:

"The so-called *ἀναλυόμενος* ('Treasury of Analysis') is, to put it shortly, a special body of doctrine provided for the use of those who, after finishing the ordinary Elements, are desirous of acquiring the power of solving problems which may be set them involving (the construction of) lines, and it is useful for this alone. It is the work of three men, Euclid the author of the Elements, Apollonius of Perga, and Aristaëus the elder, and proceeds by way of analysis and synthesis.

"**Analysis** then takes that which is sought as if it were admitted and passes from it through its successive consequences to something which is admitted as the result of synthesis: for in analysis we assume that which is sought as if it were (already) done (*γεγονός*), and we inquire what it is from which this results, and again what is the antecedent cause of the latter, and so on, until by so retracing our steps we come upon something already known or belonging to the class of first principles, and such a method we call analysis as being solution backwards (*ἀνάπαλιν λύσιν*).

"But in **synthesis**, reversing the process, we take as already done that which was last arrived at in the analysis and, by arranging in their natural order as consequences what were before antecedents, and successively connecting them one with another, we arrive finally at the construction of what was sought; and this we call synthesis.

"Now analysis is of two kinds, the one directed to searching for the truth and called *theoretical*, the other directed to finding what we are told to find and called *problematical*. (1) In the *theoretical* kind we assume what is sought as if it were existent and true, after which we pass through its successive consequences, as if they too were true and established by virtue of our hypothesis, to something admitted: then (*a*), if that something admitted is true, that which is sought will also be true and the proof will correspond in the reverse order to the analysis, but (*b*), if we come upon something admittedly false, that which is sought will also be false. (2) In the *problematical* kind we assume that which is propounded as if it were known, after which we pass through its successive consequences, taking them as true, up to something admitted: if then (*a*) what is admitted is possible and obtainable, that is, what mathematicians call *given*, what was originally proposed will also be possible, and the proof will again correspond in

¹ Pappus, VII. pp. 634—6.

reverse order to the analysis, but if (*b*) we come upon something admittedly impossible, the problem will also be impossible."

The ancient Analysis has been made the subject of careful studies by several writers during the last half-century, the most complete being those of Hankel, Duhamel and Zeuthen; others by Offerdinger and Cantor should also be mentioned¹.

The method is as follows. It is required, let us say, to prove that a certain proposition A is true. We assume as a hypothesis that A is true and, starting from this we find that, if A is true, a certain other proposition B is true; if B is true, then C; and so on until we arrive at a proposition K which is *admittedly* true. The object of the method is to enable us to infer, in the reverse order, that, since K is true, the proposition A originally assumed is true. Now Aristotle had already made it clear that false hypotheses might lead to a conclusion which is true. There is therefore a possibility of error unless a certain precaution is taken. While, for example, B may be a necessary consequence of A, it may happen that A is not a necessary consequence of B. Thus, in order that the reverse inference from the truth of K that A is true may be logically justified, it is necessary that each step in the chain of inferences should be unconditionally convertible. As a matter of fact, a very large number of theorems in elementary geometry are unconditionally convertible, so that in practice the difficulty in securing that the successive steps shall be convertible is not so great as might be supposed. But care is always necessary. For example, as Hankel says², a proposition may not be unconditionally convertible in the form in which it is generally quoted. Thus the proposition "The vertices of all triangles having a common base and constant vertical angle lie on a circle" cannot be converted into the proposition that "All triangles with common base and vertices lying on a circle have a constant vertical angle"; for this is only true if the further conditions are satisfied (1) that the circle passes through the extremities of the common base and (2) that only that part of the circle is taken as the locus of the vertices which lies on *one* side of the base. If these conditions are added, the proposition is unconditionally convertible. Or again, as Zeuthen remarks³, K may be obtained by a series of inferences in which A or some other proposition in the series is only *apparently* used; this would be the case e.g. when the method of modern algebra is being employed and the expressions on each side of the sign of equality have been inadvertently multiplied by some composite magnitude which is in reality equal to zero.

Although the above extract from Pappus does not make it clear that each step in the chain of argument must be convertible in the case taken, he almost implies this in the second part of the definition of Analysis where, instead of speaking of the consequences B, C...

¹ Hankel, *Zur Geschichte der Mathematik in Alterthum und Mittelalter*, 1874, pp. 137—150; Duhamel, *Des méthodes dans les sciences de raisonnement*, Part I., 3 ed., Paris, 1885, pp. 39—68; Zeuthen, *Geschichte der Mathematik im Alterthum und Mittelalter*, 1896, pp. 92—104; Offerdinger, *Beiträge zur Geschichte der griechischen Mathematik*, Ulm, 1860; Cantor, *Geschichte der Mathematik*, I, pp. 220—2.

² Hankel, p. 139.

³ Zeuthen, p. 103.

successively following from A, he suddenly changes the expression and says that we inquire *what it is* (B) *from which A follows* (A being thus the consequence of B, instead of the reverse), and then what (viz. C) is the antecedent cause of B; and in practice the Greeks secured what was wanted by always insisting on the *analysis* being confirmed by subsequent *synthesis*, that is, they laboriously worked backwards the whole way from K to A, reversing the order of the analysis, which process would undoubtedly bring to light any flaw which had crept into the argument through the accidental neglect of the necessary precautions.

Reductio ad absurdum a variety of analysis.

In the process of analysis starting from the hypothesis that a proposition A is true and passing through B, C... as successive consequences we may arrive at a proposition K which, instead of being admittedly true, is either admittedly false or the contradictory of the original hypothesis A or of some one or more of the propositions B, C... intermediate between A and K. Now correct inference from a true proposition cannot lead to a false proposition; and in this case therefore we may at once conclude, without any inquiry whether the various steps in the argument are convertible or not, that the hypothesis A is false, for, if it were true, all the consequences correctly inferred from it would be true and no incompatibility could arise. This method of proving that a given hypothesis is *false* furnishes an indirect method of proving that a given hypothesis A is *true*, since we have only to take the *contradictory* of A and to prove that it is false. This is the method of *reductio ad absurdum*, which is therefore a variety of analysis. The contradictory of A, or not-A, will generally include more than one case and, in order to prove its falsity, each of the cases must be separately disposed of: e.g., if it is desired to prove that a certain part of a figure is *equal* to some other part, we take separately the hypotheses (1) that it is *greater*, (2) that it is *less*, and prove that each of these hypotheses leads to a conclusion either admittedly false or contradictory to the hypothesis itself or to some one of its consequences.

Analysis as applied to problems.

It is in relation to problems that the ancient analysis has the greatest significance, because it was the one general method which the Greeks used for solving all "the more abstruse problems" (*τὰ ἀσαφέστερα τῶν προβλημάτων*)¹.

We have, let us suppose, to construct a figure satisfying a certain set of conditions. If we are to proceed at all methodically and not by mere guesswork, it is first necessary to "analyse" those conditions. To enable this to be done we must get them clearly in our minds, which is only possible by assuming all the conditions to be actually fulfilled, in other words, by supposing the problem solved. Then we have to transform those conditions, by all the means which practice in such cases has taught us to employ, into other conditions which are necessarily fulfilled if the original conditions are, and to continue this

¹ Proclus, p. 242, 16, 17.

transformation until we at length arrive at conditions which we are in a position to satisfy¹. In other words, we must arrive at some relation which enables us to *construct* a particular part of the figure which, it is true, has been hypothetically assumed and even drawn, but which nevertheless really requires to be *found* in order that the problem may be solved. From that moment the particular part of the figure becomes one of the *data*, and a fresh relation has to be found which enables a fresh part of the figure to be determined by means of the original data and the new one together. When this is done, the second new part of the figure also belongs to the data; and we proceed in this way until all the parts of the required figure are found². The first part of the analysis down to the point of discovery of a relation which enables us to say that a certain new part of the figure not belonging to the original data is *given*, Hankel calls the *transformation*; the second part, in which it is proved that all the remaining parts of the figure are "given," he calls the *resolution*. Then follows the *synthesis*, which also consists of two parts, (1) the *construction*, in the order in which it has to be actually carried out, and in general following the course of the second part of the analysis, the *resolution*; (2) the *demonstration* that the figure obtained does satisfy all the given conditions, which follows the steps of the first part of the analysis, the *transformation*, but in the reverse order. The second part of the analysis, the *resolution*, would be much facilitated and shortened by the existence of a systematic collection of *Data* such as Euclid's book bearing that title, consisting of propositions proving that, if in a figure certain parts or relations are *given*, other parts or relations are also *given*. As regards the first part of the analysis, the *transformation*, the usual rule applies that every step in the chain must be unconditionally convertible; and any failure to observe this condition will be brought to light by the subsequent synthesis. The second part, the *resolution*, can be directly turned into the *construction* since that only is *given* which can be constructed by the means provided in the *Elements*.

It would be difficult to find a better illustration of the above than the example chosen by Hankel from Pappus³.

Given a circle ABC and two points D, E external to it, to draw straight lines DB, EB from D, E to a point B on the circle such that, if DB, EB produced meet the circle again in C, A, AC shall be parallel to DE.

Analysis.

Suppose the problem solved and the tangent at *A* drawn, meeting *ED* produced in *F*.

(Part I. *Transformation*.)

Then, since *AC* is parallel to *DE*, the angle at *C* is equal to the angle *CDE*.

But, since *FA* is a tangent, the angle at *C* is equal to the angle *FAE*.

Therefore the angle *FAE* is equal to the angle *CDE*, whence *A, B, D, F* are concyclic.

¹ Zeuthen, p. 93.

² Hankel, p. 141.

³ Pappus, vii. pp. 830—2.

Therefore the rectangle AE, EB is equal to the rectangle FE, ED .

(Part II. *Resolution*.)

But the rectangle AE, EB is given, because it is equal to the square on the tangent from E .

Therefore the rectangle FE, ED is given;
and, since ED is given, FE is given (in length). [Data, 57.]

But FE is given in position also, so that F is also given. [Data, 27.]

Now FA is the tangent from a given point F to a circle ABC given in position;

therefore FA is given in position and magnitude. [Data, 90.]

And F is given; therefore A is given.

But E is also given; therefore the straight line AE is given in position. [Data, 26.]

And the circle ABC is given in position;
therefore the point B is also given. [Data, 25.]

But the points D, E are also given;
therefore the straight lines DB, BE are also given in position.

Synthesis.

(Part I. *Construction*.)

Suppose the circle ABC and the points D, E given.

Take a rectangle contained by ED and by a certain straight line EF equal to the square on the tangent to the circle from E .

From F draw FA touching the circle in A ; join ABE and then DB , producing DB to meet the circle at C . Join AC .

I say then that AC is parallel to DE .

(Part II. *Demonstration*.)

Since, by hypothesis, the rectangle FE, ED is equal to the square on the tangent from E , which again is equal to the rectangle AE, EB , the rectangle AE, EB is equal to the rectangle FE, ED .

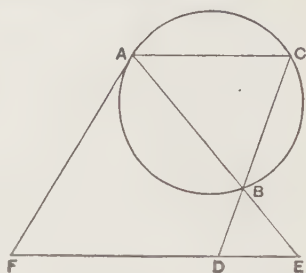
Therefore A, B, D, F are concyclic,
whence the angle FAE is equal to the angle BDE .

But the angle FAE is equal to the angle ACB in the alternate segment;

therefore the angle ACB is equal to the angle BDE .

Therefore AC is parallel to DE .

In cases where a *διορισμός* is necessary, i.e. where a solution is only possible under certain conditions, the analysis will enable those conditions to be ascertained. Sometimes the *διορισμός* is stated and proved at the end of the analysis, e.g. in Archimedes, *On the Sphere and Cylinder*, II. 7; sometimes it is stated in that place and the proof postponed till after the end of the synthesis, e.g. in the solution of the problem subsidiary to *On the Sphere and Cylinder*, II. 4, preserved in Eutocius' commentary on that proposition. The analysis should also enable us to determine the number of solutions of which the problem is susceptible.



§ 7. THE DEFINITIONS.

General. "Real" and "Nominal" Definitions.

It is necessary, says Aristotle, whenever any one treats of any whole subject, to divide the genus into its primary constituents, those which are indivisible in species respectively: e.g. number must be divided into triad and dyad; then an attempt must be made in this way to obtain definitions, e.g. of a straight line, of a circle, and of a right angle¹.

The word for definition is *ῥπος*. The original meaning of this word seems to have been "boundary," "landmark." Then we have it in Plato and Aristotle in the sense of standard or determining principle ("id quo alicuius rei natura constituitur vel definitur," *Index Aristotelicus*)²; and closely connected with this is the sense of *definition*. Aristotle uses both *ῥπος* and *ὀρισμός* for definition, the former occurring more frequently in the *Topics*, the latter in the *Metaphysics*.

Let us now first be clear as to what a definition does *not* do. There is nothing in connexion with definitions which Aristotle takes more pains to emphasise than that a definition asserts nothing as to the *existence* or *non-existence* of the thing defined. It is an answer to the question *what* a thing is (*τί ἐστι*), and does not say *that* it is (*ὅτι ἐστι*). The *existence* of the various things defined has to be *proved*, except in the case of a few primary things in each science, the existence of which is indemonstrable and must be *assumed* among the first principles of each science; e.g. points and lines in geometry must be *assumed* to exist, but the existence of everything else must be *proved*. This is stated clearly in the long passage quoted above under First Principles³. It is reasserted in such passages as the following. "The (answer to the question) *what is a man* and the *fact that a man exists* are different things⁴." "It is clear that, even according to the view of definitions now current, those who define things do not prove that they exist⁵." "We say that it is by *demonstration* that we must show that everything exists, except essence (*εἰ μὴ οὐσία εἶη*). But the *existence* of a thing is never essence; for the *existent* is not a genus. Therefore there must be demonstration that a thing exists. Thus, *what is meant by triangle* the geometer assumes, but that it exists he has to prove⁶." "Anterior knowledge of two sorts is necessary: for it is necessary to presuppose, with regard to some things, that they *exist*; in other cases it is necessary to understand *what* the thing described is, and in other cases it is necessary to do both. Thus, with the fact that one of two contradictories must be true, we must know that it exists (is true);

¹ *Anal. post.* II. 13, 96 b 15.

² Cf. *De anima*, I. 2, 404 a 9, where "breathing" is spoken of as the *ῥπος* of "life," and the many passages in the *Politics* where the word is used to denote that which gives its special character to the several forms of government (virtue being the *ῥπος* of aristocracy, wealth of oligarchy, liberty of democracy, 1294 a 10); Plato, *Republic* VIII. 551 C.

³ *Anal. post.* I. 10, 76 a 31 sqq.

⁴ *ibid.* II. 7, 92 b 10.

⁵ *ibid.* 92 b 19.

⁶ *ibid.* 92 b 12 sqq.

of the triangle we must know that it means such and such a thing; of the unit we must know both what it means and that it exists¹." What is here so much insisted on is the very fact which Mill pointed out in his discussion of earlier views of Definitions, where he says that the so-called *real* definitions or definitions of *things* do not constitute a different kind of definition from *nominal* definitions, or definitions of *names*; the former is simply the latter *plus* something else, namely a covert assertion that the thing defined exists. "This covert assertion is not a definition but a postulate. The definition is a mere identical proposition which gives information only about the use of language, and from which no conclusion affecting matters of fact can possibly be drawn. The accompanying postulate, on the other hand, affirms a fact which may lead to consequences of every degree of importance. It affirms the actual or possible existence of Things possessing the combination of attributes set forth in the definition: and this, if true, may be foundation sufficient on which to build a whole fabric of scientific truth²." This statement really adds nothing to Aristotle's doctrine³: it has even the slight disadvantage, due to the use of the word "postulate" to describe "the covert assertion" in all cases, of not definitely pointing out that there are cases where existence has to be *proved* as distinct from those where it must be *assumed*. It is true that the existence of a definiend may have to be taken for granted provisionally until the time comes for proving it; but, so far as regards any case where existence must be proved sooner or later, the provisional assumption would be for Aristotle, not a *postulate*, but a *hypothesis*. In modern times, too, Mill's account of the true distinction between *real* and *nominal* definitions had been fully anticipated by Saccheri⁴, the editor of *Euclides ab omni naevo vindicatus* (1733), famous in the history of non-Euclidean geometry. In his *Logica Demonstrativa* (to which he also refers in his *Euclid*) Saccheri lays down the clear distinction between what he calls *definitiones quid nominis* or *nominales*, and *definitiones quid rei* or *reales*, namely that the former are only intended to explain the meaning

¹ *Anal. post.* I. 1, 71 a 11 sqq.

² Mill's *System of Logic*, Bk. I. ch. viii.

³ It is true that it was in opposition to "the ideas of most of the *Aristotelian logicians*" (rather than of Aristotle himself) that Mill laid such stress on his point of view. Cf. his observation: "We have already made, and shall often have to repeat, the remark, that the philosophers who overthrew Realism by no means got rid of the consequences of Realism, but retained long afterwards, in their own philosophy, numerous propositions which could only have a rational meaning as part of a Realistic system. It had been handed down from Aristotle, and probably from earlier times, as an obvious truth, that the science of geometry is deduced from definitions. This, so long as a definition was considered to be a proposition 'unfolding the nature of the thing,' did well enough. But Hobbes followed and rejected utterly the notion that a definition declares the nature of the thing, or does anything but state the meaning of a name; yet he continued to affirm as broadly as any of his predecessors that the *ἀρχαί, principia*, or original premisses of mathematics, and even of all science, are definitions; producing the singular paradox that systems of scientific truth, nay, all truths whatever at which we arrive by reasoning, are deduced from the arbitrary conventions of mankind concerning the signification of words." Aristotle was guilty of no such paradox; on the contrary, he exposed it as plainly as did Mill.

⁴ This has been fully brought out in two papers by G. Vailati, *La teoria Aristotelica della definizione* (*Rivista di Filosofia e scienze affini*, 1903), and *Di un' opera dimenticata del P. Gerolamo Saccheri* ("Logica Demonstrativa," 1697) (in *Rivista Filosofica*, 1903).

that is to be attached to a given term, whereas the latter, besides declaring the meaning of a word, affirm at the same time the existence of the thing defined or, in geometry, the possibility of constructing it. The *definitio quid nominis* becomes a *definitio quid rei* "by means of a postulate, or when we come to the question whether the thing exists and it is answered affirmatively¹." *Definitiones quid nominis* are in themselves quite arbitrary, and neither require nor are capable of proof; they are merely provisional and are only intended to be turned as quickly as possible into *definitiones quid rei*, either (1) by means of a postulate in which it is asserted or conceded that what is defined exists or can be constructed, e.g. in the case of *straight lines* and *circles*, to which Euclid's first three postulates refer, or (2) by means of a demonstration reducing the construction of the figure defined to the successive carrying-out of a certain number of those elementary constructions, the possibility of which is *postulated*. Thus *definitiones quid rei* are in general obtained as the result of a series of demonstrations. Saccheri gives as an instance the construction of a square in Euclid I. 46. Suppose that it is objected that Euclid had no right to define a square, as he does at the beginning of the Book, when it was not certain that such a figure exists in nature; the objection, he says, could only have force if, before proving and making the construction, Euclid had assumed the aforesaid figure as given. That Euclid is not guilty of this error is clear from the fact that he never presupposes the existence of the square as defined until after I. 46.

Confusion between the *nominal* and the *real* definition as thus described, i.e. the use of the former in demonstration before it has been turned into the latter by the necessary proof that the thing defined exists, is according to Saccheri one of the most fruitful sources of illusory demonstration, and the danger is greater in proportion to the "complexity" of the definition, i.e. the number and variety of the attributes belonging to the thing defined. For the greater is the possibility that there may be among the attributes some that are *incompatible*, i.e. the simultaneous presence of which in a given figure can be proved, by means of *other* postulates etc. forming part of the basis of the science, to be impossible.

The same thought is expressed by Leibniz also. "If," he says, "we give any definition, and it is not clear from it that the idea, which we ascribe to the thing, is possible, we cannot rely upon the demonstrations which we have derived from that definition, because, if that idea by chance involves a contradiction, it is possible that even contradictories may be true of it at one and the same time, and thus our demonstrations will be useless. Whence it is clear that definitions are not arbitrary. And this is a secret which is hardly sufficiently known." Leibniz' favourite illustration was the "regular polyhedron with ten faces," the impossibility of which is not obvious at first sight.

¹ "Definitio quid nominis nata est evadere definitio quid rei per postulatum vel dum venit ad quaestionem an est et respondetur affirmative."

² *Opusculs et fragments inédits de Leibniz*, Paris, Alcan, 1903, p. 427. Quoted by Vailati.

It need hardly be added that, speaking generally, Euclid's definitions, and his use of them, agree with the doctrine of Aristotle that the definitions themselves say nothing as to the existence of the things defined, but that the existence of each of them must be proved or (in the case of the "principles") *assumed*. In geometry, says Aristotle, the existence of points and lines only must be assumed, the existence of the rest being proved. Accordingly Euclid's first three postulates declare the possibility of constructing straight lines and circles (the only "lines" except straight lines used in the *Elements*). Other things are defined and afterwards constructed and proved to exist: e.g. in Book I., Def. 20, it is explained what is meant by an equilateral triangle; then (I. 1) it is proposed to construct it, and, when constructed, it is proved to agree with the definition. When a square is defined (I. Def. 22), the question whether such a thing really exists is left open until, in I. 46, it is proposed to construct it and, when constructed, it is proved to satisfy the definition¹. Similarly with the right angle (I. Def. 10, and I. 11) and parallels (I. Def. 23, and I. 27—29). The greatest care is taken to exclude mere presumption and imagination. The transition from the subjective definition of names to the objective definition of things is made, in geometry, by means of *constructions* (the first principles of which are postulated), as in other sciences it is made by means of experience².

Aristotle's requirements in a definition.

We now come to the positive characteristics by which, according to Aristotle, scientific definitions must be marked.

First, the different attributes in a definition, when taken separately, cover more than the notion defined, but the combination of them does not. Aristotle illustrates this by the "triad," into which enter the several notions of number, odd and prime, and the last "in both its two senses (*a*) of not being measured by any (other) number (*ὡς μὴ μετρεῖσθαι ἀριθμῷ*) and (*b*) of not being obtainable by adding numbers together" (*ὡς μὴ συγκεῖσθαι ἐξ ἀριθμῶν*), a unit not being a number. Of these attributes some are present in all other odd numbers as well, while the last [primeness in the second sense] belongs also to the dyad, but in nothing but the triad are they *all* present³. The fact can be equally well illustrated from geometry. Thus, e.g. into the definition of a square (Eucl. I., Def. 22) there enter the several notions of figure, four-sided, equilateral, and right-angled, each of which covers more than the notion into which *all* enter as attributes⁴.

Secondly, a definition must be expressed in terms of things which are prior to, and better known than, the things defined⁵. This is

¹ Trendelenburg, *Elementa Logices Aristoteleae*, § 50.

² Trendelenburg, *Erläuterungen zu den Elementen der aristotelischen Logik*, 3 ed. p. 107. On construction as proof of existence in ancient geometry cf. H. G. Zeuthen, *Die geometrische Construction als "Existenzbeweis" in der antiken Geometrie* (in *Mathematische Annalen*, 47. Band).

³ *Anal. post.* II. 13, 96 a 33—b 1.

⁴ Trendelenburg, *Erläuterungen*, p. 108.

⁵ *Topics* VI. 4, 141 a 26 sqq.

clear, since the object of a definition is to give us knowledge of the thing defined, and it is by means of things prior and better known that we acquire fresh knowledge, as in the course of demonstrations. But the terms "prior" and "better known" are, as usual susceptible of two meanings; they may mean (1) *absolutely* or *logically* prior and better known, or (2) better known *relatively to us*. In the absolute sense, or from the standpoint of reason, a point is better known than a line, a line than a plane, and a plane than a solid, as also a unit is better known than number (for the unit is prior to, and the first principle of, any number). Similarly, in the absolute sense, a letter is prior to a syllable. But the case is sometimes different relatively to us; for example, a solid is more easily realised by the senses than a plane, a plane than a line, and a line than a point. Hence, while it is more scientific to begin with the *absolutely* prior, it may, perhaps, be permissible, in case the learner is not capable of following the scientific order, to explain things by means of what is more intelligible *to him*. "Among the definitions framed on this principle are those of the point, the line and the plane; all these explain what is prior by means of what is posterior, for the point is described as the extremity of a line, the line of a plane, the plane of a solid." But, if it is asserted that such definitions by means of things which are more intelligible relatively only to a particular individual are really definitions, it will follow that there may be many definitions of the same thing, one for each individual for whom a thing is being defined, and even different definitions for one and the same individual at different times, since at first sensible objects are more intelligible, while to a better trained mind they become less so. It follows therefore that a thing should be defined by means of the *absolutely* prior and not the *relatively* prior, in order that there may be one sole and immutable definition. This is further enforced by reference to the requirement that a good definition must state the *genus* and the *differentiae*, for these are among the things which are, in the absolute sense, better known than, and prior to, the species (τῶν ἀπλῶς γνωριμωτέρων καὶ προτέρων τοῦ εἶδους ἐστίν). For to destroy the genus and the differentia is to destroy the species, so that the former are *prior* to the species; they are also *better known*, for, when the species is known, the genus and the differentia must necessarily be known also, e.g. he who knows "man" must also know "animal" and "land-animal," but it does not follow, when the genus and differentia are known, that the species is known too, and hence the species is less known than they are¹. It may be frankly admitted that the scientific definition will require superior mental powers for its apprehension; and the extent of its use must be a matter of discretion. So far Aristotle; and we have here the best possible explanation why Euclid supplemented his definition of a point by the statement in I. Def. 3 that *the extremities of a line are points* and his definition of a surface by I. Def. 6 to the effect that *the extremities of a surface are lines*. The supplementary expla-

¹ *Topics* VI. 4, 141 b 25-34.

nations do in fact enable us to arrive at a better understanding of the formal definitions of a point and a line respectively, as is well explained by Simson in his note on Def. 1. Simson says, namely, that we must consider a solid, that is, a magnitude which has length, breadth and thickness, in order to understand aright the definitions of a point, a line and a surface. Consider, for instance, the boundary common to two solids which are contiguous or the boundary which divides one solid into two contiguous parts; this boundary is a surface. We can prove that it has no thickness by taking away either solid, when it remains the boundary of the other; for, if it had thickness, the thickness must either be a part of one solid or of the other, in which case to take away one or other solid would take away the thickness and therefore the boundary itself: which is impossible. Therefore the boundary or the surface has no thickness. In exactly the same way, regarding a line as the boundary of two contiguous surfaces, we prove that the line has no breadth; and, lastly, regarding a point as the common boundary or extremity of two lines, we prove that a point has no length, breadth or thickness.

Aristotle on unscientific definitions.

Aristotle distinguishes three kinds of definition which are unscientific because founded on what is *not* prior (*μὴ ἐκ προτέρων*). The *first* is a definition of a thing by means of its opposite, e.g. of "good" by means of "bad"; this is wrong because opposites are naturally evolved together, and the knowledge of opposites is not uncommonly regarded as one and the same, so that one of the two opposites cannot be better known than the other. It is true that, in some cases of opposites, it would appear that no other sort of definition is possible: e.g. it would seem impossible to define double apart from the half and, generally, this would be the case with things which in their very nature (*καθ' αὐτά*) are *relative* terms (*πρὸς τι λέγεται*), since one cannot be known without the other, so that in the notion of either the other must be comprised as well¹. The *second* kind of definition which is based on what is not prior is that in which there is a complete circle through the unconscious use in the definition itself of the notion to be defined though not of the name². Trendelenburg illustrates this by two current definitions, (1) that of magnitude as that which can be increased or diminished, which is bad because the positive and negative comparatives "more" and "less" presuppose the notion of the positive "great," (2) the famous Euclidean definition of a straight line as that which "lies evenly with the points on itself" (*ἐξ ἴσου τοῖς ἐφ' ἑαυτῆς σημείοις κεῖται*), where "lies evenly" can only be understood with the aid of the very notion of a straight line which is to be defined³. The *third* kind of vicious definition from that which is not prior is the definition of one of two coordinate species by means of its coordinate (*ἀντιδιηρημένον*), e.g. a definition of "odd" as that which exceeds the even by a unit (the second alternative in Eucl. VII. Def. 7); for "odd" and "even" are coordinates, being *differentiae* of

¹ *Topics* VI. 4, 142 a 22—31.

² *ibid.* 142 a 34—b 6.

³ Trendelenburg, *Erläuterungen*, p. 115.

number¹. This third kind is similar to the first. Thus, says Trendelenburg, it would be wrong to define a *square* as "a *rectangle* with equal sides."

Aristotle's third requirement.

A third general observation of Aristotle which is specially relevant to geometrical definitions is that "to know *what* a thing is (τί ἐστίν) is the same as knowing *why* it is (διὰ τί ἐστίν)²." "What is an eclipse? A deprivation of light from the moon through the interposition of the earth. *Why* does an eclipse take place? Or *why* is the moon eclipsed? Because the light fails through the earth obstructing it. *What* is harmony? A ratio of numbers in high or low pitch. *Why* does the high-pitched harmonise with the low-pitched? Because the high and the low have a numerical ratio to one another³." "We seek the *cause* (τὸ διότι) when we are already in possession of the *fact* (τὸ ὅτι). Sometimes they both become evident at the same time, but at all events the cause cannot possibly be known [as a cause] before the fact is known⁴." "It is impossible to know *what* a thing is if we do not know *that* it is⁵." Trendelenburg paraphrases: "The definition of the notion does not fulfil its purpose until it is made *genetic*. It is the producing cause which first reveals the essence of the thing.... The nominal definitions of geometry have only a provisional significance and are superseded as soon as they are made genetic by means of construction." E.g. the genetic definition of a parallelogram is evolved from Eucl. I. 31 (giving the construction for parallels) and I. 33 about the lines joining corresponding ends of two straight lines parallel and equal in length. Where existence is proved by *construction*, the cause and the fact appear *together*⁶.

Again, "it is not enough that the defining statement should set forth the fact, as most definitions do; it should also contain and present the cause; whereas in practice what is stated in the definition is usually no more than a conclusion (συμπέρασμα). For example, what is quadrature? The construction of an equilateral right-angled figure equal to an oblong. But such a definition expresses merely the conclusion. Whereas, if you say that quadrature is the discovery of a mean proportional, then you state the reason⁷." This is better understood if we compare the statement elsewhere that "the cause is the middle term, and this is what is sought in all cases⁸," and the illustration of this by the case of the proposition that the angle in a semi-circle is a right angle. Here the middle term which it is sought to establish by means of the figure is that the angle in the semi-circle is equal to *the half of two right angles*. We have then the syllogism: Whatever is half of two right angles is a right angle; the angle in a semi-circle is the half of two right angles; therefore (*conclusion*) the angle in a semi-circle is a right angle⁹. As with the demonstration, so

¹ *Topics* VI. 4, 142 b 7—10.

² *Anal. post.* II. 2, 90 a 15—21.

³ *ibid.* 93 a 20.

⁴ *De anima* II. 2, 413 a 13—20.

⁵ *ibid.* II. 11, 94 a 28.

⁶ *Anal. post.* II. 2, 90 a 31.

⁷ *ibid.* II. 8, 93 a 17.

⁸ Trendelenburg, *Erläuterungen*, p. 116.

⁹ *Anal. post.* II. 2, 90 a 6.

it should be with the definition. A definition which is to show the *genesis* of the thing defined should contain the middle term or cause; otherwise it is a mere statement of a conclusion. Consider, for instance, the definition of "quadrature" as "making a square equal in area to a rectangle with unequal sides." This gives no hint as to whether a solution of the problem is possible or how it is solved: but, if you add that to find the mean proportional between two given straight lines gives another straight line such that the square on it is equal to the rectangle contained by the first two straight lines, you supply the necessary middle term or cause¹.

Technical terms not defined by Euclid.

It will be observed that what is here defined, "quadrature" or "squaring" (τετραγωνισμός), is not a geometrical figure, or an attribute of such a figure or a part of a figure, but a technical term used to describe a certain problem. Euclid does not define such things; but the fact that Aristotle alludes to this particular definition as well as to definitions of *deflection* (κεκλᾶσθαι) and of *verging* (νεύειν) seems to show that earlier text-books included among definitions explanations of a number of technical terms, and that Euclid deliberately omitted these explanations from his *Elements* as surplusage. Later the tendency was again in the opposite direction, as we see from the much expanded Definitions of Heron, which, for example, actually include a definition of a *deflected line* (κεκλασμένη γραμμή)². Euclid uses the passive of κλᾶν occasionally³, but evidently considered it unnecessary to explain such terms, which had come to bear a recognised meaning.

The mention too by Aristotle of a definition of *verging* (νεύειν) suggests that the problems indicated by this term were not excluded from elementary text-books before Euclid. The type of problem (νεύσεις) was that of placing a straight line across two lines, e.g. two straight lines, or a straight line and a circle, so that it shall *verge* to a given point (i.e. pass through it if produced) and at the same time the intercept on it made by the two given lines shall be of given length.

¹ Other passages in Aristotle may be quoted to the like effect: e.g. *Anal. post.* i. 2, 71 b 9 "We consider that we know a particular thing in the absolute sense, as distinct from the sophistical and incidental sense, when we consider that we know the cause on account of which the thing is, in the sense of knowing that it is the cause of that thing and that it cannot be otherwise," *ibid.* i. 13, 79 a 2 "For here to know the *fact* is the function of those who are concerned with sensible things, to know the *cause* is the function of the mathematician; it is he who possesses the proofs of the causes, and often he does not know the fact." In view of such passages it is difficult to see how Proclus came to write (p. 202, 11) that Aristotle was the originator (Ἀριστοτέλους κατάρξαντος) of the idea of Amphinomus and others that geometry does not investigate the cause and the *why* (τὸ διὰ τί). To this Geminus replied that the investigation of the cause does, on the contrary, appear in geometry. "For how can it be maintained that it is not the business of the geometer to inquire for what reason, on the one hand, an infinite number of equilateral polygons are inscribed in a circle, but, on the other hand, it is not possible to inscribe in a sphere an infinite number of polyhedral figures, equilateral, equiangular, and made up of similar plane figures? Whose business is it to ask this question and find the answer to it if it is not that of the geometer? Now when geometers reason *per impossibile* they are content to discover the property, but when they argue by direct proof, if such proof be only partial (ἐπὶ μέρους), this does not suffice for showing the cause; if however it is general and applies to all like cases, the why (τὸ διὰ τί) is at once and concurrently made evident."

² Heron, Def. 12 (vol. IV. Heib. pp. 22-24).

³ e.g. in III. 20 and in *Data* 89.

In general, the use of conics is required for the theoretical solution of these problems, or a mechanical contrivance for their practical solution¹. Zeuthen, following Oppermann, gives reasons for supposing, not only that mechanical constructions were *practically* used by the older Greek geometers for solving these problems, but that they were *theoretically* recognised as a permissible means of solution when the solution could not be effected by means of the straight line and circle, and that it was only in later times that it was considered necessary to use conics in *every* case where that was possible². Heiberg³ suggests that the allusion of Aristotle to *νεύσεις* perhaps confirms this supposition, as Aristotle nowhere shows the slightest acquaintance with conics. I doubt whether this is a safe inference, since the problems of this type included in the elementary text-books might easily have been limited to those which could be solved by "plane" methods (i.e. by means of the straight line and circle). We know, e.g., from Pappus that Apollonius wrote two Books on *plane νεύσεις*⁴. But one thing is certain, namely that Euclid deliberately excluded this class of problem, doubtless as not being essential in a book of Elements.

Definitions not afterwards used.

Lastly, Euclid has definitions of some terms which he never afterwards uses, e.g. oblong (*ἑτερόμηκες*), rhombus, rhomboid. The "oblong" occurs in Aristotle; and it is certain that all these definitions are survivals from earlier books of Elements.

¹ Cf. the chapter on *νεύσεις* in *The Works of Archimedes*, pp. c—cxxii.

² Zeuthen, *Die Lehre von den Kegelschnitten im Altertum*, ch. 12, p. 262.

³ Heiberg, *Mathematisches zu Aristoteles*, p. 16.

⁴ Pappus VII. pp. 670—2.

TO THE
RIGHT HONOURABLE THE EARL FITZWILLIAM,

ETC. ETC. ETC.

THIS WORK IS DEDICATED

BY HIS LORDSHIP'S OBEDIENT

AND MUCH OBLIGED SERVANT,

OLIVER BYRNE.



INTRODUCTION.



THE arts and sciences have become so extensive, that to facilitate their acquirement is of as much importance as to extend their boundaries. Illustration, if it does not shorten the time of study, will at least make it more agreeable. THIS WORK has a greater aim than mere illustration; we do not introduce colours for the purpose of entertainment, or to amuse *by certain combinations of tint and form*, but to assist the mind in its researches after truth, to increase the facilities of instruction, and to diffuse permanent knowledge. If we wanted authorities to prove the importance and usefulness of geometry, we might quote every philosopher since the days of Plato. Among the Greeks, in ancient, as in the school of Pestalozzi and others in recent times, geometry was adopted as the best gymnastic of the mind. In fact, Euclid's Elements have become, by common consent, the basis of mathematical science all over the civilized globe. But this will not appear extraordinary, if we consider that this sublime science is not only better calculated than any other to call forth the spirit of inquiry, to elevate the mind, and to strengthen the reasoning faculties, but also it forms the best introduction to most of the useful and important vocations of human life. Arithmetic, land-surveying, mensuration, engineering, navigation, mechanics, hydrostatics, pneumatics, optics, physical astronomy, &c. are all dependent on the propositions of geometry.

Much however depends on the first communication of any science to a learner, though the best and most easy methods are seldom adopted. Propositions are placed before a student, who though having a sufficient understanding, is told just as much about them on entering at the very threshold of the science, as gives him a prepossession most unfavourable to his future study of this delightful subject; or “the formalities and paraphernalia of rigour are so ostentatiously put forward, as almost to hide the reality. Endless and perplexing repetitions, which do not confer greater exactitude on the reasoning, render the demonstrations involved and obscure, and conceal from the view of the student the consecution of evidence.” Thus an aversion is created in the mind of the pupil, and a subject so calculated to improve the reasoning powers, and give the habit of close thinking, is degraded by a dry and rigid course of instruction into an uninteresting exercise of the memory. To raise the curiosity, and to awaken the listless and dormant powers of younger minds should be the aim of every teacher; but where examples of excellence are wanting, the attempts to attain it are but few, while eminence excites attention and produces imitation. The object of this Work is to introduce a method of teaching geometry, which has been much approved of by many scientific men in this country, as well as in France and America. The plan here adopted forcibly appeals to the eye, the most sensitive and the most comprehensive of our external organs, and its pre-eminence to imprint its subject on the mind is supported by the incontrovertible maxim expressed in the well known words of Horace :—

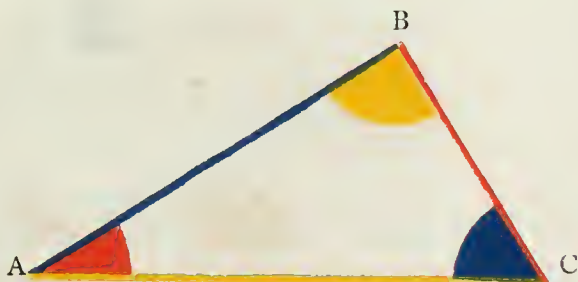
*Segnius irritant animos demissa per aurem
Quàm quæ sunt oculis subiecta fidelibus.*

A feebler impress through the ear is made,
Than what is by the faithful eye conveyed.

All language consists of representative signs, and those signs are the best which effect their purposes with the greatest precision and dispatch. Such for all common purposes are the audible signs called words, which are still considered as audible, whether addressed immediately to the ear, or through the medium of letters to the eye. Geometrical diagrams are not signs, but the materials of geometrical science, the object of which is to show the relative quantities of their parts by a process of reasoning called Demonstration. This reasoning has been generally carried on by words, letters, and black or uncoloured diagrams; but as the use of coloured symbols, signs, and diagrams in the linear arts and sciences, renders the process of reasoning more precise, and the attainment more expeditious, they have been in this instance accordingly adopted.

Such is the expedition of this enticing mode of communicating knowledge, that the Elements of Euclid can be acquired in less than one third the time usually employed, and the retention by the memory is much more permanent; these facts have been ascertained by numerous experiments made by the inventor, and several others who have adopted his plans. The particulars of which are few and obvious; the letters annexed to points, lines, or other parts of a diagram are in fact but arbitrary names, and represent them in the demonstration; instead of these, the parts being differently coloured, are made to name themselves, for their forms in corresponding colours represent them in the demonstration.

In order to give a better idea of this system, and



angled triangle, and expresses some of its properties both by colours and the method generally employed.

Some of the properties of the right angled triangle ABC, expressed by the method generally employed.

1. The angle BAC, together with the angles BCA and ABC are equal to two right angles, or twice the angle ABC.
2. The angle CAB added to the angle ACB will be equal to the angle ABC.
3. The angle ABC is greater than either of the angles BAC or BCA.
4. The angle BCA or the angle CAB is less than the angle ABC.
5. If from the angle ABC, there be taken the angle BAC, the remainder will be equal to the angle ACB.
6. The square of AC is equal to the sum of the squares of AB and BC.

The same properties expressed by colouring the different parts.

$$1. \quad \text{Red Triangle} + \text{Yellow Triangle} + \text{Blue Triangle} = 2 \text{ Yellow Triangle} = \text{Semicircle}.$$

That is, the red angle added to the yellow angle added to the blue angle, equal twice the yellow angle, equal two right angles.

$$2. \quad \text{Red Triangle} + \text{Blue Triangle} = \text{Yellow Triangle}.$$

Or in words, the red angle added to the blue angle, equal the yellow angle.

$$3. \quad \text{Yellow Triangle} \supset \text{Red Triangle} \text{ or } \supset \text{Blue Triangle}.$$

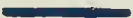
The yellow angle is greater than either the red or blue angle.

4.  or  $\sqsubset$ .

Either the red or blue angle is less than the yellow angle.

5.  minus  = .

In other terms, the yellow angle made less by the blue angle equal the red angle.

6.  ² =  ² +  ².

That is, the square of the yellow line is equal to the sum of the squares of the blue and red lines.

In oral demonstrations we gain with colours this important advantage, the eye and the ear can be addressed at the same moment, so that for teaching geometry, and other linear arts and sciences, in classes, the system is the best ever proposed, this is apparent from the examples just given.

Whence it is evident that a reference from the text to the diagram is more rapid and sure, by giving the forms and colours of the parts, or by naming the parts and their colours, than naming the parts and letters on the diagram. Besides the superior simplicity, this system is likewise conspicuous for concentration, and wholly excludes the injurious though prevalent practice of allowing the student to commit the demonstration to memory; until reason, and fact, and proof only make impressions on the understanding.

Again, when lecturing on the principles or properties of figures, if we mention the colour of the part or parts referred to, as in saying, the red angle, the blue line, or lines, &c. the part or parts thus named will be immediately seen by all in the class at the same instant; not so if we say the angle ABC, the triangle PFQ, the figure EGKt, and so on;

for the letters must be traced one by one before the students arrange in their minds the particular magnitude referred to, which often occasions confusion and error, as well as loss of time. Also if the parts which are given as equal, have the same colours in any diagram, the mind will not wander from the object before it; that is, such an arrangement presents an ocular demonstration of the parts to be proved equal, and the learner retains the data throughout the whole of the reasoning. But whatever may be the advantages of the present plan, if it be not substituted for, it can always be made a powerful auxiliary to the other methods, for the purpose of introduction, or of a more speedy reminiscence, or of more permanent retention by the memory.

The experience of all who have formed systems to impress facts on the understanding, agree in proving that coloured representations, as pictures, cuts, diagrams, &c. are more easily fixed in the mind than mere sentences unmarked by any peculiarity. Curious as it may appear, poets seem to be aware of this fact more than mathematicians; many modern poets allude to this visible system of communicating knowledge, one of them has thus expressed himself:

Sounds which address the ear are lost and die
In one short hour, but these which strike the eye,
Live long upon the mind, the faithful sight
Engraves the knowledge with a beam of light.

This perhaps may be reckoned the only improvement which plain geometry has received since the days of Euclid, and if there were any geometers of note before that time, Euclid's success has quite eclipsed their memory, and even occasioned all good things of that kind to be assigned to him; like Æsop among the writers of Fables. It may also be worthy of remark, as tangible diagrams afford the only medium through which geometry and other linear

arts and sciences can be taught to the blind, this visible system is no less adapted to the exigencies of the deaf and dumb.

Care must be taken to show that colour has nothing to do with the lines, angles, or magnitudes, except merely to name them. A mathematical line, which is length without breadth, cannot possess colour, yet the junction of two colours on the same plane gives a good idea of what is meant by a mathematical line; recollect we are speaking familiarly, such a junction is to be understood and not the colour, when we say the black line, the red line or lines, &c.

Colours and coloured diagrams may at first appear a clumsy method to convey proper notions of the properties and parts of mathematical figures and magnitudes, however they will be found to afford a means more refined and extensive than any that has been hitherto proposed.

We shall here define a point, a line, and a surface, and demonstrate a proposition in order to show the truth of this assertion.

A point is that which has position, but not magnitude; or a point is position only, abstracted from the consideration of length, breadth, and thickness. Perhaps the following description is better calculated to explain the nature of a mathematical point to those who have not acquired the idea, than the above specious definition.

Let three colours meet and cover a portion of the paper, where they meet is not blue, nor is it yellow, nor is it red, as it occupies no portion of the plane, for if it did, it would belong to the blue, the red, or the yellow part; yet it exists, and has position without magnitude, so that with a little reflection, this junc-



tion of three colours on a plane, gives a good idea of a mathematical point.

A line is length without breadth. With the assistance of colours, nearly in the same manner as before, an idea of a line may be thus given:—

Let two colours meet and cover a portion of the paper; where they meet is not red, nor is it blue; therefore the junction occupies no portion of the plane, and therefore it cannot have breadth, but only length: from which we can readily form an idea of what is meant by a mathematical line. For the purpose of illustration, one colour differing from the colour of the paper, or plane upon which it is drawn, would have been sufficient; hence in future, if we say the red line, the blue line, or lines, &c. it is the junctions with the plane upon which they are drawn are to be understood.

Surface is that which has length and breadth without thickness.

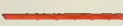
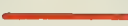
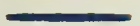


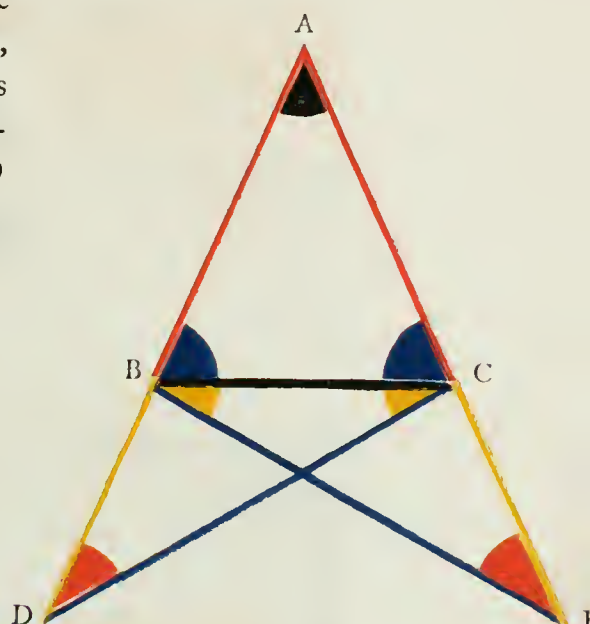
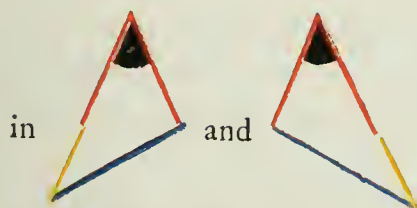
When we consider a solid body (PQ), we perceive at once that it has three dimensions, namely:—length, breadth, and thickness; suppose one part of this solid (PS) to be red, and the other part (QR) yellow, and that the colours be distinct without commingling, the blue surface (RS) which separates these parts, or which is the same thing, that which divides the solid without loss of material, must be without thickness, and only possesses length and breadth;

this plainly appears from reasoning, similar to that just employed in defining, or rather describing a point and a line.

The proposition which we have selected to elucidate the manner in which the principles are applied, is the fifth of the first Book.

In an isosceles triangle ABC , the internal angles at the base ABC , ACB are equal, and when the sides AB , AC are produced, the external angles at the base BCE , CBD are also equal.

Produce  and 
make  = 
Draw  and 
(B. 1. pr. 3.)



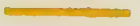
we have  = 

 =  and  common :

$\therefore$  = ,  = 

and  =  (B. 1. pr. 4.)

Again in  and ,

 = ,

$$\text{—} = \text{—},$$

$$\text{and } \triangle = \triangle ;$$

$$\therefore \triangle = \triangle$$

$$\text{and } \triangle = \triangle \quad (\text{B. 1. pr. 4}).$$

$$\text{But } \triangle = \triangle ,$$

$$\therefore \triangle = \triangle .$$

Q. E. D.

By annexing Letters to the Diagram.

LET the equal sides AB and AC be produced through the extremities BC, of the third side, and in the produced part BD of either, let any point D be assumed, and from the other let AE be cut off equal to AD (B. 1. pr. 3). Let the points E and D, so taken in the produced sides, be connected by straight lines DC and BE with the alternate extremities of the third side of the triangle.

In the triangles DAC and EAB the sides DA and AC are respectively equal to EA and AB, and the included angle A is common to both triangles. Hence (B. 1. pr. 4.) the line DC is equal to BE, the angle ADC to the angle AEB, and the angle ACD to the angle ABE; if from the equal lines AD and AE the equal sides AB and AC be taken, the remainders BD and CE will be equal. Hence in the triangles BDC and CEB, the sides BD and DC are respectively equal to CE and EB, and the angles D and E included by those sides are also equal. Hence (B. 1. pr. 4.)

the angles DBC and ECB, which are those included by the third side BC and the productions of the equal sides AB and AC are equal. Also the angles DCB and EBC are equal if those equals be taken from the angles DCA and EBA before proved equal, the remainders, which are the angles ABC and ACB opposite to the equal sides, will be equal.

Therefore in an isosceles triangle, &c.

Q. E. D.

Our object in this place being to introduce the system rather than to teach any particular set of propositions, we have therefore selected the foregoing out of the regular course. For schools and other public places of instruction, dyed chalks will answer to describe diagrams, &c. for private use coloured pencils will be found very convenient.

We are happy to find that the Elements of Mathematics now forms a considerable part of every sound female education, therefore we call the attention of those interested or engaged in the education of ladies to this very attractive mode of communicating knowledge, and to the succeeding work for its future developement.

We shall for the present conclude by observing, as the senses of sight and hearing can be so forcibly and instantaneously addressed alike with one thousand as with one, *the million* might be taught geometry and other branches of mathematics with great ease, this would advance the purpose of education more than any thing that *might* be named, for it would teach the people how to think, and not what to think; it is in this particular the great error of education originates.



Here is (gentle Reader) nothing
(the word of God onely set apart)
which so much beautifieth and a-
dorneth the soule and minde of
mā, as doth the knowledge of good
artes and sciences: as the know-
ledge of naturall and morall Phi-
losophie. The one setteth before
our eyes, the creatures of God,

both in the heauens aboue, and in the earth beneath: in which as
in a glasse, we beholde the exceding maiestie and wisedome of
God, in adorning and beautifying them as we see: in geuing vn-
to them such wonderfull and manifolde proprieties, and naturall
workinges, and that so diuersly and in such varietie: farther in
maintaining and conseruing them continually, whereby to praise
and adore him, as by S. Paule we are taught. The other tea-
cheth vs rules and preceptes of vertue, how, in common life a-
mongest men, we ought to walke vprightly: what dueties per-
taine to our selues, what pertaine to the gouernment or good or-
der both of an housholde, and also of a citie or common wealth.
The reading likewise of histories, conduceth not a litle, to the ad-
orning of the soule & minde of man, a studie of all men cōmen-
ded: by it are seene and known the artes and doinges of infinite
wise men gone before vs. In histories are contained infinite ex-
amples of heroicall vertues to be of vs followed, and horrible ex-
amples of vices to be of vs eschewed. Many other artes also
there are which beautifie the minde of man: but of all other none
do more garnishe & beautifie it, then those artes which are cal-
led Mathematicall. Vnto the knowledge of which no man can
attaine, without the perfecte knowledge and instruction of the
principles, groundes, and Elementes of Geometrie. But per-

The Translator to the Reader.

fectly to be instructed in them, requireth diligent studie and reading of olde auncient authors. Amongest which, none for a beginner is to be preferred before the most auncient Philosopher Euclide of Megara. For of all others he hath in a true methode and iuste order, gathered together whatsoeuer any before him had of these Elementes written: inuenting also and adding many thinges of his owne: wherby he hath in due forme accomplished the arte: first geuing definitions, principles, & groundes, wherof he deduceth his Propositions or conclusions, in such wonderfull wise, that that which goeth before, is of necessitie required to the proufe, of that which followeth. So that without the diligent studie of Euclides Elementes, it is impossible to attaine vnto the perfecte knowledge of Geometrie, and consequently of any of the other Mathematicall sciences. Wherefore considering the want & lacke of such good authors hitherto in our English tongue, lamenting also the negligence, and lacke of zeale to their countrey in those of our nation, to whom God hath geuen both knowledge, & also abilitie to translate into our tongue, and to publishe abroad such good authors, and bookes (the chiefe instrumentes of all learninges): seing moreouer that many good wittes both of gentlemen and of others of all degrees, much desirous and studious of these artes, and seeking for them as much as they can, sparing no paines, and yet frustrate of their intent, by no meanes attaining to that which they seeke: I haue for their sakes, with some charge & great trauaile, faithfully translated into our vulgare tongue, & set abroad in Print, this booke of Euclide. Whereunto I haue added easie and plaine declarations and examples by figures, of the definitions. In which booke also ye shall in due place finde manifolde additions, Scholies, Annotations, and Inuentions: which I haue gathered out of many of the most famous & chiefe Mathematiciēs, both of old time, and in our age: as by diligent reading it in course, ye shall well

The Tranflater to the Reader.

well perceauē. The fruite and gaine which I require for theſe my paines and trauaile, ſhall be nothing els, but onely that thou gentle reader, will gratefully accept the ſame: and that thou mayeſt thereby receaue ſome profite: and moreouer to excite and ſtirre vp others learned, to do the like, & to take paines in that behalfe. By meanes wherof, our Engliſhe tounge ſhall no leſſe be enriched with good Authors, then are other ſtraunge toungeſ: as the Dutch, French, Italian, and Spaniſhe: in which are red all good authors in a maner, found amongeſt the Grekes or Latines. Which is the chiefeſt cauſe, that amongeſt the do flouriſhe ſo many cunning and ſkilfull men, in the inuentions of ſtraunge and wonderfull thinges, as in theſe our daies we ſee there do. Which fruite and gaine if I attaine vnto, it ſhall encourage me hereafter, in ſuch like ſort to tranſlate, and ſet abroad ſome other good authors, both pertaining to religion (as partly I haue already done) and alſo pertaining to the Mathematicall Artes. Thus gentle reader farewell.

(?)

 .ij.

Faultes eſcaped.

M Aruaile not (gentle reader) that faultes here following, haue eſcaped in the correction of this booke. For, for that the matter in it containd is ſtraunge to our Printers here in England, not hauing bene accuſtomed to Print many, or rather any bookes containyng ſuch matter, which cauſeth them to be vnfurniſhed of a corrector ſkilfull in that art: I was forced, to my great trauaile and paine, to correcte the whole booke my ſelfe. And in deed ſometimes for want of *Argus* eyes, and iuſte conſideration, notwithstanding my diligence in correctyng, faultes eſcaped through me: ſometimes alſo for lacke of diligence in the Printer to amend my corrections, faultes remayned vncorrected by his meanes. So that betwene vs both theſe faultes haue eſcaped vncorrected: which faultes yet, to ſay the truth, for the moſt part are ſuch, as a very young ſtudent without noting them vnto him, might eaſily of him ſelfe finde and correcte. And this I dare boldly affirme, that not many bookes, if any, concerning this art in other toungeſ, Greke, Latine, or Italian, are with ſo fewe faults of importance Printed, as this booke is. The triall wherof I referre to them which haue red any bookes of this arte in other toungeſ, & ſhall happen hereafter to read this. And as touchyng theſe faultes to be corrected, I would wiſhe you (good reader) before you beginne to read any of the 16. bookes in this volume contained, firſt to amend the faultes in that booke which you will read, according as they are here ſignified vnto you. Where you ſhall finde in what booke, leaſe, ſide, and line, both the fault eſcaped is, & alſo how it is to be corrected. And if you happen in reading to finde any more faultes not here mentioned, as peraduenture you may, for that diuers faultes were vterly ſo eaſie and light to correcte, that I would not note them, & beſides that, no one man though he be neuer ſo diligent and circumspecte can eſpie all thinges, I truſt you will therefore impute no blame either vnto me or to the Printer, but gently amend and correct them, accepting our good minde, which was to haue had the booke paſſed to your handes vterly without fault, as touching the Printing.

TO
THE KING,
THIS EDITION
OF THE
PRINCIPAL BOOKS
OF THE
ELEMENTS OF EUCLID,
AND OF THE
BOOK OF HIS DATA,
IS MOST HUMBLY DEDICATED,
BY
HIS MAJESTY'S
MOST DUTIFUL,
AND MOST DEVOTED
SUBJECT AND SERVANT,
ROBERT SIMSON.

P R E F A C E.

THE opinions of the moderns concerning the author of the Elements of Geometry, which go under Euclid's name, are very different and contrary to one another. Peter Ramus ascribes the Propositions, as well as their Demonstrations, to Theon; others think the Propositions to be Euclid's, but that the Demonstrations are Theon's; and others maintain, that all the Propositions and their Demonstrations are Euclid's own. John Buteo and Sir Henry Savile are the authors of greatest note who assert this last; and the greater part of geometers have ever since been of this opinion, as they thought it the most probable. Sir Henry Savile, after the several arguments he brings to prove it, makes this conclusion (Page 13. Prælect.) "That, excepting a very few interpolations, explanations, and additions, Theon altered nothing in Euclid." But, by often considering and comparing together the Definitions and Demonstrations as they are in the Greek editions we now have, I found that Theon, or whoever was the editor of the present Greek text, by adding some things, suppressing others, and mixing his own with Euclid's Demonstrations, had changed more things to the worse than is commonly supposed, and those not of small moment, especially in the fifth and eleventh Books of the Elements, which this editor has greatly vitiated; for instance, by substituting a shorter, but insufficient Demonstration of the 18th Prop. of the 5th Book, in place of the legitimate one which Euclid had given; and by taking out of this Book, besides other things, the good definition which Eudoxus or Euclid had given of compound ratio, and giving an absurd one in place of it in the 5th Definition of the 6th Book, which neither Euclid, Archimedes, Apollonius, nor any geometer before Theon's time, ever made use of, and of which there is not to be found the least appearance in any of their writings; and, as this Definition did much embarrass beginners, and is quite useless, it is now thrown out of the Elements, and another, which, without doubt, Euclid had given, is put in its proper place among the Definitions

Definitions of the 5th Book, by which the doctrine of compound ratios is rendered plain and easy. Besides among the Definitions of the 11th Book, there is this, which is the tenth, viz. "Equal and similar solid figures are those which are contained by similar planes of the same number and magnitude." Now this Proposition is a Theorem, not a Definition; because the equality of figures of any kind must be demonstrated, and not assumed; and therefore, though this were a true Proposition, it ought to have been demonstrated. But, indeed, this Proposition, which makes the 10th Definition of the 11th Book, is not true universally, except in the case in which each of the solid angles of the figures is contained by no more than three plane angles; for in other cases, two solid figures may be contained by similar planes of the same number and magnitude, and yet be unequal to one another, as shall be made evident in the Notes subjoined to these Elements. In like manner, in the Demonstration of the 26th Prop. of the 11th Book, it is taken for granted, that those solid angles are equal to one another which are contained by plain angles of the same number and magnitude, placed in the same order; but neither is this universally true, except in the case in which the solid angles are contained by no more than three plane angles; nor of this case is there any Demonstration in the Elements we now have, though it be quite necessary there should be one. Now, upon the 10th Definition of this Book depend the 25th and 28th Propositions of it; and, upon the 25th and 26th depend other eight, viz. the 27th, 31st, 32d, 33d, 34th, 36th, 37th, and 40th of the same Book; and the 12th of the 12th Book depends upon the eighth of the same; and this eighth, and the Corollary of Proposition 17th and Proposition 18th of the 12th Book, depend upon the 9th Definition of the 11th Book, which is not a right definition; because there may be solids contained by the same number of similar plane figures, which are not similar to one another, in the true sense of similarity received by geometers; and all these Propositions have, for these reasons, been insufficiently demonstrated since Theon's time hitherto. Besides, there are several other things, which have nothing of Euclid's accuracy, and which plainly shew, that his Elements have been much corrupted by unskilful geometers; and though these are not so gross as the others now mentioned, they ought by no means to remain uncorrected.

Upon these accounts it appeared necessary, and I hope will prove acceptable, to all lovers of accurate reasoning, and of

mathematical learning, to remove such blemishes, and restore the principal Books of the Elements to their original accuracy, as far as I was able; especially since these Elements are the foundation of a science by which the investigation and discovery of useful truths, at least in mathematical learning, is promoted as far as the limited powers of the mind allow; and which likewise is of the greatest use in the arts both of peace and war, to many of which geometry is absolutely necessary. This I have endeavoured to do, by taking away the inaccurate and false reasonings which unskilful editors have put into the place of some of the genuine Demonstrations of Euclid, who has ever been justly celebrated as the most accurate of geometers, and by restoring to him those things which Theon or others have suppressed, and which have these many ages been buried in oblivion.

In this edition, Ptolemy's Proposition concerning a property of quadrilateral figures in a circle, is added at the end of the sixth Book. Also the Note on the 29th Proposition, Book 1st, is altered, and made more explicit, and a more general Demonstration is given, instead of that which was in the Note on the 10th Definition of Book 11th; besides, the Translation is much amended by the friendly assistance of a learned gentleman.

To which are also added, the Elements of Plane and Spherical Trigonometry, which are commonly taught after the Elements of Euclid.

ADVERTISEMENT.

THE favourable reception which former editions of Professor Simson's *Elements of Euclid* have met with from the public, induced the proprietors of the work to carry into execution every measure most likely to secure and continue general approbation. With this view, the present edition has been carefully revised throughout, by a very eminent mathematician; for the convenience of tutors, as well as students, a short treatise on the CONSTRUCTION of the TRIGONOMETRICAL CANON has now been inserted, from a late celebrated author; and to this has been added, a concise ACCOUNT of LOGARITHMS, and improved methods of calculating them, by the present Savilian Professor of Geometry in the University of Oxford.

THE

INTRODUCTORY REMARKS

THE subject of Plane Geometry is here presented to the student arranged in six books, and each book is subdivided into propositions. The propositions are of two kinds, *problems* and *theorems*. In a problem something is required to be done; in a theorem some new principle is asserted to be true.

A proposition consists of various parts. We have first the general enunciation of the problem or theorem; as for example, *To describe an equilateral triangle on a given finite straight line*, or *Any two angles of a triangle are together less than two right angles*. After the general enunciation follows the discussion of the proposition. First, the enunciation is repeated and applied to the particular figure which is to be considered; as for example, *Let AB be the given straight line: it is required to describe an equilateral triangle on AB*. The *construction* then usually follows, which states the necessary straight lines and circles which must be drawn in order to constitute the *solution* of the problem, or to furnish assistance in the *demonstration* of the theorem. Lastly, we have the demonstration itself, which shows that the problem has been solved, or that the theorem is true.

Sometimes, however, no construction is required; and sometimes the construction and demonstration are combined.

The demonstration is a process of reasoning in which we draw inferences from results already obtained. These results consist partly of truths established in former propositions, or admitted as obvious in commencing the subject, and partly of truths which follow from the *construction* that has been made, or which are given in the *supposition* of the proposition itself. The word *hypothesis* is used in the same sense as *supposition*.

To assist the student in following the steps of the reasoning, *references* are given to the results already obtained

which are required in the demonstration. Thus I. 5 indicates that we appeal to the result established in the fifth proposition of the First Book; *Constr.* is sometimes used as an abbreviation of *Construction*, and *Hyp.* as an abbreviation of *Hypothesis*.

It is usual to place the letters Q.E.F. at the end of the discussion of a problem, and the letters Q.E.D. at the end of the discussion of a theorem. Q.E.F. is an abbreviation for *quod erat faciendum*, that is, *which was to be done*; and Q.E.D. is an abbreviation for *quod erat demonstrandum*, that is, *which was to be proved*.

I. T.

1862.

Include $\frac{9-103}{4-7}$ in one
nomenclature

P R E F A C E.

IN offering to students and teachers a new edition of the Elements of Euclid, it will be proper to give some account of the plan on which it has been arranged, and of the advantages which it hopes to present.

Geometry may be considered to form the real foundation of mathematical instruction. It is true that some acquaintance with Arithmetic and Algebra usually precedes the study of Geometry; but in the former subjects a beginner spends much of his time in gaining a practical facility in the application of rules to examples, while in the latter subject he is wholly occupied in exercising his reasoning faculties.

In England the text-book of Geometry consists of the Elements of Euclid; for nearly every official programme of instruction or examination explicitly includes some portion of this work. Numerous attempts have been made to find an appropriate substitute for the Elements of Euclid; but such attempts, fortunately, have hitherto been made in vain. The advantages attending a common standard of reference in such an important subject, can hardly be over-estimated; and it is extremely improbable, if Euclid were once abandoned, that any agreement would exist as to the author who should replace him. It cannot be denied that

defects and difficulties occur in the Elements of Euclid, and that these become more obvious as we examine the work more closely; but probably during such examination the conviction will grow deeper that these defects and difficulties are due in a great measure to the nature of the subject itself, and to the place which it occupies in a course of education; and it may be readily believed that an equally minute criticism of any other work on Geometry would reveal more and graver blemishes.

Of all the editions of Euclid that of Robert Simson has been the most extensively used in England, and the present edition substantially reproduces Simson's; but his translation has been carefully compared with the original, and some alterations have been made, which it is hoped will be found to be improvements. These alterations, however, are of no great importance; most of them have been introduced with the view of rendering the language more uniform, by constantly using the same words when the same meaning is to be expressed.

As the Elements of Euclid are usually placed in the hands of young students, it is important to exhibit the work in such a form as will assist them in overcoming the difficulties which they experience on their first introduction to processes of continuous argument. No method appears to be so useful as that of breaking up the demonstrations into their constituent parts; this was strongly recommended by Professor De Morgan more than thirty years ago as a suitable exercise for students, and the plan has been adopted more or less closely in some modern editions. An excellent example of this method of exhibiting the Elements of Euclid will be found in an edition in quarto, published at the Hague, in the French language, in 1762. Two persons are named in the title-page as concerned in the work,

Koenig and Blassiere. This edition has served as the model for that which is now offered to the student: some slight modifications have necessarily been made, owing to the difference in the size of the pages.

It will be perceived then, that in the present edition each distinct assertion in the argument begins a new line; and at the ends of the lines are placed the necessary references to the preceding principles on which the assertions depend. Moreover, the longer propositions are distributed into subordinate parts, which are distinguished by breaks at the beginning of the lines.

This edition contains all the propositions which are usually read in the Universities. After the text will be found a selection of notes; these are intended to indicate and explain the principal difficulties which have been noticed in the Elements of Euclid, and to supply the most important inferences which can be drawn from the propositions. The notes relate to Geometry exclusively; they do not introduce developments involving Arithmetic and Algebra, because these latter subjects are always studied in special works, and because Geometry alone presents sufficient matter to occupy the attention of early students. After some hesitation on the point, all remarks relating to Logic have also been excluded. Although the study of Logic appears to be reviving in this country, and may eventually obtain a more assured position than it now holds in a course of liberal education, yet at present few persons take up Logic before Geometry; and it seems therefore premature to devote space to a subject which will be altogether unsuitable to the majority of those who use a work like the present.

After the notes will be found an Appendix, consisting of propositions supplemental to those in the Elements of Euclid; it is hoped that a judicious choice has been made

from the abundant materials which exist for such an Appendix. The propositions selected are worthy of notice on various grounds; some for their simplicity, some for their value as geometrical facts, and some as being problems which may naturally suggest themselves, but of which the solutions are not very obvious.

The work finishes with a collection of exercises. Geometrical deductions afford a most valuable discipline for a student of mathematics, especially in the earlier period of his course; the numerous departments of analysis which subsequently demand his attention will leave him but little time then for pure Geometry. It seems however that the habits of mind which the study of pure Geometry tends to form, furnish an advantageous corrective for some of the evils resulting from an exclusive devotion to Analysis, and it is therefore desirable to engage the attention of beginners with geometrical exercises.

Many persons whose duties have rendered them familiar with the examination of large numbers of students in elementary mathematics have noticed with regret the frequent failures in geometrical deductions. Several collections of exercises already exist, but the general complaint is that they are too difficult. Those in the present volume may be divided into two parts; the first part contains 440 exercises, which it is hoped will not be found beyond the power of early students; the second part consists of the remainder, which may be reserved for practice at a later stage. These exercises have been principally selected from College and University examination papers, and have been tested by long experience with pupils. It will be seen that they are distributed into sections according to the propositions in the Elements of Euclid on which they chiefly depend. As far as possible they are arranged in order of difficulty, but it must sometimes happen, as is the case

in the Elements of Euclid, that one example prepares the way for a set of others which are much easier than itself. It should be observed that the exercises relate to pure Geometry; all examples which would find a more suitable place in works on Trigonometry or Algebraical Geometry have been carefully rejected.

It only remains to advert to the mechanical execution of the volume, to which great attention has been devoted. The figures will be found to be unusually large and distinct, and they have been repeated when necessary, so that they always occur in immediate connexion with the corresponding text. The type and paper have been chosen so as to render the volume as clear and attractive as possible. The design of the editor and of the publishers has been to produce a practically useful edition of the Elements of Euclid, at a moderate cost; and they trust that the design has been fairly realised.

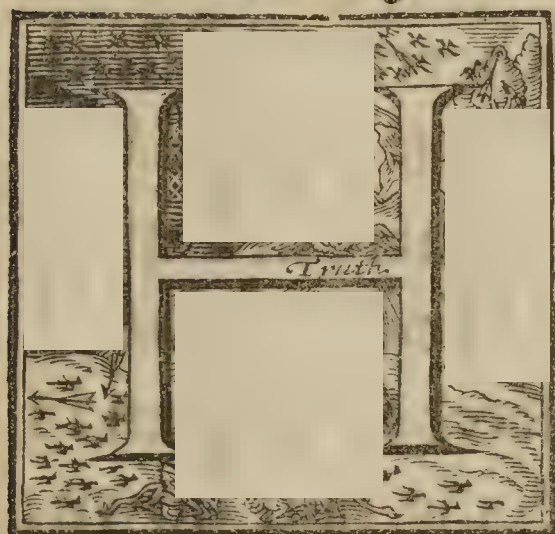
Any suggestions or corrections relating to the work will be most thankfully received.

I. TODHUNTER.

ST JOHN'S COLLEGE,
October 1862.

TO THE READER.

Friendly Reader,



Here is presented to thy view an exact and excellent *Compendium* of the first six Books of the *Elements* of *Geometry*, written by that learned *Philosopher* & *Geometrician* *Euclide*. I shal in this place say little or nothing concerning the excellencie of the *Mathematicall Sciences* in

generall, nor particularly of *Geometry*, that pious and learned Mathematician Mr. JOHN DEE in his following *Preface* (which deserves perpetual commendations) having beene so large in the explanation and use of all the parts thereof, so that in it you may discern the whole *Body* of the *Mathematicks*.

Now *Geometry* (next after *Arithmetick*) is the first thing to be studied in the course of the *Mathematicks*, and of all the *Books* of *Geometry* hitherto published, none are comparable to those of *Euclide*, *The first six* of which are here in a most compendious form *contracted* and *demonstrated*, then they have heretofore been in our *Language*: there being laid down, First, the *Probleme*, or *Theoreme* it selfe, then the *Figure*, and the *construction* thereof; and lastly, the *Demonstration* of the same.

The

The use of these *Elements* of *Euclide* are so frequent in all the parts of the *Mathematicks*, that whatsoever is done in them, hath dependance upon these for prooffe.

As he can be accounted no good *Gramarian* that cannot prove his *Latine* by *Lillies Rules*, so he can be accounted no good *Geometrician*, that cannot prove his Proposition by *Euclides Elements*.

I will not, in this place, goe about to apologize for the *Work*, only thus, the Author of this *Epitomie* having finished his Manuscript, before he could provide for the printing thereof, was transported from this life to a better, so that the *Copie*, in many places, was deficient, but some time hath been bestowed in the supplying of those defects. But notwithstanding all this, some mistakes may yet remain, which when thou chancest to meet

meet with correct with thy pen, so shalt thou do right to the deceased Author, and supply the defects of the Revisor, and in both doe good to thy self. These things thus premised, I commend you to the Work it selfe, in the perusall and practise whereof, thou shalt receive abundance both of pleasure and profit, in that before your eyes you shall see the *demonstration* and *prooffe* of all your workings, and be thereby able to discern and judge of other things of the like nature. And so I leave it with you, as a Clew which will lead you by degrees through the whole *Treasury* of the MATHEMATICKS.

Farewell.

To the READER.

IF you are desirous, Courteous Reader, to know what I have perform'd in this Edition of the Elements of Euclide, I shall here explain it to you in short, according to the nature of the Work. I have endeavor'd to attain two ends chiefly; the first, to be very perspicuous, and at the same time so very brief, that the Book may not swell to such a Bulk, as may be troublesome to carry about one, in which I think I have succeeded, unless in my absence the Printer's care should frustrate my Design. Some of a brighter Genius, and endued with greater Skill, may have demonstrated most of these Propositions with more nicety, but perhaps none with more succinctness than I have; especially since I alter'd nothing in the number and order of the Author's Propositions; nor presum'd either to take the liberty of rejecting, as less necessary, any of them, or of reducing some of the easier sort into the rank of Axioms, as several have done; and among others, that most expert Geometrician A. Tacquetus C. (whom I the more willingly name, because I think it is but civil to acknowledge that I have imitated him in some Points) after whose most accurate Edition I had no Thoughts of attempting any thing of this nature, till I consider'd that this most learned Man thought fit to publish only eight of Euclide's Books, which he took the pains to explain and embellish, having in a manner rejected and undervalued the other seven, as less appertaining to the Elements of Geometry. But my Province was

To the READER.

originally quite different, not that of writing the Elements of Geometry, after what method soever I pleas'd, but of demonstrating, in as few words as possible I cou'd, the whole Works of Euclide. As to four of the Books, viz. the seventh, eighth, ninth, and tenth, altho' they don't so nearly appertain to the Elements of plain and solid Geometry, as the six precedent and the two subsequent, yet none of the more skilful Geometricians can be so ignorant as not to know that they are very useful for Geometrical matters, not only by reason of the mighty near affinity that is between Arithmetick and Geometry, but also for the knowledge of both measurable and unmeasurable Magnitudes, so exceeding necessary for the Doctrine of both plain and solid figures. Now the noble Contemplation of the five Regular Bodies that is contain'd in the three last Books, cannot without great Injustice be pretermitted, since that for the sake thereof our σοιχωτής, being a Philosopher of the Platonic Sect, is said to have compos'd this universal System of Elements; as Proclus lib. 2. witnesseth in these Words, Ὅθεν δὴ καὶ τῆς συμπάσης σοιχωσίας τέλος προετίθετο τὴν ὑπὸ καλυμμένων πλατωνικῶν χημάτων οὖσαν. Besides, I easily perswaded my self to think, that it would not be unacceptable to any Lover of these Sciences to have in his Possession the whole Euclidean Work, as it is commonly cited and celebrated by all Men. Wherefore I resolv'd to omit no Book or Proposition of those that are found in P. Herigonius's Edition, whose Steps I was oblig'd closely to follow, by reason I took a Resolution to make use of most of the Schemes of the said Book, very well foreseeing that time would not allow me to form
new

TO the READER.

new ones, tho' sometimes I chose rather to do it. For the same reason I was willing to use for the most part Euclide's own Demonstrations, having only express'd them in a more succinct Form, unless perhaps in the second, thirteenth, and very few in the seventh, eighth, and ninth Book, in which it seem'd not worth my while to deviate in any particular from him. Therefore I am not without good hopes that as to this part I have in some measure satisfied both my own Intentions, and the Desire of the Studious. As for some certain Problems and Theorems that are added in the Scholions (or short Expositions) either appertaining (by reason of their frequent Use) to the nature of these Elements, or conducing to the ready Demonstration of those things that follow, or which do intimate the reasons of some principal Rules of practical Geometry, reducing them to their original Fountains, these I say, will not, I hope, make the Book swell to a Size beyond the design'd Proportion.

The other Butt, which I levell'd at, is to content the Desires of those who are delighted more with symbolical than verbal Demonstrations. In which kind, whereas most among us are accusom'd to the Symbols of Gulielmus Oughtredus, I therefore thought best to make use, for the most part, of his. None hitherto (as I know of) has attempted to interpret and publish Euclide after this manner, except P. Herigonius; whose Method (tho' indeed most excellent in many things, and very well accommodated for the particular purpose of that most ingenious Man) yet seems in my Opinion to labour under a double Defect. First, in regard that, altho' of two or more Propositions, produ-
ced

TO the READER.

ced for the Proof of any one Problem or Theorem, the former don't always depend of the latter, yet it don't readily enough appear either from the order of each, or by any other manner, when they agree together, and when not; wherefore for want of the Conjunctions and Adjectives, ergo, rursus, &c. many difficulties and occasions of doubt do often arise in reading, especially to those that are Novices. Besides it frequently happens, that the said Method cannot avoid superfluous Repetitions, by which the Demonstrations are oftentimes render'd tedious, and sometimes also more intricate; which Faults my Method doth easily remedy by the arbitrary mixture of both Words and Signs. Therefore let what has been said, touching the Intention and Method of this little Work, suffice. As to the rest, whoever covets to please himself with what may be said, either in Praise of the Mathematicks in general, or of Geometry in particular, or touching the History of these Sciences, and consequently of Euclide himself, (who digested those Elements) and others $\epsilon\chi\omega\tau\epsilon\alpha\iota$ of that kind, may consult other Interpreters. Neither will I (as if I were afraid least these my Endeavors may fall short of being satisfactory to all Persons) alledge as an Excuse (tho' I may very lawfully do it) the want of due time which ought to be employ'd in this Work, nor the Interruption occasion'd by other Affairs, nor yet the want of requisite help for these Studies nor several other things of the like nature. But what I have here employ'd my Labour and Study in for the Use of the ingenuous Reader, I wholly submit to his Censure and Judgment, to approve if useful, or reject if otherwise.

J. B.

Ad amicissimum Virum, I. C. de EUCLIDE
contracto, Εὐκλείδους.

Factum bene ! didicit Laconice loqui
Senex profundus, & aphorismos induit.
Immensa dudum margo commentarii
Diagramma circuit minutum ; utque Insula
Problema breve natabat in vasto mari.
Sed unda jam detumuit ; & glossa arctior
Stringit Theoremata : minoris anguli
Lateribus ecce totus Euclides jacer,
Inclusus olim velut Homerus in nuce ;
Pluteoque Jarcina modo qui incubuit, levis
En fit manipulus. Pelle in exigua latet
Ingens Mathesis, matris utero Hercules,
In glande quercus, vel Ithaca Eurus in pila.
Nec mole dum decrescit, usu fit minor ;
Quin auctior jam evadit, & cumulatius
Contracta prodest erudita pagina.
Sic ubere magis liquor è presso affluit ;
Sic pleniori vasa imundat sanguinis
Torrente cordis Systole ; sic fuscus
Procurrit aquor ex Abyla angustiiis.
Tantilli operis ars tanta referenda unice est
BAROVIANO nomini, ac solertia.
Sublimis euge mentis ingentium potens !
Cui invitum nil, arduum esse nil solet ;
Sic usque pergas prospero conamine,
Radiusque multum debeat ac abacus tibi ;
Sic crescat indies feracior seges,
Simili colonum germine assiduo beans.
Specimen futura messis hic fiet labor.
Magnaue famæ illustria hæc præhudia.
Juvenis dedit qui tanta, quid dabit senex ?

Car. Robotham, *CANTAB.*
Coll. Trin. Sen. Soc.

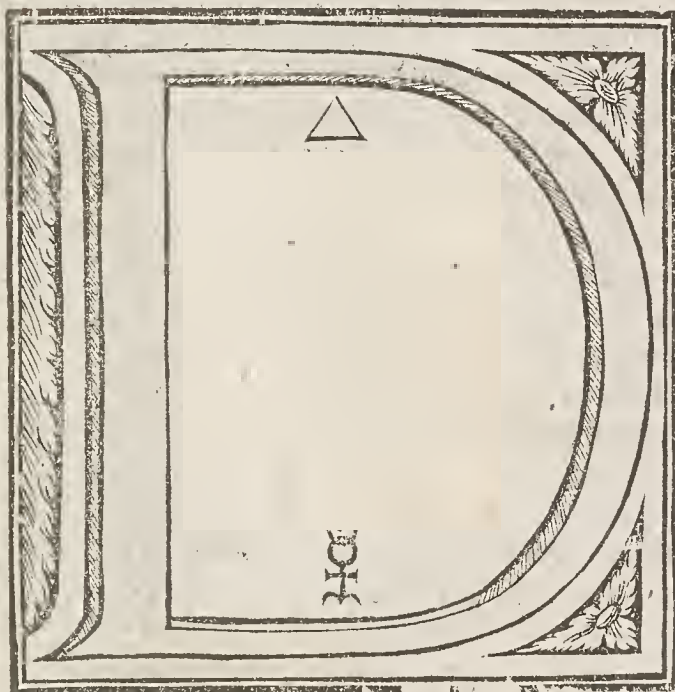
The

The Explication of the Signs or Characters.

=	}	Equal.
>	}	Greater.
<	}	Lesser.
+	}	More, or to be added.
-	}	Less, or to be subtracted.
-	}	The Differences, or Excess; Also, that all the quantities which follow, are to be subtracted, the Signs not being changed.
×	}	Multiplication, or the Drawing one side of a Rectangle into another. The same is denoted by the Conjunction of letters; as $AB = A \times B$.
√	}	The Side or Root of a Square, or Cube, &c.
Q & q	}	A Square.
C & c	}	A Cube.
Q. Q.	}	The ratio of a square number to a square number.

Other Abbreviations of words, where-ever they occur, the Reader will without trouble understand of himself; saving some few, which, being of less general use, we refer to be explained in their places.

TO THE VNFAINED LOVERS
 of truthe , and constant Studentes of Noble
Sciences, **IOHN DEE** of *London*, hartily
 wisheth grace from heauen, and most prosper-
 ous successe in all their honest attemptes and
 exercises.



Iuine *Plato*, the great Master
 of many worthy Philosophers,
 and the constant auoucher, and
 pithy perswader of *Vnum*, *Bo-*
num, and *Ens*: in his Schole and
 Academie, sundry times (besides
 his ordinary Scholers) was visited
 of a certaine kinde of men, allured
 by the noble fame of *Plato*, and
 the great commendation of hys
 profound and profitable doctrine.
 But when such Hearers, after long
 harkening to him, perceaued, that
 the drift of his discourses issued
 out, to conclude, this *Vnum*, *Bo-*
num, and *Ens*, to be Spirituall, Infi-
 nite, Æternall, Omnipotent, &c.

Nothing beyng alledged or expressed, How, worldly goods: how, worldly digni-
 tie: how, health, Stréngth or lustines of body: nor yet the meanes, how a merueilous
 sensible and bodyly blysse and felicitie hereafter, might be atteyned: Straightway,
 the fantasies of those hearers, were dampnt: their opinion of *Plato*, was clene chaun-
 ged: yea his doctrine was by them despised: and his schole, no more of them visi-
 ted. Which thing, his Scholer, *Aristotle*, narrowly cōsidering, founde the cause ther-
 of, to be, For that they had no forwarnyng and information, in generall, whereto
 his doctrine tended. For, so, might they haue had occasion, either to haue forborne
 his schole hauntyng: (if they, then, had misliked his Scope and purpose) or con-
 stantly to haue continued therein: to their full satisfaction: if such his finall scope &
 intent, had ben to their desire. Wherefore, *Aristotle*, euer, after that, vsed in brief, to
 forewarne his owne Scholers and hearers, both of what matter, and also to what
 ende, he tooke in hand to speake, or teach. While I consider the diuerse trades of
 these two excellent Philosophers (and am most sure, both, that *Plato* right well, o-
 therwise could teach: and that *Aristotle* mought boldely, with his hearers, haue
 dealt in like sorte as *Plato* did) I am in no little pang of perplexitie: Bycause, that,
 which I mislike, is most easy for me to performe (and to haue *Plato* for my exāple.)
 And that, which I know to be most commendable: and (in this first bringyng, into
 common handling, the *Artes Mathematicall*) to be most necessary: is full of great
 difficultie and sundry daungers. Yet, neither do I think it mete, for so straunge mat-
 ter (as now is ment to be published) and to so straunge an audience, to be bluntly,
 at first, put forth, without a peculiar Preface: Nor (Imitatyng *Aristotle*) well can I
 hope, that accordyng to the amplexes and dignitie of the *State Mathematicall*, I
 am able, either playnly to prescribe the materiall boundes: or precisely to expresse
 the chief purposes, and most wonderfull applications therof. And though I am
 sure, that such as did shrink from *Plato* his schole, after they had perceiued his fi-
 nall

Iohn Dee his Mathematicall Præface.

nall conclusion, would in these things haue ben his most diligent hearers) so infinitely mought their desires, in fine and at length, by our *Artes Mathematicall* be satisfied) yet, by this my Præface & forewarnyng, Aswell all such, may (to their great behofe) the soner, hither be allured: as also the *Pythagoricall*, and *Platonicall* perfect scholer, and the constant profound Philosopher, with more ease and spede, may (like the Bee,) gather, hereby, both wax and hony.

Wherfore, seying I finde great occasion (for the causes alleged, and farder, in respect of my *Art Mathematicke generall*) to vse a certaine forewarnyng and Præface, whose content shalbe, that mighty, most plesant, and frutefull *Mathematicall Tree*, with his chief armes and second (gifted) braunches: Both, what euery one is, and also, what commodity, in generall, is to be looked for, aswell of griff as stocke. And forasmuch as this enterprise is so great, that, to this our tyme, it neuer was (to my knowledge) by any achieved: And also it is most hard, in these our drery dayes, to such rare and straunge Artes, to wyn due and common credit: Neuertheles, if, for my sincere endeiuour to satisfie your honest expectation, you will but lend me your thâkefull mynde a while: and, to such matter as, for this time, my penne (with spede) is hable to deliuer, apply your eye or eare attentifely: perchance, at once, and for the first salutynge, this Preface you will finde a lesson long enough. And either you will, for a second (by this) be made much the apter: or shortly become, well hable your selues, of the Lyons claw, to coniecture his royall symmetrie, and farder propertie. Now then, gentle, my frendes, and countrey men, Turne your eyes, and bend your myndes to that doctrine, which for our present purpose, my simple talent is hable to yeld you.

All things which are, & haue beyng, are found vnder a triple diuersitie generall. For, either, they are demed Supernaturall, Naturall, or, of a third being. Things Supernaturall, are immateriall, simple, indiuisible, incorruptible, & vnchangeable. Things Naturall, are materiall, compounded, diuisible, corruptible, and chaungeable. Things Supernaturall, are, of the minde onely, comprehended: Things Naturall, of the sense exterior, are hable to be perceiued. In things Naturall, probabilitie and coniecture hath place: But in things Supernaturall, chief demonstration, & most sure Science is to be had. By which properties & comparafons of these two, more easily may be described, the state, condition, nature and property of those things, which, we before termed of a third being: which, by a peculier name also, are called *Thynges Mathematicall*. For, these, beyng (in a maner) middle, betwene things supernaturall and naturall: are not so absolute and excellent, as things supernatural: Nor yet so base and grosse, as things naturall: But are things immateriall: and neuerthelesse, by materiall things hable somewhat to be signified. And though their particular Images, by Art, are aggregable and diuisible: yet the generall *Formes*, notwithstanding, are constant, vnchaungeable, vntrâfformable, and incorruptible. Neither of the sense, can they, at any tyme, be perceiued or iudged. Nor yet, for all that, in the royall mynde of man, first conceiued. But, surmountyng the imperfectiõ of coniecture, weenyng and opinion: and commyng short of high intellectuall conceitiõ, are the Mercuriall fruite of *Dianeticall* discourse, in perfect imagination subsistynge. A meruaylous newtralitye haue these things *Mathematicall*: and also a straunge participatiõ betwene things supernaturall, immortall, intellectuall, simple and indiuisible: and thynges naturall, mortall, sensible, compounded and diuisible. Probabilitie and sensible profe, may well serue in things naturall: and is commendable: In Mathematicall reasoninges, a probable Argument, is nothyng regarded: nor yet the testimony of sense, any whit credited: But onely a perfect demonstration, of truthes certaine, necessary, and inuincible: vniuersally and necessarily concluded:

John Dee his Mathematicall Præface.

cluded: is allowed as sufficient for an Argument exactly and purely Mathematical. "

Of *Mathematicall* thinges, are two principall kinde: namely, *Number*, and *Magnitude*. *Number*, we define, to be, a certayne Mathematicall Summe, of *Units*. And, an *Unit*, is that thing Mathematicall, Indiuisible, by participation of some likenes of whose property, any thing, which is in deede, or is counted One, may resonably be called One. We account an *Unit*, a thing *Mathematicall*, though it be no *Number*, and also indiuisible: because, of it, materially, *Number* doth consist: which, principally, is a thing *Mathematicall*. *Magnitude* is a thing *Mathematicall*, by participation of some likenes of whose nature, any thing is iudged long, broad, or thicke. A thicke *Magnitude* we call a *Solide*, or a *Body*. What *Magnitude* so euer, is *Solide* or *Thicke*, is also broad, & long. A broad magnitude, we call a *Superficies* or a *Plaine*. Euery playne magnitude, hath also length. A long magnitude, we terme a *Line*. A *Line* is neither thicke nor broad, but onely long: Euery certayne *Line*, hath two endes: The endes of a line, are *Pointes* called. A *Point*, is a thing *Mathematicall*, indiuisible, which may haue a certayne determined situation. If a Poynt moue from a determined situation, the way wherein it moued, is also a *Line*: mathematically produced. whereupon, of the auncient Mathematiciens, a *Line* is called the race, or course of a *Point*. A Poynt we define, by the name of a thing *Mathematicall*: though it be no *Magnitude*, and indiuisible: because it is the propre ende, and bound of a *Line*: which is a true *Magnitude*. And *Magnitude* we may define to be that thing *Mathematicall*, which is diuisible for euer, in partes diuisible, long, broad, or thicke. Therefore though a Poynt be no *Magnitude*, yet *Terminatiuely* we reckon it a thing *Mathematicall* (as I sayd) by reason it is properly the end, and bound of a line.

Number.

Note the worde, *Unit*, to expresse the Greke *Monas*, & not *Vnitie*: as we haue all, commonly, till now, used.

Magnitude.

"

A point.

"

A Line.

Magnitude.

Neither *Number*, nor *Magnitude*, haue any Materialitie. First, we will consider of *Number*, and of the Science *Mathematicall*, to it appropriate, called *Arithmetike*: and afterward of *Magnitude*, and his Science, called *Geometrie*. But that name contenteth me not: whereof a word or two hereafter shall be sayd. How Immateriall and free from all matter, *Number* is, who doth not perceauē: yea, who doth not wonderfully wōder at it. For, neither pure *Element*, nor *Aristoteles*, *Quinta Essentia*, is hable to serue for *Number*, as his propre matter. Nor yet the puritie and simplenes of Substance *Spirituall* or *Angelicall*, will be found propre enough thereto. And therefore the great & godly Philosopher *Annius Boetius*, sayd: *Omnia quacūq; a prima rerū natura constructa sunt, Numerorum videntur ratione formata. Hoc enim fuit principale in animo Conditoris Exemplar*. That is:

All thinges (which from the very first originall being of thinges, haue bene framed and made) do appeare to be Formed by the reason of Numbers. For this was the principall example or patterne in the minde of the Creator.

O comfortable allurements, O rauishing perswasion, to deale with a Science, whose Subiect, is so Auncient, so pure, so excellent, so surmounting all creatures, so vsed of the Almighty and incomprehensible wisdom of the Creator, in the distinct creation of all creatures: in all their distinct partes, properties, natures, and vertues, by order, and most absolute number, brought from *Nothing*, to the *Formalitie* of their being and state. By *Numbers* propertie therefore, of vs, by all possible meanes, (to the perfection of the Science) learned, we may both winde and draw our selues into the inward and deepe search and vew, of all creatures distinct vertues, natures, properties, and *Formes*: And also, farder, arise, clime, ascend, and mount vp (with *Speculative winges*) in spirit, to behold in the Glas of Creation, the *Forme* of *Formes*, the *Exemplar Number* of all thinges *Numerable*: both visible and inuifible: mortall and

immortall

Iohn Dee his Mathematicall Præface.

immortall, Corporall and Spirituall. Part of this profound and diuine Science, had *Ioachim* the Prophetier atteyned vnto: by *Numbers Formall, Naturall, and Rationall*, forseyng, concludyng, and forshewyng great particular euent, long before their comming. His booke yet remainyng, hereof, are good profe: And the noble Earle of *Mirandula*, (besides that,) a sufficient witnesse: that *Ioachim*, in his propheties, proceeded by no other way, then by *Numbers Formall*. And this Earle hym selfe, in Rome, *set vp 900. Conclusions, in all kinde of Sciences, openly to be disputed of: and among the rest, in his Conclusions *Mathematicall*, (in the eleuenth Conclusion) hath in Latin, this English sentence. *By Numbers, a way is had, to the searchyng out, and understanding of euery thyng, hable to be knowne. For the verifing of which Conclusion, I promise to aunswere to the 74. Questions, vnder written, by the way of Numbers.* Which Conclusions, I omit here to rehearse: as well auoidyng superfluous prolixitie: as, by cause *Ioannes Picus*, workes, are commonly had. But, in any case, I would wish that those Conclusions were red diligently, and perceiued of such, as are earnest Obseruers and Considerers of the constant law of nūbers: which is planted in thyngs Naturall and Supernaturall: and is prescribed to all Creatures, inuiolably to be kept. For, so, besides many other thynges, in those Conclusions to be marked, it would appeare; how sincerely, & within my boundes, I disclose the wonderfull mysteries, by numbers, to be atteyned vnto.

Of my former wordes, easy it is to be gathered, that *Number* hath a treble state: One, in the Creator: an other in euery Creature (in respect of his complete constitution:) and the third, in Spirituall and Angelicall Myndes, and in the Soule of mā. In the first and third state, *Number*, is termed *Number Numbring*. But in all Creatures, otherwise, *Number*, is termed *Nūber Numbred*. And in our Soule, *Nūber* beareth such a swaye, and hath such an affinitie therewith, that some of the old *Philosophers* taught, *Māns Soule*, to be a *Number mouyng it selfe*. And in dede, in vs, though it be a very Accident: yet such an Accident it is, that before all Creatures it had perfect beyng, in the Creator, Sempiternally. *Number Numbring* therfore, is the discretion discerning, and distincting of thynges. But in God the Creator, This discretion, in the beginnyng, produced orderly and distinctly all thynges. For his *Numbring*, then, was his Creatyng of all thynges. And his Continuall *Numbring*, of all thynges, is the Conseruation of them in being. And, where and when he will lacke an *Vnit*: there and then, that particular thyng shalbe *Discreated*. Here I stay. But our Seuerallyng, distinctyng, and *Numbring*, createth nothyng: but of Multitude considered, maketh certaine and distinct determination. And albeit these thynges be waighty and truthe of great importance, yet (by the infinite goodnes of the Almighty *Ternarie*,) Artificiall Methods and easy wayes are made, by which the zealous Philosopher, may wyn here this *Riuierish Ida*, this Mountayne of Contemplation: and more then Contemplation. And also, though *Number*, be a thyng so Immateriall, so diuine, and aternall: yet by degrees, by litle and litle, stretchyng forth, and applying some likenes of it, as first, to thynges Spirituall: and then, bryngyng it lower, to thynges sensibly perceiued: as of a momentanye founde iterated: then to the least thynges that may be seen, numerable: And at length, (most grossely,) to a multitude of any corporall thynges seen, or felt: and so, of these grosse and sensible thynges, we are trayned to learne a certaine Image or likenes of numbers: and to vse Arte in them to our pleasure and proffit. So grosse is our conuersation, and dull is our apprehension: while mortall Sense, in vs, ruleth the common wealth of our litle world. Hereby we say, Three Lyons, are three: or a *Ternarie*. Three Egles, are three, or a *Ternarie*. Which **Ternaries*, are eche, the *Vnion*, knot, and *Vniformitie*, of three discrete and distinct *Vnits*. That is, we may in eche *Ternarie*, thrise, seuerally pointe, and shew a part, *One, One, and One*. Where, in *Numbring*, we say *One, two, Three,*

Iohn Dee his Mathematicall Præface.

Three. But how farre, these visible Ones, do differre from our Indiuisible Vnits (in pure *Arithmetike*, principally considered) no man is ignorant. Yet from these grosse and materiall thynges, may we be led vpwrd, by degrees, so, informing our rude Imagination, toward the cōceiuyng of *Numbers*, absolutely (: Not supposing, nor admixtyng any thyng created, Corporall or Spirituall, to support, conteyne, or represent those *Numbers* imagined:) that at length, we may be hable, to finde the number of our owne name, gloriously exemplified and registred in the booke of the *Trinitie* most blessed and aternall.

But farder vnderstand, that vulgar Practisers, haue *Numbers*, otherwise, in sundry Considerations: and extend their name farder, then to *Numbers*, whose least part is an *Vnit*. For the common Logist, Reckenmaster, or Arithmeticien; in hys vsing of *Numbers*: of an *Vnit*, imagineth lesse partes: and calleth them *Fractions*. As of an *Vnit*, he maketh an halfe, and thus noteth it, $\frac{1}{2}$. and so of other, (infinitely diuerse) partes of an *Vnit*. Yea and farder, hath, *Fractions of Fractions*. &c. And, forasmuch, as, *Addition*, *Substraction*, *Multiplication*, *Diuision* and *Extraction of Rotes*, are the chief, and sufficient partes of *Arithmetike*: which is, the Science that demonstrateth the properties, of *Numbers*, and all operatiōs, in numbers to be performed: How often, therefore, these fīue sundry sortes of Operations, do, for the most part, of their execution, differre from the fīue operations of like generall property and name, in our Whole numbers practisable, So often, (for a more distinct doctrine) we, vulgarly account and name it, an other kynde of *Arithmetike*. And by this reason: the Consideration, doctrine, and working, in whole numbers onely: where, of an *Vnit*, is no lesse part to be allowed: is named (as it were) an *Arithmetike* by it selfe. And so of the *Arithmetike of Fractions*. In lyke sorte, the necessary, wonderfull and Secret doctrine of Proportion, and proportionalitye hath purchased vnto it selfe a peculier maner of handlyng and working: and so may seme an other forme of *Arithmetike*. Moreouer, the *Astronomers*, for spede and more commodious calculation, haue deuised a peculier maner of orderyng nūbers, about theyr circular motions, by Sexagenes, and Sexagesimes. By Signes, Degrees and Minutes &c. which commonly is called the *Arithmetike of Astronomical* or *Phisicall Fractions*. That, haue I briefly noted, by the name of *Arithmetike Circular*. Bycause it is also vsed in circles, not *Astronomical*. &c. Practise hath led *Numbers* farder, and hath framed them, to take vpon them, the shew of *Magnitudes* propertie: Which is *Incommensurabilitie* and *Irrationalitie*. (For in pure *Arithmetike*, an *Vnit*, is the common Measure of all *Numbers*.) And here, Nūbers are become, as Lynes, Playnes and Solides: some tymes *Rationall*, some tymes *Irrationall*. And haue propre and peculier characters, (as $\sqrt{8}$. $\sqrt{11}$. and so of other. Which is to signifie *Rote Square*, *Rote Cubik*: and so forth.) & propre and peculier fashions in the fīue principall partes: Wherefore the practiser, este meth this, a diuerse *Arithmetike* from the other. Practise bryngeth in, here, diuerse compoundyng of *Numbers*: as some tyme, two, three, foure (or more) *Radicall* nūbers, diuently knit, by signes, of More & Lesse: as thus $\sqrt{8} \cdot 12 + \sqrt{11} \cdot 15$. Or thus $\sqrt{8} \cdot 8 \cdot 19 + \sqrt{11} \cdot 12 - \sqrt{8} \cdot 2$. &c. And some tyme with whole numbers, or fractions of whole Number, amōg them: as $20 + \sqrt{8} \cdot 24 \cdot \sqrt{11} \cdot 16 + 33 - \sqrt{8} \cdot 10 \cdot \sqrt{8} \cdot 44 + 12 - + \sqrt{11} \cdot 9$. And so, infinitely, may hap the varietie. After this: Both the one and the other hath fractions incident: and so is this *Arithmetike* greatly enlarged, by diuerse exhibiting and vse of Compositions and mixtynges. Consider how, I (beyng desirous to deliuer the student from error and Cauillation) do giue to this Practise, the name of the *Arithmetike of Radicall numbers*: Not, of *Irrationall* or *Surd Numbers*: which other while, are *Rationall*: though they haue the Signe of a Rote before them,

Iohn Dee his Mathematicall Præface.

them, which, *Arithmetike* of whole Numbers most vsuall, would say they had no such Roote: and so account them *Surd Numbers*: which, generally spokē, is vntrue: as *Euclides* tenth booke may teach you. Therefore to call them, generally, *Radical Numbers*, (by reason of the signe $\sqrt{}$. prefixed,) is a sure way: and a sufficient generall distinction from all other ordryng and vsing of Numbers: And yet (beside all this) Consider: the infinite desire of knowledge, and incredible power of mans Search and Capacitye: how, they, ioyntly haue waded farder (by mixtyng of speculation and practise) and haue found out, and atteyned to the very chief perfection (almost) of *Numbers* Practicall vse. Which thing, is well to be perceiued in that great Arithmeticall Arte of *Æquation*: commonly called the *Rule of Coss*. or *Algebra*. The Latines termed it, *Regulam Rei & Censuræ*, that is, the *Rule of the thyng and his value*. With an apt name: comprehending the first and last pointes of the worke. And the vulgar names, both in Italian, Frenche and Spanish, depend (in namyng it,) vpon the signification of the Latin word, *Res*: *A thing*: vnleaste they vse the name of *Algebra*. And therin (commonly) is a dubble error. The one, of them, which thinke it to be of *Geber* his inuentyng: the other of such as call it *Algebra*. For, first, though *Geber* for his great skill in Numbers, Geometry, Astronomy, and other maruailous Artes, mought haue seemed hable to haue first deuised the sayd Rule: and also the name carryeth with it a very nere likenes of *Geber* his name: yet true it is, that a *Greke* Philosopher and Mathematicien, named *Diophantus*, before *Geber* his tyme, wrote 13. bookes therof (of which, six are yet extant: and I had them to * vse, of the famous Mathematicien, and my great frende, *Petrus Montau-*

* Anno. 1550. *reus*;) And secondly, the very name, is *Algiebar*, and not *Algebra*: as by the Arabien *Auicen*, may be proued: who hath these precise wordes in Latine, by *Andreas Alpagus* (most perfect in the Arabik tung) so translated. *Scientia faciendi Algiebar & Almachabel. i. Scientia inueniendi numerum ignotum, per additionem Numeri, & diuisionem & æquationem*. Which is to say: *The Science of workyng Algiebar and Almachabel*, that is, the *Science of findyng an vknownen number, by Addyng of a Number, & Diuision & æquation*. Here haue you the name: and also the principall partes of the Rule, touched. To name it, *The rule, or Art of Æquation*, doth signifie the middle part and the State of the Rule. This Rule, hath his peculier Characters: and the principal partes of *Arithmetike*, to it appertayning, do differre from the other *Arithmeticall operations*. This *Arithmetike*, hath *Nubers* Simple, Cōpound, Mixt: and Fractions, accordingly. This Rule, and *Arithmetike* of *Algiebar*, is so profound, so generall and so (in maner) conteyneth the whole power of Numbers Application practicall: that mans witt, can deale with nothyng, more proffitable about numbers: nor match, with a thyng, more mete for the diuine force of the Soule, (in humane Studies, affaires, or exercises) to be tryed in. Perchaunce you looked for, (long ere now,) to haue had some particular profe, or euident testimony of the vse, proffit and Commodity of *Arithmetike* vulgar, in the Common lyfe and trade of men. Therto, then, I will now frame my selfe: But herein great care I haue, least length of sundry profes, might make you deme, that either I did misdoute your zelous mynde to vertues schole: or els mistrust your hable witts, by some, to gesse much more. A profe then, foure, fiue, or six, such, will I bryng, as any reasonable man, therewith may be perswaded, to loue & honor, yea learne and exercise the excellent Science of *Arithmetike*.

And first: who, nerer at hand, can be a better witnessse of the frute receiued by *Arithmetike*, then all kynde of Marchants? Though not all, alike, either nede it, or vse it. How could they forbear the vse and helpe of the Rule, called the Golden Rule:

John Dee his Mathematicall Præface.

Rule: Simple and Compound: both forward and backward? How might they misse *Arithmeticall* helpe in the Rules of Felowshyp: either without tyme, or with tyme: and betwene the Marchant & his Factor? The Rules of Bartering in wares onely: or part in wares, and part in money, would they gladly want? Our Marchant venturers, and Trauaylers ouer Sea, how could they order their doynge iustly and without losse, vnleaste certaine and generall Rules for Exchaunge of money, and Rechaunge, were, for their vse, deuised? The Rule of Alligation, in how sundry cases, doth it conclude for them, such precise verities, as neither by naturall witt, nor other experience, they, were hable, els, to know? And (with the Marchant then to make an end) how ample & wonderfull is the Rule of False positions? especially as it is now, by two excellent Mathematiciens (of my familier acquaintance in their life time) enlarged? I meane *Gemma Frisius*, and *Simon Iacob*. Who can either in brief conclude, the generall and Capitall Rules? or who can I imagine the Myriades of sundry Cases, and particular examples, in Act and earnest, continually wrought, tried and concluded by the forenamed Rules, onely? How sundry other *Arithmeticall practises*, are commonly in Marchantes handes, and knowledge: They them selues, can, at large, testifie.

The Mintmaster, and Goldsmith, in their Mixture of Metals, either of diuerse kindes, or diuerse values: how are they, or may they, exactly be directed, and meruailously pleased, if *Arithmetike* be their guide? And the honorable Phisiciās, will gladly confesse them selues, much beholding to the Science of *Arithmetike*, and that sundry wayes: But chiefly in their Art of Graduation, and compounde Medicines. And though *Galenus*, *Auerrois*, *Arnoldus*, *Lullus*, and other haue published their positions, aswell in the quantities of the Degrees about Temperament, as in the Rules, concluding the new *Forme* resulting: yet a more precise, commodious, and easy *Method*, is extant: by a Countreyman of ours (about 200. *R. B.* yeares ago) inuented. And forasmuch as I am vncertaine, who hath the fame: or when that litle Latin treatise, (as the Author writ it,) shall come to be Printed: (Both to declare the desire I haue to pleasure my Countrey, wherein I may: and also, for very good profe of Numbers vse, in this most subtile and frutefull, Philosophicall Conclusion,) I intend in the meane while, most briefly, and with my farther helpe, to communicate the pith therof vnto you.

First describe a circle: whose diameter let be an inch. Diuide the Circumference into foure equall partes. Frō the Center, by those 4. sections, extend 4. right lines: eche of 4. inches and a halfe long: or of as many as you liste, about 4. without the circumference of the circle: So that they shall be of 4. inches long (at the least) without the Circle. Make good euident markes, at euery inches end. If you list, you may subdiuide the inches againe into 10. or 12. smaller partes, equall. At the endes of the lines, write the names of the 4. principall elementall Qualities. *Hotte* and *Colde*, one against the other. And likewise *Moyst* and *Dry*, one against the other. And in the Circle write *Temperate*. Which *Temperature* hath a good Latitude: as appeareth by the Complexion of man. And therefore we haue allowed vnto it, the foresayd Circle: and not a point Mathematicall or Physicall.

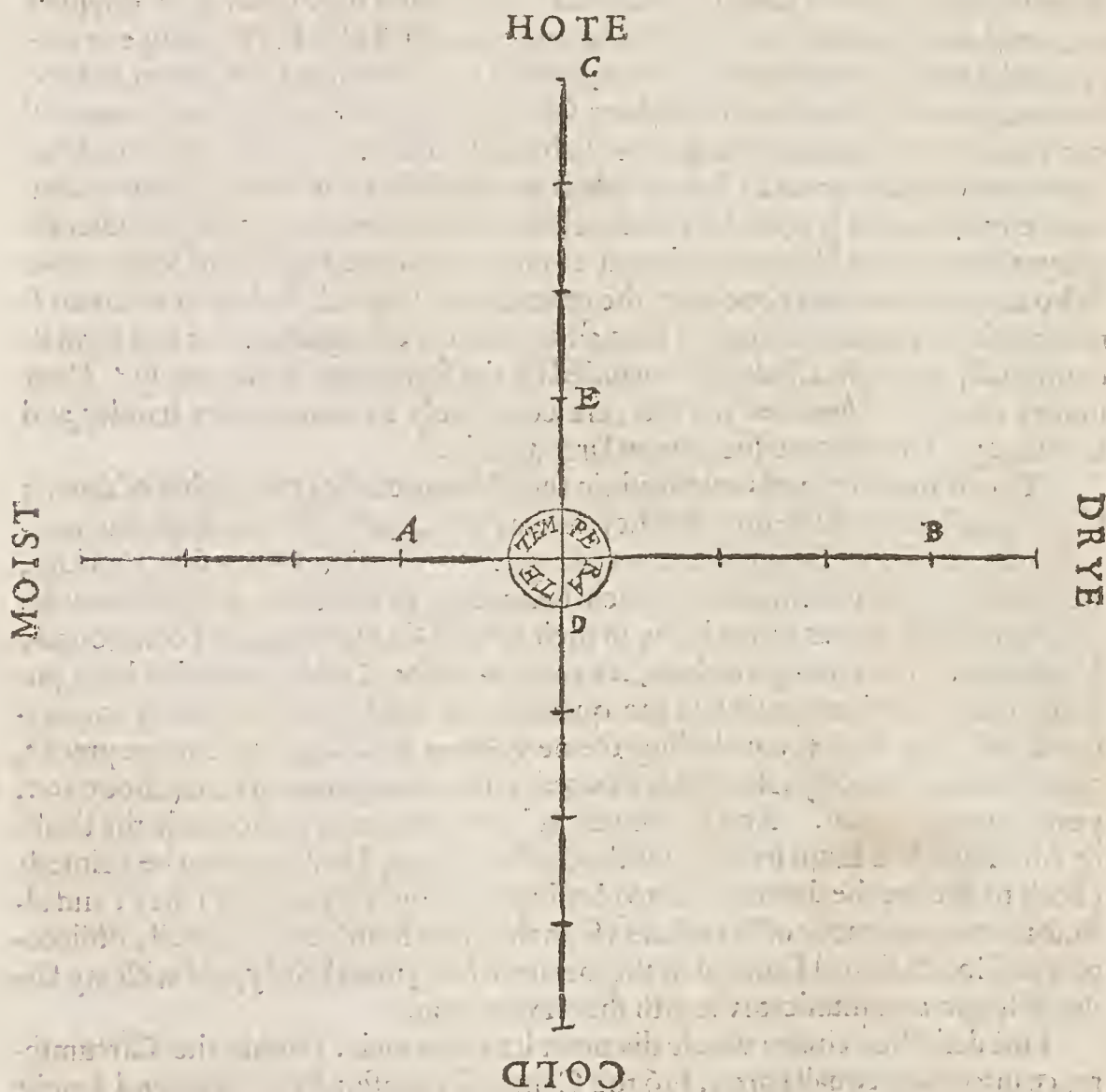
Now, when you haue two thinges Miscible, whose degrees are * truly known: Of necessitie, either they are of one Quantitie and waight, or of diuerse. If they be of one Quantitie and waight: whether their formes, be Contrary Qualities, or of one kinde (but of diuerse intentions and degrees) or a *Temperate*, and a Contrary, *The forme resulting of their Mixture, is in the Middle betwene the degrees of*

*Take some part of Lullus counsaile in his booke de Q. Essentia.

Iohn Dee his Mathematicall Præface.

the formes mixt. As for example, let *A*, be *Moist* in the first degree: and *B*, *Dry* in the third degree. Adde 1. and 3. that maketh 4: the halfe or middle of 4. is 2.

**Note.* This 2. is the middle, equally distant from *A* and *B* (for the **Temperament* is counted none. And for it, you must put a Ciphre, if at any time, it be in mixture).



Counting then from *B*, 2. degrees, toward *A*: you finde it to be *Dry* in the first degree: So is the *Forme resulting* of the Mixture of *A*, and *B*, in our example. I will geue you an other example. Suppose, you haue two thinges, as *C*, and *D*: and of *C*, the Heate to be in the 4. degree: and of *D*, the Colde, to be remisse, euen vnto the *Temperament*. Now, for *C*, you take 4: and for *D*, you take a Ciphre: which, added vnto 4, yeldeth onely 4. The middle, or halfe, whereof, is 2. Wherefore the *Forme resulting* of *C*, and *D*, is Hote in the second degree: for, 2. degrees, accounted from *C*, toward *D*, ende iuste in the 2. degree of heate. Of the third manner, I will geue also an example: which let be this: I haue a liquid Medicine whose

Note. Qualitie of heate is in the 4. degree exalted: as was *C*, in the example foregoing: and an other liquid Medicine I haue: whose Qualitie, is heate, in the first degree. Of eche of these, I mixt a like quantitie: Subtract here, the lesse frõ the more: and the residue diuide into two equall partes: whereof, the one part, either added to the lesse, or subtracted from the higher degree, doth produce the degree of the *Forme*

John Dee his Mathematicall Præface.

Forme resulting, by this mixture of C, and E. As, if from 4. ye abate 1. there resteth 3. the halfe of 3. is $1\frac{1}{2}$. Adde to 1. this $1\frac{1}{2}$. you haue $2\frac{1}{2}$. Or subtract from 4. this $1\frac{1}{2}$. you haue likewise $2\frac{1}{2}$ remayning. Which declareth, the *Forme resulting*, to be *Heate*, in the middle of the third degree.

But if the Quantities of two things Commixt, be diuerse, and the Intensions (of their Formes Miscible) be in diuerse degrees, and heighthes. (Whether those Formes be of one kinde, or of Contrary kindes, or of a Temperate and a Contrary, *What proportion is of the lesse quantitie to the greater, the same shall be of the difference, which is betwene the degree of the Forme resulting, and the degree of the greater quantitie of the thing miscible, to the difference, which is betwene the same degree of the Forme resulting, and the degree of the lesse quantitie.* As for example. Let two pound of Liquor be geuen, hote in the 4. degree: & one pound of Liquor be geuen, hote in the third degree. I would gladly know the Forme resulting, in the Mixture of these two Liquors. Set downe your nũbers in order, thus.

P. 2.	Hote. 4.
P. 1.	Hote. 3.

Now by the rule of Algiebar, haue I deuised a very easie, briefe, and generall maner of working in this case. Let vs first, suppose that *Middle Forme resulting*, to be $1ze$: as that Rule teacheth. And because (by our Rule, here geuen) as the waight of 1. is to 2: So is the difference betwene 4. (the degree of the greater quantitie) and $1ze$: to the difference betwene $1ze$ and 3. (the degree of the thing, in lesse quãtitie. And with all, $1ze$, being alwayes in a certaine middell, betwene the two heighthes or degrees). For the first difference, I set $4-1ze$: and for the second, I set $1ze-3$. And, now againe, I say, as 1. is to 2. so is $4-1ze$ to $1ze-3$. Wherefore, of these foure proportionall numbers, the first and the fourth Multiplied, one by the other, do make as much, as the second and the third Multiplied the one by the other. Let these Multiplications be made accordingly. And of the first and the fourth, we haue $1ze-3$. and of the second & the third, $8-2ze$. Wherefore, our Equation is betwene $1ze-3$: and $8-2ze$. Which may be reduced, according to the Arte of Algiebar: as, here, adding 3. to eche part, geueth the Equation, thus, $1ze=11-2ze$. And yet againe, contracting, or Reducing it: Adde to eche part, $2ze$: Then haue you $3ze$ æquall to 11: thus represented $3ze=11$. Wherefore, diuiding 11. by 3: the Quotient is $3\frac{2}{3}$: the *Valew* of our $1ze$, *Coß*, or *Thing*, first supposed. And that is the heighth, or Intension of the *Forme resulting*: which is, *Heate*, in two thirdes of the fourth degree: And here I set the shew of the worke in conclusion, thus. The proufe hereof is easie. by subtracting 3. from $3\frac{2}{3}$, resteth

$\frac{2}{3}$. Subtracte the same heighth of the *Forme resulting*, (which is $3\frac{2}{3}$) frõ 4: then resteth $\frac{10}{3}$.

P. 2.	Hote. 4.
P. 1.	Hote. 3.

You see, that

is double to $\frac{10}{3}$.

as 2. P. is double to 1. P. So should it be: by the rule here geuen. Note. As you added to eche part of the Equation, 3: so if ye first added to eche part $2ze$, it would stand, $3ze-3=8$. And now adding to eche part 3: you haue (as afore) $3ze=11$.

And though I, here, speake onely of two things Miscible: and most commonly, mo then three, foure, fīue or six, (&c.) are to be Mixed: (and in one Compound

Iohn Dee his Mathematicall Præface.

Note. to be reduced: & the Forme resultyng of the same, to serue the turne) yet these Rules are sufficient: duely repeated and iterated. In procedyng first, with any two: and then, with the Forme Resultyng, and an other: & so forth. For, the last worke, concludeth the Forme resultyng of them all. I nede nothing to speake, of the Mixture (here supposed) what it is. Common Philosophie hath defined it, saying, *Mixtio est miscibilium, alteratorum, per minima coniunctorum, Vnio.* Euery word in the definition, is of great importance. I nede not also spend any time, to shew, how, the other manner of distributing of degrees, doth agree to these Rules. Neither nede I of the farder vse belonging to the Crosse of Graduation (before described) in this place declare, vnto such as are capable of that, which I haue all ready sayd. Neither yet with examples specifie the Manifold varieties, by the foresayd two generall Rules, to be ordered. The witty and Studious, here, haue sufficient: And they which are not hable to atteine to this, without liuely teaching, and more in particular: would haue larger discoursing, then is mete in this place to be dealt withall: And other (perchaunce) with a proude snuffe will disdain this litle: and would be vnthankfull for much more. I, therefore conclude: and wish such as haue modest and earnest Philosophicall mindes, to laude God highly for this: and to Meruayle, that the profoundest and subtillest point, concerning *Mixture of Formes and Qualities Naturall*, is so Matcht and maryed with the most simple, easie, and short way of the noble Rule of *Algiebar*. Who can remaine, therefore vnpersuaded, to loue, allow, and honor the excellent Science of *Arithmetike*? For, here, you may perceiue that the litle finger of *Arithmetike*, is of more might and contriuing, then a hundred thousand mens wittes, of the middle sorte, are hable to perfourme, or truely to conclude, with out helpe thereof.

Taxilix. Now will we farder, by the wise and valiant Capitaine, be certified, what helpe he hath, by the Rules of *Arithmetike*: in one of the Artes to him appertaining: And of the Grekes named *Taxilix*. That is, the Skill of Ordryng Souldiers in Battell ray after the best maner to all purposes. This Art so much dependeth vppon Numbers vse, and the Mathematicals, that *Ælianus* (the best writer therof,) in his worke, to the *Emperour Hadrianus*, by his perfection, in the Mathematicals, (beyng greater, then other before him had,) thinketh his booke to passe all other the excellent workes, written of that Art, vnto his dayes. For, of it, had written *Æneas: Cyneas of Thessaly: Pyrrhus Epirota: and Alexander his sonne: Clearchus: Pausanias: Euangelus: Polybius*, familer frende to *Scipio: Eupolemus: Iphicrates, Possidonius*: and very many other worthy Capitaines, Philosophers and Princes of Immortall fame and memory: Whose fayrest floure of their garland (in this feat) was *Arithmetike*: and a litle perceiuerance, in *Geometricall Figures*. But in many other cases doth *Arithmetike* stand the Capitaine in great stede. As in proportionyng of vittayles, for the Army, either remaining at a stay: or suddenly to be encreased with a certaine number of Souldiers: and for a certain tyme. Or by good Art to diminish his company, to make the victuals, longer to serue the remanent, & for a certaine determined tyme: if nede so require. And so in sundry his other accountes, Reckenynges, Measurynges, and proportionynges, the wise, expert, and Circumspect Capitaine will affirme the Science of *Arithmetike*, to be one of his chief Counsaylors, directers and aiders. Which thing (by good meanes) was euident to the Noble, the Couragious, the loyall, and Curteous *Iohn*, late Earle of Warwicke. Who was a yong Gentleman, throughly knowne to very few. Albeit his lusty valiantnes, force, and Skill in Chiualrous feates and exercises: his humblenes, and frendelynes to all men, were thinges, openly, of the world perceiued. But what rotes (otherwise,) vertue had fastened in his brest, what Rules of godly and honorable life

John Dee his Mathematicall Præface.

life he had framed to him selfe: what vices, (in some then liuing) notable, he tooke great care to eschew: what manly vertues, in other noble men, (flourishing before his eyes,) he Sythingly aspired after: what prowesses he purposed and ment to achieve: with what feats and Artes, he began to furnish and fraught him selfe, for the better seruice of his Kyng and Countrey, both in peace & warre. These (I say) his Heroicall Meditations, forecastinges and determinations, no twayne, (I thinke) beside my selfe, can so perfectly, and truly report. And therefore, in Conscience, I count it my part, for the honor, preferment, & procuring of vertue (thus, briefly) to haue put his Name, in the Register of *Fame Immortall*.

To our purpose. This *John*, by one of his actes (besides many other: both in England and Fraunce, by me, in him noted.) did disclose his hartie loue to vertuous Sciences: and his noble intent, to excell in Martiall prowesse: When he, with humble request, and instant Solliciting: got the best Rules (either in time past by Greke or Romaine, or in our time vsed: and new Stratagemes therein deuised) for ordning of all Companies, summes and Numbers of mē, (Many, or few) with one kinde of weapon, or mo, appointed: with Artillery, or without: on horsebacke, or on fote: to giue, or take onset: to seem many, being few: to seem few, being many. To marche in battaile or Iornay: with many such feates, to Foughten field, Skarmoush, or Ambushe appartaining: And of all these, liuely designementes (most curiously) to be in velame parchment described: with Notes & peculier marks, as the Arte requireth: and all these Rules, and descriptions Arithmeticall, inclosed in a riche Case of Gold, he vsed to weare about his necke: as his Iuell most precious, and Counsaylour most trusty. Thus, *Arithmetike*, of him, was shryned in gold: Of *Numbers* frute, he had good hope. Now, Numbers therefore innumerable, in *Numbers* prayse, his shryne shall finde.

This noble Earle, dyed Anno. 1554. skarse of 24. yeares of age: hauing no issue by his wife: Daughter to the Duke of Somerset.

What nede I, (for farder profe to you) of the Scholemasters of Iustice, to require testimony: how nedefull, how frutefull, how skillfull a thing *Arithmetike* is? I meane, the Lawyers of all sortes. Vndoubtedly, the Ciuilians, can meruaylously declare: how, neither the Auncient Romaine lawes, without good knowledge of *Numbers art*, can be perceiued: Nor (Iustice in infinite Cases) without due proportion, (narrowly considered,) is hable to be executed. How Iustly, & with great knowledge of Arte, did *Papinianus* institute a law of partition, and allowance, betwene man and wife after a diuorce? But how *Accursius*, *Baldus*, *Bartolus*, *Iason*, *Alexander*, and finally *Alciatus*, (being otherwise, notably well learned) do iumble, gesse, and erre, from the æquity, art and Intent of the lawmaker: *Arithmetike* can detect, and conuince: and clerely, make the truth to shine. Good *Bartolus*, tyred in the examining & proportioning of the matter: and with *Accursius* Glosses, much cumbered: burst out, and sayd: *Nulla est in toto libro; hac glossa difficilior: Cuius computationem nec Scholastici nec Doctores intelligunt. &c.* That is: In the whole booke, there is no Glosses harder then this: Whose accoumpt or rekenyng, neither the Scholars, nor the Doctours vnderstand. &c. What can they say of *Iulianus* law, *Sita Scriptum. &c.* Of the Testators will iustly performing, betwene the wife, Sonne and daughter? How can they perceiue the æquitie of *Aphricanus*, *Arithmeticall* Reckening, where he treateth of *Lex Falcidia*? How can they deliuer him, from his Reprouers: and their maintainers: as *Ioannes*, *Accursius*, *Hypolitus* and *Alciatus*? How Iustly and artificially, was *Africanus* reckening made: Proportionating to the Sommes bequeathed, the Contributions of eche part? Namely, for the hundred presently receiued, $17\frac{1}{7}$. And for the hundred, receiued after ten monethes, $12\frac{6}{7}$: which make the 30: which were to be cōtributed by the legataries to the heire.

Iohn Dee his Mathematicall Præface.

For, what proportion, 100 hath to 75: the same hath 17 $\frac{1}{2}$ to 12 $\frac{1}{2}$: Which is Sequitertia: that is, as 4, to 3. which make 7. Wonderfull many places, in the Ciuile law, require an expert *Arithmeticien*, to vnderstand the deepe Iudgemēt, & Iust determination of the Auncient Romaine Lawmakers. But much more expert ought he to be, who should be hable, to decide with æquitie, the infinite varietie of Cases, which do, or may happen, vnder euery one of those lawes and ordinances Ciuile. Hereby, easely, ye may now coniecture: that in the Canon law: and in the lawes of the Realme (which with vs, beare the chief Authoritie), Iustice and equity might be greatly preferred, and skilfully executed, through due skill of *Arithmetike*, and proportions appertainyng. The worthy Philosophers, and prudent lawmakers (who haue written many bookes *De Republica*: How the best state of Common wealthes might be procured and mainteined,) haue very well determined of Iustice: (which, not onely, is the Base and foundation of Common weales: but also the totall perfection of all our workes, words, and thoughtes:) defining it, to be that vertue, by which, to euery one, is rendred, that to him appertaineth. God challengeth this at our handes, to be honored as God: to be loued, as a father: to be feared as a Lord & master. Our neighbours proportiō, is also prescribed of the Almighty lawmaker: which is, to do to other, euen as we would be done vnto. These proportions, are in Iustice necessary: in duety, commendable: and of Common wealthes, the life, strength, stay and flourishing. *Aristotle* in his *Ethikes* (to fatch the sede of Iustice, and light of direction, to vse and execute the same) was fayne to fly to the perfection, and power of Numbers: for proportions *Arithmeticall* and *Geometricall*. *Plato* in his booke called *Epinomis* (which booke, is the Threasury of all his doctrine) where, his purpose is, to seke a Science, which, when a man had it, perfectly: he might seme, and so be, in dede, *Wise*. He, briefly, of other Sciences discoursing, findeth them, not hable to bring it to passe: But of the Science of Numbers, he sayth. *Illā, quæ numerum mortalium generi dedit, id profecto efficiet. Deum autem aliquem, magis quam fortunam, ad salutem nostram, hoc munus nobis arbitror contulisse. &c. Nam ipsum bonorum omnium Authorem, cur non maximi boni, Prudentie dico, causam arbitramur?* That Science, verely, which hath taught man-kynde number, shall be able to bryng it to passe. And, I thinke, a certaine God, rather then fortune, to haue giuen vs this gift, for our blisse. For, why should we not Iudge him, who is the Author of all good things, to be also the cause of the greatest good thyng, namely, *Wisedome*? There, at length, he proueth *Wisedome* to be atteyned, by good Skill of *Numbers*. With which great Testimony, and the manifold profes, and reasons, before expressed, you may be sufficiently and fully persuaded: of the perfect Science of *Arithmetike*, to make this account: That of all Sciences, next to *Theologie*, it is most diuine, most pure, most ample and generall, most profounde, most subtile, most commodious and most necessary. Whose next Sister, is the Absolute Science of *Magnitudes*: of which (by the Direction and aide of him, whose *Magnitude* is Infinite, and of vs Incomprehensible) I now entend, so to write, that both with the *Multitude*, and also with the *Magnitude* of Meruaylous and frutefull verities, you (my frendes and Countreymen) may be stird vp, and awaked, to behold what certaine Artes and Sciences, (to our vn-speakable behofe) our heaucnly father, hath for vs prepared, and reuealed, by sundry *Philosophers* and *Mathematiciens*.

Both, *Number* and *Magnitude*, haue a certaine Originall sede, (as it were) of an incredible property: and of man, neuer hable, Fully, to be declared. Of *Number*, an Vnit, and of *Magnitude*, a Poynte, doo seeme to be much like Originall

John Dee his Mathematicall Præface.

nall causes: But the diuerfitie neuerthelesse, is great. We defined an *Vnit*, to be a thing Mathematicall Indiuifible: A Point, likewise, we sayd to be a Mathematicall thing Indiuifible. And farder, that a Point may haue a certaine determined Situation: that is, that we may assigne, and prescribe a Point, to be here, there, yonder. &c. Herein, (behold) our *Vnit* is free, and can abyde no bondage, or to be tyed to any place, or seat: diuifible or indiuifible. Agayne, by reason, a Point may haue a Situation limited to him: a certaine motion, therfore (to a place, and from a place) is to a Point: incident and appertainyng. But an *Vnit*, can not be imagined to haue any motion. A Point, by his motion, produceth; Mathematically, a line: (as we sayd before) which is the first kinde of Magnitudes, and most simple: An *Vnit*, can not produce any number. A Line, though it be produced of a Point moued, yet, it doth not consist of pointes: Number, though it be not produced of an *Vnit*, yet doth it Consist of vnits, as a materiall cause. But formally, Number, is the Vnion, and Vnitie of Vnits. Which vnytyng and knitting, is the workmanship of our minde: which, of distinct and discrete Vnits, maketh a Number: by vniformitie, resulting of a certaine multitude of Vnits. And so, euery number, may haue his least part, giuen: namely, an *Vnit*: But not of a Magnitude, (no, not of a Lyne,) the least part can be giue: bycause, infinitely, diuision thereof, may be conceiued. All Magnitude, is either a Line, a Plaine, or a Solid. Which Line, Plaine, or Solid, of no Sense, can be perceiued, nor exactly by hād (any way) represented: nor of Nature produced: But, as (by degrees) Number did come to our perceiuerance: So, by visible formes, we are holpen to imagine, what our Line Mathematicall, is. What our Point, is. So precise, are our Magnitudes, that one Line is no broader then an other: for they haue no bredth: Nor our Plaines haue any thicknes. Nor yet our Bodies, any weight: be they neuer so large of dimensiō. Our Bodyes, we can haue Smaller, then either Arte or Nature can produce any: and Greater also, then all the world can comprehend. Our least Magnitudes, can be diuided into so many partes, as the greatest. As, a Line of an inch long, (with vs) may be diuided into as many partes, as may the diameter of the whole world, from East to West: or any way extended: What priuiledges, aboue all manual Arte, and Natures might, haue our two Sciences Mathematicall, to exhibite, and to deale with thinges of such power, liberty, simplicity, puritie, and perfection: And in them, so certainly, so orderly, so precisely to procede: as, excellent is that workemā Mechanicall Iudged, who nereft can approche to the representing of workes, Mathematically demonstrated? And our two Sciences, remaining pure, and absolute, in their proper termes, and in their owne Matter: to haue, and allowe, onely such Demonstrations, as are plaine, certaine, vniuerfall, and of an æternall veritye: This Science of *Magnitude*, his properties, conditions, and appertenances: commonly, now is, and from the beginnyng, hath of all Philosophers, ben called *Geometrie*. But, veryly, with a name to base and scant, for a Science of such dignitie and amplexes. And, perchaunce, that name, by cōmon and secret consent, of all wisemen, hitherto hath ben suffred to remayne: that it might carry with it a perpetuall memorye, of the first and notablest benefite, by that Science, to common people shewed: Which was, when Boundes and meres of land and ground were lost, and confounded (as in *Egypt*, yearly, with the ouerflowyng of *Nilus*, the greatest and longest riuer in the world) or, that ground bequeathed, were to be assigned: or, ground sold, were to be layd out: or (when disorder preuailed) that Commōs were distributed into feueralties. For, where, vpon these & such like occasiōs, Some by ignorāce, some by negligēce, Some by fraude, and some by violence, did wrongfully limite, measure, encroach, or challenge (by

Numbers

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Geometrie

John Dee his Mathematicall Præface.

pretence of iust content, and measure) those landes and groundes : great losse, disquietnes, murder, and warre did (full oft) ensue: Till, by Gods mercy, and mans Industrie, The perfect Science of Lines, Plaines, and Solides (like a diuine Iusticier,) gaue vnto euery man, his owne. The people then, by this art pleased, and greatly relieued, in their landes iust measuring: & other Philosophers, writing Rules for land measuring: betwene them both, thus, confirmed the name of *Geometria*, that is, (according to the very etimologie of the word) Land measuring. Wherin, the people knew no farder, of Magnitudes vse, but in Plaines: and the Philosophers, of the, had no feet hearers, or Scholers. farder to disclose vnto, then of flat, plaine *Geometrie*. And though, these Philosophers, knew of farder vse, and best vnderstode the etymologie of the worde, yet this name *Geometria*, was of them applyed generally to all sortes of Magnitudes : vnleast, otherwhile, of *Plato*, and *Pythagoras* : When

*Plato. 7. de
Rep.

0. 7. de they would precisely declare their owne doctrine. Then, was * *Geometria*, with them, *Studium quod circa planum versatur*. But, well you may perceiue by *Euclides Elementes*, that more ample is our Science, then to measure Plaines: and nothyng lesse therin is taught (of purpose) then how to measure Land. An other name, therefore, must nedes be had, for our Mathematicall Science of Magnitudes: which regardeth neither clod, nor turff: neither hill, nor dale: neither earth nor heauen: but is absolute *Megethologia*: not creping on ground, and daffeling the eye, with pole
 33 perche, rod or lyne: but lifyng the hart aboute the heauens, by inuisible lines, and
 33 immortall beames: meteth with the reflexions, of the light incomprehensible: and
 33 so procureth Ioye, and perfection vnspeakable. Of which true vse of our *Megethica*, or *Megethologia*, *Diuine Plato* seemed to haue good taste, and iudgement: and (by the name of *Geometrie*) so noted it: and warned his Scholers therof: as, in hys seuenth *Dialog*, of the Common wealth, may euidently be sene. Where (in Latin) thus it is: right well translated: *Profecto, nobis hoc non negabunt, Quicumq; vel paululum quid Geometrie gustarunt, quin hac Scientia, contra, omnino se habeat, quam de ea loquuntur, qui in ipsa versantur*. In English, thus. *Verely* (sayth *Plato*) *whosoever haue, (but euen very litle) tasted of Geometrie, will not denye vnto vs, this: but that this Science, is of an other condicion, quite contrary to that, which they that are exercised in it, do speake of it.* And there it followeth, of our *Geometrie*, *Quod queritur cognoscendi illius gratia, quod semper est, non & eius quod oritur quandoq; & interit. Geometria, eius quod est semper, Cognitio est. Attollet igitur (o Generose vir) ad Veritatem, animum: atq; ita, ad Philosophandum preparabit cogitationem, vt ad supera conuertamus: quæ, nunc, contra quam decet, ad inferiora deijcimus. &c. Quam maximè igitur præcipiendum est, vt qui præclarissimam hanc habitât Civitatem, nullo modo, Geometriam spernant. Nam & quæ præter ipsius propositum, quodam modo esse videntur, haud exigua sunt. &c.* It must nedes be confessed (saith *Plato*) *That [Geometrie] is learned, for the knowyng of that, which is euer: and not of that, which, in tyme, both is bred and is brought to an ende. &c. Geometrie is the knowledge of that which is euerlastyng. It will lift vp therefore (O Gentle Syr) our mynde to the Veritie: and by that meanes, it will prepare the Thought, to the Philosophicall loue of wisdome: that we may turne or conuert, toward heauenly thinges [both mynde and thought] which now, otherwise then becommeth vs, we cast down on base or inferior things. &c. Chiefly, therefore, Commaundement must be giuen, that such as do inhabit this most honorable Citie, by no meanes, despise Geometrie. For euen those thinges [done by it] which, in manner, seame to be, beside the purpose of Geometrie: are of*

John Dee his Mathematicall Præface.

no small importance. &c. And besides the manifold vses of *Geometrie*, in matters appertaining to warre, he addeth more, of second vnpurposed frute, and commoditye, arising by *Geometrie*: saying: *Scimus quin etiam, ad Disciplinas omnes facilius perdiscendas, interesse omnino, attigerit ne Geometriam aliquis, an non.* &c. *Hanc ergo Doctrinam, secundo loco discendam Iuuenibus statuamus.* That is. But, also, we know, that for the more easy learning of all *Artes*, it importeth much, whether one haue any knowledge in *Geometrie*, or no. &c. Let vs therefore make an ordinance or decree, that this Science, of young men shall be learned in the second place. This was *Diuine Plato* his Iudgement, both of the purposed, chief, and perfect vse of *Geometrie*: and of his second, dependyng, deriuatiue commodities. And for vs, Christen men, a thousand thousand mo occasions are, to haue nede of the helpe of * *Megethologicall* Contemplations: wherby, to trayne our Imaginations and Myndes, by litle and litle, to forsake and abandon, the grosse and corruptible Obiectes, of our vtward senses: and to apprehend, by sure doctrine demonstratiue, Things Mathematicall. And by them, readily to be holpen and conducted to conceiue, discourse, and conclude of things Intellectual, Spirituall, æternall, and such as concerne our Blisse euerlasting: which, otherwise (without Speciall priuiledge of Illumination, or Reuelation frō heauen) No mortall mans wyt (naturally) is hable to reach vnto, or to Compasse. And, veryly, by my small Talent (from aboue) I am hable to proue and testifie, that the litterall Text, and order of our diuine Law, Oracles, and Mysteries, require more skill in Numbers, and Magnitudes: then (commonly) the expositors haue vttered: but rather onely (at the most) so warned: & shewed their own want therin. (To name any, is nedeles: and to note the places, is, here, no place: But if I be duely asked, my answer is ready.) And without the litterall, Grammaticall, Mathematicall or Naturall verities of such places, by good and certaine Arte, perceiued, no Spirituall sense (propre to those places, by Absolute *Theologie*) will thereon depend. No man, therefore, can doute, but toward the atteyning of knowledge incomparable, and Heauenly Wisedome: Mathematicall Speculations, both of Numbers and Magnitudes: are meanes, aydes, and guides: ready, certaine, and necessary. From henceforth, in this my Preface, will I frame my talke, to *Plato* his fugitiue Scholers: or, rather, to such, who well can, (and also wil,) vse their vtward senses, to the glory of God, the benefite of their Countrey, and their owne secret contentation, or honest preferment, on this earthly Scaffold. To them, I will orderly recite, describe & declare a great Number of *Artes*, from our two Mathematicall fountaines, deriued into the fieldes of *Nature*. Wherby, such Sedes, and Rotes, as lye depe hyd in the ground of *Nature*, are refreshed, quickened, and prouoked to grow, shote vp, floure, and giue frute, infinite, and incredible. And these *Artes*, shalbe such, as vpon Magnitudes properties do depende, more, then vpon Number. And by good reason we may call them *Artes*, and *Artes Mathematicall Deriuatiue*: for (at this tyme) I Define An Arte, to be a Methodicall cōplete Doctrīne, hauing abundance of sufficient, and peculier matter to deale with, by the allowance of the Metaphisicall Philosopher: the knowledge whereof, to humane state is necessarye. And that I account, An Art Mathematicall deriuatiue, which by Mathematicall demonstratiue Method, in Nūbers, or Magnitudes, ordieth and confirmeth his doctrine, as much & as perfectly, as the matter subiect will admit. And for that,

I. D.
* Herein, I
would gladly
shake of, the
earthly name,
of *Geometrie*.

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An Arte.

Art Mathematicall Deriuatiue.

John Dee his Mathematicall Præface.

A Mechanicien.

I intend to vse the name and propertie of a *Mechanicien*, otherwise, then (hitherto) it hath ben vsed, I thinke it good, (for distinction sake) to giue you also a brief description, what I meane therby. A *Mechanicien*, or a *Mechanicall* workman is he, whose skill is, without knowledge of *Mathematicall* demonstration, perfectly to worke and finishe any sensible worke, by the *Mathematicien* principall or deriuatiue, demonstrated or demonstrable. Full well I know, that he which inuenteth, or maketh these demonstrations, is generally called *A speculatiue Mechanicien*: which differreth nothing from a *Mechanicall Mathematicien*. So, in respect of diuerse actions, one man may haue the name of sundry artes: as, some tyme, of a *Logicien*, some tymes (in the same matter otherwise handled) of a *Rethoricien*. Of these trifles, I make, (as now, in respect of my Preface,) small account: to fyle the for the fine handlyng of subtile curious disputers. In other places, they may commaunde me, to giue good reason: and yet, here, I will not be vnreasonable.

Geometrie vulgar.

- First, then, from the puritie, absolutenes, and Immaterialitie of Principall *Geometrie*, is that kinde of *Geometrie* deriued, which vulgarly is counted *Geometrie*: and is the Arte of Measuring sensible magnitudes, their iust quāties and contentes. This, teacheth to measure, either at hand: and the practiser, to be by the thing Measured: and so, by due applying of Cumpase, Rule, Squire, Yarde, Ell, Perch, Pole, Line, Gaging rod, (or such like instrument) to the Length,
1. Plaine, or Solide measured, *to be certified; either of the length, perimetry, or distance lineall: and this is called, *Mecometrie*. Or *to be certified of the content of any plaine Superficies: whether it be in ground Surueyed, Borde, or Glasse measured, or such like thing: which measuring, is named *Embadometrie*. *Or els to vnderstand the Soliditie, and content of any bodily thing: as of Tymber and Stone, or the content of Pits, Pondes, Wells, Vessels, small & great, of all fashions. Where, of Wine, Oyle, Beere, or Ale vessells, &c, the Measuring, commonly, hath a peculiar name: and is called *Gaging*. And the generall name of these Solide measures,
 2. is *Stereometrie*. Or els, this *vulgar Geometrie*, hath consideration to teach the practiser, how to measure things, with good distance betwene him and the thing measured: and to vnderstand thereby, either *how Farre, a thing seene (on land or water) is from the measurer: and this may be called *Apomecometrie*: Or, how High or depe, aboue or vnder the leuel of the measurers standing, any thing is, which is sene
 3. on land or water, called *Hypsometrie*. *Or, it informeth the measurer, how Broad any thing is, which is in the measurers vew: so it be on Land or Water, situated: and may be called *Platometrie*. Though I vse here to condition, the thing measured, to be on Land, or Water Situated: yet, know for certaine, that the sundry heigthe of Cloudes, blasing Starres, and of the Mone, may (by these meanes) haue their distances from the earth: and, of the blasing Starres and Mone, the Soliditie (aswell as distances) to be measured: But because, neither these things are vulgarly taught: nor of a common practiser so ready to be executed: I, rather, let such measures be reckened incident to some of our other Artes, dealing with thinges on high, more purposely, then this vulgar Land measuring *Geometrie* doth: as in *Perspectiue* and *Astronomie*, &c.

Note.

OF these feates (Farther applied) is Sprong the feate of *Geodesie*, or Land Measuring: more cunningly to measure & Suruey Land, Woods, and Waters, a farre of. More cunningly, I say: But God knoweth (hitherto) in these Realmes of England and Ireland (whether through ignorance or fraude, I can not tell; in euery particular) how great wrong and iniurie hath (in my time) bene committed

Note.

by

John Dee his Mathematicall Præface.

by vntrue measuring and surueying of Land or Woods, any way . And, this I am sure: that the Value of the difference, betwene the truth and such Surueyes, would haue bene hable to haue found (for euer) in eche of our two Vniuersities, an excellent Mathematicall Reader: to eche, allowing (yearly) a hundred Markes of lawfull money of this realme: which, in dede, would seme requisit, here, to be had (though by other wayes prouided for) as well, as, the famous Vniuersitie of Paris, hath two Mathematicall Readers : and eche, two hundreth French Crownes yearly, of the French Kinges magnificent liberalitie onely . Now, againe, to our purpose returning : Moreouer, of the former knowledge Geometricall, are growen the Skills of *Geographie* , *Chorographie* , *Hydrographie* , and *Stratarithmetrie* .

Geographie teacheth wayes, by which, in sundry formes, (as *Spharique*, *Plaine* , or other), the Situation of Cities, Townes, Villages, Fortes, Castells, Mountaines, Woods, Hauens, Riuers, Crekes, & such other things, vpon the outface of the earthly Globe (either in the whole, or in some principall member and portion thereof contained) may be described and designed, in commensurations Analogicall to Nature and veritie: and most aptly to our view, may be represented. Of this Arte how great pleasure, and how manifold commodities do come vnto vs, daily and houely: of most men, is perceaued . While, some, to beautifie their Halls, Parlers, Chambers, Galleries, Studies, or Libraries with: other some, for thinges past, as battels fought, earthquakes, heauenly fyringes, & such occurrentes, in histories mentioned: therby liuely, as it were, to vewe the place, the region adioyning, the distance from vs : and such other circumstances . Some other, presently to vewe the large dominion of the Turke : the wide Empire of the Moschouite: and the litle morsell of ground, where Christendome (by profession) is certainly knowen. Litle, I say, in respecte of the rest, &c . Some, either for their owne iorneyes directing into farre landes: or to vnderstand of other mens trauailes . To conclude, some, for one purpose : and some, for an other, liketh, loueth, getteth, and vseth, Mappes, Chartes, & Geographical Globes . Of whose vse, to speake sufficiently, would require a booke peculiar.

Chorographie seemeth to be an vnderling, and a twig, of *Geographie*: and yet neuerthelesse, is in practise manifolde, and in vse very ample . This teacheth Analogically to describe a small portion or circuite of ground, with the contents: not regarding what commensuration it hath to the whole, or any parcell, without it, contained . But in the territory or parcell of ground which it taketh in hand to make description of, it leaueth out (or vndescribed) no notable, or odde thing, aboue the ground visible . Yea and sometimes, of thinges vnder ground, geueth some peculiar marke : or warning : as of Mettall mines, Cole pittes, Stone quarries, &c . Thus, a Dukedome, a Shiere, a Lordship, or lesse, may be described distinctly . But marueilous pleasant, and profitable it is , in the exhibiting to our eye, and commensuration, the plat of a Citie, Towne, Forte, or Pallace , in true Symmetry : not approaching to any of them : and out of Gunne shot, &c . Hereby, the *Architect* may furnishe him selfe, with store of what patterns he liketh : to his great instruction: euen in those thinges which outwardly are proportioned: either simply in them selues : or respectiue, to Hilles, Riuers, Hauens, and Woods adioyning . Some also, terme this particular description of places , *Topographie*.

Hydrographie, deliuereth to our knowledge, on Globe or in Plaine, the perfect Analogicall description of the Ocean Sea coastes, through the whole world : or in the chiefe and principall partes thereof : with the Iles and chiefe

Iohn Dee his Mathematicall Præface.

particular places of daungers, conteyned within the boundes, and Sea coastes described: as, of Quicksand, Bankes, Pittes, Rockes, Races, Countertides, Whorle-pooles, &c. This, dealeth with the Element of the water chiefly: as *Geographie* did principally take the Element of the Earthes description (with his appertinances) to taske. And besides thys, *Hydrographie*, requireth a particular Register of certaine Landmarkes (where markes may be had) from the sea, well hable to be skried, in what point of the Seacumpase they appeare, and what apparent forme, Situation, and bignes they haue, in respecte of any daungorous place in the sea, or nere vnto it, assigned: And in all Coastes, what Mone, maketh full Sea: and what way, the Tides and Ebbes, come and go, the *Hydrographer* ought to recorde. The Soundinges likewise: and the Chanelles wayes: their number, and depthes ordinarily, at ebbe and flud, ought the *Hydrographer*, by obseruation and diligence of *Measuring*, to haue certainly knowen. And many other pointes, are belonging to perfecte *Hydrographie*, and for to make a *Rutter*, by: of which, I nede not here speake: as of the describing, in any place, vpon Globe or Plaine, the 32. pointes of the Compase, truely: (wherof, scarcely foure, in England, haue right knowledge: bycause, the lines therof, are no straight lines, nor Circles.) Of making due projection of a Sphere in plaine. Of the Variacion of the Compas, from true Northe: And such like matters (of great importance, all) I leaue to speake of, in this place: bycause, I may seame (al ready) to haue enlarged the boundes, and duety of an *Hydrographer*, much more, then any man (to this day) hath noted, or prescribed. Yet am I well hable to proue, all these thinges, to appertaine, and also to be proper to the *Hydrographer*. The chief vse and ende of this Art, is the Art of Navigation: but it hath other diuerse vses: euen by them to be enioyed, that neuer lacke sight of land.

Stratarithmetrie, is the Skill, (appertainyng to the warre,) by which a man can set in figure, analogicall to any *Geometricall* figure appointed; any certaine number or summe of men: of such a figure capable: (by reason of the vsuall spaces betwene Souldiers allowed: and for that, of men, can be made no Fractions. Yet, neuertheles, he can order the giuen summe of men, for the greatest such figure, that of them, can be ordred) and certifie, of the ouerplus: (if any be) and of the next certaine summe, which, with the ouerplus, will admit a figure exactly proportionall to the figure assigned. By which Skill, also, of any army or company of men: (the figure & sides of whose orderly standing, or array, is knowen) he is able to expresse the iust number of men, within that figure contained: or (orderly) able to be contained. * And this figure, and sides therof, he is hable to know: either beyng by,

*Note: and at hand: or a farre of. Thus farre, stretcheth the description and property of

The difference betwene Stratarithmetrie and Tacticie. „ *Stratarithmetrie*: sufficient for this tyme and place. It differreth from the Feate „ *Tactickall*, *De aciebus instruendis*. bycause, there, is necessary the wisedome and fore- „ fight, to what purpose he so ordreth the men: and Skillfull hability, also, for any „ occasion, or purpose, to deuise and vse the aptest and most necessary order, array „ and figure of his Company and Summe of men. By figure, I meane: as, either of a „ *Perfect Square*, *Triangle*, *Circle*, *Ouale*, *long square*, (of the Grekes it is called *Eteromekes*) *Rhombe*, *Rhomboïd*, *Lunular*, *Ryng*, *Serpentine*, and such other *Geometricall* „ figures: Which, in warres, haue ben, and are to be vsed: for commodiousnes, ne- „ cessity, and auantage &c. And no small skill ought he to haue, that should make „ true report, or nere the truth, of the numbers and Summes, of footemen or horse- „ men, in the Enemyes ording. A farre of, to make an estimate, betwene nere „ termes of More and Lesse, is not a thyng very rife, among those that gladly would

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John Dee his Mathematicall Præface.

do it. Great pollicy may be vsed of the Capitaines, (at tymes fete, and in places conuenient) as to vse Figures, which make greatest shew, of so many as he hath: and vsing the aduantage of the three kindes of vsuall spaces: (betwene footemen or horsemen) to take the largest: or when he would seme to haue few, (beyng many:) contrarywise, in Figure, and space. The Herald, Purseuant, Sergeant Royall, Capitaine, or who soeuer is carefull to come nere the truth herein, besides the Iudgement of his expert eye, his skill of Ordering *Tacticall*, the helpe of his Geometricall instrument: Ring, or Staffe Astronomicall: (commodiously framed for cariage and vse) He may wonderfully helpe him selfe, by perspective Glasses. In which, (I trust) our posterity will proue more skillfull and expert, and to greater purposes, then in these dayes, can (almost) be credited to be possible.

Thus haue I lightly passed ouer the Artificiall Feates, chiefly dependyng vpon vulgar *Geometrie*: & commonly and generally reckened vnder the name of *Geometrie*. But there are other (very many) *Methodicall Artes*, which, declynyng from the purity, simplicitie, and Immateriality, of our Principall Science of *Magnitudes*: do yet neuertheles vse the great ayde, direction, and Method of the sayd principall Science, and haue propre names, and distinct: both from the Science of *Geometrie*, (from which they are deriued) and one from the other. As *Perspective*, *Astronomie*, *Musike*, *Cosmographie*, *Astrologie*, *Statike*, *Anthropographie*, *Trochilike*, *Helicosophie*, *Pneumatithmie*, *Menadrie*, *Hypogeiodie*, *Hydragogie*, *Horometrie*, *Zographie*, *Architecture*, *Nauigation*, *Thaumaturgike* and *Archemastrie*. I thinke it necessary, orderly, of these to giue some peculier descriptions: and withall, to touch some of their commodious vses, and so to make this Preface, to be a little swete, pleasant Nofegaye for you: to comfort your Spirites, beyng almost out of courage, and in despayre, (through brutish brute) Weenyng that *Geometrie*, had but serued for buildyng of an house, or a curious bridge, or the rouse of Westminster hall, or some witty pretty deuise, or engyn, appropriate to a Carpenter, or a Ioyner &c: That the thing is farre otherwise, then the world, (commonly) to this day, hath demed, by worde and worke, good profe wilbe made.

Among these Artes, by good reason, *Perspective* ought to be had, ere of *Astronomicall Apparences*, perfect knowledge can be atteyned. And bycause of the prerogatiue of *Light*, beyng the first of *Gods Creatures*: and the eye, the light of our body, and his Sense most mighty, and his organ most Artificiall and *Geometricall*: At *Perspective*, we will begyn therefore. *Perspective*, is an Art Mathematicall, which demonstrateth the maner, and properties, of all Radiations Direct, Broken, and Reflected. This Description, or Notation, is brief: but it reacheth so farre, as the world is wyde. It concerneth all Creatures, all Actions, and passions, by Emanation of beames perfourmed. Beames, or naturall lines, (here) I meane, not of light onely, or of colour (though they, to eye, giue shew, witnes, and profe, wherby to ground the Arte vpon) but also of other *Formes*, both *Substantiall*, and *Accidental*, the certaine and determined actiue Radiall emanations. By this Art (omitting to speake of the highest pointes) we may vse our eyes, and the light, with greater pleasure: and perfecter Iudgement: both of things, in light seene, & of other: which by like order of Lightes Radiations, worke and produce their effectes. We may be ashamed to be ignorant of the cause, why so sundry wayes our eye is deceiued, and abused: as, while the eye weeneth a round Globe or Sphere (beyng farre of) to be a flat and plaine Circle, and so likewise iud-

1.5.
Frende,
you will finde
it hard, to performe
my description of
this Feate. But by
Chorographie, you
may helpe your selfe
some what: where the
Figures knowne (in
Sides and Angles)
are not Regular:
And where, Resolu-
tion into Triangles
can serue. &c. And
yet you will finde it
strange to deale thus
generally with A-
rithmetical figures:
and, that for Bat-
taye ray. Their con-
tentes, differ so much
from like Geometri-
call Figures.

John Dee his Mathematicall Præface.

geth a plaine Square, to be roūd: supposeth walles parallels, to approche, a farre of: rose and floure parallels, the one to bend downward, the other to rise vpward, at a little distance from you. Againe, of thinges being in like swiftnes of mouing, to thinke the nerer, to moue faster: and the farder, much slower. Nay, of two thinges, wherof the one (incomparably) doth moue swifter then the other, to deme the slower to moue very swift, & the other to stand: what an error is this, of our eye? Of the Raynbow, both of his Colours, of the order of the colours, of the bignes of it, the place and heith of it, (&c) to know the causes demonstratiue; is it not pleasant, is it not necessary: of two or three Sonnes appearing: of Blasing Sterres: and such like thinges: by naturall causes, brought to passe, (and yet neuertheles, of farder matter, Significatiue) is it not commodious for man to know the very true cause, & occasion Naturall: Yea, rather, is it not, greatly, against the Souerainty of Mans nature, to be so ouerhot and abused, with thinges (at hand) before his eyes: as with a Pecockes tayle, and a Doues necke: or a whole ore, in water, holden, to seme broken. Thynges, farre of, to seme nere: and nere, to seme farre of. Small thinges, to seme great: and great, to seme small. One man, to seme an Army: Or a man to be curstly affrayed of his owne shadow. Yea, so much, to feare, that, if you, being (alone) nere a certaine glasse, and proffer, with dagger or sword, to foyne at the glasse, you shall suddenly be moued to giue backe (in maner) by reason of an Image, appearing in the ayre, betwene you & the glasse, with like hand, sword or dagger, & with like quicknes, foyning at your very eye, likewise as you do at the Glasse. Straunge, this is, to heare of: but more meruailous to behold, then these my wordes can signifie. And neuerthelesse by demonstration Opticall, the order and cause therof, is certified: euen so, as the effect is consequent. Yea, thus much more, dare I take vpon me, toward the satisfiing of the noble courage, that longeth ardently for the wisdom of Causes Naturall: as to let him vnderstand, that, in London, he may with his owne eyes, haue profe of that, which I haue sayd herein. A Gentleman, (which, for his good seruice, done to his Countrey, is famous and honorable: and for skill in the Mathematicall Sciences, and Languages, is the Od man of this land. &c.) euen he, is hable: and (I am sure) will, very willingly, let the Glasse, and profe be sene: and so I (here) request him: for the encrease of wisdom, in the honorable: and for the stopping of the mouthes malicious: and repressing the arrogancy of the ignorant. Ye may easily gesse, what I meane. This Art of *Perspectiue*, is of that excellency, and may be led, to the certifiing, and executing of such thinges, as no man would easily beleue: without Actuall profe perceiued. I speake nothing of *Naturall Philosophie*, which, without *Perspectiue*, can not be fully vnderstanded, nor perfectly attained vnto. Nor, of *Astronomie*: which, without *Perspectiue*, can not well be grounded: Nor *Astrologie*, naturally Verified, and auouched. That part hereof, which dealeth with Glasses (which name, Glasse, is a generall name, in this Arte, for any thing, from which, a Beame reboundeth) is called *Catoptrike*: and hath so many vses, both merueilous, and profitablen: that, both, it would hold me to long, to note therein the principall conclusions, all ready knowne: And also (perchaunce) some thinges, might lacke due credite with you: And I, therby, to leese my labor: and you, to slip into light Iudgement*, Before you haue learned sufficiently the powre of Nature and Arte.

Now, to procede: *Astronomie*, is an Arte Mathematicall, which demonstrateth the distance, magnitudes, and all naturall motions, apparences, and passions propre to the Planets and fixed Sterres: for

any

A merueilous
Glasse. E

S.W.P.

E

John Dee his Mathematicall Præface.

any time past, present and to come: in respect of a certaine Horizon, or without respect of any Horizon. By this Arte we are certified of the distance of the Starry Skye, and of eche *Planete* from the Centre of the Earth: and of the greatnes of any Fixed starre sene, or *Planete*, in respect of the Earthes greatnes. As, we are sure (by this Arte) that the Solidity, Massines and Body of the *Sonne*, conteineth the quantitie of the whole Earth and Sea, a hundred thre score and two times, lesse by $\frac{1}{8}$ one eight parte of the earth. But the Body of the whole earthly globe and Sea, is bigger then the body of the *Mone*, three and forty times lesse by $\frac{1}{8}$ of the *Mone*. Wherefore the *Sonne* is bigger then the *Mone*, 7000 times, lesse, by $59\frac{32}{64}$ that is, precisely $6940\frac{25}{64}$ bigger then the *Mone*. And yet the vnskillfull man, would iudge them a like bigge. Wherefore, of Necessity, the one is much farder from vs, then the other. The *Sonne*, when he is fardest from the earth (which, now, in our age, is, when he is in the 8. degree, of Cancer) is, 1179 Semidiameters of the Earth, distante. And the *Mone* when she is fardest from the earth, is 68 Semidiameters of the earth and $\frac{1}{4}$. The nerest, that the *Mone* cometh to the earth, is Semidiameters $52\frac{1}{4}$. The distance of the Starry Skye is, frō vs, in Semidiameters of the earth $20081\frac{1}{2}$. Twenty thousand fourescore, one, and almost a halfe. Subtract from this, the *Mones* nerest distance, from the Earth: and therof remaineth Semidiameters of the earth $20029\frac{1}{4}$. Twenty thousand nine and twenty and a quarter. So thicke is the heauenly Palace, that the *Planetes* haue all their exercise in, and most meruailously perfourme the Commaūdement and Charge to them giuen by the omnipotent Maiestie of the king of kings. This is that, which in *Genesis* is called *Ha Rakia*. Consider it well. The Semidiameter of the earth, cōteineth of our common miles $3436\frac{1}{4}$ three thousand, foure hundred thirty six and foure eleuenth partes of one inyle: Such as the whole earth and Sea, round about, is 21600. One and twenty thousand six hundred of our myles. Allowyng for euery degree of the greatest circle, thre score myles. Now if you way well with your selfe but this litle parcell of frute *Astronomicall*, as concerning the bignesse, Distances of *Sonne*, *Mone*, *Sterry Sky*, and the huge massines of *Ha Rakia*, will you not finde your Consciences moued, with the kingly Prophet, to sing the confession of Gods Glory, and say, *The Heauens declare the glory of God, and the Firmament [Ha Rakia] sheweth forth the workes of his bandes*. And so forth, for those fiue first staues, of that kingly Psalme. Well, well, It is time for some to lay hold on wisdome, and to Iudge truly of thinges: and not so to expound the Holy word, all by Allegories: as to Neglect the wisdome, powre and Goodnes of God, in, and by his Creatures, and Creation to be seen and learned. By parables and Analogies of whose natures and properties, the course of the Holy Scripture, also, declareth to vs very many Mysteries. The whole Frame of Gods Creatures, (which is the whole world,) is to vs, a bright glasse: from which, by reflexion, reboundeth to our knowledge and perceiuerance, Beames, and Radiations: representing the Image of his Infinite goodnes, Omnipotēcy, and wisdome. And we therby, are taught and perswaded to Glorifie our Creator, as God: and be thankfull therfore. Could the Heathenistes finde these vses, of these most pure, beawtifull, and Mighty Corporall Creatures: and shall we, after that the true *Sonne* of rightwisenesse is risen aboue the *Horizon*, of our temporall *Hemissphærie*, and hath so abundantly streamed into our hartes, the direct beames of his goodnes, mercy, and grace: Whose heat All Creatures feele: Spirituall and Corporall: Visible and

Note.

Iohn Dee his Mathematicall Præface.

Inuisible: Shall we (I say) looke vpon the *Heauen, Sterres, and Planets*, as an Oxe and an Asse doth: no furder carefull or inquisitiue, what they are: why were they Created, How do they execute that they were Created for: Seing, All Creatures, were for our sake created: and both we, and they, Created, chiefly to glorifie the Almighty Creator: and that, by all meanes, to vs possible. *Nolite ignorare* (saith *Plato in Epinomis*) *Astronomiam, Sapientissimū quiddam esse.* Be ye not ignorant, *Astronomie* to be a thyng of excellent wisdom. *Astronomie*, was to vs, from the beginning commended, and in maner commaunded by God him selfe. In asmuch as he made the *Sonne, Mone, and Sterres*, to be to vs, for *Signes*, and knowledge of Seasons, and for Distinctions of Dayes, and yeares. Many wordes nede not. But I wish, euery man should way this word, *Signes*. And besides that, conferre it also with the tenth Chapter of *Hieremie*. And though Some thinke, that there, they haue found a rod: Yet Modest Reason, will be indifferent Iudge, who ought to be beaten therewith, in respect of our purpose. Leauing that: I pray you vnderstand this: that without great diligence of Obseruation, examination and Calculation, their periods and courses (wherby *Distinction* of Seasons, yeares, and New Mones might precisely be knowne) could not exactly be certified. Which thing to performe, is that *Art*, which we here haue Defined to be *Astronomie*. Wherby, we may haue the distinct Course of Times, dayes, yeares, and Ages: aswell for Consideratiō of Sacred Prophecies, accomplished in due time, foretold: as for high Myfticall Solemnities holding: And for all other humaine affaires, Conditions, and couenantes, vpon certaine time, betwene man and man: with many other great vses: Wherin, (verely), would be great incertainty, Confusion, vntruth, and brutish Barbarousnes: without the wonderfull diligence and skill of this Arte: continually learning, and determining Times, and periodes of Time, by the Record of the heavenly booke, wherein all times are written: and to be read with an *Astronomicall staffe*, in stede of a festue.

Musike, of Motion, hath his Originall cause: Therefore, after the motions most swift, and most Slow, which are in the Firmament, of Nature performed: and vnder the *Astronomers Consideration*: now I will Speake of an other kinde of Motion, producing sound, audible, and of Man numerable. *Musike* I call here that *Science*, which of the Grekes is called *Harmonice*. Not meddling with the Controuersie betwene the auncient *Harmonistes*, and *Canonistes*. *Musike* is a Mathematicall Science, which teacheth, by sense and reason, perfectly to iudge, and order the diuersities of soundes, hye and low. *Astronomie* and *Musike* are Sisters, saith *Plato*. As, for *Astronomie*, the eyes: So, for *Harmonious Motion*, the eares were made. But as *Astronomie* hath a more diuine Contemplation, and commodity, then mortall eye can perceiue: So, is *Musike* to be considered; that the

1. * Minde may be preferred, before the eare. And from audible sound, we ought to ascende, to the examination: which numbers are *Harmonious*, and which not.
2. And why, either, the one are: or the other are not. I could at large, in the heavenly
3. * motions and distances, describe a meruailous Harmonie, of *Pythagoras Harpe*
4. with eight stringes. Also, somewhat might be sayd of *Mercurius* * two Harpes,
5. eche of foure Stringes Elementall. And very straunge matter, might be alledged
6. of the *Harmonie*, to our * Spirituall part appropriate. As in *Ptolomæus* third boke, in the fourth and sixth Chapters may appeare. * And what is the cause of the apt bonde, and frendly felowship, of the Intellectuall and Mentall part of vs, with our grosse & corruptible body: but a certaine Meane, and *Harmonious Spiritualitie*, with

both

John Dee his Mathematicall Præface.

both participatyng, & of both (in a maner) resulyng: In the * Tune of Mans voyce, and also * the sound of Instrument, what might be sayd, of Harmonie. No common Musicien would lightly beleue. But of the sundry Mixture (as I may terme it) and concurse, diuerse collation, and Application of these Harmonies: as of thre, foure, fve, or more: Maruailous haue the effectes ben: and yet may be founde, and produced the like: with some proportionall consideration for our time, and being : in respect of the State, of the thinges then : in which, and by which, the wondrous effectes were wrought. Democritus and Theophrastus affirmed, that, by Musike, griefes and diseases of the Minde, and body might be cured, or inferred. And we finde in Records, that Terpander, Arion, Ismenias, Orpheus, Amphion, David, Pythagoras, Empedocles, Asclepiades and Timotheus, by Harmonicall Consonācy, haue done, and brought to pas, thinges, more then meruailous, to here of. Of them then, making no farther discourse, in this place : Sure I am, that Common Musike, commonly vsed, is found to the Musiciens and Hearers, to be so Commodious and pleasant, That if I would say and dispute, but thus much: That it were to be otherwise vsed, then it is, I should finde more repreeuers, then I could finde priuy, or skilfull of my meaning. In thinges therfore euident, and better knowen, then I can expresse: and so allowed and liked of, (as I would wish, some other thinges, had the like hap) I will spare to enlarge my lines any farder, but consequently follow my purpose.

Of Cosmographie, I appointed briefly in this place, to geue you some intelligence. Cosmographie, is the vhole and perfect description of the heauenly, and also elementall parte of the world, and their homologall application, and mutuall collation necessarie. This Art, requireth Astronomie, Geographie, Hydrographie and Musike. Therefore, it is no small Arte, nor so simple, as in common practise, it is (slightly) considered. This matcheth Heauen, and the Earth, in one frame, and aptly applieth parts Correspondent: So, as, the Heauenly Globe, may (in practise) be duely described vpon the Geographicall, and Hydrographicall Globe. And there, for vs to consider an Equinoctiall Circle, an Ecliptike line, Colures, Poles, Sterres in their true Longitudes, Latitudes, Declinations, and Verticalitie: also Climes, and Parallels: and by an Horizon annexed, and reuolution of the earthly Globe (as the Heauen, is, by the Primouant, caried about in 24. æquall Houres) to learne the Risings and Settings of Sterres (of Virgill in his Georgikes: of Hesiod: of Hippocrates in his Medicinall Sphere, to Perdicca King of the Macedonians: of Diocles, to King Antigonus, and of other famous Philosophers prescribed) a thing necessary, for due manuring of the earth, for Navigation, for the Alteration of mans body: being, whole, Sicke, wounded, or bruised. By the Reuolution, also, or mouing of the Globe Cosmographicall, the Rising and Setting of the Sonne: the Lengthes, of dayes and nightes : the Houres and times (both night and day) are knowne: with very many other pleasant and necessary vses : Wherof, some are knowne: but better remaine, for such to know and vse: who of a sparke of true fire, can make a wonderfull bonfire, by applying of due matter, duely.

Of Astrologie, here I make an Arte, seuerall from Astronomie: not by new deuise, but by good reason and authoritie: for, Astrologie, is an Arte Mathematicall, which reasonably demonstrateth the operations and effectes, of the naturall beames, of light, and secrete influence: of the Sterres and Planets: in euery element and elementall body:

7.
8.

I.D.

Read in Aristotle his 8. booke of Politikes: the 5, 6, and 7. chapters. Where you shall haue some occasion farder to thinke of Musike, than commonly is thought.

John Dee his Mathematicall Præface.

at all times, in any Horizon assigned. This Arte is furnished with many other great Artes and experiences: As with perfecte *Perspectiue*, *Astronomic*, *Cosmographie*, *Naturall Philosophie* of the 4. Elementes, the Arte of Graduation, and some good vnderstanding in *Musike*: and yet moreouer, with an other great Arte, hereafter following, though I, here, set this before, for some considerations me mouing. Sufficient (you see) is the stuffe, to make this rare and secrete Arte, of: and hard enough to frame to the Conclusion Syllogisticall. Yet both the manifolde and continuall trauailes of the most auncient and wise Philosophers, for the atteyning of this Arte: and by examples of effectes, to confirme the same: hath left vnto vs sufficient proufe and witnesse: and we, also, daily may perceauē, That mans body, and all other Elementall bodies, are altered, disposed, ordred, pleased, and displeased, by the Influentiall working of the *Sunne*, *Mone*, and the other Starres and Planets. And therefore, sayth *Aristotle*, in the first of his *Meteorologicall* bookes, in the second Chapter: *Est autem necessario Mundus iste, supernis lationibus ferè continuus: Vt, inde, vis eius vniuersa regatur. Ea siquidem Causa prima putanda omnibus est, unde motus principium existit.* That is: This [Elementall] World is of necessitie, almost, next adioyning, to the heauenly motions: That, from thence, all his vertue or force may be gouerned. For, that is to be thought the first Cause vnto all: from which, the beginning of motion, is. And againe, in the tenth Chapter. *Oportet igitur & horum principia sumamus, & causas omnium similiter. Principium igitur ut mouens, præcipuumq; & omnium primum, Circulus ille est, in quo manifeste Solis latio, &c.* And so forth. His *Meteorologicall* bookes, are full of argumentes, and effectuell demonstrations, of the vertue, operation, and power of the heauenly bodies, in and vpon the fower Elementes, and other bodies, of them (either perfectly, or vnperfectly) composēd. And in his second booke, *De Generatione & Corruptione*, in the tenth Chapter. *Quocirca & prima latio, Ortus & Interitus causa non est: Sed obliqui Circuli latio: ea namq; & continua est, & duobus motibus fit:* In Englishē, thus. *Wherefore the vppermost motion, is not the cause of Generation and Corruption, but the motion of the Zodiake: for, that, both, is continuall, and is caused of two mouinges.* And in his second booke, and second Chapter of hys *Physikes*. *Homo namq; generat hominem, atq; Sol. For Man (sayth he) and the Sonne, are cause of mans generation.* Authorities may be brought, very many: both of 1000. 2000. yea and 3000. yeares Antiquitie: of great *Philosophers*, *Expert*, *Wise*, and godly men, for that Conclusion: which, daily and hourelly, we men, may discerne and perceauē by sense and reason: All beastes do feelee, and simply shew, by their actions and passions, outward and inward: All Plants, Herbes, Trees, Flowers, and Fruites. And finally, the Elementes, and all thinges of the Elementes composēd, do geue Testimonie (as *Aristotle* sayd) that theyr *Whole Dispositions*, *vertues*, and *naturall motions*, depend of the *Actiuitie* of the heauenly motions and Influences. Whereby, beside the specificall order and forme, due to euery seede: and beside the Nature, propre to the Indiuiduall Matrix, of the thing produced: What shall be the heauenly Impression, the perfect and circumspecte *Astrologien* hath to Conclude. Not onely (by *Apotelesmes*) τὸ ὄν, but by *Naturall* and *Mathematicall* demonstration τὸ δῖόν. Whereunto, what Sciences are requisite (without exception) I partly haue here warned: And in my *Propædeumes* (besides other matter there disclosed) I haue Mathematically furnished vp the whole Method: To this our age, not so carefully handled by any, that

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Iohn Dee his Mathematicall Præface.

cuer I saw, or heard of. I was, (for * 21. yeares ago) by certaine earnest disputations, of the Learned *Gerardus Mercator*, and *Antonius Gogaua*, (and other,) therto so prouoked: and (by my constant and inuincible zeale to the veritie) in obseruations of Heauenly Influencies (to the Minute of time,) than, so diligent: And chiefly by the Supernaturall influence, from the Starre of Iacob, so directed: That any Modest and Sober Student, carefully and diligently seeking for the Truth, will both finde & cōfesse, therein, to be the Veritie, of these my wordes: And also become a Reasonable Reformer, of three Sortes of people: about these Influentiall Operations, greatly erring from the truth. Wherof, the one, is Light Beleuers, the other, Light Despisers, and the third Light Practisers. The first, & most cōmon Sort, thinke the Heauen and Sterres, to be answerable to any their doubts or desires: which is not so: and, in dede, they, to much, ouer reache. The Second sorte thinke no Influentiall vertue (frō the heauenly bodies) to beare any Sway in Generation and Corruption, in this Elementall world. And to the *Sunne*, *Mone* and *Sterres* (being so many, so pure, so bright, so wonderfull bigge, so farre in distance, so manifold in their motions, so constant in their periodes. &c.) they assigne a sleight, simple office or two, and so allow vnto the (according to their capacities) as much vertue, and power Influentiall, as to the Signe of the *Sunne*, *Mone*, and seuen *Sterres*, hanged vp (for Signes) in London, for distinction of houses, & such grosse helpes, in our wordly affaires: And they vnderstand not (or will not vnderstand) of the other workinges, and vertues of the Heauenly *Sunne*, *Mone*, and *Sterres*: not so much, as the Mariner, or Husband man: no, not so much, as the *Elephant* doth, as the *Cynocephalus*, as the *Porpentine* doth: nor will allow these perfect, and incorruptible mighty bodies, so much vertuall Radiation, & Force, as they see in a litle peece of a *Magnes Stone*: which, at great distance, sheweth his operation. And perchance they thinke, the Sea & Riuers (as the *Thames*) to be some quicke thing, and so to ebbe, and flow, run in and out, of them selues, at their owne fantasies. God helpe, God helpe. Surely, these men, come to short: and either are to dull: or willfully blind: or, perhaps, to malicious. The third man, is the common and vulgare *Astrologien*, or Practiser: who, being not duely, artificially, and perfectly furnished: yet, either for vaine glory, or gayne: or like a simple dolt, & blinde Bayard, both in matter and maner, erreth: to the discredit of the *Wary*, and modest *Astrologien*: and to the robbing of those most noble corporall Creatures, of their Naturall Vertue: being most mighty: most beneficiall to all elementall Generation, Corruption and the appartenances: and most Harmonious in their Monarchie: For which thinges, being knowen, and modestly vsed: we might highly, and continually glorifie God, with the princely Prophet, saying. *The Heauens declare the Glorie of God: who made the Heauens in his wisedome: who made the Sonne, for to haue dominion of the day: the Mone and Sterres to haue dominion of the nyght: whereby, Day to day vttereth talke: and night, to night declareth knowledge: Prayse him, all ye Sterres, and Light. Amen.*

IN order, now foloweth, of Statike, somewhat to say, what we meane by that name: and what commodity, doth, on such Art, depend. Statike, is an Arte Mathematicall, which demonstrateth the causes of heauynes, and lightnes of all thynges: and of motions and properties, to heauynes and lightnes, belonging. And for asmuch as, by the Bilanx, or Balance (as the chieffensible Instrument,) Experience of these demonstrations may

* Anno. 1548
and 1549.^{ms}
Louayn.

Note.

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3.

Iohn Dee his Mathematicall Præface.

be had: we call this Art, *Statike*: that is, *the Experimentes of the Balance*. Oh, that men wist, what profit, (all maner of wayes) by this Arte might grow, to the hable examiner, and diligent practiser. Thou onely knowest all thinges precisely (O God) who hast made weight and Balance, thy Iudgement: who hast created all thinges in *Number, Waight, and Measure*: and hast wayed the mountaines and hils in a Balance: who hast peysed in thy hand, both Heauen and earth. We therefore warned by the Sacred word, to Consider thy Creatures: and by that consideration, to wyne a glyms (as it were,) or shaddow of perceiuerance, that thy wisdome, might, and goodnes is infinite, and ynspeakable, in thy Creatures declared: And being farther aduertised, by thy mercifull goodnes, that, three principall wayes, were, of the, vsed in Creation of all thy Creatures, namely, *Number, Waight and Measure*, And for as much as, of *Number and Measure*, the two Artes (auncient, famous, and to humaine vses most necessary,) are, all ready, sufficiently knowen and extant: This third key, we beseeche thee (through thy accustomed goodnes,) that it may come to the nedefull and sufficient knowledge, of such thy Seruauntes, as in thy workemanship, would gladly finde, thy true occasions (purposely of the vsed) whereby we should glorifie thy name, and shew forth (to the weaklinges in faith) thy wondrous wisdome and Goodnes. Amen.

Meruaile nothing at this pang (godly frend, you Gentle and zelous Student.) An other day, perchaunce, you will perceiue, what occasion moued me. Here, as now, I will giue you some ground, and withall some shew, of certaine commodities, by this Arte arising. And bycause this Arte is rare, my wordes and practises might be to darke: vnleast you had some light, holden before the matter: and that, best will be, in giuing you, out of *Archimedes* demonstrations, a few principal Conclusions, as foloweth.

1.

The Superficies of euery Liquor, by it selfe consistyng, and in quyet, is Sphæricall: the centre whereof, is the same, which is the centre of the Earth.

2.

If Solide Magnitudes, being of the same bignes, or quãtitie, that any Liquor is, and hauyng also the same Waight: be let downe into the same Liquor, they will settle downeward, so, that no parte of them, shall be aboue the Superficies of the Liquor: and yet neuertheles, they will not sinke vtterly downe, or drowne.

3.

If any Solide Magnitude beyng Lighter then a Liquor, be let downe into the same Liquor, it will settle downe, so farre into the same Liquor, that so great a quantitie of that Liquor, as is the parte of the Solid Magnitude, settled downe into the same Liquor: is in Waight, æquall, to the waight of the whole Solid Magnitude.

4.

Any Solide Magnitude, Lighter then a Liquor, forced downe into

John Dee his Mathematicall Præface.

into the same Liquor, will moue vpward, with so great a power, by how much, the Liquor hauyng æquall quantitie to the whole Magnitude, is heauyer then the same Magnitude.

5.

Any Solid Magnitude, heauyer then a Liquor, beyng let downe into the same Liquor, will sinke downe vtterly: And wilbe in that Liquor, Lighter by so much, as is the waight or heauynes of the Liquor, hauing bygnes or quantitie, æquall to the Solid Magnitude.

6.

If any Solide Magnitude, Lighter then a Liquor, be let downe into the same Liquor, the waight of the same Magnitude, will be, to the Waight of the Liquor. (Which is æquall in quantitie to the whole Magnitude,) in that proportion, that the parte, of the Magnitude settled downe, is to the whole Magnitude.

I.D.

The Cutting of a Sphere according to any proportion assigned may by this proposition be done Mechanically by tempering Liquor to a certayne waight in respect of the waight of the Sphere therein Swymming.

BY these verities, great Errors may be reformed, in Opinion of the Naturall Motion of thinges, Light and Heauy. Which errors, are in Naturall Philosophie (almost) of all men allowed: to much trusting to Authority: and false Suppositions. As, Of any two bodyes, the heauyer, to moue downward faster then the lighter. This error, is not first by me, Noted: but by one *John Baptist de Benedictis*. The chief of his propositions, is this: which seemeth a Paradox.

A common error, noted.

If there be two bodyes of one forme, and of one kynde, æquall in quantitie or vnæquall, they will moue by æquall space, in æquall tyme: So that both theyr mouynges be in ayre, or both in water: or in any one Middle.

A paradox.

Hereupon, in the feate of Gunnyng, certaine good discourses (otherwise) may receiue great amendement, and furdurance. In the entended purpose, also, allowing somewhat to the imperfection of Nature: not aunswerable to the precisenes of demonstration. Moreouer, by the foresaid propositions (wisely vsed.) The Ayre, the water, the Earth, the Fire, may be nerely, knowen, how light or heauy they are (Naturally) in their assigned partes: or in the whole. And then, to thinges Elementall, turning your practise: you may deale for the proportion of the Elementes, in the thinges Compounded. Then, to the proportions of the Humours in Man: their waightes: and the waight of his bones, and flesh. &c. Than, by waight, to haue consideration of the Force of man, any maner of way: in whole or in part. Then, may you, of Ships water drawing, diuersly, in the Sea and in fresh water, haue pleasant consideration: and of waying vp of any thing, sonken in Sea or in fresh water &c. And (to lift vp your head a loft:) by waight, you may, as precisely, as by any instrument els, measure the Diameters of *Sonne* and *Mone*. &c. Frende, I pray you, way these thinges, with the iust Balance of Reason. And you will finde Meruailes vpon Meruailes: And esteeme one Drop of Truth (yea in Naturall Philosophie) more worth, then whole Libraries of Opinions, vndemonstrated: or not aunswering to Natures Law, and your experience. Leauing these

N. T.

The wonderfull use of these Propositions.

c.j.

thinges,

Iohn Dee his Mathematicall Præface.

*The practise
Staticall, to
know the pro-
portion, be-
twene the
Cube, and the
Sphere*

*I. D.
*For, so, haue
you. 256.
partes of a
Graine.*

things, thus: I will giue you two or three, light practises, to great purpose: and so finish my Annotation *Staticall*. In Mathematicall matters, by the Mechaniciens ayde, we will behold, here, the Cominodity of waight. Make a Cube, of any one Vniforme: and through like heauy stuffe: of the same Stuffe, make a Sphære or Globe, precisely, of a Diameter æquall to the Radicall side of the Cube. Your stuffe, may be wood, Copper, Tinne, Lead, Siluer. &c. (being, as I sayd, of like nature, condition, and like waight throughout.) And you may, by Say Balance, haue prepared a great number of the smallest waightes: which, by those Balance can be discerned or tryed: and so, haue proceeded to make you a perfect Pyle, company & Number of waightes: to the waight of six, eight, or twelue pound waight: most diligently tryed, all. And of euery one, the Content knowen, in your least waight, that is wayable. [They that can not haue these waightes of precisenes: may, by Sand, Vniforme, and well dusted, make them a number of waightes, somewhat nere precisenes: by halving euer the Sand: they shall, at length, come to a least common waight. Therein, I leaue the farder matter, to their discretion, whom nede shall pinche.] The *Venetians* consideration of waight, may seme precise enough: by eight descentes progressionall, * halving, from a grayne. Your Cube, Sphære, apt Balance, and conuenient waightes, being ready: fall to worke. First, way your Cube. Note the Number of the waight. Way, after that, your Sphære. Note likewise, the Nüber of the waight. If you now find the waight of your Cube, to be to the waight of the Sphære, as 21. is to 11. Then you see, how the Mechanicien and *Experimenter*, without Geometrie and Demonstration, are (as nerely in effect) taught the proportion of the Cube to the Sphere: as I haue demonstrated it, in the end of the twelfth boke of *Euclide*. Often, try with the same Cube and Sphære. Then, chaunge, your Sphære and Cube, to an other matter: or to an other bignes: till you haue made a perfect vniuersall Experience of it. Possible it is, that you shall wyne to nerer termes, in the proportion.

When you haue found this one certaine Drop of Naturall veritie, procede on, to Inferre, and duely to make assay, of matter depending. As, bycause it is well demonstrated, that a Cylinder, whose heith, and Diameter of his base, is æquall to the Diameter of the Sphære, is Sesquialter to the same Sphære (that is, as 3. to 2.) To the number of the waight of the Sphære, adde halfe so much, as it is: and so haue you the number of the waight of that Cylinder. Which is also Comprehended of our former Cube: So, that the base of that Cylinder, is a Circle described in the Square, which is the base of our Cube. But the Cube and the Cylinder, being both of one heith, haue their Bases in the same proportion, in the which, they are, one to an other, in their Massines or Soliditie. But, before, we haue two numbers, expressing their Massines, Solidities, and Quantities, by waight: wherfore, we haue * the proportion of the Square, to the Circle, inscribed in the same Square. And so are we fallen into the knowledge sensible, and Experimentall of *Archimedes* great Secret: of him, by great trauaile of minde, sought and found. Wherfore, to any Circle giuen, you can giue a Square æquall: * as I haue taught, in my Annotation, vpon the first proposition of the twelfth boke, And likewise, to any Square giuen, you may giue a Circle æquall: * If you describe a Circle, which shall be in that proportion, to your Circle inscribed, as the Square is to the same Circle. This, you may do, by my Annotations, vpon the second proposition of the twelfth boke of *Euclide*, in my third Probleme there. Your diligence may come to a proportion, of the Square to the Circle inscribed, nerer the truth, then is the proportion of 14. to 11. And consider, that you may begyn at the Circle and Square, and so come to conclude of the Sphære, & the Cube, what their

**The proportion of
the Square to the
Circle inscribed.*

**The Squaring of
the Circle, Mecha-
nically:*

**To any Square
giuen, to geue a
Circle, æquall.*

John Dee his Mathematicall Bræface.

their proportion is: as now, you came from the Sphere to the Circle. For, of Silver, or Gold, or Latton Lamyns or plates (thorough one hole drawe, as the manner is) if you make a Square figure: & way it: and then, describing thereon, the Circle inscribed: & cut of, & file away, precisely (to the Circle) the ouerplus of the Square: you shall then, waying your Circle, see, whether the waight of the Square, be to your Circle, as 14. to 11. As I haue Noted, in the beginning of *Euclides* twelfth booke. &c. after this resort to my last proposition, vpon the last of the twelfth. And there, helpe your selfe, to the end. And, here, Note this, by the way. That we may Square the Circle, without hauing knowledge of the proportion, of the Circumference to the Diameter: as you haue here perceined. And otherwayes also, I can demonstrate it. So that, many haue cumberd them selues superfluously, by trauiailing in that point first, which was not of necessitie, first: and also very intricate. And easily, you may, (and that diuersly) come to the knowledge of the Circumference: the Circles Quantitie, being first knowen. Which thing, I leaue to your consideration: making hast to despatch an other Magistrall Probleme: and to bring it, nerer to your knowledge, and readier dealing with, then the world (before this day,) had it for you, that I can tell of. And that is, *A Mechanicall Dubbling of the Cube: &c.* Which may, thus, be done: Make of Copper plates, or Tyn plates, a foursquare vpright Pyramis, or a Cone: perfectly fashioned in the holow, within. Wherin, let great diligence be vsed, to approche (as nere as may be) to the Mathematicall perfection of those figures. At their bases, let them be all open: euery where, els, most close, and iust to. From the vertex, to the Circumference of the base of the Cone: & to the sides of the base of the Pyramis: Let 4. straight lines be drawen, in the inside of the Cone and Pyramis: making at their fall, on the perimeters of the bases, equall angles on both sides them selues, with the sayd perimeters. These 4. lines (in the Pyramis: and as many, in the Cone) diuide: one, in 12. æquall partes: and an other, in 24. an other, in 60, and an other, in 100. (reckenyng vp from the vertex.) Or vse other numbers of diuision, as experience shall teach you. Then, * set your Cone or Pyramis, with the vertex downward, perpendicularly, in respect of the Base. (Though it be otherwayes, it hindreth nothyng.) So let the most stedily be stayed. Now, if there be a Cube, which you wold haue Dubbled. Make you a prety Cube of Copper, Silver, Lead, Tynne, Wood, Stone, or Bone. Or els make a hollow Cube, or Cubik coffen, of Copper, Silver, Tynne, or Wood &c. These, you may so proportio in respect of your Pyramis or Cone, that the Pyramis or Cone, will be hable, to containe the waight of them, in water, 3. or 4. times: at the least: what stuff so euer they be made of. Let not your Solid angle, at the vertex, be to sharpe: but that the water may come with ease, to the very vertex, of your hollow Cone or Pyramis. Put one of your Solid Cubes in a Balance apt: take the waight therof exactly in water. Powre that water, (without losse) into the hollow Pyramis or Cone, quietly. Marke in your lines, what numbers the water Cutterth: Take the waight of the same Cube againe: in the same kinde of water, which you had before: put that * also, into the Pyramis or Cone, where you did put the first. Marke now againe, in what number or place of the lines, the water Cutterth them. Two

Note
Squaring of
the Circle
without know-
ledge of the
proportion be-
twene Cir-
cumference
and Diamo-
ter.

*To Dubble
the Cubere-
dily: by Art
Mechanicall:
depending vpon
Demon-
stration Ma-
thematicall.*

I. D.
The 4. sides of this
Pyramis must be 4.
Isosceles Triangles &
like and equall.

I. D.
* In all workinges
with this Pyramis
or Cone, Let their
Situations be in all
Pointes and Condi-
tions, a like, or all
one: while you are
about one worke. Els
you will erre.

I. D.
* Consider well when
you must put your
waters together: and
when, you must emp-
ty your first water,
out of your Pyramis
or Cone. Els you
will erre.

John Dee his Mathematicall Præface.

wayes you may conclude your purpose : it is to wete, either by numbers or lines. By numbers : as, if you diuide the side of your Fundamentall Cube into so many æquall partes, as it is capable of, conueniently, with your ease, and precisenes of the diuision. For, as the number of your first and lesse line (in your hollow Pyramis or Cone,) is to the second or greater (both being counted from the vertex) so shall the number of the side of your Fundamentall Cube, be to the nūber belonging to the Radical side, of the Cube, dubble to your Fundamentall Cube: Which being multiplied Cubik wise, will sone shew it selfe, whether it be dubble or no, to the Cubik number of your Fundamentall Cube. By lines, thus: As your lesse and first line, (in your hollow Pyramis or Cone,) is to the second or greater, so let the Radical side of your Fundamētall Cube, be to a fourth proportionall line, by the 12. proposition, of the sixth boke of *Euclide*. Which fourth line, shall be the Rote Cubik, or Radical side of the Cube, dubble to your Fundamentall Cube: which is the thing we desired. For this, may I (with ioy) say, ΕΥΡΗΚΑ, ΕΥΡΗΚΑ, ΕΥΡΗΚΑ: thanking the holy and glorious Trinity: hauing greater cause therto, then * *Archimedes* had (for finding the fraude vsed in the Kinges Crowne, of Gold): as all men may easily Iudge: by the diuersitie of the frute following of the one, and the other. Where I spake before, of a hollow Cubik Coffen: the like vse, is of it: and without waight. Thus. Fill it with water, precisely full, and poure that water into your Pyramis or Cone. And here note the lines cutting in your Pyramis or Cone. Againe, fill your coffen, like as you did before. Put that Water, also, to the first. Marke the second cutting of your lines. Now, as you proceded before, so must you here procéde. * And if the Cube, which you should Double, be neuer so great: you haue, thus, the proportion (in small) betwene your two litle Cubes: And then, the side, of that great Cube (to be doubled) being the third, will haue the fourth, found, to it proportionall: by the 12. of the sixth of *Euclide*.

* Vitruvius.
Lib. 9. Cap. 3.

God be than-
ked for this
Inuention, &
the fruite en-
suing.

*Note.

Note, as concerning the Spharicall Superficies of the Water.



*Note.

Note this A-
bridgement of
Dubbling the
Cube. &c.

Note, that all this while, I forget not my first Proposition Staticall, here rehearsed: that, the Superficies of the water, is Sphaericall. Wherein, vse your discretion: to the first line, adding a small heere breadth, more: and to the second, halfe a heere breadth more, to his length. For, you will easily perceave, that the difference can be no greater, in any Pyramis or Cone, of you to be handled. Which you shall thus trye. *For finding the swelling of the water aboue leuell.* Square the Semidiameter, from the Centre of the earth, to your first Waters Superficies. Square then, halfe the Subtendent of that watry Superficies (which Subtendent must haue the equall partes of his measure, all one, with those of the Semidiameter of the earth to your watry Superficies) : Subtracte this square, from the first : Of the residue, take the Rote Square. That Rote, Subtracte from your first Semidiameter of the earth to your watry Superficies : that, which remaineth, is the heith of the water, in the middle, aboue the leuell. Which, you will finde, to be a thing insensible. And though it were greatly sensible,* yet, by helpe of my sixt Theoreme vpon the last Proposition of Euclides twelfth booke, noted : you may reduce all, to a true Leuell. But, farther diligence, of you is to be vsed, against accidentall causes of the waters swelling: as by hauing (somewhat) with a moyst Sponge, before, made moyst your hollow Pyramis or Cone, will preuent an accidentall cause of Swelling, &c. Experience will teach you abundantly : with great ease, pleasure, and comoditie.

Thus, may you Double the Cube Mechanically, Treble it, and so forth, in any proportion. Now will I Abridge your paine, cost, and Care herein. Without all preparing of your Fundamentall Cubes: you may (alike) worke this Conclusion. For, that, was rather a kinde of Experimentall demonstration, then the shortest way:

and

John Dee his Mathematicall Preface.

and all, vpon one Mathematicall Demonstration, depending . Take water (as much as conueniently will serue your turne : as I warned before of your Fundamentall Cubes bignes) Way it precisely . Put that water, into your Pyramis or Cone . Of the same kinde of water, then take againe, the same waight you had before : put that likewise into the Pyramis or Cone . For, in eche time, your marking of the lines, how the Water doth cut them, shall geue you the proportion between the Radicall sides, of any two Cubes, wherof the one is Double to the other: working as before I haue taught you: *sauiug that for you Fundamentall Cube his Radicall side: here, you may take a right line, at pleasure.

Note. * 

Yet farther proceeding with our droppe of Naturall truth : you may (now) geue Cubes, one to the other, in any proportiō geue: Rationall or Irrationall : on this maner. Make a hollow Parallelipipedon of Copper or Tinne: with one Base wating, or open: as in our Cubike Coffen. Frō the bottome of that Parallelipipedon, raise vp, many perpendiculars, in euery of his fower sides. Now if any proportion be assigned you, in right lines: Cut one of your perpendiculars (or a line equall to it, or lesse then it) likewise : by the 10. of the sixth of Euclide. And those two partes, set in two sundry lines of those perpendiculars (or you may set them both, in one line) making their beginniges, to be, at the base : and so their lengthes to extend vpward . Now, set your hollow Parallelipipedon, vp right, perpendicularly, steadie . Poure in water, handsomly, to the heith of your shorter line . Poure that water, into the hollow Pyramis or Cone . Marke the place of the rising . Settle your hollow Parallelipipedon againe . Poure water into it: vnto the heith of the second line, exactly . Poure that water *duely into the hollow Pyramis or Cone : Marke now againe, where the water cutteth the same line which you marked before . For, there, as the first marked line, is to the second : So shall the two Radicall sides be, one to the other, of any two Cubes: which, in their Soliditie, shall haue the same proportion, which was at the first assigned : were it Rationall or Irrationall.

To giue Cubes one to the other in any proportion, Rationall or Irrationall.

* Emptying the first.

Thus, in sundry waies you may furnishe your selfe with such straunge and profitable matter: which, long hath bene wished for. And though it be Naturally done and Mechanically : yet hath it a good Demonstration Mathematicall . Which is this : Alwaies, you haue two Like Pyramids : or two Like Cones, in the proportions assigned : and like Pyramids or Cones, are in proportion, one to the other, in the proportion of their Homologall sides (or lines) tripled. Wherefore, if to the first, and second lines, found in your hollow Pyramis or Cone, you ioyne a third and a fourth, in continuall proportion : that fourth line, shall be to the first, as the greater Pyramis or Cone, is to the lesse : by the 33. of the eleuenth of Euclide . If Pyramis to Pyramis, or Cone to Cone, be double, then shall * Line to Line, be also double, &c. But, as our first line, is to the second, so is the Radicall side of our Fundamentall Cube, to the Radicall side of the Cube to be made, or to be doubled : and therefore, to those twaine also, a third and a fourth line, in continuall proportion, ioyned : will geue the fourth line in that proportion to the first, as our fourth Pyramidall, or Conike line, was to his first : but that was double, or treble, &c. as the Pyramids or Cones were, one to an other (as we haue proued) therefore, this fourth, shalbe also double or treble to the first, as the Pyramids or Cones were one to an other : But our made Cube, is described of the second in proportion, of the fower proportionall lines : therefore * as the fourth line, is to the first, so is that Cube, to the first Cube : and we haue proued the fourth line, to be to the first, as the Pyramis or Cone, is to the Pyramis or Cone : Wherefore the Cube is

The demonstrations of this Dubbling of the Cube, and of the rest.

I.D.
* Hereby, helpe your selfe to become a precise practiser. And so consider, how, nothing at all, you are hindered (sensibly) by the Conueniencie of the water.

* By the 33. of the eleuenth booke of Euclide.

John Dee his Mathematicall Præface.

I.D.
* And your diligence
in practise, can so
(in waight of wa-
ter) performe it:
Therefore, now, you
are able to geue good
reason of your whole
doing.

* Note this
Corollary.

* The great
Commodities
following of
these new In-
ventions.

☞*

Such is the
Fruite of the
Mathemati-
call Sciences
and Artes.

to the Cube, as Pyramis is to Pyramis, or Cone is to Cone. But we* Suppose Pyramis to Pyramis, or Cone to Cone, to be double or treble. &c. Therefore Cube, is to Cube, double, or treble, &c. Which was to be demonstred. And of the Parallelipipedō, it is euident, that the water Solide Parallelipipedons, are one to the other, as their heithes are, seing they haue one base. Wherefore the Pyramids or Cones, made of those water Parallelipipedons, are one to the other, as the lines are (one to the other) betwene which, our proportion was assigned. But the Cubes made of lines, after the proportiō of the Pyramidal or Conik *homologall* lines, are one to the other, as the Pyramides or Cones are, one to the other (as we before did proue) therefore, the Cubes made, shalbe one to the other, as the lines assigned, are one to the other: Which was to be demonstred. Note. * This, my Demonstratiō is more generall, then onely in Square Pyramis or Cone: Consider well. Thus, haue I, both Mathematically and Mechanically, ben very long in wordes: yet (I trust) nothing tedious to them, who, to these thinges, are well affected. And verily I am forced (auoiding prolixitie) to omit sundry such things, easie to be practised: which to the Mathematicien, would be a great Threasure: and to the Mechanicien, no small gaine. * Now may you, Betwene two lines giuen, finde two middle proportionals, in Continuall proportion: by the hollow Parallelipipedon, and the hollow Pyramis, or Cone. Now, any Parallelipipedon rectangle being giuen: thre right lines may be found, proportionall in any proportion assigned, of which, shal be produced a Parallelipipedon, aequal to the Parallelipipedon giuen. Hereof, I noted somwhat, vpon the 36. proposition, of the 11. booke of *Euclide*. Now, all those thinges, which *Vitruuius* in his Architecture, specified hable to be done, by dubbling of the Cube. Or, by finding of two middle proportionall lines, betwene two lines giuen, may easely be performed. Now, that Probleme, which I noted vnto you, in the end of my Addition, vpon the 34. of the 11. booke of *Euclide*, is proued possible. Now, may any regular body, be Transformed into an other, &c. Now, any regular body: any Sphere, yea any Mixt Solid: and (that more is) Irregular Solides, may be made (in any proportiō assigned) like vnto the body, first giuen. Thus, of a *Manneken*, (as the *Dutch* Painters terme it) in the same *Symmetrie*, may a Giant be made: and that, with any gesture, by the *Manneken* vsed: and contrarywise. Now, may you, of any Mould, or Modell of a Ship, make one, of the same Mould (in any assigned proportion) bigger or lesser. Now, may you, of any* Gunne, or little peece of ordinaunce, make an other, with the same *Symmetrie* (in all pointes) as great, and as little, as you will. Marke that: and thinke on it. Infinitely, may you apply this, so long sought for, and now so easily concluded: and withall, so willingly and frankly communicated to such, as faithfully deale with vertuous studies. Thus, can the Mathematicall minde, deale Speculatiuely in his own Arte: and by good meanes, Mount aboue the cloudes and sterres: And thirdly, he can, by order, Descend, to frame Naturall thinges, to wonderfull vses: and when he list, retire home into his owne Centre: and there, prepare more Meanes, to Ascend or Descend by; and, all, to the glory of God, and our honest delectation in earth.

Although, the Printer, hath looked for this Præface, a day or two, yet could I not bring my pen from the paper, before I had giuen you comfortable warning, and brief instructions, of some of the Commodities, by *Statike*, hable to be reaped: In the rest, I will therefore, be as brief, as it is possible: and with all, describing them, somwhat accordingly. And that, you shall perceiue, by this, which in order com-
meth

Iohn Dee his Mathematicall Præface.

meth next. For, wheras, it is so ample and wonderfull, that, an whole yeare long, one might finde fruitfull matter therin, to speake of: and also in practise, is a Threasure endeles: yet will I glanse ouer it, with wordes very few.

THIS do I call **Anthropographie**. Which is an Art restored, and of my preferment to your Seruice. I pray you, thinke of it, as of one of the chief pointes, of Humane knowledge. Although it be, but now, first Cōfirmed, with this new name: yet the matter, hath from the beginning, ben in consideration of all perfect Philosophers. **Anthropographie**, is the description of the Number, Measure, Waight, figure, Situation, and colour of euery diuerse thing, conteyned in the perfect body of MAN: with certain knowledge of the Symmetrie, figure, waight, Characterization, and due locall motion, of any parcell of the sayd body, asigned: and of Nūbers, to the sayd parcell appertainyng. This, is the one part of the Definition, mete for this place: Sufficient to notifie, the particularitie, and excellency of the Arte: and why it is, here, ascribed to the Mathematics. Yf the description of the heauenly part of the world, had a peculier Art, called *Astronomie*: If the description of the earthly Globe, hath his peculier arte, called *Geographie*. If the Matching of both, hath his peculier Arte, called *Cosmographie*: Which is the Descriptiō of the whole, and vniuersall frame of the world: Why should not the description of him, who is the Lesse world: and, frō the beginning, called *Microcosmus* (that is. *MAN* is the Lesse World.) And for whose sake, and seruice, all bodily creatures els, were created: Who, also, participateth with Spirites, and Angels: and is made to the Image and similitude of God: haue his peculier Art: and be called the *Arte of Artes*: rather, then, either to want a name, or to haue to base and improprie a name? You must of sundry professions, borow or challenge home, peculier partes hereof: and farder procede: as, God, Nature, Reason and Experience shall informe you. The Anatomistes will restore to you, some part: The Physiognomistes, some: The Chyromantistes some. The Metaposcopistes, some: The excellent, *Albert Durer*, a good part: the Arte of Perspectiue, will somwhat, for the Eye, helpe forward: *Pythagoras*, *Hipocrates*, *Plato*, *Galenus*, *Meletius*, & many other (in certaine thinges) will be Contributaries. And farder, the Heauen, the Earth, and all other Creatures, will eche shew, and offer their Harmonious seruice, to fill vp, that, which wanteth hereof: and with your own Experience, concluding: you may Methodically register the whole, for the posteritie: Whereby, good profe will be had, of our Harmonious, and Microcosmicall constitution. The outward Image, and vew hereof: to the Art of *Zographie* and Painting, to Sculpture, and Architecture: (for Church, House, Fort, or Ship) is most necessary and profitable: for that, it is the chiefe base and foundation of them. Looke in * *Vitruuius*, whether I deale sincerely for your behoufe, or no. Looke in *Albertus Durerus*, *De Symmetria humani Corporis*. Looke in the 27. and 28. Chapters, of the second booke, *De occulta Philosophia*. Consider the Arke of *Noe*. And by that, wade farther. Remember the *Delphicall Oracle* *NO SCETE IPSVM* (*Knowe thy selfe*) so long agoe pronounced: of so many a Philosopher repeated: and of the *Wiseſt* attempted: And then, you will perceauē, how long agoe, you haue bene called to the Schole, where this Arte might be learned. Well. I am nothing affrayde, of the disdayne of some such, as thinke Sciences and Artes, to be but Seuen. Perhaps, those Such, may, with ignorance, and shame enough, come short of them Seuen also: and yet neuerthelesse

MAN is the Lesse World.

* *Microcosmus.*
* *Lib. 3.*
* *Cap. 1.*

Iohn Dee his Mathematicall Præface.

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they can not prescribe a certaine number of Artes: and in eche, certaine vnpassable boundes, to God, Nature, and mans Industrie. New Artes, dayly rise vp: and there was no such order taken, that, All Artes, should in one age, or in one land, or of one man, be made knowne to the world. Let vs embrace the giftes of God; and wayes to wisdome, in this time of grace, from aboue, continually bestowed on them, who thankfully will receiue them: *Et bonis Omnia Cooperabuntur in bonum.*

Trochilike, is that Art Mathematicall, which demonstrateth the properties of all Circular motions, Simple and Compounde. And bycause the frute hercof, vulgarly receiued, is in Wheles, it hath the name of *Trochilike*: as a man would say, *Whele Art*. By this art, a Whele may be geuen which shall moue ones about, in any tyme assigned. Two Wheles may be giuen, whose turnynges about in one and the same tyme, (or equall tymes), shall haue, one to the other, any proportion appointed. By Wheles, may a straight line be described: Likewise, a Spirall line in plaine, Conicall Section lines, and other Irregular lines, at pleasure, may be drawn. These, and such like, are principall Conclusions of this Arte: and helpe forward many pleasant and profitable Mechanicall workes: As Milles, to Saw great and very long Deale bordes, no man being by. Such haue I seene in Germany: and in the Citie of Prage: in the kingdome of Bohemia: Coyning Milles, Hand Milles for Corne grinding: And all maner of Milles, and Whele worke: By Winde, Smoke, Water, Waight, Spring, Man or Beast, moued. Take in your hand, *Agricola De re Metallica*: and then shall you (in all Mines) perceaue, how great nede is, of Whele worke. By Wheles, straunge workes and incredible, are done: as will, in other Artes hereafter, appeare. A wonderfull example of farther possibilitie, and present commoditie, was sene in my time, in a certaine Instrument: which by the Inuenter and Artificer (before) was solde for xx. Talentes of Golde: and then had (by misfortune) receaued some iniurie and hurt: And one *Ianellus* of *Cremona* did mend the same, and presented it vnto the Emperour *Charles* the fifth. *Hieronymus Cardanus*, can be my witnesse, that therein, was one Whele, which moued, and that, in such rate, that, in 7000. yeares onely, his owne periode should be finished. A thing almost incredible: But how farre, I keepe me within my boundes: very many men (yet aliue) can tell.

Helicosophie, is nere Sister to *Trochilike*: and is, An Arte Mathematicall, which demonstrateth the designing of all Spirall lines in Plaine, on Cylinder, Cone, Sphære, Conoid, and Sphæroid, and their properties appertayning. The vse hereof, in *Architecture*, and diuerse Instrumentes and Engines, is most necessary. For, in many thinges, the Skruie worketh the feate, which, els, could not be performed. By helpe hereof, it is recorded, that, where all the power of the Citie of *Syracusa*, was not hable to moue a certaine Ship (being on ground) mightie *Archimedes*, setting to, his Skruish Engine, caused *Hiero* the king, by him self, at ease, to remoue her, as he would. Wherat, the King wondring: *Από τούτης τῆς ἡμέρας, περὶ πάντος, Ἀρχιμήδης λέγοντι πεισευτέρω.* From this day, forward (said the King) Credit ought to be giuen to *Archimedes*, what soeuer he sayth.

Pneumatithmie demonstrateth by close hollow Geometricall Figures, (regular and irregular) the straunge properties (in motion or stay) of the Water, Ayre, Smoke, and Fire, in theyr cōtinuitie, and

Saw Milles.

* *Athenens*
Lib. 5. cap. 8.

Proclus.
Pag. 18.

Iohn Dee his Mathematicall Præface.

and as they are ioyned to the Elementes next them. This Arte, to the Naturall Philosopher, is very profitable: to proue, that *Vacuum*, or *Emptines* is not in the world. And that, all Nature, abhorreth it so much: that, contrary to ordinary law, the Elementes will moue or stand. As, Water to ascend: rather then betwene him and Ayre, Space or place should be left, more then (naturally) that quantitie of Ayre requireth, or can fill. Againe, Water to hang, and not descend: rather then by descending, to leaue Emptines at his backe. The like, is of Fire and Ayre: they will descend: when, either, their Cōtinuitie should be dissolued: or their next Element forced from them. And as they will not be extended, to discontinuitie: So, will they not, nor yet of mans force, can be prest or pent, in space, not sufficient and aunswerable to their bodily substance. Great force and violence will they vse, to enioy their naturall right and libertie. Hereupon, two or three men together, by keping Ayre vnder a great Cauldron, and forcyng the same downe, orderly, may without harme descend to the Sea bottome: and continue there a tyme &c. Where, Note, how the thicker Element (as the Water) giueth place to the thynner (as, is the ayre:) and receiue violence of the thinner, in maner. &c. Pumps and all maner of Bellowes, haue their ground of this Art: and many other straunge deuises. As, *Hydraulica*, Organes goyng by water. &c. Of this Feat, (called commonly *Pneumatica*,) goodly workes are extant, both in Greke, and Latin. With old and learned Schole men, it is called *Scientia de pleno & vacuo*.

To go to the
bottom of the
Sea without
daunger.

Menadrie, is an Arte Mathematicall, which demonstrateth, how, aboue Natures vertue and power simple: Vertue and force may be multiplied: and so, to direct, to lift, to pull to, and to put or cast fro, any multiplied or simple, determined Vertue, Waight or Force: naturally, not, so, directible or moueable. Very much is this Art furdred by other Artes: as, in some pointes, by *Perspectiue*: in some, by *Statike*: in some, by *Trochilike*: and in other, by *Helicosophie*: and *Pneumatikie*. By this Art, all Cranes, Gybbettes, & Ingines to lift vp, or to force any thing, any maner way, are ordred: and the certaine cause of their force, is knowne. As, the force which one man hath with the Duche wagen Racke: therwith, to set vp agayne, a mighty wagen laden, being ouerthrowne. The force of the Crossebow Racke, is certainly, here, demonstrated. The reason, why one mā, doth with a leauer, lift that, which Sixe men, with their handes onely, could not, so easily do. By this Arte, in our common Cranes in London, where powre is to Crane vp, the waight of 2000. pound: by two Wheles more (by good order added) Arte concludeth, that there may be Craned vp 200000. pound waight &c. So well knew *Archimedes* this Arte: that he alone, with his deuises and engynes, (twise or thrise) spoyled and discomfited the whole Army and Hoste of the Romaines, besieging *Syracusa*, *Marcus Marcellus the Consul*, being their Generall Capitaine. Such huge Stones, so many, with such force, and so farre, did he with his engynes hayle among them, out of the Citie. And by Sea likewise: though their Ships might come to the walls of *Syracusa*, yet hee vtterly confounded the Romaine Nauye. What with his mighty Stones hurlyng: what with Pikes of * 18 fote long, made like shaftes: which he forced almost a quarter of a myle. What, with his catchyng hold of their Shyps, and hoysing them vp aboue the water, and suddenly letting them fall into the Sea againe: what with his* Burning Glasses: by which he fired their other Shippes a far-off: what, with his other pollicies, deuises, and engines, he so manfully acquit himselfe: that all the Force, courage, and pollicie of the Romaines (for a great season)

*Plutarchus in Mar-
co Marcello.
Synesius in Episto-
lis.
Polybius.
Plinius.
Quintilianus.
T. Liuius.
* Athenaus,*

* *Galenus.
Anthemius.*

John Dee his Mathematicall Præface.

Burning
Glasses.

Gunnes.

could nothing preuaile, for the winning of Syracusa. Wherupon, the Romanes named *Archimedes*, *Briareus*, and *Centimanus*. *Zonaras* maketh mention of one *Proclus*, who so well had perceiued *Archimedes* Arte of *Menadrie*, and had so well inuented of his owne, that with his Burning Glasses, being placed vpon the walles of Byfance, he multiplied so the heate of the Sunne, and directed the beames of the same against his enemies Nauie with such force, and so sodeinly (like lightening) that he burned and destroyed both man and ship. And *Dion* specifieth of *Priscus*, a *Geometricien* in Byfance, who inuented and vsed sondry Engines, of Force multiplied: Which was cause, that the *Emperour Seuerus* pardoned him, his life, after he had wonne Byfance: Bycause he honored the Arte, wytt, and rare industrie of *Priscus*. But nothing inferior to the inuention of these engines of Force, was the inuention of Gunnes. Which, from an English man, had the occasion and order of first inuenting: though in an other land, and by other men, it was first executed. And they that should see the record, where the occasion and order generall, of Gunning, is first discoursed of, would thinke: that, small thinges, slight, and comon: comming to wise mens consideration, and industrious mens handling, may grow to be of force incredible.

Hypogeiodie, is an Arte Mathematicall, demonstratyng, how, vnder the Sphæricall Superficies of the earth, at any depth, to any perpendicular line assigned (whose distance from the perpendicular of the entrance: and the Azimuth, likewise, in respect of the said entrance, is knowen) certaine way may be præscribed and gone: And how, any way aboue the Superficies of the earth designed, may vnder earth, at any depth limited, be kept: goyng alwayes, perpendicularly, vnder the way, on earth designed: And, contrarywise, Any way, (straight or croked,) vnder the earth, beyng giuen: vppon the vtface, or Superficies of the earth, to Lyne out the same: So, as, from the Centre of the earth, perpendiculars drawen to the Sphæricall Superficies of the earth, shall precisely fall in the Correspondent pointes of those two wayes. This, with all other Cases and circumstances herein, and appertenances, this Arte demonstrateth. This Arte, is very ample in varietie of Conclusions: and very profitable sundry wayes to the Common Wealth. The occasion of my Inuenting this Arte, was at the request of two Gentlemen, who had a certaine worke (of gaine) vnder ground: and their groundes did ioine ouer the worke: and by reason of the crokednes, diuers depthes, and heithes of the way vnder ground, they were in doubt, and at controuersie, vnder whose ground, as then, the worke was. The name onely (before this) was of me published, *De Itinere Subterraneo*: The rest, be at Gods will. For Pioners, Miners, Diggers for Mettalls, Stone, Cole, and for secrete passages vnder ground, betwene place and place (as this land hath diuerse) and for other purposes, any man may easily perceauē, both the great fruite of this Arte, and also in this Arte, the great aide of Geometrie.

Hydragogie, demonstrateth the possible leading of Water, by Natures lawe, and by artificiall helpe, from any head (being a Spring, standing, or running Water) to any other place assigned.

Long

John Dee his Mathematicall Præface.

Long, hath this Arte bene in vse : and much thereof written : and very marueilous workes therein, performed : as may yet appeare, in Italy: by the Ruynes remaining of the Aqueductes . In other places, of Riuers leading through the Maine land, Nauigable many a Mile. And in other places, of the marueilous forcings of Water to Ascend . which all, declare the great skill, to be required of him, who should in this Arte be perfecte, for all occasions of waters possible leading . To speake of the allowance of the Fall, for euery hundred foote: or of the Ventills (if the waters labour be farre, and great) I neede not : Seing, at hand (about vs) many expert men can sufficiently testific, in effecte, the order : though the Demonstration of the Necessitie thereof, they know not : Nor yet, if they should be led, vp and downe, and about Mountaines, from the head of the Spring: and then, a place being assigned : and of them, to be demaunded, how low or high, that last place is, in respecte of the head, from which (so crokedly, and vp and downe) they be come: Perhaps, they would not, or could not, very redily, or nerely assoyle that question. Geometrie therefore, is necessary to *Hydragogie* . Of the sundry wayes to force water to ascend, cyther by *Tympane*, *Kettell mills*, *Skrue*, *Ctesibike*, or such like : in *Vitruuius*, *Agricola*, (and other,) fully, the maner may appeare . And so, thereby, also be most euident, how the Artes, of *Pneumatithmie*, *Helicosophie*, *Statike*, *Trochilike*, and *Menadrie*, come to the furniture of this, in Speculation, and to the Commoditie of the Common Wealth, in practise.

Horometrie, is an Arte Mathematicall, which demonstrateth, how, at all times appointed, the precise vsuall denominatiō of time, may be knowen, for any place assigned . These wordes, are sinoth and plaine easie Englishe, but the reach of their meaning, is farther, then you woulde lightly imagine . Some part of this Arte, was called in olde time, *Gnomonice*: and of late, *Horologiographia* : and in Englishe, may be termed, *Dialling* . Auncient is the vse, and more auncient, is the Inuention . The vse, doth well appeare to haue bene (at the least) aboue two thousand and three hundred yeare agoe : in * King *Achaz* Diall, then, by the Sunne, shewing the distinction of time . By Sunne, Mone, and Sterres, this Dialling may be performed, and the precise Time of day or night knowen . But the demonstratiue delineation of these Dials, of all sortes, requireth good skill, both of *Astronomie*, and *Geometrie* Elementall, Sphæricall, Phænomenall, and Conikall . Then, to vse the groundes of the Arte, for any regular Superficies, in any place offred : and (in any possible apt position therof) theron, to describe (all maner of wayes) how, vsuall howers, may be (by the *Sunnes* shadow) truely determined : will be found no sleight Painters worke . So to Paint, and prescribe the *Sunnes* Motion, to the breadth of a heare . In this Feate (in my youth) I Inuented a way, How in any Horizontall, Murall, or *Æquinoctiall* Diall, &c. At all howers (the *Sunne* shining) the Signe and Degree ascendent, may be knowen . Which is a thing very necessary for the Rising of those fixed Sterres : whose Operation in the Ayre, is of great might, euidently . I speake no further, of the vse hereof . But forasmuch as, Mans affaires require knowledge of Times & Momentes, when, neither *Sunne*, *Mone*, or *Sterre*, can be sene: Therefore, by Industrie Mechanicall, was inuented, first, how, by Water, running orderly, the Time and howers might be knowen: whereof, the famous *Ctesibius*, was Inuentor : a man, of *Vitruuius*, to the Skie (iustly) extolled . Then, after that, by Sand running, were howers measured : Then, by *Trochilike* with waight : And of late time, by *Trochilike* with Spring : without waight . All these,

4. Reg. 20.

d.ij.

by

John Dee his Mathematicall Præface.

*A perpetual
Motion.*

by Sunne or Sterres direction (in certaine time) require ouersight and reformation, according to the heauenly Æquinoctiall Motion : besides the inæqualitie of their owne Operation . There remayneth (without parabolicall meaning herein) among the Philosophers, a more excellent, more commodious, and more marueilous way, then all these : of hauing the motion of the Primouant (or first æquinoctiall motion,) by Nature and Arte, Imitated : which you shall (by furder search in waightier studyes) hereafter, vnderstand more of. And so, it is tyme to finish this Annotation, of Tymes distinction, vsed in our common, and priuate affaires : The commoditie wherof, no man would want, that can tell, how to bestow his tyme.

Zographie, is an Arte Mathematicall, which teacheth and demonstrateth, how, the Interfection of all visuall Pyramides, made by any playne assigned, (the Centre, distance, and lightes, beyng determined) may be, by lynes, and due propre colours, represented. A notable Arte, is this : and would require a whole Volume, to declare the property thereof : and the Commodities ensuyng . Great skill of *Geometrie*, *Arithmetike*, *Perspectiue*, and *Anthropographie*, with many other particular Artes, hath the *Zographer*, nede of, for his perfection. For, the most excellent Painter, (who is but the propre Mechanicien, & Imitator sensible, of the *Zographer*) hath attained to such perfection, that Sense of Man and beast, haue iudged thinges painted, to be things naturall, and not artificiall : aliue, and not dead. This Mechanicall *Zographer* (commonly called the Painter) is meruailous in his skill : and seemeth to haue a certaine diuine power : As, of frendes absent, to make a frendly, present comfort : yea, and of frendes dead, to giue a continuall, silent presence : not onely with vs, but with our posteritie, for many Ages. And so procedyng, Consider, How, in Winter, he can shew you, the liuely vew of Sommers Ioy, and riches : and in Sommer, exhibite the countenance of Winters dolefull State, and nakednes. Cities, Townes, Fortes, Woodes, Armyes, yea whole Kingdomes (be they neuer so farre, or greate) can he, with ease, bring with him, home (to any mans Iudgement) as Paternes liuely, of the thinges rehearsed. In one little house, can he, enclose (with great pleasure of the beholders,) the portrayture liuely, of all visible Creatures, either on earth, or in the earth, liuing : or in the waters lying, Creeping, flyding, or swimming : or of any foule, or fly, in the ayre flying. Nay, in respect of the Starres, the Skie, the Cloudes : yea, in the shew of the very light it selfe (that Diuine Creature) can he match our eyes Iudgement, most nerely. What a thing is this : thinges not yet being, he can represent so, as, at their being, the Picture shall seame (in maner) to haue Created them. To what Artificer, is not Picture, a great pleasure and Commoditie : Which of them all, will refuse the Direction and ayde of Picture : The Architect, the Goldsmith, and the Arras Weauer : of Picture, make great account. Our liuely Herbals, our portraitures of birdes, beastes, and fishes : and our curious Anatomies, which way, are they most perfectly made, or with most pleasure, of vs beholden : Is it not, by Picture onely ? And if Picture, by the Industry of the Painter, be thus commodious and meruailous : what shall be thought of *Zographie*, the Scholemaster of Picture, and chief gouernor : Though I mencion not *Sculpture*, in my Table of Artes Mathematicall : yet may all men perceiue, How, that *Picture* and *Sculpture*, are Sisters germaine : and both, right profitable, in a Commō wealth, and of *Sculpture*, as well as of Picture, excellent Artificers haue written great bokes in commendation. Witnesse I take, of *Georgio Vasari*, *Pittore Aretino* : of *Pomponius Gauricus* : and other. To these two Artes, (with other,) is a certaine od Arte, called *Althamafat*, much beholdyng : more, then the common *Sculptor*, *Entayler*, *Keruer*, *Cutter*, *Grauer*, *Founder*,

John Dee his Mathematicall Præface.

der, or Paynter (&c) know their Arte, to be commodious.

Architecture, to many may seme not worthy, or not mete, to be reckned among the *Artes Mathematicall*. To whom, I thinke good, to giue some account of my so doying. Not worthy, (will they say,) bycause it is but for building, of a house, Pallace, Church, Forte, or such like, grosse workes. And you, also, defined the *Artes Mathematicall*, to be such, as dealt with no Materiall or corruptible thing: and also did demonstratiuely procéde in their faculty, by Number or Magnitude. First, you see, that I count, here, *Architecture*, among those *Artes Mathematicall*, which are Deriued from the Principals: and you know, that such, may deale with Naturall thinges, and sensible matter. Of which, some draw nerer, to the Simple and absolute Mathematicall Speculation, then other do. And though, the *Architect* procureth, enformeth, & directeth, the *Mechanicien*, to handworke, & the building actuall, of house, Castell, or Pallace, and is chief Iudge of the same: yet, with him selfe (as chief *Master* and *Architect*,) remaineth the Demonstratiue reason and cause, of the *Mechanicien*s worke: in Lyne, plaine, and Solid: by *Geometricall*, *Arithmeticall*, *Opticall*, *Musicall*, *Astronomicall*, *Cosmographicall* (& to be brief) by all the former Deriued *Artes Mathematicall*, and other Naturall Artes, hable to be confirmed and stablished. If this be so: then, may you thinke, that *Architecture*, hath good and due allowance, in this honest Company of *Artes Mathematicall* Deriuatiue. I will, herein, craue Iudgement of two most perfect *Architectes*: the one, being *Vitruuius*, the Romaine: who did write ten bookes thereof, to the Emperour *Augustus* (in whose daies our Heauenly Archemaster, was borne): and the other, *Leo Baptista Albertus*, a Florentine: who also published ten bookes therof. *Architectura* (sayth *Vitruuius*) est *Scientia pluribus disciplinis & varijs eruditionibus ornata: cuius Iudicio probantur omnia, quæ ab ceteris Artificibus perficiuntur opera*. That is. Architecture, is a Science garnished with many doctrines & diuerse instructions: by whose Iudgement, all workes, by other workmen finished, are Iudged. It followeth. *Ea nascitur ex Fabrica, & Ratiocinatione. &c. Ratiocinatio autem est, quæ res fabricatas, Solertia ac ratione proportionis, demonstrare atq; explicare potest*. Architecture groweth of Framing, and Reasoning. &c. Reasoning, is that, which of thinges framed, with forecast, and proportion: can make demonstration, and manifest declaration. Againe. *Cum, in omnibus enim rebus, tum maxime etiam in Architectura, hæc duo insunt: quod significatur, & quod significat. Significatur proposita res, de qua dicitur: hanc autem Significat Demonstratio, rationibus doctrinarum explicata*. Forasmuch as, in all thinges: therefore chiefly in *Architecture*, these two thinges are: the thing signified: and that which signifieth. The thing propounded, whereof we speake, is the thing Signified. But Demonstration, expressed with the reasons of diuerse doctrines, doth signifie the same thing. After that. *Vt literatus sit, peritus Graphidos, eruditus Geometrie, & Optices non ignarus: instructus Arithmetica: historias complures nouerit, Philosophos diligenter audiuerit: Musicam sciuerit: Medicinæ non sit ignarus, responsa Iuris peritorum nouerit: Astrologiam, Calig. rationes cognitæ habeat*. An *Architect* (sayth he) ought to vnderstand Languages, to be skilfull of Painting, well instructed in Geometrie, not ignorant of Perspective, furnished with Arithmetike, haue knowledge of many histories, and diligently haue heard Philosophers, haue skill of Musike, not ignorant of Physike, know the aunsweres of Lawyers, and haue Astro-

John Dee his Mathematicall Præface.

nomie, and the courses Cæleſtiall, in good knowledge. He geueth reason, orderly, wherefore all theſe Artes, Doctrines, and Inſtructions, are requiſite in an excellent *Architeēt*. And (for breuitie) omitting the Latin text, thus he hath. Secondly, it is behofefull for an *Architeēt* to haue the knowledge of Painting: that he may the more eaſilie faſhion out, in patternes painted, the forme of what worke he liketh. And Geometrie, geueth to *Architeēture* many helpes: and firſt teacheth the *Uſe* of the Rule, and the Cumpaſſe: wherby (chiefly and eaſilie) the deſcriptions of Buildinges, are deſpatched in Groundplats: and the direCTIONS of Squires, Leuells, and Lines. Likewise, by *Perspective*, the Lightes of the heauen, are well led, in the buildinges: from certaine quarters of the world. By *Arithmetike*, the charges of Buildinges are ſummed together: the meaſures are expreſſed, and the hard queſtions of Symmetries, are by Geometricall Meanes and Methods diſcourſed on. &c. Beſides this, of the Nature of thinges (which in Greke is called *φυσικολογία*) *Philophie* doth make declaration. Which, it is neceſſary, for an *Architeēt*, with diligence to haue learned: becauſe it hath many and diuers naturall queſtions: as ſpecially, in *Aqueduētes*. For in their courses, leadinges about, in the leuell ground, and in the mountinges, the naturall Spirites or breathes are ingendred diuers wayes: The hindrances, which they cauſe, no man can helpe, but he, which out of *Philophie*, hath learned the originall cauſes of thinges. Likewise, who ſoener ſhall read *Cteſibius*, or *Archimedes* bookes, (and of others, who haue written ſuch Rules) can not thinke, as they do: vnleſſe he ſhall haue receaued of *Philophers*, inſtructions in theſe thinges. And Muſike he muſt nedes know: that he may haue vnderſtanding, both of Regular and Mathematicall Muſike: that he may temper well his Baſiſtes, Catapultes, and Scorpions. &c. Moreouer, the Braſen Veſſels, which in Theatres, are placed by Mathematicall order, in ambries, vnder the ſteppes: and the diuerſities of the ſoundes (which y Greicians call *ἡρμία*) are ordred according to Muſicall Symphonies & Harmonies: being diſtributed in y Circuites, by *Diateſſaron*, *Diapente*, and *Diapaſon*. That the conuenient voyce, of the players ſound, whē it came to theſe preparations, made in order, there being increaſed: with y increaſing, might come more cleare & pleaſant, to y eares of the lokers on. &c. And of *Aſtronomie*, is knowē y Eaſt, Weſt, South, and North. The faſhion of the heauen, the *Æquinox*, the *Solſticie*, and the course of the ſterres. Which thinges, vnleaſt one know: he can not perceiue, any thyng at all, the reaſon of *Horologies*. Seyng therfore this ample Science, is garniſhed, beautified and ſtored, with ſo many and ſundry ſkils and knowledges: I thinke, that none can iuſtly account them ſelues *Architeētes*, of the ſuddeyne. But they onely, who from their childeſ yeares, aſcendyng by theſe degrees of knowledges, beyng foſtered vp with the atteynnyng of many Languages and Artes, haue wonne to the high Tabernacle of *Architeēture*. &c. And to whom Nature hath giuen ſuch quicke Circumſpection, ſharpnes of witt, and Memorie, that they may be very abſolutely ſkillfull in *Geometrie*, *Aſtronomie*, *Muſike*, and the reſt of the Artes Mathematicall.

John Dee his Mathematicall Praeface.

call: Such surmount and passe the calling, and state, of Architec̃tes: and are become Mathematiciens: &c. And they are found seldome. As, in tymes past, was Aristarchus Samius: Philolaus, and Archytas, Tarentynes: Apollonius Pergens: Eratosthenes Cyreneus: Archimedes, and Scopas, Syracusians. Who also, left to theyr posteritie, many Engines and Gnomonicall workes: by numbers and naturall meanes, inuented and declared.

Thus much, and the same wordes (in sence) in one onely Chapter of this Incōparable *Architect Vitruuius*, shall you finde. And if you should, but take his boke in your hand, and slightly loke thorough it, you would say straight way: This is *Geometrie, Arithmetike, Astronomie, Musike, Anthropographie, Hydragogie, Horometrie*. &c. and (to cōclude) the Storehouse of all workmāship. Now, let vs listen to our other Iudge, our Florentine, *Leo Baptista*: and narrowly consider, how he doth determine of *Architect̃ure*. *Sed anteq̃ ultra progrediar*. &c. But before I procede any further

(sayth he) I thinke, that I ought to expresse, what man I would haue to bee allowed an *Architect̃*. For, I will not bryng in place a Carpenter: as though you might Compare him to the Chief Masters of other Artes. For the hand of the Carpenter, is the *Architec̃tes* Instrument. But I will appoint the *Architect̃* to be

that man, who hath the skill, (by a certaine and meruailous meanes and way,) both in minde and Imagination to determine: and also in worke to finish: what

workes so euer, by motion of waight, and cuppling and framynge together of bodies, may most aptly be Commodious for the worthiest Vses of Man. And that he

may be able to performe these thinges, he hath nede of atteynynge and knowledge of the best, and most worthy thynges. &c. The whole Feate of *Architect̃ure* in building, consisteth in Lineamentes, and in Framynge. And the whole power and skill of Lineamentes, tendeth to this: that the right and absolute way may

be had, of Coptyng and ioyning Lines and angles: by which, the face of the building or frame, may be comprehended and concluded. And it is the property of Lineamentes, to prescribe vnto buildynges, and euery part of them, an apt place,

& certaine nūber: a worthy maner, and a femely order: that, so, y whole forme and figure of the building, may rest in the very Lineamentes. &c. And we may

prescribe in mynde and imagination the whole formes, all materiall stuffe being secluded. Which point we shall atteyne, by Notynge and forepointynge the angles, and lines, by a sure and certaine direction and connexion. Seyng then, these

thinges, are thus: Lineamente, shalbe the certaine and constant prescribing, conceived in mynde: made in lines and angles: and finished with a learned minde and wyt. We thanke you Master Baptist, that you haue so aptly brought your

Arte, and phrased therof, to haue some Mathematicall perfection: by certaine order, nūber, forme, figure, and Symmetrie mentall: all naturall & sensible stuffe set a part. Now, then, it is euident, (Gentle reader) how aptely and worthely, I haue

preferred *Architect̃ure*, to be bred and fostered vp in the Dominion of the percles Princeſſe, *Mathematica*: and to be a naturall Subiect of hers. And the name of *Architect̃ure*, is of the principalitie, which this Science hath, aboue all other Artes. And Plato affirmeth, the *Architect̃* to be Master ouer all, that make any worke.

Wherupon, he is neither Smith, nor Builder: nor, separately, any Artificer: but the

A Mathematicien.

Vitruuius.

Who is an Architect.

* The Immaterialitie of perfect Architecture.

What, Lineament is.

Note.

John Dee his Mathematicall Præface.

Hed, the Prouest, the Director, and Iudge of all Artificiall workes, and all Artificers. For, the true *Architect*, is hable to teach, Demonstrate, distribute, describe, and Iudge all workes wrought. And he, onely, searcheth out the causes and reasons of all Artificiall thynges. Thus excellent, is *Architecture*: though few (in our dayes) attayne thereto: yet may not the Arte, be otherwise thought on, then in very dede it is worthy. Nor we may not, of auncient Artes, make new and imperfect Definitions in our dayes: for scarcitie of Artificers: No more, than we may pynche in, the Definitions of *Wisedome*, or *Honestie*, or of *Friendeshyp* or of *Iustice*: No more will I consent, to Diminish any whit, of the perfection and dignitie; (by iust cause) allowed to absolute *Architecture*. Vnder the Direction of this Arte, are thre principall, necessary *Mechanicall Artes*. Namely, *Howsing*, *Fortification*, and *Naupegie*. *Howsing*, I vnderstand, both for Diuine Seruice, and Mans common vsage: publike, and priuate. Of *Fortification* and *Naupegie*, straunge matter might be told you: But perchaunce, some will be tyred, with this Bederoll, all ready rehearsed: and other some, will nycely nip my grosse and homely discoursing with you: made in post hast: for feare you should wante this true and frendly warnyng, and tast giuyng, of the *Power Mathematicall*. Lyfe is short, and vncertaine: Tymes are perilouse: &c. And still the Printer awayting, for my pen staying: All these thinges, with farder matter of Ingratefulnes, giue me occasion to passe away, to the other Artes remainyng, with all spede possible.

THE Arte of *Nauigation*, demonstrateth how, by the shortest good way, by the aptest Directiō, & in the shortest time, a sufficient Ship, betwene any two places (in passage Nauigable,) assigned: may be cōducted: and in all stormes, & naturall disturbances chauncyng, how, to vse the best possible meanes, whereby to recouer the place first assigned. What nede, the *Master Pilote*, hath of other Artes, here before recited, it is easie to know: as, of *Hydrographie*, *Astronomie*, *Astrologie*, and *Horometrie*. Presupposing continually, the common Base, and foundation of all: namely, *Arithmetike* and *Geometrie*. So that, he be hable to vnderstand, and Iudge his own necessary Instrumentes, and furniture Necessary: Whether they be perfectly made or no: and also can, (if nede be) make them, hym selfe. As Quadrantes, The Astronomers Ryng, The Astronomers staffe, The Astrolabe vniuersall. An Hydrographicall Globe. Charts Hydrographicall, true, (not with parallel Meridians). The Common Sea Compas: The Compas of variacion: The Proportionall, and Paradoxall Compasses (of me Inuented, for our two Moscouy Master Pilotes, at the request of the Company) Clockes with spryng: houre, halfe houre, and three houre Sandglassses: & sundry other Instrumētes: And also, be hable, on Globe, or Playne to describe the Paradoxall Compasse: and duely to vse the same, to all maner of purposes, whereto it was inuented. And also, be hable to Calculate the Planetes places for all tymes.

Moreouer, with Sonne Mone or Sterre (or without) be hable to define the Longitude & Latitude of the place, which he is in: So that, the Longitude & Latitude of the place, from which he sayled, be giuen: or by him, be knowne. whereto, appertayneth expert meanes, to be certified euer, of the Ships way. &c. And by foreseeing the Rising, Settyng, Nonestedyng, or Midnightyng of certaine tempestuous fixed Sterres: or their Coniunctions, and Anglynges with the Planetes, &c. he ought to haue expert coniecture of Stormes, Tempestes, and Spoutes: and such lyke Meteorologicall effectes, daungerous on Sea. For (as *Plato* sayth,) *Mutationes,*

oppor-

John Dee his Mathematicall Præface.

opportunitatesq; temporum presentire, non minus rei militari, quàm Agricultura, Navigationiq; conuenit. To foresee the alterations and opportunities of tymes, is conuenient, no lesse to the Art of Warre, then to Husbandry and Nauigation. And besides such cunningg meanes, more euident tokens in Sonne and Mone, ought of hym to be knowen: such as (the Philosophicall Poëte) Virgilius teacheth, in hys Georgikes. Where he sayth,

*Sol quoq; & exoriens & quum se condet in undas,
Signa dabit, Solem certissima signa sequuntur. &c.*

*Nam sæpe videmus,
Ipsius in vultu varios errare colores.*

Ceruleus, pluuiam denunciat, igneus Euros.

Sin macula incipient rutilo immiscerier igni,

Omnia tum pariter vento, nimbisq; videbis

Feruere: non illa quisquam me nocte per altum

Ire, neq; a terra moueat conuellere funem. &c.

Sol tibi signa dabit. Solem quis dicere falsum

Audeat: ————— &c.

Georgic. I.

And so of Mone, Sterres, Water, Ayre, Fire, Wood, Stones, Birdes, and Beastes, and of many thynges els, a certaine Sympathicall forewarnyng may be had: sometymes to great pleasure and proffit, both on Sea and Land. Sufficiently, for my present purpose, it doth appeare, by the premisses, how *Mathematicall*, the *Arte of Nauigation*, is: and how it nedeth and also vseth other *Mathematicall Artes*: And now, if I would go about to speake of the manifold Commodities, commyng to this Land, and others, by Shypps and *Nauigation*, you might thinke, that I catch at occasions, to vse many wordes, where no nede is.

Yet, this one thyng may I, (iustly) say. In *Nauigation*, none ought to haue greater care, to be skillfull, then our English Pylotes. And perchaunce, Some, would more attempt: And other Some, more willingly would be aydyng, if they wist certainly, What Priuiledge, God had endued this Iland with, by reason of Situation, most commodious for *Nauigation*, to Places most Famous & Riche. And though, (of* Late) a young Gentleman, a Courragious Capitaine, was in a great readiness, with good hope, and great causes of persuation, to haue ventured, for a Discouerye, (either *Westerly*, by *Cape de Paramantia*: or *Esterly*, aboue *Noua Zemla*, and the *Cyremisses*) and was, at the very nere tyme of Attemptyng, called and employed otherwise (both then, and since,) in great good seruice to his Countrey, as the Irish Rebels haue* tasted: Yet, I say, (though the same Gentleman, doo not hereafter, deale therewith) Some one, or other, should listen to the Matter: and by good aduise, and discrete Circumspection, by little, and little, wyne to the sufficient knowledge of that Trade and Voyage: Which, now, I would be sory, (through Carelesnesse, want of Skill, and Courrage,) should remayne Vnknowne and vnheard of. Seyng, also, we are herein, halfe Challenged, by the learned, by halfe request, published. Therof, verely, might grow Commoditye, to this Land chiefly, and to the rest of the Christen Common wealth, farre passing all riches and worldly Threasure.

** Anno. 1567
S. H. G.*

** Anno. 1569*

Thaumaturgike, is that Art Mathematicall, which giueth certaine order to make straunge workes, of the sense to be perceiued, and of men greatly to be wondred at. By sundry meanes, this *Wonderworke* is wrought. Some, by *Pneumatithmie*. As the workes of *Ctesibius* and *Hero*,

A.j.

Some

Iohn Dee his Mathematicall Praeface.

Some by waight: wherof *Timaeus* speaketh. Some, by Stringes strayned, or Springs, therewith Imitating liuely Motions. Some, by other meanes, as the Images of Mercurie: and the brasen hed, made by *Albertus Magnus*, which dyd seme to speake. *Boethius* was excellent in these feates. To whom, *Cassiodorus* writyng, sayth. Your purpose is to know profound thynges: and to shew meruayles. By the disposition of your *Arte*, Metals do low: *Diomedes* of brasse, doth blow a Trumpet loude: a brasen Serpent hisseth: byrdes made, sing sweetely. Small thynges we rehearse of you, who can Imitate the heauen. &c.

* Anno. 1551

Of the straunge Selfmouyng, which, at Saint Denys, by Paris, * I saw, ones or twise (*Orontius* beyng then with me, in Company) it were to straunge to tell. But some haue written it. And yet, (I hope) it is there, of other to be sene. And by *Perspectiue* also straunge thinges, are done. As partly (before) I gaue you to vnderstand in *Perspectiue*. As, to see in the Ayre, a loft, the lyuely Image of an other man, either walkyng to and fro: or standyng still. Likewise, to come into an house, and there to see the liuely shew of Gold, Siluer or precious stones: and commyng to take them in your hand, to finde nought but Ayre. Hereby, haue some men (in all other matters counted wise) fouly ouershot the selues: misdeaming of the meanes. Therefore sayd *Claudius Celestinus*. *Hodie magna literaturae uis & magnae reputationis uidemus, opera quaedam quasi miranda, supra Naturā putare: de quibus in Perspectiua doctus causam facilliter reddidisset.* That is. Now a dayes, we see some men, yea of great learnyng and reputation, to Iudge certain workes as meruaylous, aboue the power of Nature: Of which workes, one that were skillfull in *Perspectiue* might easely haue giuen the Cause.

De his qua
Mundo mi-
rabiliter eue-
niunt. cap. 8.

Of *Archimedes* Sphere, *Cicero* witnesseth. Which is very straunge to thinke on. For when *Archimedes* (sayth he) did fasten in a Sphere, the mouynges of the Sonne, Mone, and of the fine other Planets, he did, as the God, which (in *Timaeus* of *Plato*) did make the world. That, one turnyng, should rule motions most vnlike in slownes, and swiftnes. But a greater cause of meruayling we haue by *Claudianus* report hereof. Who affirmeth this *Archimedes* worke, to haue ben of Glasse. And discourseth of it more at large: which I omit. The Doue of wood, which the Mathematicien *Archytas* did make to flye, is by *Agellius* spoken of. Of *Daedalus* straunge Images, *Plato* reporteth. *Homere* of *Vulcans* Selfmouers, (by secrete wheles) leaueth in writyng. *Aristotle*, in hys *Politikes*, of both, maketh mention. Meruaylous was the workmanship, of late dayes, performed by good skill of *Trochilike*. &c. For in Noremberge, A flye of Iern, beyng let out of the Artificers hand, did (as it were) fly about by the gestes, at the table; and at length, as though it were weary, retourne to his masters hand agayne. Moreouer, an Artificiall Egle, was ordred, to fly out of the same Towne, a mighty way, and that a loft in the Ayre, toward the Emperour comming thether: and followed hym, beyng come to the gate of the towne. * Thus, you see, what, *Arte Mathematicall* can performe, when Skill, will, Industry, and Hability, are duely applyed to profe.

Tusc. 1.

✎ *

A Digression.
Apologeticall.

AND for these, and such like marueilous Actes and Feates, Naturally, Mathematically, and Mechanically, wrought and contriued: ought any honest Student, and Modest Christian Philosopher, be counted, & called a Coniurer? Shall the folly of Idiotes, and the Mallice of the Scornfull, so much preuaile, that He, who seeketh no worldly gaine or glory at their handes: But onely, of God, the threasor of heauenly wisedome, & knowledge of pure veritie: Shall he (I say) in the meane

space,

John Dee his Mathematicall Præface.

space, be robbed and spoiled of his honest name and fame : He that seeketh (by S. Pauls aduertisement) in the Creatures Properties, and wonderfull vertues, to finde iuste cause, to glorifie the Æternall, and Almighty Creator by : Shall that man, be (in hugger mugger) condemned, as a Companion of the Helhoundes, and a Caller, and Coniurer of wicked and damned Spirites : He that bewaileth his great want of time, sufficient (to his contentation) for learning of Godly wisdom, and Godly Verities in : and onely therein setteth all his delight : Will that mā leese and abuse his time, in dealing with the Chiefe enimie of Christ our Redemer: the deadly foe of all mankind : the subtile and impudent peruerter of Godly Veritie: the Hypocriticall Crocodile: the Enuious Basiliske, continually desirous, in the twinke of an eye, to destroy all Mankind, both in Body and Soule, æternally : Surely (for my part, somewhat to say herein) I haue not learned to make so brutish, and so wicked a Bargaine. Should I, for my xx. or xxv. yeares Studie : for two or three thousand Markes spending : seuen or eight thousand Miles going and trauailing, onely for good learninges sake : And that, in all maner of wethers : in all maner of waies and passages : both early and late : in daunger of violence by man : in daunger of destruction by wilde beastes : in hunger : in thirst : in perillous heates by day, with toyle on foote : in dangerous dampes of colde, by night, almost bereuing life : (as God knoweth) : with lodgings, oft times, to small ease : and sometime to lesse securitie. And for much more (then all this) done & suffred, for Learning and attaining of Wisedome : Should I (I pray you) for all this, no otherwise, nor more warily : or (by Gods mercifulnes) no more luckily, haue fished, with so large, and costly, a Nette, so long time in drawing (and that with the helpe and aduise of Lady Philosophie, & Queene Theologie) : but at length, to haue caught, and drawen vp, * a Frog : Nay, a Deuill : For, so, doth the Common peuissh Pratler. Imagine and Iangle: And, so, doth the Malicious skorner, secretly wishe, & brauely and boldly face down, behinde my backe. Ah, what a miserable thing, is this kinde of Men : How great is the blindnes & boldnes, of the Multitude, in thinges aboue their Capacitie : What a Land : what a People : what Maners : what Times are these : Are they become Deuils, them selues: and, by false witnesse bearing against their Neighbour, would they also, become Murderers : Doth God, so long geue them respite, to reclaime them selues in, from this horrible slaundering of the guiltlesse : contrary to their owne Consciences : and yet will they not cease : Doth the Innocent, forbear the calling of them, Iuridically to aunswere him, according to the rigour of the Lawes : and will they despise his Charitable pacience : As they, against him, by name, do forge, fable, rage, and raise slaunder, by Worde & Print: Will they prouoke him, by worde and Print, likewise, to Note their Names to the World : with their particular deuises, fables, beastly Imaginations, and vnchristenlike slaunders : Well : Well. O (you such) my vnkinde Countrey men. O vn-naturall Countrey men. O vnthankfull Countrey men. O Brainicke, Rashe, Spitefull, and Disdainfull Countrey men. Why oppresse you me, thus violently, with your slaundering of me : Contrary to Veritie : and contrary to your owne Consciences : And I, to this hower, neither by worde, deede, or thought, haue bene, any way, hurtfull, damageable, or iniurious to you, or yours : Haue I, so long, so dearly, so farre, so carefully, so painfully, so daungerously sought & trauailed for the learning of Wisedome, & atteyning of Vertue : And in the end (in your iudgemēt) am I become, worse, then when I begā : Worse, thē a Mad man? A dangerous Member in the Common Wealth: and no Member of the Church of Christ? Call you this, to be Learned? Call you this, to be a Philosopher? and a louer of Wisedome? To forsake the straight heavenly way : and to wallow in the broad way of

* A prouerb.
Fayre fisht,
and caught a
Frog.

Iohn Dee his Mathematicall Præface.

damnation? To forsake the light of heauenly Wisedome: and to lurke in the dungeon of the Prince of darkenesse? To forsake the Veritie of God, & his Creatures: and to fawne vpon the Impudent, Craftie, Obstinate Lier, and continuall disgracer of Gods Veritie, to the vttermoſt of his power? To forsake the Life & Blisse Æternall: and to cleaue vnto the Author of Death euerlasting? that Murderous Tyrant, most greedily awaiting the Pray of Mans Soule? Well: I thanke God and our Lorde Iesus Christ, for the Comfort which I haue by the Examples of other men, before my time: To whom, neither in godlines of life, nor in perfection of learning, I am worthy to be compared: and yet, they sustained the very like Iniuries, that I do: or rather, greater. Patient *Socrates*, his *Apologie* will testifie: *Apuleius* his *Apologies*, will declare the Brutishnesse of the Multitude. *Ioannes Picus*, Earle of Mirandula, his *Apologie* will teach you, of the Raging slaunder of the Malicious Ignorant against him. *Ioannes Trithemius*, his *Apologie* will specifie, how he had occasion to make publike Protestation: as well by reason of the Rude Simple: as also, in respect of such, as were counted to be of the wisest sort of men. Ma-

- ny could I recite: But I deferre the precise and determined handling of this matter: being loth to detect the Folly & Mallice of my Natiue Countrey men.* Who, so hardly, can digest or like any extraordinary course of Philosophicall Studies: not falling within the Cumpasse of their Capacitie: or where they are not made priuie of the true and secrete cause, of such wonderfull Philosophicall Feates. These men, are of fower sortes, chiefly. The first, I may name, *Vaine pratling busie bodies*: The second, *Fond Frenches*: The third, *Imperfectly zelous*: and the fourth, *Malicious Ignorant*. To eche of these (briefly, and in charitie) I will say a word
1. or two, and so returne to my Præface. *Vaine pratling busie bodies*, vse your idle assemblies, and conferences, otherwise, then in talke of matter, either aboue your Capacities, for hardnesse: or contrary to your Consciences, in Veritie. *Fond Frenches*, leaue of, so to commend your vnacquainted friend, vpon blinde affection: As, because he knoweth more, then the common Student: that, therefore, he must needes be skilfull, and a doer, in such matter and maner, as you terme *Coniuring*. Weening, thereby, you aduaunce his fame: and that you make other men, great marueilers of your hap, to haue such a learned friend. Cease to ascribe Impietie, where you preterid Amitie. For, if your tounge were true, then were that your friend, *Vntrue*, both to God, and his Soueraigne. Such *Frenches* and *Fondlinges*, I shake of, and renounce you: Shake you of, your Folly. *Imperfectly zelous*, to you,
 3. do I say: that (perhaps) well, do you Meane: But farre you misse the Marke: If a Lambe you will kill, to feede the flocke with his bloud. Sheepe, with Lambes bloud, haue no naturall sustenance: No more, is Christes flocke, with horrible slaunders, duely ædified. Nor your faire pretense, by such rashe ragged Rhetorike, any whit, well graced. But such, as so vse me, will finde a fowle Cracke in their Credite. Speake that you know: And know, as you ought: Know not, by Heare say, when life lieth in daunger. Search to the quicke, & let Charitie be your guide.
 4. *Malicious Ignorant*, what shall I say to thee? *Prohibe linguam tuam a malo. A detractione parcite lingua. Cause thy tounge to refraine frō euill. Refraine your tounge from slaunder*. Though your tounge be sharpned, Serpent like, & Adders poison lye in your lippes: yet take heede, and thinke, betimes, with your selfe, *Vir linguosus non stabilietur in terra. Virum violentum venabitur malum, donec precipitetur*. For, sure I am, *Quia faciet Dominus Iudicium afflicti: & vindictam pauperum*.

Psal. 140.

Thus, I require you, my assured friendes, and Countrey men (you Mathematiciens, Mechaniciens, and Philosophers, Charitable and discrete) to deale in my behalfe,

John Dee his Mathematicall Præface.

behalf, with the light & vnttrue tounge, my enuious Aduersaries, or Fond friends. And farther, I would wishe, that at leysor, you would consider, how *Basilius Magnus*, layeth *Moses* and *Daniel*; before the eyes of those, which count all such Studies Philosophicall (as mine hath bene) to be vngodly, or vnprofitable. Waye well *S. Stephen* his witnesse of *Moses*. *Eruditus est Moses omni Sapientia Egyptiorū: & erat potens in verbis & operibus suis.* *Moses* was instructed in all maner of wisedom of the *Ægyptians*: and he was of power both in his wordes, and workes. You see this Philosophicall Power & Wisedom, which *Moses* had, to be nothing misliked of the Holy Ghost. Yet *Plinius* hath recorded, *Moses* to be a wicked Magicien. And that (of force) must be, either for this Philosophicall wisedom, learned, before his calling to the leading of the Children of *Israel*: or for those his wonders, wrought before King *Pharao*, after he had the conducting of the *Israelites*. As concerning the first, you perceave, how *S. Stephen*, at his Martyrdome (being full of the Holy Ghost) in his Recapitulation of the olde Testament, hath made mention of *Moses* Philosophie: with good liking of it: And *Basilius Magnus* also, auoucheth it, to haue bene to *Moses* profitable (and therefore, I say, to the Church of God, necessary). But as cōcerning *Moses* wonders, done before King *Pharao*: God, him selfe, sayd: *Vide ut omnia ostenta, quæ posui in manu tua, facias coram Pharaone.* See that thou do all those wonders before *Pharao*, which I haue put in thy hand. Thus, you euidently perceave, how rashly, *Plinius* hath slandered *Moses*, of vayne fraudulent *Magike*, saying: *Est & alia Magices Factio, a Mose, Iamne, & Iotape, Iudeis pendens: sed multis millibus annorum post Zoroastrem. &c.* Let all such, therefore, who, in Iudgement and Skill of Philosophie, are farre Inferior to *Plinie*, take good heede, least they ouershoot them selues rashly, in Iudging of *Philosophers* *straunge Actes*: and the Meanes, how they are done. But, much more, ought they to beware of forging, deuising, and imagining monstrous feates, and wonderfull workes, when and where, no such were done: no, not any sparke or likelihode, of such, as they, without all shame, do report. And (to conclude) most of all, let them be ashamed of Man; and afraide of the dreadfull and Iuste Iudge: both Follishly or Maliciously to deuise: and then, deuilishly to father their new fond Monsters on me: Innocent, in hand and hart: for trespassing either against the lawe of God, or Man, in any my Studies or Exercises, Philosophicall, or Mathematicall: As in due time, I hope, will be more manifest.

Now end I, with *Archemastrie*. Which name, is not so new, as this Arte is rare. For an other Arte, vnder this, a degree (for skill and power) hath bene indued with this English name before. And yet, this, may serue for our purpose, sufficiently, at this present. This Arte, teacheth to bryng to actuall experience sensible, all worthy conclusions by all the Artes Mathematicall purposed; & by true Naturall Philosophie concluded: & both addeth to them a farder scope, in the termes of the same Artes, & also by hys propre Method, and in peculier termes, procedeth, with helpe of the foresayd Artes, to the performance of complet Experiēces, which of no particular Art, are hable (Formally) to be challenged. If you remember, how we considered *Architecture*, in respect of all common handworkes: some light may you haue, therby, to vnderstand the Souerainty and propriētie of this Science. Science I may call it, rather, then an Arte: for the excellency and Mastershyp it hath, ouer so many, and so mighty Artes and

Art. 7. C.

Lib. 30.

Cap. 1.

1.

2.

3.

2.

3.

A. iij.

Sciences.

John Dee his Mathematicall Præface.

Sciences. And bycause it procedeth by *Experiences*, and searcheth forth the causes of Conclusions, by *Experiences*: and also putteth the Conclusions them selues, in *Experience*, it is named of some, *Scientia Experimentalis*. The *Experimentall Science*. *Nicolaus Cusanus* termeth it so, in hys *Experimentes Statikall*. And an other *Philosopher*, of this land Natiue (the floure of whose worthy fame, can neuer dye nor wither) did write therof largely, at the request of *Clement the sixth*. The Arte carrieth with it, a wonderfull Credit: By reason, it certifieth, sensibly, fully, and completely to the vtmost power of Nature, and Arte. This Arte, certifieth by *Experience* complete and absolute: and other Artes, with their Argumentes, and Demonstrations, persuaide: and in wordes, proue very well their Conclusions. * But wordes, and Argumentes, are no sensible certifying: nor the full and finall frute of Sciences practisable. And though some Artes, haue in them, *Experiences*, yet they are not complete, and brought to the vttermost, they may be stretched vnto, and applyed sensibly. As for example: the Naturall *Philosopher* disputeth and maketh goodly shew of reason: And the *Astronomer*, and the *Opticall Mechanicien*, put some thynges in *Experience*: but neither, all, that they may: nor yet sufficiently, and to the vtmost, those, which they do. There, then, the *Archemaster* steppeth in, and leadeth forth on, the *Experiences*, by order of his doctrine *Experimentall*, to the chief and finall power of Naturall and Mathematicall Artes. Of two or three men, in whom, this Description of *Archemastrie* was *Experimentally*, verified, I haue read and hard: and good record, is of their such perfection. So that, this Art, is no fantastickall Imagination: as some Sophister, might, *Cum suis Insolubilibus*, make a flourish: and daseill your Imagination: and dash your honest desire and Courage, from beleuing these thinges, so vnheard of, so meruaylous, & of such Importance. Well: as you will. I haue forewarned you. I haue done the part of a frende: I haue discharged my Duety toward God: for my small Talent, at hys most mercyfull handes receiued. To this Science, doth the *Science Alnirangiat*, great Seruice. Muse nothyng of this name. I chaunge not the name, so vsed, and in Print published by other: beyng a name, propre to the Science. Vnder this, commeth *Ars Sintrillia*, by *Artephius*, briefly written. But the chief Science, of the *Archemaster*, (in this world) as yet knowen, is an other (as it were) OPTICAL Science: wherof, the name shall be told (God willyng) when I shall haue some, (more iust) occasion, therof, to Discourse.

Here, I must end, thus abruptly (Gentle frende, and vnfayned louer of honest and necessary verities.) For, they, who haue (for your sake, and vertues cause) requested me, (an old forworne Mathematicien) to take pen in hand: (through the confidence they reposed in my long experience: and tryed sincerity) for the declaring and reportyng somewhat, of the frute and commodity, by the Artes Mathematicall, to be atteyned vnto: euen they, Sore agaynst their willes, are forced, for sundry causes, to satisfie the workemans request, in endyng forthwith: He, so feareth this, so new an attempt, & so costly: And in matter so slenderly (hitherto) among the common Sorte of Studentes, considered or esteemed.

And where I was willed, somewhat to alledge, why, in our vulgare Speche, this part of the Principall Science of *Geometrie*, called *Euclides Geometricall Elementes*, is published, to your handlyng: being vnlatined people, and not Vniuersitie Scholers: Verily, I thinke it nedelesse.

I. For, the Honour, and Estimation of the Vniuersities, and Graduates, is, hereby, nothing diminished. Seing, from, and by their Nurse Children, you receaue all this Benefite: how great soeuer it be.

Neither

Iohn Dee his Mathematicall Præface.

Neither are their Studies, hereby, any whit hindred. No more, then the Italian *Vniuersities*, as *Academia Bononiensis, Ferrariensis, Florentina, Mediolanensis, Patauina, Papiensis, Perusina, Pisana, Romiana, Senensis*, or any one of them, finde them selues, any deale, disgraced, or their Studies any thing hindred, by *Frater Lucas de Burgo*, or by *Nicolaus Tartalea*, who in vulgar Italian language, haue published, not onely *Euclides Geometrie*, but of *Archimedes* somewhat: and in Arithmetike and Practicall Geometrie, very large volumes, all in their vulgar speche. Nor in Germany haue the famous *Vniuersities*, any thing bene discontent with *Albertus Durerus*, his Geometricall Institutions in Dutch: or with *Gulielmus Xylander*, his learned translation of the first sixe bookes of *Euclide*, out of the Greke into the high Dutch. Nor with *Gualterus H. Riffius*, his Geometricall Volume: very diligently translated into the high Dutch tounge, and published. Nor yet the *Vniuersities* of Spaine, or Portugall, thinke their reputation to be decayed: or suppose any their Studies to be hindred by the Excellent *P. Nonnius*, his Mathematicall workes, in vulgare speche by him put forth. Haue you not, likewise, in the French tounge, the whole Mathematicall Quadriue? and yet neither Paris, Orleance, or any of the other *Vniuersities* of Fraunce, at any time, with the Translators, or Publishers offended: or any mans Studie thereby hindred?

And surely, the Common and Vulgar Scholer (much more, the Gramarian) before his comming to the *Vniuersitie*, shall (or may) be, now (according to *Plato* his Counsell) sufficiently instructed in *Arithmetike* and *Geometrie*, for the better and easier learning of all maner of *Philosophie, Academicall, or Peripateticall*. And by that meanes, goe more cherefully, more skilfully, and speedily forward, in his Studies, there to be learned. And, so, in lesse time, profite more, then (otherwise) he should, or could do.

Also many good and pregnant Englishe wittes, of young Gentlemen, and of other, who neuer intend to meddle with the profound search and Studie of *Philosophie* (in the *Vniuersities* to be learned) may neuerthelesse, now, with more ease and libertie, haue good occasion, vertuously to occupie the sharpnesse of their wittes: where, els (perchance) otherwise, they would in fond exercises, spend (or rather leese) their time: neither seruing God: nor furduring the Weale, common or priuate.

And great Comfort, with good hope, may the *Vniuersities* haue, by reason of this *Englishe Geometrie, and Mathematicall Præface*, that they (hereafter) shall be the more regarded, esteemed, and resorted vnto. For, when it shall be knownen and reported, that of the *Mathematicall Sciences* onely, such great Commodities are ensuing (as I haue specified): and that in dede, some of you vnlained Studentes, can be good witnesse, of such rare fruite by you enioyed (thereby): as either, before this, was not heard of: or els, not so fully credited: Well, may all men coniecture, that farre greater ayde, and better furniture, to winne to the Perfection of all *Philosophie*, may in the *Vniuersities* be had: being the Storehouses & Threasury of all Sciences, and all Artes, necessary for the best, and most noble State of Common Wealthes.

Besides this, how many a Common Artificer, is there, in these Realmes of England and Ireland, that dealeth with Numbers, Rule, & Cumpasse: Who, with their owne Skill and experience, already had, will be hable (by these good helpes and informations) to finde out, and deuise, new workes, straunge Engines, and Instrumentes: for sundry purposes in the Common Wealth? or for priuate pleasure? and for the better maintayning of their owne estate? I will not (therefore)

John Dee his Mathematicall Præface.

fight against myne owne shadowe. For, no man (I am sure) will open his mouth against this Enterprise. No mā (I say) who either hath Charitie toward his brother (and would be glad of his furtherance in vertuous knowledge) : or that hath any care & zeale for the bettering of the Cōmon state of this Realme. Neither any, that make accompt, what the wiser sort of men (Sage and Stayed) do thinke of them. To none (therefore) will I make any *Apologie*, for a vertuous acte doing : and for cōmending, or setting forth, Profitable Artes to English men, in the English tounge.

„ But, vnto God our Creator, let vs all be thankefull : for that, *As he, of his Good-*
„ *nes, by his Powre, and in his wisdom, hath Created all thynges, in Number,*
„ *Waight, and Measure*: So, to vs, of hys great Mercy, he hath reuealed Meanes,
„ whereby, to attayne the sufficient and necessary knowledge of the foresayd hys
„ three principall Instrumentes : Which Meanes, I haue abundantly proued vnto
„ you, to be the *Sciences and Artes Mathematicall*.

And though I haue ben pinched with straightsnes of tyme: that, no way, I could so pen downe the matter (in my Mynde) as I determined: hopng of conuenient laysure. Yet, if vertuous zeale, and honest Intent prouoke and bryng you to the readyng and examinyng of this Compendious treatise, I do not doute, but, as the veritie therof (accordng to our purpose) will be euident vnto you : So the pith and force therof, will persuaide you : and the wonderfull frute therof, highly pleasure you. And that you may the easier perceiue, and better remember, the principall pointes, whereof my Preface treateth, I will giue you the Groundplatt of my whole discourse, in a Table annexed: from the first to the last, somewhat Methodically contriued.

*The Ground
platt of this
Preface in a
Table.*

If Haft, hath caused my poore pen, any where, to stumble : You will, (I am sure) in part of recompence, (for my earnest and sincere good will to pleasure you), Consider the rockish huge mountaines, and the perilous vnbeaten wayes, which (both night and day, for the while) it hath toyled and labored through, to bryng you this good Newes, and Comfortable profe, of Vertues frute.

So, I Commit you vnto Gods Mercyfull direction, for the rest : hartely besechyng hym, to prosper your Studyes, and honest Intentes: to his Glory, & the Commodity of our Countrey. *Amen.*

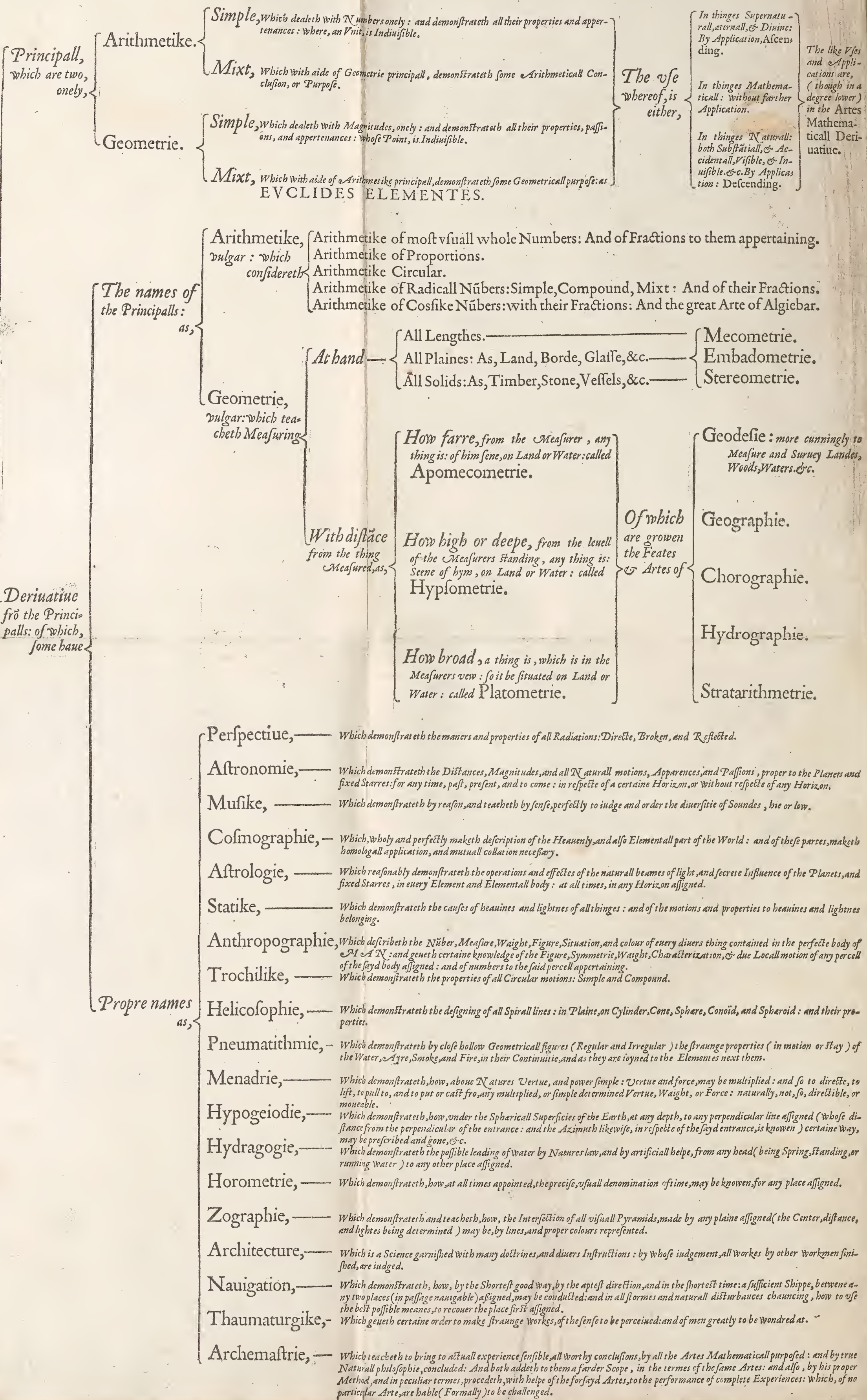
*Written at my poore House
At Mortlake.*

Anno. 1579. Februarij. 9.



Here haue you (according to my promise) the Groundplat of my MATHEMATICALL Praeface: annexed to *Euclide* (now first) published in our Englishetounge. An. 1570. Febr. 3.

Sciences,
and Artes
Mathe-
maticall,
are, either



BOOK V.

INTRODUCTORY NOTE.

The anonymous author of a scholium to Book v. (Euclid, ed. Heiberg, Vol. v. p. 280), who is perhaps Proclus, tells us that "some say" this Book, containing the general theory of proportion which is equally applicable to geometry, arithmetic, music, and all mathematical science, "is the discovery of Eudoxus, the teacher of Plato." Not that there had been no theory of proportion developed before his time; on the contrary, it is certain that the Pythagoreans had worked out such a theory with regard to *numbers*, by which must be understood commensurable and even whole numbers (a number being a "multitude made up of units," as defined in Eucl. vii). Thus we are told that the Pythagoreans distinguished three sorts of *means*, the arithmetic, the geometric and the harmonic mean, the geometric mean being called proportion (*ἀναλογία*) *par excellence*; and further Iamblichus speaks of the "most perfect proportion consisting of four terms and specially called *harmonic*," in other words, the proportion

$$a : \frac{a+b}{2} = \frac{2ab}{a+b} : b,$$

which was said to be a discovery of the Babylonians and to have been first introduced into Greece by Pythagoras (Iamblichus, *Comm. on Nicomachus*, p. 118). Now the principle of similitude is one which is presupposed by all the arts of design from their very beginnings; it was certainly known to the Egyptians, and it must certainly have been thoroughly familiar to Pythagoras and his school. This consideration, together with the evidence of the employment by him of the *geometric proportion*, makes it indubitable that the Pythagoreans used the theory of proportion, in the form in which it was known to them, i.e. as applicable to commensurables only, in their geometry. But the discovery, also by the Pythagoreans, of the incommensurable would of course be seen to render the proofs which depended on the theory of proportion as then understood inconclusive; as Tannery observes (*La Géométrie grecque*, p. 98), "the discovery of incommensurability must have caused a veritable logical scandal in geometry and, in order to avoid it, they were obliged to restrict as far as possible the use of the principle of similitude, pending the discovery of a means of establishing it on the basis of a theory of proportion independent of commensurability." The glory of the latter discovery belongs then most probably to Eudoxus. Certain it is that the complete theory was already familiar to Aristotle, as we shall see later.

It seems probable, as indicated by Tannery (*loc. cit.*), that the theory of proportions and the principle of similitude took, in the earliest Greek geometry, an earlier place than they do in Euclid, but that, in consequence of the discovery of the incommensurable, the treatment of the subject was fundamentally remodelled in the period between Pythagoras and Eudoxus. An indication of this is afforded by the clever device used in Euclid I. 44 for applying to a given straight line a parallelogram equal to a given triangle; the equality of the "complements" in a parallelogram is there used for doing what is practically finding a fourth proportional to three given straight lines. Thus Euclid was no doubt following for the subject-matter of Books I.—IV. what had become the traditional method, and this is probably one of the reasons why proportions and similitude are postponed till as late as Books V., VI.

It is a remarkable fact that the theory of proportions is twice treated in Euclid, in Book V. with reference to magnitudes in general, and in Book VII. with reference to the particular case of numbers. The latter exposition referring only to commensurables may be taken to represent fairly the theory of proportions at the stage which it had reached before the great extension of it made by Eudoxus. The differences between the definitions etc. in Books V. and VII. will appear as we go on; but the question naturally arises, why did Euclid not save himself so much repetition and treat numbers merely as a particular case of magnitude, referring back to the corresponding more general propositions of Book V. instead of proving the same propositions over again for numbers? It could not have escaped him that numbers fall under the conception of magnitude. Aristotle had plainly indicated that magnitudes may be numbers when he observed (*Anal. post.* I. 7, 75 b 4) that you cannot adapt the arithmetical method of proof to the properties of magnitudes if the magnitudes are not numbers. Further Aristotle had remarked (*Anal. post.* I. 5, 74 a 17) that the proposition that the terms of a proportion can be taken alternately was at one time proved separately for numbers, lines, solids and times, though it was possible to prove it for all by one demonstration; but, because there was no common name comprehending them all, namely numbers, lengths, times and solids, and their character was different, they were taken separately. Now however, he adds, the proposition is proved generally. Yet Euclid says nothing to connect the two theories of proportion even when he comes in X. 5 to a proportion two terms of which are magnitudes and two are numbers ("Commensurable magnitudes have to one another the ratio which a number has to a number"). The probable explanation of the phenomenon is that Euclid simply followed tradition and gave the two theories as he found them. This would square with the remark in Pappus (VII. p. 678) as to Euclid's fairness to others and his readiness to give them credit for their work.

DEFINITIONS.

1. A magnitude is a **part** of a magnitude, the less of the greater, when it measures the greater.
2. The greater is a **multiple** of the less when it is measured by the less.

3. A **ratio** is a sort of relation in respect of size between two magnitudes of the same kind.

4. Magnitudes are said to **have a ratio** to one another which are capable, when multiplied, of exceeding one another.

5. Magnitudes are said to **be in the same ratio**, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.

6. Let magnitudes which have the same ratio be called **proportional**.

7. When, of the equimultiples, the multiple of the first magnitude exceeds the multiple of the second, but the multiple of the third does not exceed the multiple of the fourth, then the first is said to **have a greater ratio** to the second than the third has to the fourth.

8. A proportion in three terms is the least possible.

9. When three magnitudes are proportional, the first is said to have to the third the **duplicate ratio** of that which it has to the second.

10. When four magnitudes are <continuously> proportional, the first is said to have to the fourth the **triplicate ratio** of that which it has to the second, and so on continually, whatever be the proportion.

11. The term **corresponding magnitudes** is used of antecedents in relation to antecedents, and of consequents in relation to consequents.

12. **Alternate ratio** means taking the antecedent in relation to the antecedent and the consequent in relation to the consequent.

13. **Inverse ratio** means taking the consequent as antecedent in relation to the antecedent as consequent.

14. **Composition of a ratio** means taking the antecedent together with the consequent as one in relation to the consequent by itself.

15. **Separation of a ratio** means taking the excess by which the antecedent exceeds the consequent in relation to the consequent by itself.

16. **Conversion of a ratio** means taking the antecedent in relation to the excess by which the antecedent exceeds the consequent.

17. A ratio **ex aequali** arises when, there being several magnitudes and another set equal to them in multitude which taken two and two are in the same proportion, as the first is to the last among the first magnitudes, so is the first to the last among the second magnitudes;

Or, in other words, it means taking the extreme terms by virtue of the removal of the intermediate terms.

18. A **perturbed proportion** arises when, there being three magnitudes and another set equal to them in multitude, as antecedent is to consequent among the first magnitudes, so is antecedent to consequent among the second magnitudes, while, as the consequent is to a third among the first magnitudes, so is a third to the antecedent among the second magnitudes.

DEFINITION I.

Μέρος ἐστὶ μέγεθος μεγέθους τὸ ἔλασσον τοῦ μείζονος; ὅταν καταμετρήῃ τὸ μείζον.

The word *part* (μέρος) is here used in the restricted sense of a *submultiple* or an *aliquot part* as distinct from the more general sense in which it is used in the Common Notion (5) which says that "the whole is greater than the part." It is used in the same restricted sense in VII. Def. 3, which is the same definition as this with "number" (ἀριθμός) substituted for "magnitude." VII. Def. 4, keeping up the restriction, says that, when a number does not measure another number, it is *parts* (in the plural), not *a part* of it. Thus, 1, 2, or 3, is *a part* of 6, but 4 is not *a part* of 6 but *parts*. The same distinction between the restricted and the more general sense of the word *part* appears in Aristotle, *Metaph.* 1023 b 12: "In one sense a part is that into which quantity (τὸ ποσόν) can anyhow be divided; for that which is taken away from quantity, *quâ* quantity, is always called a 'part' of it, as e.g. two is said to be in a sense a part of three. But in another sense a 'part' is only *what measures* (τὰ καταμετροῦντα) such quantities. Thus two is in one sense said to be a part of three, in the other not."

DEFINITION 2.

Πολλαπλάσιον δὲ τὸ μείζον τοῦ ἐλάττωνος, ὅταν καταμετρήται ὑπὸ τοῦ ἐλάττωνος.

DEFINITION 3.

Λόγος ἐστὶ δύο μεγεθῶν ὁμογενῶν ἡ κατὰ πηλικότητα ποιά σχέσις.

The best explanation of the definitions of *ratio* and *proportion* that I have seen is that of De Morgan, which will be found in the articles under those titles in the *Penny Cyclopaedia*, Vol. xix. (1841); and in the following notes I shall draw largely from these articles. Very valuable also are the notes on the definitions of Book v. given by Hankel (fragment on Euclid published as an appendix to his work *Zur Geschichte der Mathematik in Alterthum und Mittelalter*, 1874).

There has been controversy as to what is the proper translation of the word *πηλικότης* in the definition. *σχέσις κατὰ πηλικότητα* has generally been translated "relation in respect of *quantity*." Upon this De Morgan remarks that it makes nonsense of the definition; "for magnitude has hardly a different meaning from quantity, and a relation of magnitudes with respect to quantity may give a clear idea to those who want a word to convey a notion of architecture with respect to building or of battles with respect to fighting, and to no others." The true interpretation De Morgan, following Wallis and Gregory, takes to be *quantuplicity*, referring to the number of times one magnitude is contained in the other. For, he says, we cannot describe magnitude in language without quantuplicative reference to other magnitude; hence he supposes that the definition simply conveys the fact that the mode of expressing quantity in terms of quantity is entirely based upon the notion of quantuplicity or that relation of which we take cognizance when we find how many times one is contained in the other. While all the rest of De Morgan's observations on the definition are admirable, it seems to me that on this question of the proper translation of *πηλικότης* he is in error. He supports his view mainly by reference (1) to the definition of a compounded ratio usually given as the 5th definition of Book vi., which speaks of the *πηλικότητες* of two ratios being multiplied together, and (2) to the comments of Eutocius and a scholiast on this definition. Eutocius says namely (Archimedes, ed. Heiberg, III. p. 120) that "the term *πηλικότης* is evidently used of the number from which the given ratio is called, as (among others) Nicomachus says in his first book on music and Heron in his commentary on the Introduction to Arithmetic." But it now appears certain that this definition is an interpolation; it is never used, it is not found in Campanus, and Peyrard's ms. only has it in the margin. At the same time it is clear that, if the definition is admitted at all, any commentator would be obliged to explain it in the way that Eutocius does, whether the explanation was consistent with the proper meaning of *πηλικότης* or not. Hence we must look elsewhere for the meaning of *πηλικός* and *πηλικότης*. If we do this, I think we shall find no case in which the words have the sense attributed to them by De Morgan. The real meaning of *πηλικός* is *how great*. It is so used by Aristotle, e.g. in *Eth. Nic.* v. 10, 1134 b 11, where he speaks of a man's child being as it were a part of him so long as he is of a certain age (*ἕως αὖν ἢ πηλικόν*). Again Nicomachus, to whom Eutocius appeals, himself (I. 2, 5, p. 5, ed. Hoche) distinguishes *πηλικός* as referring to *magnitude*, while *ποσός* refers to *multitude*. So does Iamblichus in his commentary on Nicomachus (p. 8, 3—5); besides which Iamblichus distinguishes *πηλικόν* as the subject of geometry, being *con-*

tinuous, and ποσόν as the subject of arithmetic, being *discrete*, and speaks of a point being the origin of *πηλικόν* as a unit is of ποσόν, and so on. Similarly, Ptolemy (*Syntaxis*, ed. Heiberg, p. 31) speaks of the *size* (πηλικότης) of the chords in a circle (περί τῆς πηλικότητος τῶν ἐν τῷ κύκλῳ εὐθειῶν). Consequently I think we can only translate πηλικότης in the definition as *size*. This corresponds to Hankel's translation of it as "Grösse," though he uses this same word for a concrete "magnitude" as well; *size* seems to me to give the proper distinction between πηλικότης and μέγεθος, as *size* is the attribute, and a *magnitude* (in its ordinary mathematical sense) is the thing which possesses the attribute of size.

The view that "relation in respect of *size*" is meant by the words in the text is also confirmed, I think, by a later remark of De Morgan himself, namely that a synonym for the word *ratio* may be found in the more intelligible term *relative magnitude*. In fact σχέσις in the definition corresponds to *relative* and πηλικότης to *magnitude*. (By *magnitude* De Morgan here means the attribute and not the thing possessing it.)

Of the definition as a whole Simson and Hankel express the opinion that it is an interpolation. Hankel points to the fact that it is unnecessary and moreover so vague as to be of no practical use, while the very use of the expression κατὰ πηλικότητητα seems to him suspicious, since the only other place in which the word πηλικότης occurs in Euclid is the 5th definition of Book vi., which is admittedly not genuine. Yet the definition of ratio appears in all the mss., the only variation being that some add the words πρὸς ἄλληλα, "to one another," which are rejected by Heiberg as an interpolation of Theon; and on the whole there seems to be no sufficient ground for regarding it as other than genuine. The true explanation of its presence would appear to be substantially that given by Barrow (*Lectiones Cantabrig.*, London, 1684, Lect. III. of 1666), namely that Euclid inserted it for completeness' sake, more for ornament than for use, intending to give the learner a general notion of ratio by means of a metaphysical, rather than a mathematical definition; "for metaphysical it is and not, properly speaking, mathematical, since nothing depends on it or is deduced from it by mathematicians, nor, as I think, can anything be deduced." This is confirmed by the fact that there is no definition of λόγος in Book vii., and it could equally have been dispensed with here. Similarly De Morgan observes that Euclid never attempts this vague sort of definition except when, dealing with a well-known term of common life, he wishes to bring it into geometry with something like an expressed meaning which may aid the conception of the thing, though it does not furnish a perfect criterion. Thus we may compare the definition with that of a straight line, where Euclid merely calls the reader's attention to the well-known term εὐθεία γραμμή, tries how far he can present the conception which accompanies it in other words, and trusts for the correct use of the term to the axioms (or postulates) which the universal conception of a straight line makes self-evident.

We have now to trace as clearly as possible the development of the conception of λόγος, *ratio*, or relative magnitude. In its primitive sense λόγος was only used of a ratio between commensurables, i.e. a ratio which could be *expressed*, and the manner of expressing it is indicated in the proposition, Eucl. x. 5, which proves that *commensurable magnitudes have to one another the ratio which a number has to a number*. That this was the primitive meaning of λόγος is proved by the use of the term ἄλογος for the incommensurable, which means *irrational* in the sense of *not having a ratio* to something taken as rational (ῥητός).

Euclid himself shows us how we are to set about finding the ratio, or relative magnitude, of two commensurable magnitudes. He gives, in x. 3, practically our ordinary method of finding the greatest common measure. If A, B be two magnitudes of which B is the less, we cut off from A a part equal to B , from the remainder a part equal to B , and so on, until we leave a remainder less than B , say R_1 . We measure off R_1 from B in the same way until a remainder R_2 is left which is less than R_1 . We repeat the process with R_1, R_2 , and so on, until we find a remainder which is contained in the preceding remainder a certain number of times exactly. If account is taken of the number of times each magnitude is contained (with something over, except at the last) in that upon which it is measured, we can calculate how many times the last remainder is contained in A and how many times the last remainder is contained in B ; and we can thus express the ratio of A to B as the ratio of one number to another.

But it may happen that the two magnitudes have no common measure, i.e. are incommensurable, in which case the process described would never come to an end and the means of expression would fail; the magnitudes would then *have no ratio* in the primitive sense. But the word λόγος, ratio, acquires in Euclid, Book v., a wider sense covering the *relative magnitude* of incommensurables as well as commensurables; as stated in Euclid's 4th definition, "magnitudes are said to have a *ratio* to one another which can, when multiplied, exceed one another," and finite incommensurables have this property as much as commensurables. De Morgan explains the manner of transition from the narrower to the wider signification of *ratio* as follows. "Since the relative magnitude of two quantities is always shown by the quantuplicative mode of expression, when that is possible, and since proportional quantities (pairs which have the same relative magnitude) are pairs which have the same mode (if possible) of expression by means of each other; in all such cases sameness of relative magnitude leads to sameness of mode of expression; or proportion is sameness of ratios (in the primitive sense). But sameness of relative magnitude may exist where quantuplicative expression is impossible; thus the diagonal of a larger square is the same compared with its side as the diagonal of a smaller square compared with *its* side. It is an easy transition to speak of sameness of ratio even in this case; that is, to use the term ratio in the sense of relative magnitude, that word having originally only a reference to the mode of expressing relative magnitude, in cases which allow of a particular mode of expression. The word *irrational* (ἄλογος) does not make any corresponding change but continues to have its primitive meaning, namely, incapable of quantuplicative expression."

It remains to consider how we are to describe the *relative magnitude* of two incommensurables of the same kind. That they have a definite relation is certain. Suppose, for precision, that S is the side of a square, D its diagonal; then, if S is given, any alteration in D or any error in D would make the figure cease to be a square. At the same time, a person altogether ignorant of the relative magnitude of D and S might say that drawing two straight lines of length S so as to form a right angle and joining the ends by a straight line, the length of which would accordingly be D , does not help him to realise the relative magnitude, but that he would like to know how many diagonals make an exact number of sides. We should have to reply that no number of diagonals whatever makes an exact number of sides; but that he may mention any fraction of the side, a hundredth, a thousandth or a millionth, and that we will then express the diagonal with an error not so great as that fraction. We then tell him that 1,000,000 diagonals exceed

1,414,213 sides but fall short of 1,414,214 sides; consequently the diagonal lies between 1'414213 and 1'414214 times the side, and these differ only by one-millionth of the side, so that the error in the diagonal is less still. To enable him to continue the process further, we show him how to perform the arithmetical operation of approximating to the value of $\sqrt{2}$. This gives the means of carrying the approximation to any degree of accuracy that may be desired. In the power, then, of carrying approximations of this kind as far as we please lies that of expressing the ratio, so far as expression is possible, and of comparing the ratio with others as accurately as if expression had been possible.

Euclid was of course aware of this, as were probably others before him; though the actual approximations to the values of ratios of incommensurables of which we find record in the works of the great Greek geometers are very few. The history of such approximations up to Archimedes is, so far as material was available, sketched in *The Works of Archimedes* (pp. lxxvii and following); and it is sufficient here to note the facts (1) that Plato, and even the Pythagoreans, were familiar with $\frac{7}{5}$ as an approximation to $\sqrt{2}$, (2) that the method of finding any number of successive approximations by the system of *side-* and *diagonal-*numbers described by Theon of Smyrna was also Pythagorean (cf. the note above on Euclid, II. 9, 10), (3) that Archimedes, without a word of preliminary explanation, gives out that

$$\frac{1351}{780} > \sqrt{3} > \frac{265}{153},$$

gives approximate values for the square roots of several large numbers, and proves that the ratio of the circumference of a circle to its diameter is less than $3\frac{1}{7}$ but greater than $3\frac{10}{71}$, (4) that the first approach to the rapidity with which the decimal system enables us to approximate to the value of surds was furnished by the method of sexagesimal fractions, which was almost as convenient to work with as the method of decimals, and which appears fully developed in Ptolemy's *σύνταξις*. A number consisting of a whole number and any fraction was under this system represented as so many units, so many of the fractions which we should denote by $\frac{1}{60}$, so many of those which we should write $(\frac{1}{60})^2$, $(\frac{1}{60})^3$, and so on. Theon of Alexandria shows us how to extract the square root of 4500 in this sexagesimal system, and, to show how effective it was, it is only necessary to mention that Ptolemy gives $\frac{103}{60} + \frac{55}{60^2} + \frac{23}{60^3}$ as an approximation to $\sqrt{3}$, which approximation is equivalent to 1.7320509 in the ordinary decimal notation and is therefore correct to 6 places.

Between Def. 3 and Def. 4 two manuscripts and Campanus insert "Proportion is the sameness of ratios" (*ἀναλογία δὲ ἡ τῶν λόγων ταυτότης*), and even the best ms. has it in the margin. It would be altogether out of place, since it is not till Def. 5 that it is explained what sameness of ratios is. The words are an interpolation later than Theon (Heiberg, Vol. v. pp. xxxv, lxxxix), and are no doubt taken from arithmetical works (cf. Nicomachus and Theon of Smyrna). It is true that Aristotle says similarly, "Proportion is equality of ratios" (*Eth. Nic.* v. 6, 1131 a 31), and he appears to be quoting from the Pythagoreans; but the reference is to *numbers*.

Similarly two mss. (inferior) insert after Def. 7 "Proportion is the similarity (*ὁμοιότης*) of ratios." Here too we have a mere interpolation.

DEFINITION 4.

Λόγον ἔχειν πρὸς ἄλληλα μεγέθη λέγεται, ἃ δύναται πολλαπλασιαζόμενα ἀλλήλων ὑπερέχειν.

This definition supplements the last one. De Morgan says that it amounts to saying that the magnitudes are of the same species. But this can hardly be all; the definition seems rather to be meant, on the one hand, to exclude the relation of a finite magnitude to a magnitude of the same kind which is either infinitely great or infinitely small, and, even more, to emphasise the fact that the term *ratio*, as defined in the preceding definition, and about to be used throughout the book, includes the relation between any two *incommensurable* as well as between any two commensurable finite magnitudes of the same kind. Hence, while De Morgan seems to regard the extension of the meaning of *ratio* to include the relative magnitude of incommensurables as, so to speak, taking place between Def. 3 and Def. 5, the 4th definition appears to show that it is ratio in its extended sense that is being defined in Def. 3.

DEFINITION 5.

Ἐν τῷ αὐτῷ λόγῳ μεγέθη λέγεται εἶναι πρῶτον πρὸς δεύτερον καὶ τρίτον πρὸς τέταρτον, ὅταν τὰ τοῦ πρῶτου καὶ τρίτου ἰσάκεις πολλαπλάσια τῶν τοῦ δευτέρου καὶ τετάρτου ἰσάκεις πολλαπλασιῶν καθ' ὅποιον οὖν πολλαπλασιασμὸν ἑκάτερον ἑκατέρου ἢ ἅμα ὑπερέχῃ ἢ ἅμα ἴσα ἢ ἅμα ἐλλείπῃ ληφθέντα κατάλληλα.

In my translation of this definition I have compromised between an attempted literal translation and the more expanded version of Simson. The difficulty in the way of an exactly literal translation is due to the fact that the words (καθ' ὅποιον οὖν πολλαπλασιασμὸν) signifying that the equimultiples *in each case* are any equimultiples *whatever* occur only once in the Greek, though they apply *both* to τὰ...ἰσάκεις πολλαπλάσια in the nominative and τῶν...ἰσάκεις πολλαπλασιῶν in the genitive. I have preferred "alike" to "simultaneously" as a translation of ἅμα because "simultaneously" might suggest that time was of the essence of the matter, whereas what is meant is that any particular comparison made between the equimultiples must be made between *the same* equimultiples of the two pairs respectively, not that they need to be compared *at the same time*.

Aristotle has an allusion to a definition of "the same ratio" in *Topics* VIII. 3, 158 b 29: "In mathematics too some things appear to be not easy to prove (γράφεσθαι) for want of a definition, e.g. that the parallel to the side which cuts a plane [a parallelogram] divides the straight line [the other side] and the area similarly. But, when the definition is expressed, the said property is immediately manifest; for the areas and the straight lines *have the same ἀνταναίρεσις*, and this is the definition of 'the same ratio.'" Upon this passage Alexander says similarly, "This is the definition of proportionals which the ancients used: *magnitudes are proportional to one another which have* (or show) *the same ἀνθυφαίρεσις*, and Aristotle has called the latter ἀνταναίρεσις." Heiberg (*Mathematisches zu Aristoteles*, p. 22) thinks that Aristotle is alluding to the fact that the proposition referred to could not be rigorously proved so long as the Pythagorean definition applicable to commensurable magnitudes only was adhered to, and is quoting the definition belonging to the complete theory of Eudoxus; whence, in view of the positive statement of Aristotle that the definition quoted *is* the definition of "the same ratio," it would appear that the Euclidean definition (which Heiberg describes as a careful and exact paraphrase of ἀνταναίρεσις) is Euclid's own. I do not

feel able to subscribe to this view, which seems to me to involve very grave difficulties. The Euclidean definition is regularly appealed to in Book v. as the criterion of magnitudes being in proportion, and the use of it would appear to constitute the whole essence of the new general theory of proportion; if then this theory is due to Eudoxus, it seems impossible to believe that the definition was not also due to him. Certainly the definition given by Aristotle would be no substitute for it; ἀνθυφαίρεσις and ἀνταναίρεσις are words almost as vague and "metaphysical" (as Barrow would say) as the words used to define *ratio*, and it is difficult to see how any mathematical facts could be deduced from such a definition. Consider for a moment the etymology of the words. ὑφαίρεσις or ἀναίρεσις means "removal," "taking away" or "destruction" of a thing; and the prefix ἀντὶ indicates that the "taking away" from one magnitude *answers to*, corresponds with, alternates with, the "taking away" from the other. So far therefore as the etymology goes, the word seems rather to suggest the "taking away" of corresponding *fractions*, and therefore to suit the old imperfect theory of proportion rather than the new one. Thus Waitz (*ad loc.*) paraphrases the definition as meaning that "as many parts as are taken from one magnitude, so many are at the same time taken from the other as well." A possible explanation would seem to be that, though Eudoxus had formulated the new definition, the old one was still current in the text-books of Aristotle's time, and was taken by him as being a good enough illustration of what he wished to bring out in the passage of the *Topics* referred to.

From the revival of learning in Europe onwards the Euclidean definition of proportion was the subject of much criticism. Campanus had failed to understand it, had in fact misinterpreted it altogether, and he may have misled others such as Ramus (1515—72), always a violently hostile critic of Euclid. Among the objectors to it was no less a person than Galileo. For particulars of the controversies on the subject down to Thomas Simpson (*Elem. of Geometry*, Lond. 1800) the reader is referred to the Excursus at the end of the second volume of Camerer's *Euclid* (1825). For us it is interesting to note that the unsoundness of the usual criticisms of the definition was never better exposed than by Barrow. Some of the objections, he pointed out (*Lect. Cantabr.* vii. of 1666), are due to misconception on the part of their authors as to the nature of a definition. Thus Euclid is required by these objectors (e.g. Tacquet) to do the impossible and to show that what is predicated in the definition is true of the thing defined, as if any one should be required to show that the name "circle" was applicable to those figures alone which have their radii all equal! As we are entitled to assign to such figures and such figures only the name of "circle," so Euclid is entitled ("quamvis non temere nec imprudenter at certis de causis iustis illis et idoneis") to describe a certain property which four magnitudes may have, and to call magnitudes possessing that property magnitudes "in the same ratio." Others had argued from the occurrence of the other definition of proportion in vii. Def. 20 that Euclid was dissatisfied with the present one; Barrow pointed out that, on the contrary, it was the fact that vii. Def. 20 was not adequate to cover the case of incommensurables which made Euclid adopt the present definition here. Lastly, he maintains, against those who descant on the "obscurity" of v. Def. 5, that the supposed obscurity is due, partly no doubt to the inherent difficulty of the subject of incommensurables, but also to faulty translators, and most of all to lack of effort in the learner to grasp thoroughly the meaning of words which, in themselves, are as clearly expressed as they could be.

To come now to the merits of the case, the best defence and explanation

of the definition that I have seen is that given by De Morgan. He first translates it, observes that it applies equally to commensurable or incommensurable quantities because no attempt is made to measure one by an aliquot part of another, and then proceeds thus.

"The two questions which must be asked, and satisfactorily answered, previously to its [the definition's] reception, are as follows:

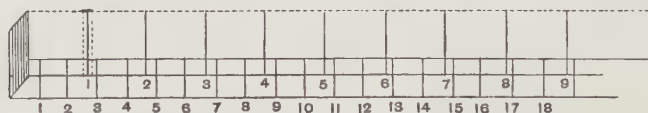
1. What right had Euclid, or any one else, to expect that the preceding most prolix and unwieldy statement should be received by the beginner as the definition of a relation the perception of which is one of the most common acts of his mind, since it is performed on every occasion where similarity or dissimilarity of figure is looked for or presents itself?

2. If the preceding question should be clearly answered, how can the definition of proportion ever be used; or how is it possible to compare every one of the infinite number of multiples of A with every one of the multiples of B ?

To the first question we reply that not only is the test proposed by Euclid tolerably simple, when more closely examined, but that it is, or might be made to appear, an easy and natural consequence of those fundamental perceptions with which it may at first seem difficult to compare it."

To elucidate this De Morgan gives the following illustration.

Suppose there is a straight colonnade composed of equidistant columns (which we will understand to mean the vertical lines forming the axes of the columns respectively), the first of which is at a distance from a bounding wall equal to the distance between consecutive columns. In front of the colonnade let there be a straight row of equidistant railings (regarded as meaning their axes), the first being at a distance from the bounding wall equal to the distance between consecutive railings. Let the columns be numbered from the wall, and also the railings. We suppose of course that the column distance (say, C) and the railing distance (say, R) are different and that they may bear to each other any ratio, commensurable or incommensurable; i.e. that there need not go any exact number of railings to any exact number of columns.



If the construction be supposed carried on to any extent, a spectator can, by mere inspection, and without measurement, compare C with R to any degree of accuracy. For example, since the 10th railing falls between the 4th and 5th columns, $10R$ is greater than $4C$ and less than $5C$, and therefore R lies between $\frac{4}{10}$ ths of C and $\frac{5}{10}$ ths of C . To get a more accurate notion, the ten-thousandth railing may be taken; suppose it falls between the 4674th and 4675th columns. Therefore $10,000R$ lies between $4674C$ and $4675C$, or R lies between $\frac{4674}{10000}$ and $\frac{4675}{10000}$ of C . There is no limit to the degree of accuracy thus obtainable; and the ratio of R to C is determined when the order of distribution of the railings among the columns is assigned *ad infinitum*; or, in other words, when the position of *any* given railing can be found, as to the numbers of the columns between which it lies. Any alteration, however small, in the place of the first railing must at last affect the order of distribution. Suppose e.g. that the first railing is moved from the wall by one part in a thousand of the distance between the columns; then the second railing is pushed forward by $\frac{2}{1000}C$, the third by $\frac{3}{1000}C$, and so on, so that

the railings after the thousandth are pushed forward by more than C ; i.e. the order with respect to the columns is disarranged.

Now let it be proposed to make a model of the preceding construction in which c shall be the column distance and r the railing distance. It needs no definition of proportion, nor anything more than the conception which we have of that term prior to definition (and with which we must show the agreement of any definition that we may adopt), to assure us that C must be to R in the same proportion as c to r if the model be truly formed. Nor is it drawing too largely on that conception of proportion to assert that the distribution of the railings among the columns in the model must be everywhere the same as in the original; for example, that the model would be *out of proportion* if its 37th railing fell between the 18th and 19th columns, while the 37th railing of the original fell between the 17th and 18th columns. Thus the dependence of Euclid's definition upon common notions is settled; for the obvious relation between the construction and its model which has just been described contains the collection of conditions, the fulfilment of which, according to Euclid, constitutes proportion. According to Euclid, whenever mC exceeds, equals, or falls short of nR , then mc must exceed, equal, or fall short of nr ; and, by the most obvious property of the constructions, according as the m th column comes after, opposite to, or before the n th railing in the original, the m th column must come after, opposite to, or before the n th railing in the correct model.

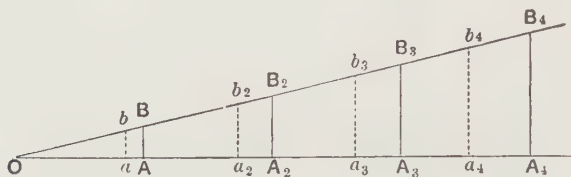
Thus the test proposed by Euclid is necessary. It is also sufficient. For admitting that, to a given original with a given column-distance in the model, there is one correct model railing distance (which must therefore be that which distributes the railings among the columns as in the original), we have seen that any other railing distance, however slightly different, would at last give a different distribution; that is, the correct distance, and the correct distance only, satisfies all the conditions required by Euclid's definition.

The use of the word *distribution* having been well learnt, says De Morgan, the following way of stating the definition will be found easier than that of Euclid. "Four magnitudes, A and B of one kind, and C and D of the same or another kind, are proportional when all the multiples of A can be distributed among the multiples of B in the same intervals as the corresponding multiples of C among those of D ." Or, whatever numbers m, n may be, if mA lies between nB and $(n+1)B$, mC lies between nD and $(n+1)D$.

It is important to note that, if the test be always satisfied from and after any given multiples of A and C , it must be satisfied before those multiples. For instance, let the test be always satisfied from and after $100A$ and $100C$; and let $5A$ and $5C$ be instances for examination. Take any multiple of 5 which will exceed 100 , say 50 times five; and let it be found on examination that $250A$ lies between $678B$ and $679B$; then $250C$ lies between $678D$ and $679D$. Divide by 50 , and it follows that $5A$ lies between $13\frac{28}{50}B$ and $13\frac{29}{50}B$, and *a fortiori* between $13B$ and $14B$. Similarly, $5C$ lies between $13\frac{28}{50}D$ and $13\frac{29}{50}D$, and therefore between $13D$ and $14D$. Or $5A$ lies in the same interval among the multiples of B in which $5C$ lies among the multiples of D . And so for any multiple of A, C less than $100A, 100C$.

There remains the second question relating to the infinite character of the definition; four magnitudes A, B, C, D are not to be called proportional until it is shown that *every* multiple of A falls in the same intervals among the multiples of B in which the same multiple of C is found among the multiples of D . Suppose that the distribution of the railings among the

columns should be found to agree in the model and the original as far as the millionth railing. This proves only that the railing distance of the model does not err by the millionth part of the corresponding column distance. We can thus fix limits to the disproportion, if any, and we may make those limits as small as we please, by carrying on the method of *observation*; but we cannot *observe* an infinite number of cases and so enable ourselves to affirm proportion absolutely. Mathematical methods however enable us to avoid the difficulty. We can take *any multiples whatever* and work with them as if they were particular multiples. De Morgan gives, as an instance to show that the definition of proportion can in practice be used, notwithstanding its infinite character, the following proof of a proposition to the same effect as Eucl. vi. 2.



"Let OAB be a triangle to one side AB of which ab is drawn parallel, and on OA produced set off AA_2, A_2A_3 etc. equal to OA , and aa_2, a_2a_3 etc. equal to Oa .

Through every one of the points so obtained draw parallels to AB , meeting OB produced in b_2, B_2 etc.

Then it is easily proved that bb_2, b_2b_3 etc. are severally equal to Ob , and BB_2, B_2B_3 etc. to OB .

Consequently a distribution of the multiples of OA among the multiples of Oa is made on one line, and of OB among those of Ob on the other.

The examination of this distribution in all its extent (which is impossible, and hence the apparent difficulty of using the definition) is rendered unnecessary by the known property of parallel lines. For, since A_3 lies between a_3 and a_4 , B_3 must lie between b_3 and b_4 ; for, if not, the line A_3B_3 would cut either a_3b_3 or a_4b_4 .

Hence, without inquiring where A_m does fall, we know that, if it fall between a_n and a_{n+1} , B_m must fall between b_n and b_{n+1} ; or, if $m.OA$ fall in magnitude between $n.Oa$ and $(n+1)Oa$, then $m.OB$ must fall between $n.Ob$ and $(n+1)Ob$."

Max Simon remarks (*Euclid und die sechs planimetrischen Bücher*, p. 110), after Zeuthen, that Euclid's definition of equal ratios is word for word the same as Weierstrass' definition of equal numbers. So far from agreeing in the usual view that the Greeks saw in the irrational no *number*, Simon thinks it is clear from Eucl. v. that they possessed a notion of number in all its generality as clearly defined as, nay almost identical with, Weierstrass' conception of it.

Certain it is that there is an exact correspondence, almost coincidence, between Euclid's definition of equal ratios and the modern theory of irrationals due to Dedekind. Premising the ordinal arrangement of natural numbers in ascending order, then enlarging the sphere of numbers by including (1) negative numbers as well as positive, (2) fractions, as a/b , where a, b may

be any natural numbers, provided that b is not zero, and arranging the fractions ordinally among the other numbers according to the definition :

$$\text{let } \frac{a}{b} \text{ be } <=> \frac{c}{d} \text{ according as } ad \text{ is } <=> bc,$$

Dedekind arrives at the following definition of an irrational number.

An irrational number α is defined whenever a law is stated which will assign every given rational number to one and only one of two classes A and B such that (1) every number in A precedes every number in B , and (2) there is no last number in A and no first number in B ; the definition of α being that it is the one number which lies between all numbers in A and all numbers in B .

Now let x/y and x'/y' be equal ratios in Euclid's sense.

Then $\frac{x}{y}$ will divide all rational numbers into two groups A and B ;

$$\frac{x'}{y'} \quad \quad \quad \text{,,} \quad \quad \quad \text{,,} \quad \quad \quad \text{,,} \quad A' \text{ and } B'.$$

Let $\frac{a}{b}$ be any rational number in A , so that

$$\frac{a}{b} < \frac{x}{y}.$$

This means that $ay < bx$.

But Euclid's definition asserts that in that case $ay' < bx'$ also.

Hence also
$$\frac{a}{b} < \frac{x'}{y'};$$

therefore every member of group A is also a member of group A' .

Similarly every member of group B is a member of group B' .

For, if $\frac{a}{b}$ belong to B ,

$$\frac{a}{b} > \frac{x}{y},$$

which means that $ay > bx$.

But in that case, by Euclid's definition, $ay' > bx'$;

therefore also
$$\frac{a}{b} > \frac{x'}{y'}.$$

Thus, in other words, A and B are coextensive with A' and B' respectively ;

therefore $\frac{x}{y} = \frac{x'}{y'}$, according to Dedekind, as well as according to Euclid.

If x/y , x'/y' happen to be *rational*,
then one of the groups, say A , includes x/y ,
and one of the groups, say A' , includes x'/y' .

In this case $\frac{a}{b}$ might *coincide* with $\frac{x}{y}$;

that is
$$\frac{a}{b} = \frac{x}{y},$$

which means that
$$ay = bx.$$

Therefore, by Euclid's definition, $ay' = bx'$;

so that

$$\frac{a}{b} = \frac{x'}{y'}.$$

Thus the groups are again coextensive.

In a word, Euclid's definition divides all rational numbers into two *coextensive* classes, and therefore defines equal ratios in a manner exactly corresponding to Dedekind's theory.

Alternatives for Eucl. V. Def. 5.

Saccheri records in his *Euclides ob omni naevo vindicatus* that a distinguished geometer of his acquaintance proposed to substitute for Euclid's the following definition :

"A first magnitude has to a second the same ratio that a third has to a fourth when the first contains the aliquot parts of the second, *according to any number* [i.e. with any denominator] *whatever*, the same number of times as the number of times the third contains the same aliquot parts of the fourth" ; on which Saccheri remarks that he sees no advantage in this definition, which presupposes the notion of *division*, over that of Euclid which uses *multiplication* and the notions of *greater*, *equal*, and *less*.

This definition was, however, practically adopted by Faifofer (*Elementi di geometria*, 3 ed., 1882) in the following form :

"Four magnitudes taken in a certain order form a proportion when, by measuring the first and the third respectively by any equi-submultiples whatever of the second and of the fourth, equal quotients are obtained."

Ingrami (*Elementi di geometria*, 1904) takes multiples of the first and third instead of submultiples of the second and fourth :

"Given four magnitudes in predetermined order, the first two homogeneous with one another, and likewise also the last two, the magnitudes are said to form a proportion (or to be in proportion) when *any multiple* of the first contains the second the same number of times that the equimultiple of the third contains the fourth."

Veronese's definition (*Elementi di geometria*, Pt. II., 1905) is like that of Faifofer ; Enriques and Amaldi (*Elementi di geometria*, 1905) adhere to Euclid's.

Proportionals of VII. Def. 20 a particular case.

It has already been observed that Euclid has nowhere proved (though the fact cannot have escaped him) that the proportion of numbers is included in the proportion of magnitudes as a special case. This is proved by Simson as being necessary to the 5th and 6th propositions of Book x. Simson's proof is contained in his propositions C and D inserted in the text of Book v. and in the notes thereon. Proposition C and the note on it prove that, *if four magnitudes are proportionals according to VII. Def. 20, they are also proportionals according to v. Def. 5*. Prop. D and the note prove the partial converse, namely that, *if four magnitudes are proportionals according to the 5th definition of Book v., and if the first be any multiple, or any part, or parts, of the second, the third is the same multiple, part, or parts, of the fourth*. The proofs use certain results obtained in Book v.

Prop. C is as follows :

If the first be the same multiple of the second, or the same part of it, that the third is of the fourth, the first is to the second as the third to the fourth.

Let the first A be the same *multiple* of B the second that C the third is of the fourth D ;

A is to B as C is to D .

A —————	E —————
B —————	G —————
C —————	F —————
D —————	H —————

Take of A, C any equimultiples whatever E, F ; and of B, D any equimultiples whatever G, H .

Then, because A is the same multiple of B that C is of D ,
and E is the same multiple of A that F is of C ,

E is the same multiple of B that F is of D . [v. 3]

Therefore E, F are the same multiples of B, D .

But G, H are equimultiples of B, D ;

therefore, if E be a greater multiple of B than G is, F is a greater multiple of D than H is of D ;

that is, if E be greater than G , F is greater than H .

In like manner,

if E be equal to G , or less, F is equal to H , or less than it.

But E, F are equimultiples, any whatever, of A, C ;

and G, H any equimultiples whatever of B, D .

Therefore A is to B as C is to D . [v. Def. 5]

Next, let the first A be the same *part* of the second B that the third C is of the fourth D :

A is to B as C is to D .

For B is the same multiple of A that D is of C ;

wherefore, by the preceding case,

B is to A as D is to C ;

and, *inversely*, A is to B as C is to D .

[For this last inference Simson refers to his Proposition B. That proposition is very simply proved by taking any equimultiples E, F of B, D and any equimultiples G, H of A, C and then arguing as follows:

Since A is to B as C is to D ,

G, H are *simultaneously* greater than, equal to, or less than E, F respectively; so that

E, F are *simultaneously* less than, equal to, or greater than G, H respectively,

and therefore [Def. 5] B is to A as D is to C .]

We have now only to add to Prop. C the case where AB contains the *same parts* of CD that EF does of GH :

in this case likewise AB is to CD as EF to GH .

Let CK be a part of CD , and GL the same part of GH ; let AB be the same multiple of CK that EF is of GL .

Therefore, by Prop. C,

AB is to CK as EF to GL .



And CD , GH are equimultiples of CK , GL , the second and fourth.

Therefore AB is to CD as EF to GH [Simson's Cor. to v. 4, which however is the particular case of v. 4 in which the "equimultiples" of one pair are the pair itself, i.e. the pair multiplied by unity].

To prove the partial converse we begin with Prop. D.

If the first be to the second as the third to the fourth, and if the first be a multiple or part of the second, the third is the same multiple or the same part of the fourth.

Let A be to B as C is to D ;

and, first, let A be a multiple of B ;

C is the same multiple of D .

Take E equal to A , and whatever multiple A or E is of B , make F the same multiple of D .

Then, because A is to B as C is to D ,

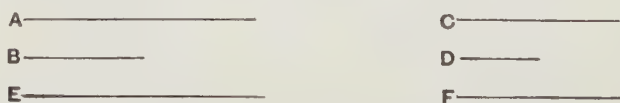
and of B the second and D the fourth equimultiples have been taken E and F ,

A is to E as C is to F . [v. 4, Cor.]

But A is equal to E ;

therefore C is equal to F .

[In support of this inference Simson cites his Prop. A, which however we can directly deduce from v. Def. 5 by taking any, but *the same*, equimultiples of all four magnitudes.]



Now F is the same multiple of D that A is of B ;

therefore C is the same multiple of D that A is of B .

Next, let the first A be a part of the second B ;

C the third is the same part of the fourth D .

Because A is to B as C is to D ,

inversely, B is to A as D is to C .

[Prop. B]

But A is a part of B ; therefore B is a multiple of A ;

and, by the preceding case, D is the same multiple of C ,

that is, C is the same part of D that A is of B .

We have, again, only to add to Prop. D the case where AB contains any parts of CD , and AB is to CD as EF to GH ;

then shall EF contain the same parts of GH that AB does of CD .

For let CK be a part of CD , and GL the same part of GH ; and let AB be a multiple of CK .

EF shall be the same multiple of GL .

Take M the same multiple of GL that AB is of CK ;
therefore AB is to CK as M is to GL .

[Prop. C]



And CD , GH are equimultiples of CK , GL ;

therefore AB is to CD as M is to GH .

But, by hypothesis, AB is to CD as EF is to GH ;

therefore M is equal to EF ,

[v. 9]

and consequently EF is the same multiple of GL that AB is of CK .

DEFINITION 6.

Τὰ δὲ τὸν αὐτὸν ἔχοντα λόγον μεγέθη ἀνάλογον καλεῖσθαι.

'*Ανάλογον*, though usually written in one word, is equivalent to *ἀνὰ λόγον*, *in proportion*. It comes however in Greek mathematics to be used practically as an indeclinable adjective, as here; cf. αἱ τέσσαρες εὐθείαι ἀνάλογον ἔσονται, "the four straight lines will be proportional," τρίγωνα τὰς πλευρὰς ἀνάλογον ἔχοντα, "triangles having their sides proportional." Sometimes it is used adverbially: ἀνάλογον ἄρα ἐστὶν ὡς ἡ ΒΑ πρὸς τὴν ΑΓ, οὕτως ἡ ΗΔ πρὸς τὴν ΔΖ, "proportionally therefore, as BA is to AC , so is GD to DF "; so too, apparently, in the expression ἡ μέση ἀνάλογον (εὐθεία), "the mean proportional." I do not follow the objection of Max Simon (Euclid, p. 110) to "proportional" as a translation of *ἀνάλογον*. "We ask," he says, "in vain, what is proportional to what? We say e.g. that weight is proportional to price because double, treble etc. weight corresponds to double, treble etc. price. But here the meaning must be 'standing in a relation of proportion.'" Yet he admits that the Latin word *proportionalis* is an adequate expression. He translates by "in proportion" in the text of this definition. But I do not see that "in proportion" is better than "proportional." The fact is that both expressions are elliptical when used of four magnitudes "in proportion"; but there is surely no harm in using either when the meaning is so well understood.

The use of the word καλεῖσθαι, "*let* magnitudes having the same ratio *be called* proportional," seems to indicate that this definition is Euclid's own.

DEFINITION 7.

Ὅταν δὲ τῶν ἰσάκεις πολλαπλασίων τὸ μὲν τοῦ πρώτου πολλαπλάσιον ὑπερέχῃ τοῦ τοῦ δευτέρου πολλαπλασίου, τὸ δὲ τοῦ τρίτου πολλαπλάσιον μὴ ὑπερέχῃ τοῦ τετάρτου πολλαπλασίου, τότε τὸ πρῶτον πρὸς τὸ δεύτερον μείζονα λόγον ἔχειν λέγεται, ἥπερ τὸ τρίτον πρὸς τὸ τέταρτον.

As De Morgan observes, the practical test of *disproportion* is simpler than that of proportion. For, whereas no examination of individual cases, however

extensive, will enable an observer of the construction and its model (the illustration by means of columns and railings described above) to affirm proportion or deny disproportion, and all it enables us to do is to fix limits (as small as we please) to the disproportion (if any), a single instance may enable us to deny proportion or affirm disproportion, and also to state which way the disproportion lies. Let the 19th railing in the original fall beyond the 11th column, while the 19th railing of the (so-called) model does not come up to the 11th column. It follows from this one instance that the railing distance of the model is too small relatively to the column distance, or that the column distance is too great relatively to the railing distance. That is, the ratio of r to c is less than that of R to C , or the ratio of c to r is greater than that of C to R .

Saccheri (*op. cit.*) remarks (as Commandinus had done) that the ratio of the first magnitude to the second will also be greater than that of the third to the fourth if, while the multiple of the first is *equal* to the multiple of the second, the multiple of the third is *less* than that of the fourth: a case not mentioned in Euclid's definition. Saccheri speaks of this case being included in Clavius' interpretation of the definition. I have, however, failed to find a reference to the case in Clavius, though he adds, as a sort of corollary, in his note on the definition, that if, on the other hand, the multiple of the first is *less* than the multiple of the second, while the multiple of the third is *not less* than that of the fourth, the ratio of the first to the second is *less* than that of the third to the fourth.

Euclid presumably left out the second possible criterion for a greater ratio, and the definition of a less ratio, because he was anxious to reduce the definitions to the minimum necessary for his purpose, and to leave the rest to be inferred as soon as the development of the propositions of Book v. enabled this to be done without difficulty.

Saccheri tried to reduce the second possible criterion for a greater ratio to that given by Euclid in his definition without recourse to anything coming later in the Book, but, in order to do this, he has to use "multiples" produced by multipliers which are not integral numbers, but integral numbers *plus* proper fractions, so that Euclid's Def. 7 becomes inapplicable.

De Morgan notes that "proof should be given that the same pair of magnitudes can never offer both tests [i.e. the test in the definition for a greater ratio and the corresponding test for a less ratio, with "less" substituted for "greater" in the definition] to another pair; that is, the test of greater ratio from one set of multiples, and that of less ratio from another." In other words, if m, n, p, q are integers and A, B, C, D four magnitudes, none of the pairs of equations

$$(1) \quad mA > nB, \quad mC = \text{or} < nD,$$

$$(2) \quad mA = nB, \quad mC < nD$$

can be satisfied simultaneously with any one of the pairs of equations

$$(3) \quad pA = qB, \quad pC > qD,$$

$$(4) \quad pA < qB, \quad pC > \text{or} = qD.$$

There is no difficulty in proving this with the help of two simple assumptions which are indeed obvious.

We need only take in illustration one of the numerous cases. Suppose, if possible, that the following pairs of equations are simultaneously true:

$$(1) \quad mA > nB, \quad mC < nD$$

and

$$(2) \quad pA < qB, \quad pC > qD.$$

Multiply (1) by q and (2) by n .

(We need here to assume that, where rX , rY are any equimultiples of any magnitudes X , Y ,

according as $X > = < Y$, $rX > = < rY$.

This is contained in Simson's Axioms 1 and 3.)

We have then the pairs of equations

$$\begin{aligned}mqA > nqB, \quad mqC < nqD, \\ n\phi A < nqB, \quad n\phi C > nqD.\end{aligned}$$

From the second equations in each pair it follows that

$$mqC < n\phi C.$$

(We now need to assume that, if rX , sX are any multiples of X , and rY , sY the same multiples of Y , then,

according as $rX > = < sX$, $rY > = < sY$.

Simson uses this same assumption in his proof of v. 18.)

Therefore $mqA < n\phi A$.

But it follows from the first equations in each pair that

$$mqA > n\phi A :$$

which is impossible.

Nor can Euclid's criterion for a greater ratio coexist with that for equal ratios.

DEFINITION 8.

Ἀναλογία δὲ ἐν τρισὶν ὁροῖς ἐλαχίστη ἐστίν.

This is the reading of Heiberg and Camerer (who follow Peyrard's ms.) and is that translated above. The other reading has *ἐλαχίστοις*, which can only be translated "consists in three terms *at least*." Hankel regards the definition as a later interpolation, because it is superfluous, and because the word *ὁρος* for a *term* in a proportion is nowhere else used by Euclid, though it is common in later writers such as Nicomachus and Theon of Smyrna. The genuineness of the definition is however supported by the fact that Aristotle not only uses *ὁρος* in this sense (*Eth. Nic.* v. 6, 7, 1131 b 5, 9), but has a similar remark (*ibid.* 1131 a 31) that a "proportion is in *four* terms at least." The difference from Euclid is only formal; for Aristotle proceeds: "The *discrete* (*διηρημένη*) (proportion) is clearly in four (terms), but so also is the *continuous* (*συνεχής*). For it uses one as two and mentions it twice, e.g. (in stating) that, as α is to β , so also is β to γ ; thus β is mentioned twice, so that, if β be twice put down, the proportionals are four." The distinction between *discrete* and *continuous* seems to have been Pythagorean (cf. Nicomachus, II. 21, 5; 23, 2, 3; where however *συνημμένη* is used instead of *συνεχής*): Euclid does not use the words *διηρημένη* and *συνεχής* in this connexion.

So far as they go, the first words of the next definition (9), "When three magnitudes are proportionals," which seemingly refer to Def. 8, also support the view that the latter is, at least in substance, genuine.

DEFINITIONS 9, 10.

9. "Όταν δὲ τρία μεγέθη ἀνάλογον ᾗ, τὸ πρῶτον πρὸς τὸ τρίτον διπλασίονα λόγον ἔχειν λέγεται ἥπερ πρὸς τὸ δεύτερον.

10. "Όταν δὲ τέσσαρα μεγέθη ἀνάλογον ᾗ, τὸ πρῶτον πρὸς τὸ τέταρτον τριπλασίονα λόγον ἔχειν λέγεται ἥπερ πρὸς τὸ δεύτερον, καὶ αἱ ἐξῆς ὁμοίως, ὡς ἀν ἡ ἀναλογία ὑπάρχη.

Here, and in connexion with the definitions of duplicate, triplicate, etc. ratios, would be the place to expect a definition of "*compound ratio*." None such is however forthcoming, and the only "definition" of it that we find is that forming VI. Def. 5, which is an interpolation made, perhaps, even before Theon's time. According to the interpolated definition, "A ratio is said to be compounded of ratios when the sizes (*πηλικότητες*) of the ratios multiplied together make some (? ratio)." But the multiplication of the *sizes* (or magnitudes) of two ratios of incommensurable, and even of commensurable, magnitudes is an operation unknown to the classical Greek geometers. Eutocius (Archimedes, ed. Heiberg, III. p. 120) is driven to explain the definition by making *πηλικότης* mean the *number* from which the given ratio is called, or, in other words, the number which multiplied into the consequent of the ratio gives the antecedent. But he is only able to work out his idea with reference to ratios between numbers, or between *commensurable* magnitudes; and indeed the definition is quite out of place in Euclid's theory of proportion.

There is then only one statement in Euclid's text as we have it indicating what is meant by compound ratio; this is in VI. 23, where he says abruptly "But the ratio of *K* to *M* is compounded of the ratio of *K* to *L* and that of *L* to *M*." Simson accordingly gives a definition (A of Book v.) of compound ratio directly suggested by the statement in VI. 23 just quoted.

"When there are any number of magnitudes of the same kind, the first is said to have to the last of them the ratio compounded of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth, and so on unto the last magnitude.

For example, if *A*, *B*, *C*, *D* be four magnitudes of the same kind, the first *A* is said to have to the last *D* the ratio compounded of the ratio of *A* to *B*, and of the ratio of *B* to *C*, and of the ratio of *C* to *D*; or the ratio of *A* to *D* is said to be compounded of the ratios of *A* to *B*, *B* to *C*, and *C* to *D*.

And if *A* has to *B* the same ratio which *E* has to *F*; and *B* to *C* the same ratio that *G* has to *H*; and *C* to *D* the same that *K* has to *L*; then, by this definition, *A* is said to have to *D* the ratio compounded of ratios which are the same with the ratios of *E* to *F*, *G* to *H*, and *K* to *L*: and the same thing is to be understood when it is more briefly expressed, by saying, *A* has to *D* the ratio compounded of the ratios of *E* to *F*, *G* to *H*, and *K* to *L*.

In like manner, the same things being supposed, if *M* has to *N* the same ratio which *A* has to *D*; then, for shortness' sake, *M* is said to have to *N* the ratio compounded of the ratios of *E* to *F*, *G* to *H*, and *K* to *L*."

De Morgan has some admirable remarks on compound ratio, which not only give a very clear view of what is meant by it but at the same time

supply a plausible explanation of the *origin* of the term. "Treat ratio," says De Morgan, "as an engine of operation. Let that of A to B suggest the power of altering any magnitude in that ratio." (It is true that it is not yet proved that, B being any magnitude, and P and Q two magnitudes of the same kind, there does exist a magnitude A which is to B in the same ratio as P to Q . It is not till vi. 12 that this is proved, by construction, in the particular case where the three magnitudes are straight lines. The proof in the Greek text of v. 18 which assumes the truth of the more general proposition is, by reason of that assumption, open to objection; see the note on that proposition.) Now "every alteration of a magnitude is alteration in some ratio, two or more successive alterations are jointly equivalent to but one, and the ratio of the initial magnitude to the terminal one is as properly said to be the *compound* ratio of alteration as 13 to be the compound addend in lieu of 8 and 5, or 28 the compound multiple for 7 and 4. Composition is used here, as elsewhere, for the process of detecting one single alteration which produces the joint effect of two or more. The composition of the ratios of P to R , R to S , T to U , is performed by assuming A , altering it in the first ratio into B , altering B in the second ratio into C , and C in the third ratio into D . The joint effect turns A into D , and the ratio of A to D is the compounded ratio."

Another word for *compounded* ratio is *συννημμένος* (*συνάπτω*) which is common in Archimedes and later writers.

It is clear that duplicate ratio, triplicate ratio etc. defined in v. Deff. 9 and 10 are merely particular cases of compound ratio, being in fact the ratios compounded of two, three etc. *equal* ratios. The use which the Greek geometers made of compounded, duplicate, triplicate ratios etc. is well illustrated by the discovery of Hippocrates that the problem of the duplication of the cube (or, more generally, the construction of a cube which shall be to a given cube in any given ratio) reduces to that of finding "two mean proportionals in continued proportion." This amounted to seeing that, if x, y are two mean proportionals in continued proportion between any two lines a, b , in other words, if a is to x as x to y , and x is to y as y to b , then a cube with side a is to a cube with side x as a is to b ; and this is equivalent to saying that a has to b the triplicate ratio of a to x .

Euclid is careful to use the forms *διπλασίων*, *τριπλασίων*, etc. to express what we translate as *duplicate*, *triplicate* etc. ratios; the Greek mathematicians, however, commonly used *διπλάσιος λόγος*, "double ratio," *τριπλάσιος λόγος*, "triple ratio" etc. in the sense of the ratios of 2 to 1, 3 to 1 etc. The effort, if such it was, to keep the one form for the one signification and the other for the other was only partially successful, as there are several instances of the contrary use, e.g. in Archimedes, Nicomachus and Pappus.

The expression for having the ratio which is "duplicate (triplicate) of that which it has to the second" is curious—*διπλασίονα* (*τριπλασίονα*) *λόγον ἔχειν ἥπερ πρὸς τὸ δεύτερον*—*ἥπερ* being used as if *διπλασίονα* or *τριπλασίονα* were a sort of comparative, in the same way as it is used after *μείζονα* or *ἐλάσσονα*. Another way of expressing the same thing is to say *λόγος διπλασίων* (*τριπλασίων*) *τοῦ, ὃν ἔχει...* the ratio "duplicate of that (ratio) which..." The explanation of both constructions would seem to be that *διπλάσιος* or *διπλασίων* is, as Hultsch translates it in his edition of Pappus (cf. p. 59, 17), *duplo maior*, where the ablative *duplo* implies not a difference but a proportion.

The four magnitudes in Def. 10 must of course be in *continued* proportion (*κατὰ τὸ συνεχές*). The Greek text as it stands does not state this.

DEFINITION II.

Ὁμόλογα μεγέθη λέγεται τὰ μὲν ἡγούμενα τοῖς ἡγούμενοις τὰ δὲ ἐπόμενα τοῖς ἐπομένοις.

It is difficult to express the meaning of the Greek in as few words. A translation more literal, but conveying less, would be, "Antecedents are called *corresponding magnitudes* to antecedents, and consequents to consequents."

I have preferred to translate ὁμόλογος by "corresponding" rather than by "homologous." I do not agree with Max Simon when he says (Euclid, p. 111) that the technical term "homologous" is not the adjective ὁμόλογος, and does not mean "corresponding," "agreeing," but "like in respect of the proportion" ("ähnlich in Bezug auf das Verhältniss"). The definition seems to me to be for the purpose of appropriating to a technical use precisely the ordinary adjective ὁμόλογος, "agreeing" or "corresponding."

Antecedents, ἡγούμενα, are literally "*leading* (terms)," and *consequents*, ἐπόμενα, "*following* (terms)."

DEFINITION 12.

Ἐναλλὰξ λόγος ἐστὶ λήψις τοῦ ἡγούμενου πρὸς τὸ ἡγούμενον καὶ τοῦ ἐπομένου πρὸς τὸ ἐπόμενον.

We now come to a number of expressions for the transformation of ratios or proportions. The first is ἐναλλάξ, *alternately*, which would be better described with reference to a proportion of four terms than with reference to a ratio. But probably Euclid defined all the terms in Deff. 12—16 with reference to *ratios* because to define them with reference to proportions would look like assuming what ought to be proved, namely the legitimacy of the various transformations of proportions (cf. v. 16, 7 Por., 18, 17, 19 Por.). The word ἐναλλάξ is of course a common term which has no exclusive reference to mathematics. But this same use of it with reference to proportions already occurs in Aristotle: *Anal. post.* 1. 5, 74 a 18, καὶ τὸ ἀνάλογον ὅτι ἐναλλάξ, "and that a proportion (is true) alternately, or *alternando*." Used with λόγος, as here, the adverb ἐναλλάξ has the sense of an adjective, "alternate"; we have already had it similarly used of "alternate angles" (αἱ ἐναλλάξ γωνίαι) in the theory of parallels.

DEFINITION 13.

Ἀνάπαλιν λόγος ἐστὶ λήψις τοῦ ἐπομένου ὡς ἡγούμενου πρὸς τὸ ἡγούμενον ὡς ἐπόμενον.

Ἀνάπαλιν, "inversely," "the other way about," is also a general term with no exclusive reference to mathematics. For this use of it with reference to proportion cf. Aristotle, *De Caelo* 1. 6, 273 b 32 τὴν ἀναλογίαν ἣν τὰ βάρη ἔχει, οἱ χρόνοι ἀνάπαλιν ἔξουσιν, "the proportion which the weights have, the times will have *inversely*." As here used with λόγος, ἀνάπαλιν is, exceptionally, adjectival.

DEFINITION 14.

Σύνθεσις λόγου ἐστὶ λήψις τοῦ ἡγούμενου μετὰ τοῦ ἐπομένου ὡς ἐνὸς πρὸς αὐτὸ τὸ ἐπόμενον.

The *composition* of a ratio is to be distinguished from the *compounding* of ratios and *compounded ratio* (συγκείμενος λόγος) as explained above in the note

on Def. 9, 10. The fact is that *συντίθῃμι* and what serves for the passive of it (*σύγκειμαι*) are used for *adding* as well as *compounding* in the sense of compounding ratios. In order to distinguish the two senses, I have always used the word *componendo* where the sense is that of this definition, though this requires a slight departure from the literal rendering of some passages. Thus the enunciation of v. 17 says, literally, "if magnitudes compounded be in proportion they will also be in proportion separated" (*ἐὰν συγκείμενα μεγέθη ἀνάλογον ᾗ, καὶ διαιεθέντα ἀνάλογον ἔσται*). This practically means that, if $A + B$ is to B as $C + D$ is to D , then A is to B as C is to D . I have accordingly translated as follows: "if magnitudes be proportional *componendo*, they will also be proportional *separando*." (It will be observed that *separando*, a term explained in the next note, is here used, not relatively to the proportion A is to B as C is to D , but relatively to the proportion *componendo*, viz. $A + B$ is to B as $C + D$ is to D .) The corresponding term for *componendo* in the Greek mathematicians is *συνθέντι*, literally "to one who has compounded," i.e. "if we compound." (For this absolute use of the dative of the participle cf. Nicomachus i. 8, 9 ἀπὸ μονάδος...κατὰ τὸν διπλάσιον λόγον προχωροῦντι μέχρις ἄπειρου, ὅσοι καὶ ἂν γίνωνται, οὗτοι πάντες ἀρτιάκεις ἀρτιοὶ εἰσιν. A very good instance from Aristotle is *Eth. Nic.* i. 5, 1097 b 12 ἐπεκτείνονται γὰρ ἐπὶ τοὺς γονεῖς καὶ τοὺς ἀπογόνους καὶ τῶν φίλων τοὺς φίλους εἰς ἄπειρον πρόεισιν.) A variation for *συνθέντι* found in Archimedes is *κατὰ σύνθεσιν*. Perhaps the more exclusive use of the form *συνθέντι* by geometers later than Euclid to denote the *composition of a ratio*, as compared with Euclid's more general use of *σύνθεσις* and other parts of the verb *συντίθῃμι* or *σύγκειμαι*, may point to a desire to get rid of ambiguity of terms and to make the terminology of geometry more exact.

DEFINITION 15.

Διαιρέσεις λόγου ἐστὶ λήψις τῆς ὑπεροχῆς, ἣ ὑπερέχει τὸ ἡγούμενον τοῦ ἐπομένου, πρὸς αὐτὸ τὸ ἐπόμενον.

As *composition of a ratio* means the transformation, e.g., of the ratio of A to B into the ratio of $A + B$ to B , so the *separation of a ratio* indicates the transformation of it into the ratio of $A - B$ to B . Thus, as the new antecedent is in one case got by *adding* the original antecedent to the original consequent, so the antecedent in the other case is obtained by *subtracting* the original consequent from the original antecedent (it being assumed that the latter is greater than the former). Hence the literal translations of *διαίρεσις λόγου*, "division of a ratio," and of *διελόντι* (the corresponding term to *συνθέντι*) as *dividendo*, scarcely give a sufficiently obvious explanation of the meaning. Heiberg accordingly translates by "subtractio rationis," which again may be thought to depart too far from the Greek. Perhaps "separation" and *separando* may serve as a compromise.

DEFINITION 16.

Ἀναστροφὴ λόγου ἐστὶ λήψις τοῦ ἡγούμενου πρὸς τὴν ὑπεροχήν, ἣ ὑπερέχει τὸ ἡγούμενον τοῦ ἐπομένου.

Conversion of a ratio means taking, e.g., instead of the ratio of A to B , the ratio of A to $A - B$ (A being again supposed greater than B). As *ἀναστροφή* is used for *conversion*, so *ἀναστρέψαντι* is used for *convertendo* (corresponding to the terms *συνθέντι* and *διελόντι*).

DEFINITION 17.

Δι' ἴσου λόγος ἐστὶ πλειόνων ὄντων μεγεθῶν καὶ ἄλλων αὐτοῖς ἴσων τὸ πλῆθος σύνδνο λαμβανομένων καὶ ἐν τῷ αὐτῷ λόγῳ, ὅταν ἢ ὡς ἐν τοῖς πρώτοις μεγέθεσι τὸ πρῶτον πρὸς τὸ ἔσχατον, οὕτως ἐν τοῖς δευτέροις μεγέθεσι τὸ πρῶτον πρὸς τὸ ἔσχατον ἢ ἄλλως· Ἀήψις τῶν ἄκρων καθ' ὑπεξάφρσιν τῶν μέσων.

δι' ἴσον, *ex aequali*, must apparently mean *ex aequali distantia*, at an equal distance or interval, i.e. after an equal number of intervening terms. The wording of the definition suggests that it is rather a *proportion ex aequali* than a *ratio ex aequali* which is being defined (cf. Def. 12). The meaning is clear enough. If $a, b, c, d \dots$ be one set of magnitudes, and $A, B, C, D \dots$ another set of magnitudes, such that

a is to b as A is to B ,

b is to c as B is to C ,

and so on, the last proportion being, e.g.,

k is to l , as K is to L ,

then the inference *ex aequali* is that

a is to l as A is to L .

The *fact* that this is so, or the *truth* of the inference from the hypothesis, is not proved until v. 22. The definition is therefore merely verbal; it gives a convenient *name* to a certain inference which is of constant application in mathematics. But *ex aequali* could not be intelligibly defined except with reference to two sets of ratios respectively equal.

DEFINITION 18.

Τεταραγμένη δὲ ἀναλογία ἐστίν, ὅταν τριῶν ὄντων μεγεθῶν καὶ ἄλλων αὐτοῖς ἴσων τὸ πλῆθος γίνηται ὡς μὲν ἐν τοῖς πρώτοις μεγέθεσιν ἡγούμενον πρὸς ἐπόμενον, οὕτως ἐν τοῖς δευτέροις μεγέθεσιν ἡγούμενον πρὸς ἐπόμενον, ὡς δὲ ἐν τοῖς πρώτοις μεγέθεσιν ἐπόμενον πρὸς ἄλλο τι, οὕτως ἐν τοῖς δευτέροις ἄλλο τι πρὸς ἡγούμενον.

Though the words δι' ἴσου, *ex aequali*, are not in this definition, it gives a description of a case in which the inference *ex aequali* is still true, as will be hereafter proved in v. 23. A *perturbed proportion* is an expression for the case when, there being three magnitudes a, b, c and three others A, B, C ,

a is to b as B is to C ,

and

b is to c as A is to B .

Another description of this case is found in Archimedes, "*the ratios being dissimilarly ordered*" (ἀνομοίως τεταγμένων τῶν λόγων). The full description of the *inference* in this case (as proved in v. 23), namely that

a is to c as A is to C ,

is *ex aequali in perturbed proportion* (δι' ἴσου ἐν τεταραγμένη ἀναλογίᾳ). Archimedes sometimes omits the δι' ἴσου, first giving the two proportions and proceeding thus: "therefore, the proportions being dissimilarly ordered, a has to c the same ratio as A has to C ."

The fact that Def. 18 describes a particular case in which the inference δι' ἴσου will be proved true seems to have suggested to some one after Theon's time the interpolation of another definition between 17 and 18 to

describe the *ordinary* case where the argument *ex aequali* holds good. The interpolated definition runs thus: "an *ordered proportion* (τεταγμένη ἀναλογία) arises when, as antecedent is to consequent, so is antecedent to consequent, and, as consequent is to something else, so is consequent to something else." This case needed no description after Def. 17 itself; and the supposed definition is never used.

After the definitions of Book v. Simson supplies the following *axioms*.

1. Equimultiples of the same or of equal magnitudes are equal to one another.
2. Those magnitudes of which the same or equal magnitudes are equimultiples are equal to one another.
3. A multiple of a greater magnitude is greater than the same multiple of a less.
4. That magnitude of which a multiple is greater than the same multiple of another is greater than that other magnitude.

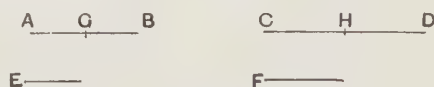
BOOK V. PROPOSITIONS.

PROPOSITION I.

If there be any number of magnitudes whatever which are, respectively, equimultiples of any magnitudes equal in multitude, then, whatever multiple one of the magnitudes is of one, that multiple also will all be of all.

Let any number of magnitudes whatever AB , CD be respectively equimultiples of any magnitudes E , F equal in multitude ;

I say that, whatever multiple AB is of E , that multiple will AB , CD also be of E , F .



For, since AB is the same multiple of E that CD is of F , as many magnitudes as there are in AB equal to E , so many also are there in CD equal to F .

Let AB be divided into the magnitudes AG , GB equal to E ,

and CD into CH , HD equal to F ;

then the multitude of the magnitudes AG , GB will be equal to the multitude of the magnitudes CH , HD .

Now, since AG is equal to E , and CH to F , therefore AG is equal to E , and AG , CH to E , F .

For the same reason

GB is equal to E , and GB , HD to E , F ;

therefore, as many magnitudes as there are in AB equal to E , so many also are there in AB , CD equal to E , F ;

therefore, whatever multiple AB is of E , that multiple will AB , CD also be of E , F .

Therefore etc.

Q. E. D.

De Morgan remarks of v. 1—6 that they are “simple propositions of concrete arithmetic, covered in language which makes them unintelligible to modern ears. The first, for instance, states no more than that *ten* acres and *ten* roods make *ten* times as much as one acre and one rood.” One aim therefore of notes on these as well as the other propositions of Book v. should be to make their purport clearer to the learner by setting them side by side with the same truths expressed in the much shorter and more familiar modern (algebraical) notation. In doing so, we shall express magnitudes by the first letters of the alphabet, a, b, c etc., adopting small instead of capital letters so as to avoid confusion with Euclid’s lettering; and we shall use the small letters m, n, p etc. to represent integral numbers. Thus ma will always mean m times a or the m^{th} multiple of a (counting $1 \cdot a$ as the first, $2 \cdot a$ as the second multiple, and so on).

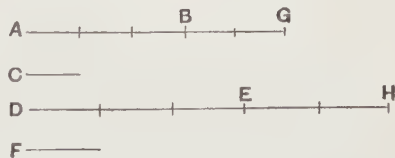
Prop. 1 then asserts that, if ma, mb, mc etc. be any equimultiples of a, b, c etc., then

$$ma + mb + mc + \dots = m(a + b + c + \dots).$$

PROPOSITION 2.

If a first magnitude be the same multiple of a second that a third is of a fourth, and a fifth also be the same multiple of the second that a sixth is of the fourth, the sum of the first and fifth will also be the same multiple of the second that the sum of the third and sixth is of the fourth.

Let a first magnitude, AB , be the same multiple of a second, C , that a third, DE , is of a fourth, F , and let a fifth, BG , also be the same multiple of the second, C , that a sixth, EH , is of the fourth F ;



I say that the sum of the first and fifth, AG , will be the same multiple of the second, C , that the sum of the third and sixth, DH , is of the fourth, F .

For, since AB is the same multiple of C that DE is of F , therefore, as many magnitudes as there are in AB equal to C , so many also are there in DE equal to F .

For the same reason also, as many as there are in BG equal to C , so many are there also in EH equal to F ;

therefore, as many as there are in the whole AG equal to C , so many also are there in the whole DH equal to F .

Therefore, whatever multiple AG is of C , that multiple also is DH of F .

Therefore the sum of the first and fifth, AG , is the same multiple of the second, C , that the sum of the third and sixth, DH , is of the fourth, F .

Therefore etc.

Q. E. D.

To find the corresponding formula for the result of this proposition, we may suppose a to be the "second" magnitude and b the "fourth." If now the "first" magnitude is ma , the "third" is, by hypothesis, mb ; and, if the "fifth" magnitude is na , the "sixth" is nb . The proposition then asserts that $ma + na$ is the same multiple of a that $mb + nb$ is of b .

More generally, if $pa, qa \dots$ and $pb, qb \dots$ be any further equimultiples of a, b respectively, $ma + na + pa + qa + \dots$ is the same multiple of a that $mb + nb + pb + qb + \dots$ is of b . This extension is stated in Simson's corollary to v. 2 thus:

"From this it is plain that, if any number of magnitudes AB, BG, GH be multiples of another C ; and as many DE, EK, KL be the same multiples of F , each of each; the whole of the first, viz. AH , is the same multiple of C that the whole of the last, viz. DL , is of F ."

The course of the proof, which separates m into its units and also n into its units, practically tells us that the multiple of a arrived at by adding the *two* multiples is the $(m + n)$ th multiple; or practically we are shown that

$$ma + na = (m + n)a,$$

or, more generally, that

$$ma + na + pa + \dots = (m + n + p + \dots)a.$$

PROPOSITION 3.

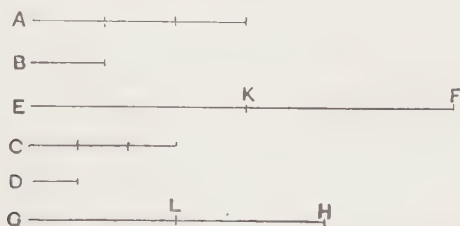
If a first magnitude be the same multiple of a second that a third is of a fourth, and if equimultiples be taken of the first and third, then also ex aequali the magnitudes taken will be equimultiples respectively, the one of the second and the other of the fourth.

Let a first magnitude A be the same multiple of a second B that a third C is of a fourth D , and let equimultiples EF, GH be taken of A, C ;

I say that EF is the same multiple of B that GH is of D .

For, since EF is the same multiple of A that GH is of C , therefore, as many magnitudes as there are in EF equal to A , so many also are there in GH equal to C .

Let EF be divided into the magnitudes EK , KF equal to A , and GH into the magnitudes GL , LH equal to C ; then the multitude of the magnitudes EK , KF will be equal to the multitude of the magnitudes GL , LH .



And, since A is the same multiple of B that C is of D , while EK is equal to A , and GL to C , therefore EK is the same multiple of B that GL is of D .

For the same reason

KF is the same multiple of B that LH is of D .

Since, then, a first magnitude EK is the same multiple of a second B that a third GL is of a fourth D , and a fifth KF is also the same multiple of the second B that a sixth LH is of the fourth D ,

therefore the sum of the first and fifth, EF , is also the same multiple of the second B that the sum of the third and sixth, GH , is of the fourth D . [v. 2]

Therefore etc.

Q. E. D.

Heiberg remarks of the use of *ex aequali* in the enunciation of this proposition that, strictly speaking, it has no reference to the definition (17) of a *ratio ex aequali*. But the uses of the expression here and in the definition are, I think, sufficiently parallel, as may be seen thus. The proposition asserts that, if

na , nb are equimultiples of a , b ,

and if

$m \cdot na$, $m \cdot nb$ are equimultiples of na , nb ,

then $m \cdot na$ is the same multiple of a that $m \cdot nb$ is of b . Clearly the proposition can be extended by taking further equimultiples of the last equimultiples and so on; and we can prove that

$p \cdot q \dots m \cdot na$ is the same multiple of a that $p \cdot q \dots m \cdot nb$ is of b ,

where the series of numbers $p \cdot q \dots m \cdot n$ is exactly the same in both expressions;

and *ex aequali* ($\delta\iota' \text{ } \iota\sigma\sigma\upsilon$) expresses the fact that the equimultiples are *at the same distance* from a , b in the series na , $m \cdot na \dots$ and nb , $m \cdot nb \dots$ respectively.

Here again the proof breaks m into its units, and then breaks n into its units; and we are practically shown that the multiple of a arrived at, viz. $m \cdot na$, is the multiple denoted by the product of the numbers m, n , i.e. the (mn) th multiple, or in other words that

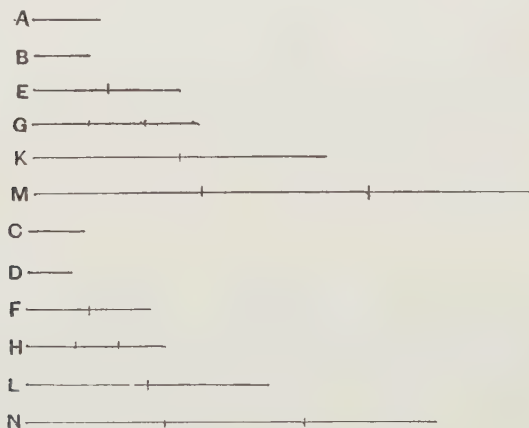
$$m \cdot na = mn \cdot a.$$

PROPOSITION 4.

If a first magnitude have to a second the same ratio as a third to a fourth, any equimultiples whatever of the first and third will also have the same ratio to any equimultiples whatever of the second and fourth respectively, taken in corresponding order.

For let a first magnitude A have to a second B the same ratio as a third C to a fourth D ; and let equimultiples E, F be taken of A, C , and G, H other, chance, equimultiples of B, D ;

I say that, as E is to G , so is F to H .



For let equimultiples K, L be taken of E, F , and other, chance, equimultiples M, N of G, H .

Since E is the same multiple of A that F is of C , and equimultiples K, L of E, F have been taken, therefore K is the same multiple of A that L is of C . [v. 3]

For the same reason

M is the same multiple of B that N is of D .

And, since, as A is to B , so is C to D ,
 and of A , C equimultiples K , L have been taken,
 and of B , D other, chance, equimultiples M , N ,
 therefore, if K is in excess of M , L also is in excess of N ,
 if it is equal, equal, and if less, less. [v. Def. 5]

And K , L are equimultiples of E , F ,
 and M , N other, chance, equimultiples of G , H ;
 therefore, as E is to G , so is F to H . [v. Def. 5]

Therefore etc.

Q. E. D.

This proposition shows that, if a , b , c , d are proportionals, then

ma is to nb as mc is to nd ;

and the proof is as follows:

Take pma , pmc any equimultiples of ma , mc , and qnb , qnd any equimultiples of nb , nd .

Since $a : b = c : d$, it follows [v. Def. 5] that,
 according as $pma > < qnb$, $pmc > < qnd$.

But the p - and q -equimultiples are *any* equimultiples;
 therefore [v. Def. 5]

$ma : nb = mc : nd$.

It will be observed that Euclid's phrase for taking *any* equimultiples of A , C and *any* other equimultiples of B , D is "let there be taken equimultiples E , F of A , C , and G , H other, *chance*, equimultiples of B , D ," E , F being called *ισάκεις πολλαπλάσια* simply, and G , H *ἄλλα, ἃ ἐτυχεῖν, ἰσάκεις πολλαπλάσια*. And similarly, when *any* equimultiples (K , L) of E , F come to be taken, and *any* other equimultiples (M , N) of G , H . But later on Euclid uses the same phrases about the *new* equimultiples with reference to the original magnitudes, reciting that "there have been taken, of A , C , equimultiples K , L and of B , D , other, *chance*, equimultiples M , N "; whereas M , N are not *any* equimultiples *whatever* of B , D , but are *any* equimultiples of the *particular* multiples (G , H) which have been taken of B , D respectively, though *these latter* have been taken at random. Simson would, in the first place, add *ἃ ἐτυχεῖν* in the passages where *any* equimultiples E , F are taken of A , C and *any* equimultiples K , L are taken of E , F , because the words are "wholly necessary" and, in the second place, would leave them out where M , N are called *ἄλλα, ἃ ἐτυχεῖν, ἰσάκεις πολλαπλάσια* of B , D , because it is not true that of B , D have been taken "any equimultiples whatever (*ἃ ἐτυχεῖν*), M , N ." Simson adds: "And it is strange that neither Mr Briggs, who did right to leave out these words in one place of Prop. 13 of this book, nor Dr Gregory, who changed them into the word 'some' in three places, and left them out in a fourth of that same Prop. 13, did not also leave them out in this place of Prop. 4 and in the second of the two places where they occur in Prop. 17 of this book, in neither of which they can stand consistent with truth: And in none of all these places, even in those which they corrected in their Latin translation, have they cancelled the words *ἃ ἐτυχεῖν* in the Greek text, as they ought to have done. The same words *ἃ ἐτυχεῖν* are found in

four places of Prop. 11 of this book, in the first and last of which they are necessary, but in the second and third, though they are true, they are quite superfluous; as they likewise are in the second of the two places in which they are found in the 12th prop. and in the like places of Prop. 22, 23 of this book; but are wanting in the last place of Prop. 23, as also in Prop. 25, Book xi."

As will be seen, Simson's emendations amount to alterations of the text so considerable as to suggest doubt whether we should be justified in making them in the absence of MS. authority. The phrase "equimultiples of A, C and other, chance, equimultiples of B, D " recurs so constantly as to suggest that it was for Euclid a quasi-stereotyped phrase, and that it is equally genuine wherever it occurs. Is it then absolutely necessary to insert ἀ ἑνυχῆ in places where it does not occur, and to leave it out in the places where Simson holds it to be wrong? I think the text can be defended as it stands. In the first place to say "take equimultiples of A, C " is a fair enough way of saying take *any* equimultiples *whatever* of A, C . The other difficulty is greater, but may, I think, be only due to the adoption of *any whatever* as the translation of ἀ ἑνυχῆ. As a matter of fact, the words only mean *chance* equimultiples, equimultiples which are the result of random selection. Is it not justifiable to describe the product of two chance numbers, numbers selected at random, as being a "*chance* number," since it is the result of two random selections? I think so, and I have translated ἀ ἑνυχῆ accordingly as implying, in the case in question, "other equimultiples whatever they may happen to be."

To this proposition Theon added the following :

"Since then it was proved that, if K is in excess of M , L is also in excess of N , if it is equal, (the other is) equal, and if less, less, it is clear also that, if M is in excess of K , N is also in excess of L , if it is equal, (the other is) equal, and if less, less; and for this reason,

as G is to E , so also is H to F .

PORISM. From this it is manifest that, if four magnitudes be proportional, they will also be proportional inversely."

Simson rightly pointed out that the demonstration of what Theon intended to prove, viz. that, if E, G, F, H be proportionals, they are proportional inversely, i.e. G is to E as H is to F , does not in the least depend upon this 4th proposition or the proof of it; for, when it is said that, "if K exceeds M , L also exceeds N etc.," this is not proved from the fact that E, G, F, H are proportionals (which is the conclusion of Prop. 4), but from the fact that A, B, C, D are proportionals.

The proposition that, if A, B, C, D are proportionals, they are also proportionals inversely is not given by Euclid, but Simson supplies the proof in his Prop. B. The fact is really obvious at once from the 5th definition of Book v. (cf. p. 127 above), and Euclid probably omitted the proposition as unnecessary.

Simson added, in place of Theon's corollary, the following :

"Likewise, if the first has the same ratio to the second which the third has to the fourth, then also any equimultiples whatever of the first and third have the same ratio to the second and fourth : And, in like manner, the first and the third have the same ratio to any equimultiples whatever of the second and fourth."

The proof, of course, follows exactly the method of Euclid's proposition itself, with the only difference that, instead of one of the two pairs of equimultiples, the magnitudes themselves are taken. In other words, the conclusion that

$$ma \text{ is to } nb \text{ as } mc \text{ is to } nd$$

is equally true when either m or n is equal to unity.

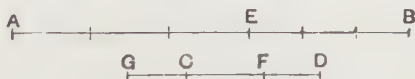
As De Morgan says, Simson's corollary is only necessary to those who will not admit M into the list $M, 2M, 3M$ etc.; the exclusion is grammatical and nothing else. The same may be said of Simson's Prop. A to the effect that, "If the first of four magnitudes has to the second the same ratio which the third has to the fourth: then, if the first be greater than the second, the third is also greater than the fourth; and if equal, equal; if less, less." This is needless to those who believe *once* A to be a proper component of the list of multiples, in spite of *multus* signifying many.

PROPOSITION 5.

If a magnitude be the same multiple of a magnitude that a part subtracted is of a part subtracted, the remainder will also be the same multiple of the remainder that the whole is of the whole.

5 For let the magnitude AB be the same multiple of the magnitude CD that the part AE subtracted is of the part CF subtracted;

I say that the remainder EB is also the same multiple of the remainder FD that the whole AB is of the whole CD .



10 For, whatever multiple AE is of CF , let EB be made that multiple of CG .

Then, since AE is the same multiple of CF that EB is of GC ,

therefore AE is the same multiple of CF that AB is of GF .

[v. 1]

15 But, by the assumption, AE is the same multiple of CF that AB is of CD .

Therefore AB is the same multiple of each of the magnitudes GF, CD ;

therefore GF is equal to CD .

20 Let CF be subtracted from each;

therefore the remainder GC is equal to the remainder FD .

And, since AE is the same multiple of CF that EB is of GC ,

and GC is equal to DF ,

25 therefore AE is the same multiple of CF that EB is of FD .

But, by hypothesis,

AE is the same multiple of CF that AB is of CD ;

therefore EB is the same multiple of FD that AB is of CD .

That is, the remainder EB will be the same multiple of
30 the remainder FD that the whole AB is of the whole CD .

Therefore etc.

Q. E. D.

10. let EB be made that multiple of CG , *τοσαυταπλάσιον γεγενέτω καὶ τὸ EB τοῦ $ΓΗ$* . From this way of stating the construction one might suppose that CG was given and EB had to be found equal to a certain multiple of it. But in fact EB is what is given and CG has to be found, i.e. CG has to be constructed as a certain *submultiple* of EB .

This proposition corresponds to v. 1, with subtraction taking the place of addition. It proves the formula

$$ma - mb = m(a - b).$$

Euclid's construction assumes that, if AE is any multiple of CF , and EB is any other magnitude, a fourth straight line can be found such that EB is the same multiple of it that AE is of CF , or in other words that, given any magnitude, we can divide it into any number of equal parts. This is however not proved, even of straight lines, much less other magnitudes, until vi. 9. Peletarius had already seen this objection to the construction. The difficulty is not got over by regarding it merely as a *hypothetical* construction; for hypothetical constructions are not in Euclid's manner. The remedy is to substitute the alternative construction given by Simson, after Peletarius and Campanus' translation from the Arabic, which only requires us to add a magnitude to itself a certain number of times. The demonstration follows Euclid's line exactly.

"Take AG the same multiple of FD that AE is of CF ;

therefore AE is the same multiple of CF that EG is of CD .

[v. 1]

But AE , by hypothesis, is the same multiple of CF that AB is of CD ; therefore EG is the same multiple of CD that AB is of CD ;

wherefore EG is equal to AB .

Take from them the common magnitude AE ; the remainder AG is equal to the remainder EB .

Wherefore, since AE is the same multiple of CF that AG is of FD , and since AG is equal to EB ,
therefore AE is the same multiple of CF that EB is of FD .

But AE is the same multiple of CF that AB is of CD ;
therefore EB is the same multiple of FD that AB is of CD ."

Q. E. D.



Euclid's proof amounts to this.

Suppose a magnitude x taken such that

$$ma - mb = mx, \text{ say.}$$

Add mb to each side, whence (by v. 1)

$$ma = m(x + b).$$

Therefore

$$a = x + b, \text{ or } x = a - b,$$

so that

$$ma - mb = m(a - b).$$

Simson's proof, on the other hand, argues thus.

Take $x = m(a - b)$, the same multiple of $(a - b)$ that mb is of b .

Then, by addition of mb to both sides, we have [v. 1]

$$x + mb = ma,$$

or

$$x = ma - mb.$$

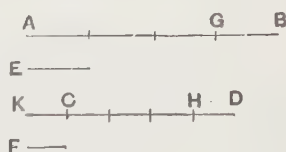
That is,

$$ma - mb = m(a - b).$$

PROPOSITION 6.

If two magnitudes be equimultiples of two magnitudes, and any magnitudes subtracted from them be equimultiples of the same, the remainders also are either equal to the same or equimultiples of them.

For let two magnitudes AB , CD be equimultiples of two magnitudes E , F , and let AG , CH subtracted from them be equimultiples of the same two E , F ; I say that the remainders also, GB , HD , are either equal to E , F or equimultiples of them.



For, first, let GB be equal to E ;
I say that HD is also equal to F .

For let CK be made equal to F .

Since AG is the same multiple of E that CH is of F , while GB is equal to E and KC to F , therefore AB is the same multiple of E that KH is of F .

[v. 2]

But, by hypothesis, AB is the same multiple of E that CD is of F ;

therefore KH is the same multiple of F that CD is of F .

Since then each of the magnitudes KH , CD is the same multiple of F ,

therefore KH is equal to CD .

Let CH be subtracted from each ;
therefore the remainder KC is equal to the remainder HD .

But F is equal to KC ;
therefore HD is also equal to F .

Hence, if GB is equal to E , HD is also equal to F .

Similarly we can prove that, even if GB be a multiple of E , HD is also the same multiple of F .

Therefore etc.

Q. E. D.

This proposition corresponds to v. 2, with subtraction taking the place of addition. It asserts namely that, if n is less than m , $ma - na$ is the same multiple of a that $mb - nb$ is of b . The enunciation distinguishes the cases in which $m - n$ is equal to 1 and greater than 1 respectively.

Simson observes that, while only the first case (the simpler one) is proved in the Greek, both are given in the Latin translation from the Arabic ; and he supplies accordingly the proof of the second case, which Euclid leaves to the reader. The fact is that it is exactly the same as the other except that, in the construction, CK is made the same multiple of F that GB is of E , and at the end, when it has been proved that KC is equal to HD , instead of concluding that HD is equal to F , we have to say "Because GB is the same multiple of E that KC is of F , and KC is equal to HD , therefore HD is the same multiple of F that GB is of E ."

PROPOSITION 7.

Equal magnitudes have to the same the same ratio, as also has the same to equal magnitudes.

Let A, B be equal magnitudes and C any other, chance, magnitude ;

I say that each of the magnitudes A, B has the same ratio to C , and C has the same ratio to each of the magnitudes A, B .



For let equimultiples D, E of A, B be taken, and of C another, chance, multiple F .

Then, since D is the same multiple of A that E is of B ,
while A is equal to B ,

therefore D is equal to E .

But F is another, chance, magnitude.

If therefore D is in excess of F , E is also in excess of F , if equal to it, equal; and, if less, less.

And D , E are equimultiples of A , B , while F is another, chance, multiple of C ;

therefore, as A is to C , so is B to C . [v. Def. 5]

I say next that C also has the same ratio to each of the magnitudes A , B .

For, with the same construction, we can prove similarly that D is equal to E ;

and F is some other magnitude.

If therefore F is in excess of D , it is also in excess of E , if equal, equal; and, if less, less.

And F is a multiple of C , while D , E are other, chance, equimultiples of A , B ;

therefore, as C is to A , so is C to B . [v. Def. 5]

Therefore etc.

PORISM. From this it is manifest that, if any magnitudes are proportional, they will also be proportional inversely.

Q. E. D.

In this proposition there is a similar use of $\delta \epsilon \tau \rho \chi \epsilon \nu$ to that which has been discussed under Prop. 4. Any multiple F of C is taken and then, four lines lower down, we are told that " F is another, chance, magnitude." It is of course not *any magnitude whatever*, and Simson leaves out the sentence, but this time without calling attention to it.

Of the Porism to this proposition Heiberg says that it is properly put here in the best ms.; for, as August had already observed, if it was in its right place where Theon put it (at the end of v. 4), the second part of the proof of this proposition would be unnecessary. But the truth is that the Porism is no more in place here. The most that the proposition proves is that, if A , B are equal, and C any other magnitude, then two conclusions are simultaneously established, (1) that A is to C as B is to C and (2) that C is to A as C is to B . The second conclusion is not established from the first conclusion (as it ought to be in order to justify the inference in the Porism), but from a hypothesis on which the first conclusion itself depends; and moreover it is not a proportion in its general form, i.e. between *four* magnitudes, that is in question, but only the particular case in which the consequents are equal.

Aristotle tacitly assumes *inversion* (combined with the solution of the problem of Eucl. vi. 11) in *Meteorologica* III. 5, 376 a 14—16.

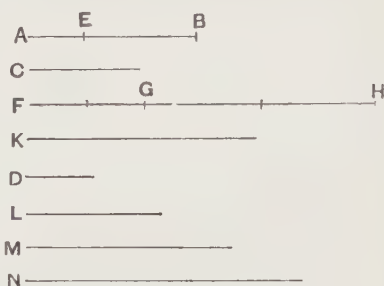
PROPOSITION 8.

Of unequal magnitudes, the greater has to the same a greater ratio than the less has; and the same has to the less a greater ratio than it has to the greater.

Let AB, C be unequal magnitudes, and let AB be greater ;
let D be another, chance,
magnitude ;

I say that AB has to D a
greater ratio than C has to
 D , and D has to C a greater
ratio than it has to AB .

For, since AB is greater
than C , let BE be made equal
to C ;
then the less of the magni-
tudes AE, EB , if multiplied,
will sometime be greater than D .



[v. Def. 4]

[Case 1.]

First, let AE be less than EB ;
let AE be multiplied, and let FG be a multiple of it which is
greater than D ;
then, whatever multiple FG is of AE , let GH be made the
same multiple of EB and K of C ;
and let L be taken double of D , M triple of it, and successive
multiples increasing by one, until what is taken is a multiple
of D and the first that is greater than K . Let it be taken,
and let it be N which is quadruple of D and the first
multiple of it that is greater than K .

Then, since K is less than N first,
therefore K is not less than M .

And, since FG is the same multiple of AE that GH is of
 EB ,

therefore FG is the same multiple of AE that FH is of AB .

[v. 1]

But FG is the same multiple of AE that K is of C ;

therefore FH is the same multiple of AB that K is of C ;

therefore FH, K are equimultiples of AB, C .

Again, since GH is the same multiple of EB that K is
of C ,

and EB is equal to C ,

therefore GH is equal to K .

But K is not less than M ;
therefore neither is GH less than M .

And FG is greater than D ;

therefore the whole FH is greater than D , M together.

But D , M together are equal to N , inasmuch as M is triple of D , and M , D together are quadruple of D , while N is also quadruple of D ; whence M , D together are equal to N .

But FH is greater than M , D ;

therefore FH is in excess of N ,

while K is not in excess of N .

And FH , K are equimultiples of AB , C , while N is another, chance, multiple of D ;

therefore AB has to D a greater ratio than C has to D .

[v. Def. 7]

I say next, that D also has to C a greater ratio than D has to AB .

For, with the same construction, we can prove similarly that N is in excess of K , while N is not in excess of FH .

And N is a multiple of D ,

while FH , K are other, chance, equimultiples of AB , C ;

therefore D has to C a greater ratio than D has to AB .

[v. Def. 7]

[Case 2.]

Again, let AE be greater than EB .

Then the less, EB , if multiplied, will sometime be greater than D .

[v. Def. 4]

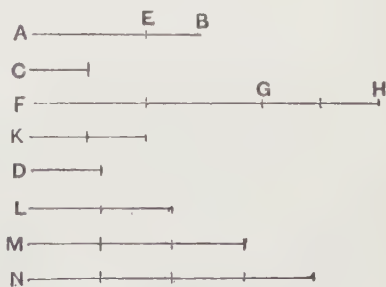
Let it be multiplied, and let GH be a multiple of EB and greater than D ;

and, whatever multiple GH is of EB , let FG be made the same multiple of AE , and K of C .

Then we can prove similarly that FH , K are equimultiples of AB , C ;

and, similarly, let N be taken a multiple of D but the first that is greater than FG ,

so that FG is again not less than M .



But GH is greater than D ;

therefore the whole FH is in excess of D , M , that is, of N .

Now K is not in excess of N , inasmuch as FG also, which is greater than GH , that is, than K , is not in excess of N .

And in the same manner, by following the above argument, we complete the demonstration.

Therefore etc.

Q. E. D.

The two separate cases found in the Greek text of the demonstration can practically be compressed into one. Also the expositor of the two cases makes them differ more than they need. It is necessary in each case to select the *smaller* of the two segments AE , EB of AB with a view to taking a multiple of it which is greater than D ; in the first case therefore AE is taken, in the second EB . But, while in the first case successive multiples of D are taken in order to find the first multiple that is greater than GH (or K), in the second case the multiple is taken which is the first that is greater than FG . This difference is not necessary; the first multiple of D that is greater than GH would equally serve in the second case. Lastly, the use of the magnitude K might have been dispensed with in both cases; it is of no practical use and only lengthens the proofs. For these reasons Simson considers that Theon, or some other unskilful editor, has vitiated the proposition. This however seems an unsafe assumption; for, while it was not the habit of the great Greek geometers to discuss separately a number of different cases (e.g. in i. 7 and i. 35 Euclid proves one case and leaves the others to the reader), there are many exceptions to prove the rule, e.g. Eucl. iii. 25 and 33; and we know that many fundamental propositions, afterwards proved generally, were first discovered in relation to particular cases and then generalised, so that Book v., presenting a comparatively new theory, might fairly be expected to exhibit more instances than the earlier books do of unnecessary subdivision. The use of the K is no more conclusive against the genuineness of the proofs.

Nevertheless Simson's version of the proof is certainly snorter, and moreover it takes account of the case in which AE is *equal* to EB , and of the case in which AE , EB are both greater than D (though these cases are scarcely worth separate mention).

"If the magnitude which is not the greater of the two AE , EB be (1) not less than D , take FG , GH the doubles of AE , EB .

But if that which is not the greater of the two AE , EB be (2) less than D , this magnitude can be multiplied so as to become greater than D whether it be AE or EB .

Let it be multiplied until it becomes greater than D , and let the other be multiplied as often; let FG be the multiple thus taken of AE and GH the same multiple of EB ,

therefore FG and GH are each of them greater than D .

And, in every one of the cases, take L the double of D , M its triple and so on, till the multiple of D be that which first becomes greater than GH .

Let N be that multiple of D which is first greater than GH , and M the multiple of D which is next less than N ,

Then, because N is the multiple of D which is the first that becomes greater than GH ,

the next preceding multiple is not greater than GH ;

that is, GH is not less than M .

And, since FG is the same multiple of AE that GH is of EB ,

GH is the same multiple of EB that FH is of AB ;

[v. 1]

wherefore FH , GH are equimultiples of AB , EB .

And it was shown that GH was not less than M ;

and, by the construction, FG is greater than D ;

therefore the whole FH is greater than M , D together.

But M , D together are equal to N ;

therefore FH is greater than N .

But GH is not greater than N ;

and FH , GH are equimultiples of AB , BE ,

and N is a multiple of D ;

therefore AB has to D a greater ratio than BE (or C) has to D . [v. Def. 7]

Also D has to BE a greater ratio than it has to AB .

For, having made the same construction, it may be shown, in like manner, that N is greater than GH but that it is not greater than FH ;

and N is a multiple of D ,

and GH , FH are equimultiples of EB , AB ;

Therefore D has to EB a greater ratio than it has to AB ." [v. Def. 7]

The proof may perhaps be more readily grasped in the more symbolical form thus.

Take the m th equimultiples of C , and of the excess of AB over C (that is, of AE), such that each is greater than D ;

and, of the multiples of D , let pD be the first that is greater than mC , and nD the next less multiple of D .

Then, since mC is not less than nD ,

and, by the construction, $m(AE)$ is greater than D ,

the sum of mC and $m(AE)$ is greater than the sum of nD and D .

That is, $m(AB)$ is greater than pD .

And, by the construction, mC is less than pD .

Therefore [v. Def. 7] AB has to D a greater ratio than C has to D .

Again, since pD is less than $m(AB)$,

and pD is greater than mC ,

D has to C a greater ratio than D has to AB .

PROPOSITION 9.

Magnitudes which have the same ratio to the same are equal to one another; and magnitudes to which the same has the same ratio are equal.

For let each of the magnitudes A , B have the same ratio to C ;

I say that A is equal to B .

For, otherwise, each of the magnitudes A , B would not have had the same ratio to C ; but it has;



[v. 8]

therefore A is equal to B .

Again, let C have the same ratio to each of the magnitudes A , B ;

I say that A is equal to B .

For, otherwise, C would not have had the same ratio to each of the magnitudes A , B ; but it has;

therefore A is equal to B .

Therefore etc.

Q. E. D.

If A is to C as B is to C ,

or if C is to A as C is to B , then A is equal to B .

Simson gives a more explicit proof of this proposition which has the advantage of referring back to the fundamental 5th and 7th definitions, instead of quoting the results of previous propositions, which, as will be seen from the next note, may be, in the circumstances, unsafe.

“Let A , B have each of them the same ratio to C ;

A is equal to B .

For, if they are not equal, one of them is greater than the other; let A be the greater.

Then, by what was shown in the preceding proposition, there are some equimultiples of A and B , and some multiple of C , such that the multiple of A is greater than the multiple of C , but the multiple of B is not greater than that of C .

Let such multiples be taken, and let D , E be the equimultiples of A , B , and F the multiple of C , so that D may be greater than F , and E not greater than F .

But, because A is to C as B is to C ,

and of A , B are taken equimultiples D , E , and of C is taken a multiple F , and D is greater than F ,

E must also be greater than F .

[v. Def. 5]

But E is not greater than F : which is impossible.

Next, let C have the same ratio to each of the magnitudes A and B ; A is equal to B .

For, if not, one of them is greater than the other; let A be the greater.

Therefore, as was shown in Prop. 8, there is some multiple F of C , and some equimultiples E and D of B and A , such that F is greater than E and not greater than D .

But, because C is to B as C is to A ,
and F the multiple of the first is greater than E the multiple of the second,
 F the multiple of the third is greater than D the multiple of the fourth.
[v. Def. 5]

But F is not greater than D : which is impossible.
Therefore A is equal to B ."

PROPOSITION 10.

Of magnitudes which have a ratio to the same, that which has a greater ratio is greater; and that to which the same has a greater ratio is less.

For let A have to C a greater ratio than B has to C ;
I say that A is greater than B .



For, if not, A is either equal to B or less.

Now A is not equal to B ;

for in that case each of the magnitudes A , B would have had the same ratio to C ;

[v. 7]

but they have not;

therefore A is not equal to B .

Nor again is A less than B ;

for in that case A would have had to C a less ratio than B has to C ;

[v. 8]

but it has not;

therefore A is not less than B .

But it was proved not to be equal either;

therefore A is greater than B .

Again, let C have to B a greater ratio than C has to A ;

I say that B is less than A .

For, if not, it is either equal or greater.

Now B is not equal to A ;

for in that case C would have had the same ratio to each of the magnitudes A , B ;

[v. 7]

but it has not;

therefore A is not equal to B .

Nor again is B greater than A ;
 for in that case C would have had to B a less ratio than it
 has to A ; [v. 8]
 but it has not;

therefore B is not greater than A .

But it was proved that it is not equal either;
 therefore B is less than A .

Therefore etc.

Q. E. D.

No better example can, I think, be found of the acuteness which Simson brought to bear in his critical examination of the *Elements*, and of his great services to the study of Euclid, than is furnished by the admirable note on this proposition where he points out a serious flaw in the proof as given in the text.

For the first time Euclid is arguing about *greater* and *less* ratios, and it will be found by an examination of the steps of the proof that he assumes more with regard to the meaning of the terms than he is entitled to assume, having regard to the fact that the definition of greater ratio (Def. 7) is all that, as yet, he has to go upon. That we cannot argue, at present, about *greater* and *less* as applied to *ratios* in the same way as about the same terms in relation to *magnitudes* is indeed sufficiently indicated by the fact that Euclid does not assume for ratios what is in Book I. an axiom, viz. that things which are equal to the same thing are equal to one another; on the contrary, he *proves*, in Prop. 11, that ratios which are the same with the same ratio are the same with one another.

Let us now examine the steps of the proof in the text. First we are told that

" A is greater than B .

For, if not, it is either equal to B or less than it.

Now A is not equal to B ;

for in that case each of the two magnitudes A , B would have had the
 same ratio to C ; [v. 7]

but they have not:

therefore A is not equal to B ."

As Simson remarks, the force of this reasoning is as follows.

If A has to C the same ratio as B has to C ,

then—supposing any equimultiples of A , B to be taken and any multiple of C —

by Def. 5, if the multiple of A be greater than the multiple of C , the multiple of B is also greater than that of C .

But it follows from the hypothesis (that A has a greater ratio to C than B has to C) that,

by Def. 7, there must be *some* equimultiples of A , B and *some* multiple of C such that the multiple of A is greater than the multiple of C , but the multiple of B is *not* greater than the same multiple of C .

And this directly contradicts the preceding deduction from the supposition that A has to C the same ratio as B has to C ;

therefore that supposition is impossible.

The proof now goes on thus :

“Nor again is A less than B ;

for, in that case, A would have had to C a less ratio than B has to C ;
[v. 8]

but it has not ;

therefore A is not less than B .”

It is here that the difficulty arises. As before, we must use Def. 7. “ A would have had to C a less ratio than B has to C ,” or the equivalent statement that B would have had to C a greater ratio than A has to C , means that there would have been *some* equimultiples of B , A and *some* multiple of C such that

(1) the multiple of B is greater than the multiple of C , but

(2) the multiple of A is *not* greater than the multiple of C ,

and it ought to have been proved that this can never happen if the hypothesis of the proposition is true, viz. that A has to C a greater ratio than B has to C : that is, it should have been proved that, in the latter case, the multiple of A is *always* greater than the multiple of C whenever the multiple of B is greater than the multiple of C (for, when this is demonstrated, it will be evident that B cannot have a greater ratio to C than A has to C). But this is not proved (cf. the remark of De Morgan quoted in the note on v. Def. 7, p. 130), and hence it is not proved that the above inference from the supposition that A is less than B is inconsistent with the hypothesis in the enunciation. The proof therefore fails.

Simson suggests that the proof is not Euclid's, but the work of some one who apparently “has been deceived in applying what is manifest, when understood of magnitudes, unto ratios, viz. that a magnitude cannot be both greater and less than another.”

The proof substituted by Simson is satisfactory and simple.

“Let A have to C a greater ratio than B has to C ;

A is greater than B .

For, because A has a greater ratio to C than B has to C , there are some equimultiples of A , B and some multiple of C such that

the multiple of A is greater than the multiple of C , but the multiple of B is *not* greater than it. [v. Def. 7]

Let them be taken, and let D , E be equimultiples of A , B , and F a multiple of C , such that

D is greater than F ,

but

E is not greater than F .

Therefore D is greater than E .

And, because D and E are equimultiples of A and B , and D is greater than E ,

therefore A is greater than B . [Simson's 4th Ax.]

Next, let C have a greater ratio to B than it has to A ;

B is less than A .

For there is some multiple F of C and some equimultiples E and D of B and A such that

F is greater than E but not greater than D .

[v. Def. 7]

Therefore E is less than D ;

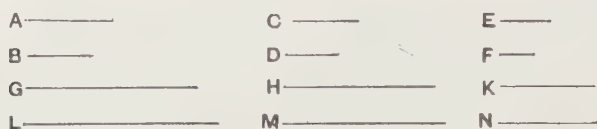
and, because E and D are equimultiples of B and A ,

therefore B is less than A .”

PROPOSITION 11.

Ratios which are the same with the same ratio are also the same with one another.

For, as A is to B , so let C be to D ,
and, as C is to D , so let E be to F ;
I say that, as A is to B , so is E to F .



For of A, C, E let equimultiples G, H, K be taken, and of B, D, F other, chance, equimultiples L, M, N .

Then since, as A is to B , so is C to D ,
and of A, C equimultiples G, H have been taken,
and of B, D other, chance, equimultiples L, M ,
therefore, if G is in excess of L , H is also in excess of M ,
if equal, equal,
and if less, less.

Again, since, as C is to D , so is E to F ,
and of C, E equimultiples H, K have been taken,
and of D, F other, chance, equimultiples M, N ,
therefore, if H is in excess of M , K is also in excess of N ,
if equal, equal,
and if less, less.

But we saw that, if H was in excess of M , G was also in excess of L ; if equal, equal; and if less, less;
so that, in addition, if G is in excess of L , K is also in excess of N ,
if equal, equal,
and if less, less.

And G, K are equimultiples of A, E ,
while L, N are other, chance, equimultiples of B, F ;
therefore, as A is to B , so is E to F .
Therefore etc.

Algebraically, if $a : b = c : d$,
 and $c : d = e : f$,
 then $a : b = e : f$.

The idiomatic use of the imperfect in quoting a result previously obtained is noteworthy. Instead of saying "But *it was proved that*, if H is in excess of M , G is also in excess of L ," the Greek text has "But if H *was* in excess of M , G was also in excess of L ," ἀλλὰ εἰ ὑπερεῖχε τὸ Θ τοῦ Μ, ὑπερεῖχε καὶ τὸ Η τοῦ Λ.

This proposition is tacitly used in combination with v. 16 and v. 24 in the geometrical passage in Aristotle, *Meteorologica* III. 5, 376 a 22—26.

PROPOSITION 12.

If any number of magnitudes be proportional, as one of the antecedents is to one of the consequents, so will all the antecedents be to all the consequents.

Let any number of magnitudes A, B, C, D, E, F be proportional, so that, as A is to B , so is C to D and E to F ;

I say that, as A is to B , so are A, C, E to B, D, F .

A —————	B —————	C —————
D —————	E —————	F —————
G —————		L —————
H —————		M —————
K —————		N —————

For of A, C, E let equimultiples G, H, K be taken, and of B, D, F other, chance, equimultiples L, M, N .

Then since, as A is to B , so is C to D , and E to F , and of A, C, E equimultiples G, H, K have been taken, and of B, D, F other, chance, equimultiples L, M, N , therefore, if G is in excess of L , H is also in excess of M , and K of N ,

if equal, equal,

and if less, less;

so that, in addition,

if G is in excess of L , then G, H, K are in excess of L, M, N ,

if equal, equal,

and if less, less.

Now G and G, H, K are equimultiples of A and A, C, E , since, if any number of magnitudes whatever are respectively equimultiples of any magnitudes equal in multitude, whatever multiple one of the magnitudes is of one, that multiple also will all be of all. [v. 1]

For the same reason

L and L, M, N are also equimultiples of B and B, D, F ;

therefore, as A is to B , so are A, C, E to B, D, F .

[v. Def. 5]

Therefore etc.

Q. E. D.

Algebraically, if $a : a' = b : b' = c : c'$ etc., each ratio is equal to the ratio $(a + b + c + \dots) : (a' + b' + c' + \dots)$.

This theorem is quoted by Aristotle, *Eth. Nic.* v. 7, 1131 b 14, in the shortened form "the whole is to the whole what each part is to each part (respectively)."

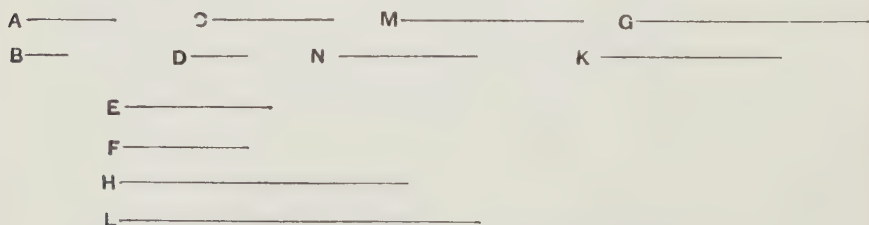
PROPOSITION 13.

If a first magnitude have to a second the same ratio as a third to a fourth, and the third have to the fourth a greater ratio than a fifth has to a sixth, the first will also have to the second a greater ratio than the fifth to the sixth.

For let a first magnitude A have to a second B the same ratio as a third C has to a fourth D ,

and let the third C have to the fourth D a greater ratio than a fifth E has to a sixth F ;

I say that the first A will also have to the second B a greater ratio than the fifth E to the sixth F .



For, since there are some equimultiples of C, E , and of D, F other, chance, equimultiples, such that the multiple of C is in excess of the multiple of D ,

while the multiple of E is not in excess of the multiple of F ,
 [v. Def. 7]
 let them be taken,
 and let G, H be equimultiples of C, E ,
 and K, L other, chance, equimultiples of D, F ,
 so that G is in excess of K , but H is not in excess of L ;
 and, whatever multiple G is of C , let M be also that multiple of A ,
 and, whatever multiple K is of D , let N be also that multiple of B .

Now, since, as A is to B , so is C to D ,
 and of A, C equimultiples M, G have been taken,
 and of B, D other, chance, equimultiples N, K ,
 therefore, if M is in excess of N , G is also in excess of K ,
 if equal, equal,
 and if less, less. [v. Def. 5]

But G is in excess of K ;
 therefore M is also in excess of N .

But H is not in excess of L ;
 and M, H are equimultiples of A, E ,
 and N, L other, chance, equimultiples of B, F ;
 therefore A has to B a greater ratio than E has to F .
 [v. Def. 7]

Therefore etc.

Q. E. D.

Algebraically, if	$a : b = c : d,$
and	$c : d > e : f,$
then	$a : b > e : f.$

After the words "for, since" in the first line of the proof, Theon added " C has to D a greater ratio than E has to F ," so that "there are some equimultiples" began, with him, the principal sentence.

The Greek text has, after "of D, F other, chance, equimultiples," "*and* the multiple of C is in excess of the multiple of D" The meaning being "such that," I have substituted this for "and," after Simson.

The following will show the method of Euclid's proof.

Since $c : d > e : f,$

there will be some equimultiples mc, me of c, e , and some equimultiples nd, nf of d, f , such that

$$mc > nd, \text{ while } me \nless nf.$$

But, since

$$a : b = c : d,$$

therefore, according as $ma > = < nb$, $mc > = < nd$.

And $mc > nd$;

therefore $ma > nb$, while (from above) $me \nless nf$.

Therefore

$$a : b > e : f.$$

Simson adds as a corollary the following :

"If the first have a greater ratio to the second than the third has to the fourth, but the third the same ratio to the fourth which the fifth has to the sixth, it may be demonstrated in like manner that the first has a greater ratio to the second than the fifth has to the sixth."

This however scarcely seems to be worth separate statement, since it only amounts to changing the order of the two parts of the hypothesis.

PROPOSITION 14.

If a first magnitude have to a second the same ratio as a third has to a fourth, and the first be greater than the third, the second will also be greater than the fourth; if equal, equal; and if less, less.

For let a first magnitude A have the same ratio to a second B as a third C has to a fourth D ; and let A be greater than C ;

I say that B is also greater than D .

$$\begin{array}{cc} A \text{ —————} & C \text{ —————} \\ B \text{ —————} & D \text{ —————} \end{array}$$

For, since A is greater than C ,
and B is another, chance, magnitude,
therefore A has to B a greater ratio than C has to B . [v. 8]

But, as A is to B , so is C to D ;

therefore C has also to D a greater ratio than C has to B .
[v. 13]

But that to which the same has a greater ratio is less ;

therefore D is less than B ;

so that B is greater than D .
[v. 10]

Similarly we can prove that, if A be equal to C , B will also be equal to D ;

and, if A be less than C , B will also be less than D .

Therefore etc.

Algebraically, if

$$a : b = c : d,$$

then, according as $a > = < c$, $b > = < d$.

Simson adds the specific proof of the second and third parts of this proposition, which Euclid dismisses with "Similarly we can prove...."

"Secondly, if A be equal to C , B is equal to D ; for A is to B as C , that is A , is to D ;

therefore B is equal to D .

[v. 9]

Thirdly, if A be less than C , B shall be less than D .

For C is greater than A ;

and, because C is to D as A is to B ,

D is greater than B , by the first case.

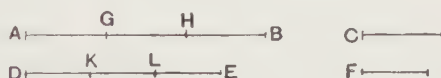
Wherefore B is less than D ."

Aristotle, *Meteorol.* III. 5, 376 a 11—14, quotes the equivalent proposition that, if $a > b$, $c > d$.

PROPOSITION 15.

Parts have the same ratio as the same multiples of them taken in corresponding order.

For let AB be the same multiple of C that DE is of F ;
I say that, as C is to F , so is AB to DE .



For, since AB is the same multiple of C that DE is of F , as many magnitudes as there are in AB equal to C , so many are there also in DE equal to F .

Let AB be divided into the magnitudes AG , GH , HB equal to C ,

and DE into the magnitudes DK , KL , LE equal to F ;

then the multitude of the magnitudes AG , GH , HB will be equal to the multitude of the magnitudes DK , KL , LE .

And, since AG , GH , HB are equal to one another,

and DK , KL , LE are also equal to one another,

therefore, as AG is to DK , so is GH to KL , and HB to LE .

[v. 7]

Therefore, as one of the antecedents is to one of the consequents, so will all the antecedents be to all the consequents;

[v. 12]

therefore, as AG is to DK , so is AB to DE .

But AG is equal to C and DK to F ;

therefore, as C is to F , so is AB to DE .

Therefore etc.

Q. E. D.

Algebraically,

$$a : b = ma : mb.$$

PROPOSITION 16.

If four magnitudes be proportional, they will also be proportional alternately.

Let A, B, C, D be four proportional magnitudes,
so that, as A is to B , so is C to D ;

I say that they will also be so alternately, that is, as A is to C , so is B to D .



For of A, B let equimultiples E, F be taken,
and of C, D other, chance, equimultiples G, H .

Then, since E is the same multiple of A that F is of B ,
and parts have the same ratio as the same multiples of
them, [v. 15]

therefore, as A is to B , so is E to F .

But as A is to B , so is C to D ;

therefore also, as C is to D , so is E to F . [v. 11]

Again, since G, H are equimultiples of C, D ,

therefore, as C is to D , so is G to H . [v. 15]

But, as C is to D , so is E to F ;

therefore also, as E is to F , so is G to H . [v. 11]

But, if four magnitudes be proportional, and the first be
greater than the third,

the second will also be greater than the fourth ;

if equal, equal ;

and if less, less.

[v. 14]

Therefore, if E is in excess of G , F is also in excess of H ,

if equal, equal,

and if less, less.

Now E, F are equimultiples of A, B ,
and G, H other, chance, equimultiples of C, D ;

therefore, as A is to C , so is B to D . [v. Def. 5]

Therefore etc.

Q. E. D.

3. "Let A, B, C, D be four proportional magnitudes, so that, as A is to B , so is C to D ." In a number of expressions like this it is absolutely necessary, when translating into English, to interpolate words which are not in the Greek. Thus the Greek here is: "Ἐστω τέσσαρα μεγέθη ἀνάλογον τὰ A, B, Γ, Δ , ὡς τὸ A πρὸς τὸ B , οὕτως τὸ Γ πρὸς τὸ Δ , literally "Let A, B, C, D be four proportional magnitudes, as A to B , so C to D ." The same remark applies to the corresponding expressions in the next propositions, v. 17, 18, and to other forms of expression in v. 20—23 and later propositions: e.g. in v. 20 we have a phrase meaning literally "Let there be magnitudes...which taken two and two are in the same ratio, as A to B , so D to E ," etc.: in v. 21 "(magnitudes)...which taken two and two are in the same ratio, and let the proportion of them be perturbed, as A to B , so E to F ," etc. In all such cases (where the Greek is so terse as to be almost ungrammatical) I shall insert the words necessary in English, without further remark.

Algebraically, if

$$a : b = c : d,$$

then

$$a : c = b : d.$$

Taking equimultiples ma, mb of a, b , and equimultiples nc, nd of c, d , we have, by v. 15,

$$a : b = ma : mb,$$

$$c : d = nc : nd.$$

And, since

$$a : b = c : d,$$

we have [v. 11]

$$ma : mb = nc : nd.$$

Therefore [v. 14], according as $ma > = < nc$, $mb > = < nd$,
so that

$$a : c = b : d.$$

Aristotle tacitly uses the theorem in *Meteorologica* III. 5, 376 a 22—24.

The four magnitudes in this proposition must all be of the same kind, and Simson inserts "of the same kind" in the enunciation.

This is the first of the propositions of Eucl. v. which Smith and Bryant (*Euclid's Elements of Geometry*, 1901, pp. 298 sqq.) prove by means of vi. 1 so far as the only geometrical magnitudes in question are *straight lines* or *rectilinear areas*; and certainly the proofs are more easy to follow than Euclid's. The proof of this proposition is as follows.

To prove that, *If four magnitudes of the same kind [straight lines or rectilinear areas] be proportionals, they will be proportionals when taken alternately.*

Let P, Q, R, S be the four magnitudes of the same kind such that

$$P : Q = R : S;$$

then it is required to prove that

$$P : R = Q : S.$$

First, let all the magnitudes be areas.

Construct a rectangle $abcd$ equal to the area P , and to bc apply the rectangle $bcef$ equal to Q .

Also to ab, bf apply rectangles ag, bk equal to R, S respectively.

Then, since the rectangles ac , be have the same height, they are to one another as their bases. [vi. 1]

Hence $P : Q = ab : bf$.

But $P : Q = R : S$.

Therefore $R : S = ab : bf$, [v. 11]

i.e. rect. ag : rect. bk = $ab : bf$.

Hence (by the converse of vi. 1) the rectangles ag , bk have the same height, so that k is on the line hg .

Hence the rectangles ac , ag have the same height, namely ab ; also be , bk have the same height, namely bf .

Therefore rect. ac : rect. ag = $bc : bg$,

and rect. be : rect. bk = $bc : bg$. [vi. 1]

Therefore rect. ac : rect. ag = rect. be : rect. bk . [v. 11]

That is, $P : R = Q : S$.

Secondly, let the magnitudes be straight lines AB , BC , CD , DE . Construct the rectangles Ab , Bc , Cd , De with the same height.



Then $Ab : Bc = AB : BC$,

and $Cd : De = CD : DE$. [vi. 1]

But $AB : BC = CD : DE$.

Therefore $Ab : Bc = Cd : De$. [v. 11]

Hence, by the first case,

$Ab : Cd = Bc : De$,

and, since these rectangles have the same height,

$AB : CD = BC : DE$.

PROPOSITION 17.

If magnitudes be proportional componendo, they will also be proportional separando.

Let AB , BE , CD , DF be magnitudes proportional *componendo*, so that, as AB is to BE , so is CD to DF ;

I say that they will also be proportional *separando*, that is, as AE is to EB , so is CF to DF .

For of AE , EB , CF , FD let equimultiples GH , HK , LM , MN be taken,

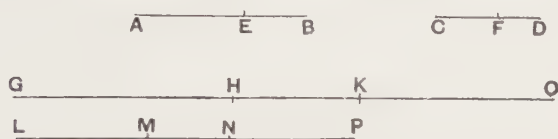
and of EB , FD other, chance, equimultiples, KO , NP ,

Then, since GH is the same multiple of AE that HK is of EB ,

therefore GH is the same multiple of AE that GK is of AB .

[v. 1]

But GH is the same multiple of AE that LM is of CF ;
therefore GK is the same multiple of AB that LM is of CF .



Again, since LM is the same multiple of CF that MN is of FD ,

therefore LM is the same multiple of CF that LN is of CD .

[v. 1]

But LM was the same multiple of CF that GK is of AB ;
therefore GK is the same multiple of AB that LN is of CD .

Therefore GK , LN are equimultiples of AB , CD .

Again, since HK is the same multiple of EB that MN is of FD ,

and KO is also the same multiple of EB that NP is of FD ,
therefore the sum HO is also the same multiple of EB that MP is of FD .

[v. 2]

And, since, as AB is to BE , so is CD to DF ,
and of AB , CD equimultiples GK , LN have been taken,
and of EB , FD equimultiples HO , MP ,
therefore, if GK is in excess of HO , LN is also in excess of MP ,

if equal, equal,

and if less, less.

Let GK be in excess of HO ;

then, if HK be subtracted from each,

GH is also in excess of KO .

But we saw that, if GK was in excess of HO , LN was also in excess of MP ;

therefore LN is also in excess of MP ,

and, if MN be subtracted from each,

LM is also in excess of NP ;

so that, if GH is in excess of KO , LM is also in excess of NP .

Similarly we can prove that,

if GH be equal to KO , LM will also be equal to NP ,

and if less, less.

And GH , LM are equimultiples of AE , CF ,

while KO , NP are other, chance, equimultiples of EB , FD ;

therefore, as AE is to EB , so is CF to FD .

Therefore etc.

Q. E. D.

Algebraically, if $a : b = c : d$,

then $(a - b) : b = (c - d) : d$.

I have already noted the somewhat strange use of the participles of *συγκείσθαι* and *διαίρεισθαι* to convey the sense of the technical *σύνθεσις* and *διαίρεσις λόγου*, or what we denote by *componendo* and *separando*. *ἐὰν συγκείμενα μεγέθη ἀνάλογον ᾗ, καὶ διαιρεθέντα ἀνάλογον ἔσται* is, literally, "if magnitudes compounded be proportional, they will also be proportional separated," by which is meant "if one magnitude made up of two parts is to one of its parts as another magnitude made up of two parts is to one of its parts, the remainder of the first whole is to the part of it first taken as the remainder of the second whole is to the part of it first taken." In the algebraical formula above a , c are the wholes and b , $a - b$ and d , $c - d$ are the parts and remainders respectively. The formula might also be stated thus:

If $a + b : b = c + d : d$,

then $a : b = c : d$,

in which case $a + b$, $c + d$ are the wholes and b , a and d , c the parts and remainders respectively. Looking at the last formula, we observe that "separated," *διαιρεθέντα*, is used with reference not to the magnitudes a , b , c , d but to the *compounded* magnitudes $a + b$, b , $c + d$, d .

As the proof is somewhat long, it will be useful to give a conspectus of it in the more symbolical form. To avoid minuses, we will take for the hypothesis the form

$a + b$ is to b as $c + d$ is to d .

Take any equimultiples of the four magnitudes a , b , c , d , viz.

ma , mb , mc , md ,

and any other equimultiples of the consequents, viz.

nb and nd .

Then, by v. 1, $m(a + b)$, $m(c + d)$ are equimultiples of $a + b$, $c + d$,

and, by v. 2, $(m + n)b$, $(m + n)d$ are equimultiples of b , d .

Therefore, by Def. 5, since $a + b$ is to b as $c + d$ is to d ,

according as $m(a + b) > = < (m + n)b$, $m(c + d) > = < (m + n)d$.

Subtract from $m(a+b)$, $(m+n)b$ the common part mb , and from $m(c+d)$, $(m+n)d$ the common part md ; and we have,
according as $ma > < nb$, $mc > < nd$.

But ma , mc are any equimultiples of a , c , and nb , nd any equimultiples of b , d ,

therefore, by v. Def. 5,

a is to b as c is to d .

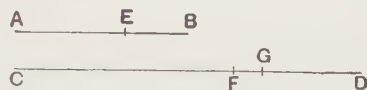
Smith and Bryant's proof follows, *mutatis mutandis*, their alternative proof of the next proposition (see pp. 173—4 below).

PROPOSITION 18.

If magnitudes be proportional separando, they will also be proportional componendo.

Let AE , EB , CF , FD be magnitudes proportional *separando*, so that, as AE is to EB , so is CF to FD ;

I say that they will also be proportional *componendo*, that is, as AB is to BE , so is CD to FD .



For, if CD be not to DF as AB to BE , then, as AB is to BE , so will CD be either to some magnitude less than DF or to a greater.

First, let it be in that ratio to a less magnitude DG .

Then, since, as AB is to BE , so is CD to DG , they are magnitudes proportional *componendo*;

so that they will also be proportional *separando*. [v. 17]

Therefore, as AE is to EB , so is CG to GD .

But also, by hypothesis,

as AE is to EB , so is CF to FD .

Therefore also, as CG is to GD , so is CF to FD . [v. 11]

But the first CG is greater than the third CF ;

therefore the second GD is also greater than the fourth FD . [v. 14]

But it is also less: which is impossible.

Therefore, as AB is to BE , so is not CD to a less magnitude than FD .

Similarly we can prove that neither is it in that ratio to a greater ;

it is therefore in that ratio to FD itself.

Therefore etc.

Q. E. D.

Algebraically, if $a : b = c : d$,
then $(a + b) : b = (c + d) : d$.

In the enunciation of this proposition there is the same special use of *διηρημένα* and *συντεθέντα* as there was of *συγκείμενα* and *διαιρεθέντα* in the last enunciation. Practically, as the algebraical form shows, *διηρημένα* might have been left out.

The following is the method of proof employed by Euclid.

Given that $a : b = c : d$,
suppose, if possible, that $(a + b) : b = (c + d) : (d \pm x)$.

Therefore, *separando* [v. 17],
 $a : b = (c \mp x) : (d \pm x)$,
whence, by v. 11, $(c \mp x) : (d \pm x) = c : d$.

But $(c - x) < c$, while $(d + x) > d$,
and $(c + x) > c$, while $(d - x) < d$,

which relations respectively contradict v. 14.

Simson pointed out (as Saccheri before him saw) that Euclid's demonstration is not legitimate, because it assumes without proof that *to any three magnitudes, two of which, at least, are of the same kind, there exists a fourth proportional*. Clavius and, according to him, other editors made this an axiom. But it is far from axiomatic ; it is not till vi. 12 that Euclid shows, by construction, that it is true even in the particular case where the three given magnitudes are all straight lines.

In order to remove the defect it is necessary either (1) to prove beforehand the proposition thus assumed by Euclid or (2) to prove v. 18 independently of it.

Saccheri ingeniously proposed that the assumed proposition should be proved, for areas and straight lines, by means of Euclid vi. 1, 2 and 12. As he says, there was nothing to prevent Euclid from interposing these propositions immediately after v. 17 and then proving v. 18 by means of them. vi. 12 enables us to *construct* the fourth proportional when the three given magnitudes are straight lines ; and vi. 12 depends only on vi. 1 and 2. "Now," says Saccheri, "when we have once found the means of constructing a straight line which is a fourth proportional to three given straight lines, we obviously have the solution of the general problem 'To construct a straight line which shall have to a given straight line the same ratio which two *polygons* have (to one another).'" For it is sufficient to transform the polygons into two triangles of equal height and then to construct a straight line which shall be a fourth proportional to the bases of the triangles and the given straight line.

The method of Saccheri is, as will be seen, similar to that adopted by

Smith and Bryant (*loc. cit.*) in proving the theorems of Euclid v. 16, 17, 18, 22, so far as straight lines and rectilinear areas are concerned, by means of vi. 1.

De Morgan gives a sketch of a general proof of the assumed proposition that, B being any magnitude, and P and Q two magnitudes of the same kind, there does exist a magnitude A which is to B in the same ratio as P to Q .

"The right to reason upon any aliquot part of any magnitude is assumed; though, in truth, aliquot parts obtained by continual bisection would suffice: and it is taken as previously proved that the tests of greater and of less ratio are never both presented in any one scale of relation as compared with another" (see note on v. Def. 7 *ad fin.*).

"(1) If M be to B in a greater ratio than P to Q , so is every magnitude greater than M , and so are *some less magnitudes*; and if M be to B in a less ratio than P to Q , so is every magnitude less than M , and so are *some greater magnitudes*. Part of this is in every system: the rest is proved thus. If M be to B in a greater ratio than P to Q , say, for instance, we find that $15M$ lies between $22B$ and $23B$, while $15P$ lies before $22Q$. Let $15M$ exceed $22B$ by Z ; then, if N be less than M by anything less than the 15th part of Z , $15N$ is between $22B$ and $23B$: or N , less than M , is in a greater ratio to B than P to Q . And similarly for the other case.

(2) M can certainly be taken so small as to be in a less ratio to B than P to Q , and so large as to be in a greater; and since we can never pass from the greater ratio back again to the smaller by increasing M , it follows that, while we pass from the first designated value to the second, we come upon an intermediate magnitude A such that every smaller is in a less ratio to B than P to Q , and every greater in a greater ratio. Now A cannot be in a less ratio to B than P to Q , for then some greater magnitudes would also be in a less ratio; nor in a greater ratio, for then some less magnitudes would be in a greater ratio; therefore A is in the same ratio to B as P to Q . The previously proved proposition above mentioned shows the three alternatives to be the only ones."

Alternative proofs of V. 18.

Simson bases his alternative on v. 5, 6. As the 18th proposition is the converse of the 17th, and the latter is proved by means of v. 1 and 2, of which v. 5 and 6 are converses, the proof of v. 18 by v. 5 and 6 would be natural; and Simson holds that Euclid must have proved v. 18 in this way because "the 5th and 6th do not enter into the demonstration of any proposition in this book as we have it, nor can they be of any use in any proposition of the Elements," and "the 5th and 6th have undoubtedly been put into the 5th book for the sake of some propositions in it, as all the other propositions about equimultiples have been."

Simson's proof is however, as it seems to me, intolerably long and difficult to follow unless it be put in the symbolical form as follows.

Suppose that a is to b as c is to d ;

it is required to prove that $a + b$ is to b as $c + d$ is to d .

Take any equimultiples of the last four magnitudes, say

$$m(a + b), \quad mb, \quad m(c + d), \quad md,$$

and any equimultiples of b, d , as

$$nb, \quad nd.$$

Clearly, if nb is greater than mb ,
 nd is greater than md ;

if equal, equal; and if less, less.

I. Suppose nb not greater than mb , so that nd is also not greater than md .

Now $m(a+b)$ is greater than mb :

therefore $m(a+b)$ is greater than nb .

Similarly $m(c+d)$ is greater than nd .

II. Suppose nb greater than mb .

Since $m(a+b)$, mb , $m(c+d)$, md are equimultiples of $(a+b)$, b , $(c+d)$, d ,

ma is the same multiple of a that $m(a+b)$ is of $(a+b)$,

and mc is the same multiple of c that $m(c+d)$ is of $(c+d)$,

so that ma , mc are equimultiples of a , c .

[v. 5]

Again nb , nd are equimultiples of b , d ,

and so are mb , md ;

therefore $(n-m)b$, $(n-m)d$ are equimultiples of b , d and, whether $n-m$ is equal to unity or to any other integer [v. 6], it follows, by Def. 5, that, since a , b , c , d are proportionals,

if ma is greater than $(n-m)b$,

then mc is greater than $(n-m)d$;

if equal, equal; and if less, less.

(1) If now $m(a+b)$ is greater than nb , subtracting mb from each, we have

ma is greater than $(n-m)b$;

therefore mc is greater than $(n-m)d$,

and, if we add md to each,

$m(c+d)$ is greater than nd .

(2) Similarly it may be proved that,

if $m(a+b)$ is equal to nb ,

then $m(c+d)$ is equal to nd ,

and (3) that, if $m(a+b)$ is less than nb ,

then $m(c+d)$ is less than nd .

But (under I. above) it was proved that, in the case where nb is not greater than mb ,

$m(a+b)$ is always greater than nb ,

and $m(c+d)$ is always greater than nd .

Hence, whatever be the values of m and n , $m(c+d)$ is always greater than, equal to, or less than nd according as $m(a+b)$ is greater than, equal to, or less than nb .

Therefore, by Def. 5,

$a+b$ is to b as $c+d$ is to d .

Todhunter gives the following short demonstration from Austin (*Examination of the first six books of Euclid's Elements*).

"Let AE be to EB as CF is to FD :

AB shall be to BE as CD is to DF .

For, because AE is to EB as CF is to FD ,
therefore, alternately,

$$AE \text{ is to } CF \text{ as } EB \text{ is to } FD.$$

[v. 16]

And, as one of the antecedents is to its consequent, so is the sum of the antecedents to the sum of the consequents: [v. 12]

therefore, as EB is to FD , so are AE, EB together to CF, FD together;

that is, AB is to CD as EB is to FD .

Therefore, alternately,

$$AB \text{ is to } BE \text{ as } CD \text{ is to } FD."$$

The objection to this proof is that it is only valid in the case where the proposition v. 16 used in it is valid, i.e. where all four magnitudes are of the same kind.

Smith and Bryant's proof avails where all four magnitudes are straight lines, where all four magnitudes are rectilinear areas, or where one antecedent and its consequent are straight lines and the other antecedent and its consequent rectilinear areas.

Suppose that $A : B = C : D$.

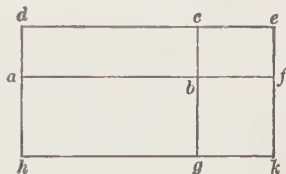
First, let all the magnitudes be areas.

Construct a rectangle $abcd$ equal to A , and to bc apply the rectangle $bcef$ equal to B .

Also to ab, bf apply the rectangles ag, bk equal to C, D respectively.

Then, since the rectangles ac, be have equal heights bc , they are to one another as their bases. [vi. 1]

$$\begin{aligned} \text{Hence } ab : bf &= \text{rect. } ac : \text{rect. } be \\ &= A : B \\ &= C : D \\ &= \text{rect. } ag : \text{rect. } bk. \end{aligned}$$



Therefore [vi. 1, converse] the rectangles ag, bk have the same height, so that k is on the straight line hg .

$$\begin{aligned} \text{Hence } A + B : B &= \text{rect. } ae : \text{rect. } be \\ &= af : bf \\ &= \text{rect. } ak : \text{rect. } bk \\ &= C + D : D. \end{aligned}$$

Secondly, let the magnitudes A, B be straight lines and the magnitudes C, D areas.

Let ab, bf be equal to the straight lines A, B , and to ab, bf apply the rectangles ag, bk equal to C, D respectively.

Then, as before, the rectangles ag, bk have the same height.

$$\begin{aligned} \text{Now } A + B : B &= af : bf \\ &= \text{rect. } ak : \text{rect. } bk \\ &= C + D : D. \end{aligned}$$

Thirdly, let all the magnitudes be straight lines.

Apply to the straight lines C, D rectangles P, Q having the same height.

Then $P : Q = C : D$. [vi. 1]

Hence, by the second case,

$$A + B : B = P + Q : Q.$$

Also $P + Q : Q = C + D : D$.

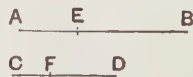
Therefore $A + B : B = C + D : D$.

PROPOSITION 19.

If, as a whole is to a whole, so is a part subtracted to a part subtracted, the remainder will also be to the remainder as whole to whole.

For, as the whole AB is to the whole CD , so let the part AE subtracted be to the part CF subtracted;

I say that the remainder EB will also be to the remainder FD as the whole AB to the whole CD .



For since, as AB is to CD , so is AE to CF ,

alternately also, as BA is to AE , so is DC to CF . [v. 16]

And, since the magnitudes are proportional *componendo*, they will also be proportional *separando*, [v. 17]

that is, as BE is to EA , so is DF to CF ,

and, alternately,

as BE is to DF , so is EA to FC . [v. 16]

But, as AE is to CF , so by hypothesis is the whole AB to the whole CD .

Therefore also the remainder EB will be to the remainder FD as the whole AB is to the whole CD . [v. 11]

Therefore etc.

[PORISM. From this it is manifest that, if magnitudes be proportional *componendo*, they will also be proportional *convertendo*.]

Q. E. D.

Algebraically, if $a : b = c : d$ (where $c < a$ and $d < b$), then

$$(a - c) : (b - d) = a : b.$$

The "Porism" at the end of this proposition is led up to by a few lines which Heiberg brackets because it is not Euclid's habit to explain a Porism, and indeed a Porism, from its very nature, should not need any

explanation, being a sort of by-product appearing without effort or trouble, *ἀπαραμεινώς* (Proclus, p. 303, 6). But Heiberg thinks that Simson does wrong in finding fault with the argument leading to the "Porism," and that it does contain the true demonstration of *conversion* of a ratio. In this it appears to me that Heiberg is clearly mistaken, the supposed proof on the basis of Prop. 19 being no more correct than the similar attempt to prove the *inversion* of a ratio from Prop. 4. The words are: "And since it was proved that, as AB is to CD , so is EB to FD ,

alternately also, as AB is to BE , so is CD to FD ;

therefore magnitudes when compounded are proportional.

But it was proved that, as BA is to AE , so is DC to CF , and this is *convertendo*."

It will be seen that this amounts to proving *from the hypothesis* $a : b = c : d$ that the following transformations are simultaneously true, viz.:

$$a : a - c = b : b - d,$$

and

$$a : c = b : d.$$

The former is not proved from the latter as it ought to be if it were intended to prove *conversion*.

The inevitable conclusion is that both the "Porism" and the argument leading up to it are interpolations, though no doubt made, as Heiberg says, before Theon's time.

The *conversion* of ratios does not depend upon v. 19 at all but, as Simson shows in his Proposition E (containing a proof already given by Clavius), on Props. 17 and 18. Prop. E is as follows.

If four magnitudes be proportionals, they are also proportionals by conversion, that is, the first is to its excess above the second as the third is to its excess above the fourth.

Let AB be to BE as CD to DF :

then BA is to AE as DC to CF .

Because AB is to BE as CD to DF ,
by division [*separando*],

$$AE \text{ is to } EB \text{ as } CF \text{ to } FD,$$

[v. 17]

and, by inversion,

$$BE \text{ is to } EA \text{ as } DF \text{ to } FC.$$

[Simson's Prop. B directly obtained from v. Def. 5]

Wherefore, by composition [*componendo*],

$$BA \text{ is to } AE \text{ as } DC \text{ to } CF$$

[v. 18]



PROPOSITION 20.

If there be three magnitudes, and others equal to them in multitude, which taken two and two are in the same ratio, and if ex aequali the first be greater than the third, the fourth will also be greater than the sixth; if equal, equal; and, if less, less.

Let there be three magnitudes A , B , C , and others D , E , F equal to them in multitude, which taken two and two are in the same ratio, so that,

as A is to B , so is D to E ,

and

as B is to C , so is E to F ;

and let A be greater than C *ex aequali*;

I say that D will also be greater than F ; if A is equal to C , equal; and, if less, less.



For, since A is greater than C ,

and B is some other magnitude,

and the greater has to the same a greater ratio than the less has, [v. 8]

therefore A has to B a greater ratio than C has to B .

But, as A is to B , so is D to E ,

and, as C is to B , inversely, so is F to E ;

therefore D has also to E a greater ratio than F has to E . [v. 13]

But, of magnitudes which have a ratio to the same, that which has a greater ratio is greater; [v. 10]

therefore D is greater than F .

Similarly we can prove that, if A be equal to C , D will also be equal to F ; and if less, less.

Therefore etc.

Q. E. D.

Though, as already remarked, Euclid has not yet given us any definition of *compounded ratios*, Props. 20—23 contain an important part of the theory of such ratios. The term “compounded ratio” is not used, but the propositions connect themselves with the definitions of *ex aequali* in its two forms, the ordinary form defined in Def. 17 and that called *perturbed proportion* in Def. 18. The compounded ratios dealt with in these propositions are those compounded of successive ratios in which the consequent of one is the antecedent of the next, or the antecedent of one is the consequent of the next.

Prop. 22 states the fundamental proposition about the ratio *ex aequali* in its ordinary form, to the effect that,

if	a is to b as d is to e ,
and	b is to c as e is to f ,
then	a is to c as d is to f ,

with the extension to any number of such ratios; Prop. 23 gives the corresponding theorem for the case of *perturbed proportion*, namely that,

if a is to b as e is to f ,
 and b is to c as d is to e ,
 then a is to c as d is to f .

Each depends on a preliminary proposition, Prop. 22 on Prop. 20 and Prop. 23 on Prop. 21. The course of the proof will be made most clear by using the algebraic notation.

The preliminary Prop. 20 asserts that,

if $a : b = d : e$,
 and $b : c = e : f$,

then, according as $a > = < c$, $d > = < f$.

For, according as a is greater than, equal to, or less than c ,
 the ratio $a : b$ is greater than, equal to, or less than the ratio $c : b$, [v. 8 or v. 7]

or (since $d : e = a : b$,
 and $c : b = f : e$)

the ratio $d : e$ is greater than, equal to, or less than the ratio $f : e$,
 [by aid of v. 13 and v. 11]

and therefore d is greater than, equal to, or less than f . [v. 10 or v. 9]

It is next proved in Prop. 22 that, by v. 4, the given proportions can be transformed into

$$ma : nb = md : ne,$$

and $nb : pc = ne : pf$,

whence, by v. 20,

according as ma is greater than, equal to, or less than pc ,
 md is greater than, equal to, or less than pf ,

so that, by Def. 5,

$$a : c = d : f.$$

Prop. 23 depends on Prop. 21 in the same way as Prop. 22 on Prop. 20, but the transformation of the ratios in Prop. 23 is to the following :

$$(1) \quad ma : mb = ne : nf$$

(by a double application of v. 15 and by v. 11),

$$(2) \quad mb : nc = md : ne$$

(by v. 4, or equivalent steps),

and Prop. 21 is then used.

Simson makes the proof of Prop. 20 slightly more explicit, but the main difference from the text is in the addition of the two other cases which Euclid dismisses with "Similarly we can prove." These cases are :

"Secondly, let A be equal to C ; then shall D be equal to F .

Because A and C are equal to one another,

A is to B as C is to B . [v. 7]

But A is to B as D is to E ,

and C is to B as F is to E ,

wherefore D is to E as F to E ; [v. 11]

and therefore D is equal to F . [v. 9]

Next, let A be less than C ; then shall D be less than F .

For C is greater than A ,

and, as was shown in the first case,

C is to B as F to E ,

and, in like manner,

B is to A as E to D ;

therefore F is greater than D , by the first case; and therefore D is less than F ."

PROPOSITION 21.

If there be three magnitudes, and others equal to them in multitude, which taken two and two together are in the same ratio, and the proportion of them be perturbed, then, if ex aequali the first magnitude is greater than the third, the fourth will also be greater than the sixth; if equal, equal; and if less, less.

Let there be three magnitudes A, B, C , and others D, E, F equal to them in multitude, which taken two and two are in the same ratio, and let the proportion of them be perturbed, so that,

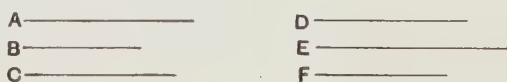
as A is to B , so is E to F ,

and,

as B is to C , so is D to E ,

and let A be greater than C *ex aequali*;

I say that D will also be greater than F ; if A is equal to C , equal; and if less, less.



For, since A is greater than C ,

and B is some other magnitude,

therefore A has to B a greater ratio than C has to B . [v. 8]

But, as A is to B , so is E to F ,

and, as C is to B , inversely, so is E to D .

Therefore also E has to F a greater ratio than E has to D .

[v. 13]

But that to which the same has a greater ratio is less;

[v. 10]

therefore F is less than D ;

therefore D is greater than F .

Similarly we can prove that,
if A be equal to C , D will also be equal to F ;
and if less, less.

Therefore etc.

Q. E. D.

Algebraically, if
and

$$\begin{aligned} a : b &= e : f, \\ b : c &= d : e, \end{aligned}$$

then, according as $a > = < c$, $d > = < f$.

Simson's alterations correspond to those which he makes in Prop. 20. After the first case he proceeds thus.

"Secondly, let A be equal to C ; then shall D be equal to F .

Because A and C are equal,

A is to B as C is to B . [v. 7]

But A is to B as E is to F ,

and C is to B as E is to D :

wherefore E is to F as E to D , [v. 11]

and therefore D is equal to F . [v. 9]

Next, let A be less than C ; then shall D be less than F .

For C is greater than A ,

and, as was shown;

C is to B as E to D ,

and, in like manner,

B is to A as F to E ;

therefore F is greater than D , by the first case,

and therefore D is less than F ."

The proof may be shown thus.

According as $a > = < c$, $a : b > = < c : b$.

But $a : b = e : f$, and, by inversion, $c : b = e : d$.

Therefore, according as $a > = < c$, $e : f > = < e : d$,

and therefore $d > = < f$.

PROPOSITION 22.

If there be any number of magnitudes whatever, and others equal to them in multitude, which taken two and two together are in the same ratio, they will also be in the same ratio ex aequali.

Let there be any number of magnitudes A , B , C , and others D , E , F equal to them in multitude, which taken two and two together are in the same ratio, so that,

as A is to B , so is D to E ,

and, as B is to C , so is E to F ;

I say that they will also be in the same ratio *ex aequali*,

< that is, as A is to C , so is D to F >.

For of A, D let equimultiples G, H be taken,
and of B, E other, chance, equimultiples K, L ;
and, further, of C, F other, chance, equimultiples M, N .



Then, since, as A is to B , so is D to E ,
and of A, D equimultiples G, H have been taken,
and of B, E other, chance, equimultiples K, L ,
therefore, as G is to K , so is H to L . [v. 4]

For the same reason also,

as K is to M , so is L to N .

Since, then, there are three magnitudes G, K, M , and
others H, L, N equal to them in multitude, which taken two
and two together are in the same ratio,

therefore, *ex aequali*, if G is in excess of M , H is also in excess
of N ;

if equal, equal ; and if less, less.

[v. 20]

And G, H are equimultiples of A, D ,

and M, N other, chance, equimultiples of C, F .

Therefore, as A is to C , so is D to F .

[v. Def. 5]

Therefore etc.

Q. E. D.

Euclid enunciates this proposition as true of *any number of magnitudes whatever* forming two sets connected in the manner described, but his proof is confined to the case where each set consists of three magnitudes only. The extension to any number of magnitudes is, however, easy, as shown by Simson.

"Next let there be four magnitudes A, B, C, D , and other four E, F, G, H , which two and two have the same ratio, viz. :

as A is to B , so is E to F ,
and as B is to C , so is F to G ,
and as C is to D , so is G to H ;
 A shall be to D as E to H .

A	B	C	D
E	F	G	H

Because A, B, C are three magnitudes, and E, F, G other three, which taken two and two have the same ratio,

by the foregoing case,

A is to C as E to G .

But C is to D as G is to H ;
wherefore again, by the first case,

A is to D as E to H .

And so on, whatever be the number of magnitudes."

PROPOSITION 23.

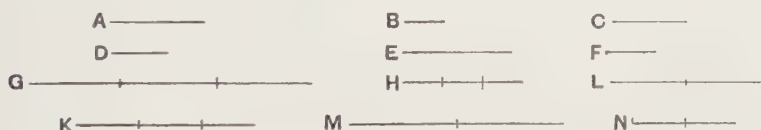
If there be three magnitudes, and others equal to them in multitude, which taken two and two together are in the same ratio, and the proportion of them be perturbed, they will also be in the same ratio ex aequali.

Let there be three magnitudes A, B, C , and others equal to them in multitude, which, taken two and two together, are in the same proportion, namely D, E, F ; and let the proportion of them be perturbed, so that,

as A is to B , so is E to F ,

and, as B is to C , so is D to E ;

I say that, as A is to C , so is D to F .



Of A, B, D let equimultiples G, H, K be taken,
and of C, E, F other, chance, equimultiples L, M, N .

Then, since G, H are equimultiples of A, B ,
and parts have the same ratio as the same multiples of
them, [v. 15]

therefore, as A is to B , so is G to H .

For the same reason also,

as E is to F , so is M to N .

And, as A is to B , so is E to F ;

therefore also, as G is to H , so is M to N . [v. 11]

Next, since, as B is to C , so is D to E ,

alternately, also, as B is to D , so is C to E . [v. 16]

And, since H, K are equimultiples of B, D ,

and parts have the same ratio as their equimultiples,

therefore, as B is to D , so is H to K . [v. 15]

But, as B is to D , so is C to E ;

therefore also, as H is to K , so is C to E . [v. 11]

Again, since L , M are equimultiples of C , E ,

therefore, as C is to E , so is L to M . [v. 15]

But, as C is to E , so is H to K ;

therefore also, as H is to K , so is L to M , [v. 11]

and, alternately, as H is to L , so is K to M . [v. 16]

But it was also proved that,

as G is to H , so is M to N .

Since, then, there are three magnitudes G , H , L , and others equal to them in multitude K , M , N , which taken two and two together are in the same ratio,

and the proportion of them is perturbed,

therefore, *ex aequali*, if G is in excess of L , K is also in excess of N ;

if equal, equal; and if less, less. [v. 21]

And G , K are equimultiples of A , D ,

and L , N of C , F .

Therefore, as A is to C , so is D to F .

Therefore etc.

Q. E. D.

There is an important difference between the version given by Simson of one part of the proof of this proposition and that found in the Greek text of Heiberg. Peyrard's ms. has the version given by Heiberg, but Simson's version has the authority of other mss. The Basel *editio princeps* gives both versions (Simson's being the first). After it has been proved by means of v. 15 and v. 11 that,

as G is to H , so is M to N ,

or, with the notation used in the note on Prop. 20,

$$ma : mb = ne : nf,$$

it has to be proved further that,

as H is to L , so is K to M ,

or

$$mb : nc = md : ne,$$

and it is clear that the latter result may be directly inferred from v. 4. The reading translated by Simson makes this inference:

"And because, as B is to C , so is D to E ,

and H , K are equimultiples of B , D ,

and L , M of C , E ,

therefore, as H is to L , so is K to M ." [v. 4]

The version in Heiberg's text is not only much longer (it adopts the roundabout method of using each of three Propositions v. 11, 15, 16 twice

over), but it is open to the objection that it uses v. 16 which is only applicable if the four magnitudes are *of the same kind*; whereas v. 23, the proposition now in question, is not subject to this restriction.

Simson rightly observes that in the last step of the proof it should be stated that "*G, K are any equimultiples whatever of A, D, and L, N any whatever of C, F.*"

He also gives the extension of the proposition to any number of magnitudes, enunciating it thus :

"If there be any number of magnitudes, and as many others, which, taken two and two, in a cross order, have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last";

and adding to the proof as follows :

"Next, let there be four magnitudes *A, B, C, D*, and other four *E, F, G, H*, which, taken two and two in a cross order, have the same ratio, viz. :

A to B as G to H,

B to C as F to G,

and *C to D as E to F;*

then *A* is to *D* as *E* to *H*.

Because *A, B, C* are three magnitudes, and *F, G, H* other three which, taken two and two in a cross order, have the same ratio,

by the first case, *A* is to *C* as *F* to *H*.

But *C* is to *D* as *E* is to *F*;

wherefore again, by the first case,

A is to *D* as *E* to *H*.

And so on, whatever be the number of magnitudes."

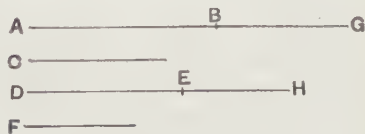
A	B	C	D
E	F	G	H

PROPOSITION 24.

If a first magnitude have to a second the same ratio as a third has to a fourth, and also a fifth have to the second the same ratio as a sixth to the fourth, the first and fifth added together will have to the second the same ratio as the third and sixth have to the fourth.

Let a first magnitude *AB* have to a second *C* the same ratio as a third *DE* has to a fourth *F*;

and let also a fifth *BG* have to the second *C* the same ratio as a sixth *EH* has to the fourth *F*;



I say that the first and fifth added together, *AG*, will have to the second *C* the same ratio as the third and sixth, *DH*, has to the fourth *F*.

For since, as BG is to C , so is EH to F ,
inversely, as C is to BG , so is F to EH .

Since, then, as AB is to C , so is DE to F ,

and, as C is to BG , so is F to EH ,

therefore, *ex aequali*, as AB is to BG , so is DE to EH . [v. 22]

And, since the magnitudes are proportional *separando*, they
will also be proportional *componendo*; [v. 18]

therefore, as AG is to GB , so is DH to HE .

But also, as BG is to C , so is EH to F ;

therefore, *ex aequali*, as AG is to C , so is DH to F . [v. 22]

Therefore etc.

Q. E. D.

Algebraically, if

$$a : c = d : f,$$

and

$$b : c = e : f,$$

then

$$(a + b) : c = (d + e) : f.$$

This proposition is of the same character as those which precede the
propositions relating to compounded ratios; but it could not be placed earlier
than it is because v. 22 is used in the proof of it.

Inverting the second proportion to

$$c : b = f : e,$$

it follows, by v. 22, that

$$a : b = d : e,$$

whence, by v. 18,

$$(a + b) : b = (d + e) : e,$$

and from this and the second of the two given proportions we obtain, by a
fresh application of v. 22,

$$(a + b) : c = (d + e) : f.$$

The first use of v. 22 is important as showing that the opposite process to
compounding ratios, or what we should now call *division* of one ratio by
another, does not require any new and separate propositions.

Aristotle tacitly uses v. 24 in combination with v. 11 and v. 16, *Meteorologica*
III. 5, 376 a 22—26.

Simson adds two corollaries, one of which (Cor. 2) notes the extension to
any number of magnitudes.

“The proposition holds true of two ranks of magnitudes whatever be their
number, of which each of the first rank has to the second magnitude the same
ratio that the corresponding one of the second rank has to a fourth magnitude;
as is manifest.”

Simson's Cor. 1 states the corresponding proposition to the above with
separando taking the place of *componendo*, viz., that corresponding to the
algebraical form

$$(a - b) : c = (d - e) : f.$$

“COR. 1. If the same hypothesis be made as in the proposition, the
excess of the first and fifth shall be to the second as the excess of the third
and sixth to the fourth. The demonstration of this is the same with that of
the proposition if division be used instead of composition.” That is, we use
v. 17 instead of v. 18, and conclude that

$$(a - b) : b = (d - e) : e.$$

PROPOSITION 25.

If four magnitudes be proportional, the greatest and the least are greater than the remaining two.

Let the four magnitudes AB , CD , E , F be proportional so that, as AB is to CD , so is E to F , and let AB be the greatest of them and F the least;

I say that AB , F are greater than CD , E .

For let AG be made equal to E , and CH equal to F .

Since, as AB is to CD , so is E to F ,

and E is equal to AG , and F to CH ,

therefore, as AB is to CD , so is AG to CH .

And since, as the whole AB is to the whole CD , so is the part AG subtracted to the part CH subtracted,

the remainder GB will also be to the remainder HD as the whole AB is to the whole CD . [v. 19]

But AB is greater than CD ;

therefore GB is also greater than HD .

And, since AG is equal to E , and CH to F , therefore AG , F are equal to CH , E .

And if, GB , HD being unequal, and GB greater, AG , F be added to GB and CH , E be added to HD ,

it follows that AB , F are greater than CD , E .

Therefore etc.

Q. E. D.

Algebraically, if

$$a : b = c : d,$$

and a is the greatest of the four magnitudes and d the least,

$$a + d > b + c.$$

Simson is right in inserting a word in the setting-out, "let AB be the greatest of them and *consequently* F the least." This follows from the particular case, really included in Def. 5, which Simson makes the subject of his proposition A, the case namely where the equimultiples taken are *once* the several magnitudes.

The proof is as follows.

Since

$$a : b = c : d,$$

$$a - c : b - d = a : b.$$

[v. 19]

But $a > b$; therefore $(a - c) > (b - d)$.

[v. 16 and 14]

Add to each $(c + d)$;

therefore

$$(a + d) > (b + c).$$

There is an important particular case of this proposition, which is, however, not mentioned here, viz. the case where $b = c$. The result shows, in this case, that *the arithmetic mean between two magnitudes is greater than their geometric mean*. The truth of this is proved for straight lines in vi. 27 by "geometrical algebra," and the theorem forms the *διορισμός* for equations of the second degree.

Simson adds at the end of Book v. four propositions, F, G, H, K, which, however, do not seem to be of sufficient practical use to justify their inclusion here. But he adds at the end of his notes to the Book the following paragraph which deserves quotation word for word.

"The 5th book being thus corrected, I most readily agree to what the learned Dr Barrow says, 'that there is nothing in the whole body of the elements of a more subtle invention, nothing more solidly established, and more accurately handled than the doctrine of proportionals.' And there is some ground to hope that geometers will think that this could not have been said with as good reason, since Theon's time till the present."

Simson's claim herein will readily be admitted by all readers who are competent to form a judgment upon his criticisms and elucidations of Book v.

BOOK I.

DEFINITIONS.

1. A **point** is that which has no part.
2. A **line** is breadthless length.
3. The extremities of a line are points.
4. A **straight line** is a line which lies evenly with the points on itself.
5. A **surface** is that which has length and breadth only.
6. The extremities of a surface are lines.
7. A **plane surface** is a surface which lies evenly with the straight lines on itself.
8. A **plane angle** is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.
9. And when the lines containing the angle are straight, the angle is called **rectilineal**.
10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is **right**, and the straight line standing on the other is called a **perpendicular** to that on which it stands.
11. An **obtuse angle** is an angle greater than a right angle.
12. An **acute angle** is an angle less than a right angle.
13. A **boundary** is that which is an extremity of anything.
14. A **figure** is that which is contained by any boundary or boundaries.
15. A **circle** is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another ;

16. And the point is called the **centre** of the circle.

17. A **diameter** of the circle is any straight line drawn through the centre and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.

18. A **semicircle** is the figure contained by the diameter and the circumference cut off by it. And the centre of the semicircle is the same as that of the circle.

19. **Rectilineal figures** are those which are contained by straight lines, **trilateral** figures being those contained by three, **quadrilateral** those contained by four, and **multilateral** those contained by more than four straight lines.

20. Of trilateral figures, an **equilateral triangle** is that which has its three sides equal, an **isosceles triangle** that which has two of its sides alone equal, and a **scalene triangle** that which has its three sides unequal.

21. Further, of trilateral figures, a **right-angled triangle** is that which has a right angle, an **obtuse-angled triangle** that which has an obtuse angle, and an **acute-angled triangle** that which has its three angles acute.

22. Of quadrilateral figures, a **square** is that which is both equilateral and right-angled; an **oblong** that which is right-angled but not equilateral; a **rhombus** that which is equilateral but not right-angled; and a **rhomboid** that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called **trapezia**.

23. **Parallel** straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

POSTULATES.

Let the following be postulated :

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.

5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

COMMON NOTIONS.

1. Things which are equal to the same thing are also equal to one another.

2. If equals be added to equals, the wholes are equal.

3. If equals be subtracted from equals, the remainders are equal.

[7] 4. Things which coincide with one another are equal to one another.

[8] 5. The whole is greater than the part.

DEFINITION I.

Σημείον ἔστιν, οὐ μέρος οὐθέν.

A point is that which has no part.

An exactly parallel use of μέρος (ἔστι) in the singular is found in Aristotle, *Metaph.* 1035 b 32 μέρος μὲν οὖν ἔστι καὶ τοῦ εἶδους, literally "There is a *part* even of the form"; Bonitz translates as if the plural were used, "Theile giebt es," and the meaning is simply "even the form is *divisible* (into parts)." Accordingly it would be quite justifiable to translate in this case "A point is that which is *indivisible into parts*."

Martianus Capella (5th c. A.D.) alone or almost alone translated differently, "Punctum est cuius pars *nil* est," "a point is that a part of which is *nothing*." Notwithstanding that Max Simon (*Euclid und die sechs planimetrischen Bücher*, 1901) has adopted this translation (on grounds which I shall presently mention), I cannot think that it gives any sense. If a part of a point is *nothing*, Euclid might as well have said that a point is *itself* "nothing," which of course he does not do.

Pre-Euclidean definitions.

It would appear that this was not the definition given in earlier textbooks; for Aristotle (*Topics* vi. 4, 141 b 20), in speaking of "the definitions" of point, line, and surface, says that they *all* define the prior by means of the posterior, a point as an extremity of a line, a line of a surface, and a surface of a solid.

The first definition of a point of which we hear is that given by the Pythagoreans (cf. Proclus, p. 95, 21), who defined it as a "monad having position" or "with position added" (μονὰς προσλαβοῦσα θέσιν). It is frequently used by Aristotle, either in this exact form (cf. *De anima* i. 4, 409 a 6) or its equivalent: e.g. in *Metaph.* 1016 b 24 he says that that which is indivisible every way in respect of magnitude and *quâ* magnitude but has not position is a *monad*, while that which is similarly indivisible and has position is a *point*.

Plato appears to have objected to this definition. Aristotle says (*Metaph.*

992 a 20) that he objected "to this genus [that of points] as being a geometrical fiction (γεωμετρικὸν δόγμα), and called a point the beginning of a line (ἀρχὴ γραμμῆς), while again he frequently spoke of 'indivisible lines.'" To which Aristotle replies that even "indivisible lines" must have extremities, so that the same argument which proves the existence of *lines* can be used to prove that *points* exist. It would appear therefore that, when Aristotle objects to the definition of a point as the extremity of a line (πέρας γραμμῆς) as unscientific (*Topics* vi. 4, 141 b 21), he is aiming at Plato. Heiberg conjectures (*Mathematisches zu Aristoteles*, p. 8) that it was due to Plato's influence that the word for "point" generally used by Aristotle (στιγμή) was replaced by σημεῖον (the regular term used by Euclid, Archimedes and later writers), the latter term (= *nota*, a conventional mark) probably being considered more suitable than στιγμή (a *puncture*) which might appear to claim greater *reality* for a point.

Aristotle's conception of a point as that which is indivisible and has position is further illustrated by such observations as that a point is not a *body* (*De caelo* ii. 13, 296 a 17) and has no *weight* (*ibid.* iii. 1, 299 a 30); again, we can make no distinction between a point and the *place* (τόπος) where it is (*Physics* iv. 1, 209 a 11). He finds the usual difficulty in accounting for the transition from the indivisible, or infinitely small, to the finite or divisible magnitude. A point being *indivisible*, no accumulation of points, however far it may be carried, can give us anything divisible, whereas of course a line is a divisible magnitude. Hence he holds that points cannot make up anything continuous like a line, point cannot be continuous with point (οὐ γάρ ἐστιν ἐχόμενον σημεῖον σημεῖον ἢ στιγμὴ στιγμῆς, *De gen. et corr.* i. 2, 317 a 10), and a line is not *made up* of points (οὐ σύγκειται ἐκ στιγμῶν, *Physics* iv. 8, 215 b 19). A point, he says, is like the *now* in time: *now* is indivisible and is not a *part* of time, it is only the beginning or end, or a division, of time, and similarly a point may be an extremity, beginning or division of a line, but is not *part* of it or of magnitude (cf. *De caelo* iii. 1, 300 a 14, *Physics* iv. 11, 220 a 1—21, vi. 1, 231 b 6 sqq.). It is only by *motion* that a point can generate a line (*De anima* i. 4, 409 a 4) and thus be the origin of magnitude.

Other ancient definitions.

According to an-Nairizi (ed. Curtze, p. 3) one "Herundes" (not so far identified) defined a point as "the indivisible beginning of all magnitudes," and Posidonius as "an extremity which has no dimension, or an extremity of a line."

Criticisms by commentators.

Euclid's definition itself is of course practically the same as that which Aristotle's frequent allusions show to have been then current, except that it omits to say that the point must have position. Is it then sufficient, seeing that there are other things which are without parts or indivisible, e.g. the *now* in time, and the *unit* in number? Proclus answers (p. 93, 18) that the point is the only thing in the *subject-matter of geometry* that is indivisible. Relatively therefore to the particular science the definition is sufficient. Secondly, the definition has been over and over again criticised because it is purely negative. Proclus' answer to this is (p. 94, 10) that negative descriptions are appropriate to first principles, and he quotes Parmenides as having described his first and last cause by means of negations merely. Aristotle too admits that it may sometimes be necessary for one framing a definition to use negations, e.g. in defining privative terms such as "blind"; and he seems to accept as proper

the negative element in the definition of a point, since he says (*De anima* III. 6, 430 b 20) that "the point and every division [e.g. in a length or in a period of time], and that which is indivisible in this sense, is exhibited as privation (*δηλοῦνται ὡς στέρησις*)."

Simplicius (quoted by an-Nairizī) says that "a point is the beginning of magnitudes and that from which they grow; it is also the only thing which, having position, is not divisible." He, like Aristotle, adds that it is by its *motion* that a point can generate a magnitude: the particular magnitude can only be "of one dimension," viz. a line, since the point does not "spread itself" (*dimittat*). Simplicius further observes that Euclid defined a point negatively because it was arrived at by detaching surface from body, line from surface, and finally point from line. "Since then body has three dimensions it follows that a point [arrived at after successively eliminating all three dimensions] has *none of the dimensions*, and has no part." This of course reappears in modern treatises (cf. Rausenberger, *Elementar-geometrie des Punktes, der Geraden und der Ebene*, 1887, p. 7).

An-Nairizī adds an interesting observation. "If any one seeks to know the essence of a point, a thing more simple than a line, let him, in the sensible world, think of the centre of the universe and the *poles*." But there is nothing new under the sun: the same idea is mentioned, in an Aristotelian treatise, in controverting those who imagine that the poles have some influence in the motion of the sphere, "when the poles have no magnitude but are extremities and points" (*De motu animalium* 3, 699 a 21).

Modern views.

In the new geometry represented by the excellent treatises which start from new systems of postulates or axioms, the result of the profound study of the fundamental principles of geometry during recent years (I need only mention the names of Pasch, Veronese, Enriques and Hilbert), points come before lines, but the vain effort to define them *a priori* is not made; instead of this, the nearest material things in nature are mentioned as illustrations, with the remark that it is from them that we can get the abstract idea. Cf. the full statement as regards the notion of a point in Weber and Wellstein, *Encyclopädie der elementaren Mathematik*, II., 1905, p. 9. "This notion is evolved from the notion of the real or supposed *material* point by the process of limits, i.e. by an act of the mind which sets a term to a series of presentations in itself unlimited. Suppose a grain of sand or a mote in a sunbeam, which continually becomes smaller and smaller. In this way vanishes more and more the possibility of determining still smaller atoms in the grain of sand, and there is evolved, so we say, with growing certainty, the presentation of the point as a definite position in space which is one and is incapable of further division. But this view is untenable; we have, it is true, some idea how the grain of sand gets smaller and smaller, but only so long as it remains just visible; after that we are completely in the dark, and we cannot see or imagine the further diminution. That this procedure comes to an end is unthinkable; that nevertheless there exists a term beyond which it cannot go, we must believe or postulate without ever reaching it. . . . It is a pure act of *will*, not of the understanding." Max Simon observes similarly (*Euclid*, p. 25) "The notion 'point' belongs to the limit-notions (*Grenzbegriffe*), the necessary conclusions of continued, and in themselves unlimited, series of presentations." He adds, "The point is the limit of localisation; if this is more and more energetically continued, it leads to the limit-notion 'point,'

better 'position,' which at the same time involves a change of notion. Content of *space* vanishes, relative *position* remains. 'Point' then, according to our interpretation of Euclid, is the extremest limit of that which we can still think of (not observe) as a *spatial* presentation, and if we go further than that, not only does extension cease but even relative *place*, and in this sense the 'part' is *nothing*." I confess I think that even the meaning which Simon intends to convey is better expressed by "it has *no* part" than by "the part is nothing," since to take a "part" of a thing in Euclid's sense of the result of a simple division, corresponding to an arithmetical fraction, would not be to change the *notion* from that of the thing divided to an entirely different one.

DEFINITION 2.

Γραμμὴ δὲ μῆκος ἀπλατές.

A line is breadthless length.

This definition may safely be attributed to the Platonic School, if not to Plato himself. Aristotle (*Topics* vi. 6, 143 b 11) speaks of it as open to objection because it "divides the genus by negation," length being necessarily either breadthless or possessed of breadth; it would seem however that the objection was only taken in order to score a point against the Platonists, since he says (*ibid.* 143 b 29) that the argument is "of service *only* against those who assert that the genus [sc. length] is one numerically, that is, those who assume *ideas*," e.g. the idea of length (αὐτὸ μῆκος) which they regard as a genus: for if the genus, being one and self-existent, could be divided into two species, one of which asserts what the other denies, it would be self-contradictory (Waitz).

Proclus (pp. 96, 21—97, 3) observes that, whereas the definition of a point is merely negative, the line introduces the first "dimension," and so its definition is to this extent positive, while it has also a negative element which denies to it the other "dimensions" (διαστάσεις). The negation of both breadth and depth is involved in the single expression "breadthless" (ἀπλατές), since everything that is without breadth is also destitute of depth, though the converse is of course not true.

Alternative definitions.

The alternative definition alluded to by Proclus, μέγεθος ἐφ' ἓν διαστατόν "magnitude in one dimension" or, better perhaps, "magnitude extended one way" (since διάστας is used with reference to line, surface and solid scarcely corresponds to our use of "dimension" when we speak of "one," "two," or "three dimensions"), is attributed by an-Nairizi to "Heromides," who must presumably be the same as "Herundes," to whom he attributes a certain definition of a point. It appears however in substance in Aristotle, though Aristotle does not use the adjective διαστατόν, nor does he apparently use διάστας except of *body* as having *three* "dimensions" or "having dimension (or extension) *all* ways (πάντη)," the "dimensions" being in his view (1) up and down, (2) before and behind, and (3) right and left, and "up" being the principle or beginning of *length*, "right" of *breadth*, and "before" of *depth* (*De caelo* ii. 2, 284 b 24). A line is, according to Aristotle, a magnitude "divisible in one way only" (μοναχῇ διαιρετόν), in contrast to a magnitude divisible in *two* ways (διχῇ διαιρετόν), or a surface, and a magnitude divisible "in all or in three ways" (πάντῃ καὶ τριχῇ διαιρετόν), or a body (*Metaph.* 1016 b 25—27); or it is a magnitude "continuous one way (or in one direction)," as compared with magnitudes continuous *two* ways or *three* ways,

which curiously enough he describes as "breadth" and "depth" respectively (*μέγεθος δὲ τὸ μὲν ἐφ' ἐν συνεχὲς μήκος, τὸ δ' ἐπὶ δύο πλάτος, τὸ δ' ἐπὶ τρία βάθος*, *Metaph.* 1020 a 11), though he immediately adds that "length" means a line, "breadth" a surface, and "depth" a body.

Proclus gives another alternative definition as "*flux of a point*" (*ῥύσις σημείου*), i.e. the path of a point when moved. This idea is also alluded to in Aristotle (*De anima* 1. 4, 409 a 4 above quoted): "they say that a line by its motion produces a surface, and a point by its motion a line." "This definition," says Proclus (p. 97, 8—13), "is a perfect one as showing the essence of the line: he who called it the flux of a point seems to define it from its genetic cause, and it is not every line that he sets before us, but only the immaterial line; for it is this that is produced by the point, which, though itself indivisible, is the cause of the existence of things divisible."

Proclus (p. 100, 5—19) adds the useful remark, which, he says, was current in the school of Apollonius, that we have the notion of a line when we ask for the length of a road or a wall measured merely as length; for in that case we mean something irrespective of breadth, viz. distance in one "dimension." Further we can obtain sensible perception of a line if we look at the division between the light and the dark when a shadow is thrown on the earth or the moon; for clearly the division is without breadth, but has length.

Species of "lines."

After defining the "line" Euclid only mentions *one* species of line, the straight line, although of course another species appears in the definition of a circle later. He doubtless omitted all *classification* of lines as unnecessary for his purpose, whereas, for example, Heron follows up his definition of a line by a division of lines into (1) those which are "straight" and (2) those which are not, and a further division of the latter into (a) "circular circumferences," (b) "spiral-shaped" (*ἐλικοειδεῖς*) lines and (c) "curved" (*καμπύλαι*) lines generally, and then explains the four terms. Aristotle tells us (*Metaph.* 986 a 25) that the Pythagoreans distinguished straight (*εὐθύ*) and curved (*καμπύλον*), and this distinction appears in Plato (cf. *Republic* x. 602 c) and in Aristotle (cf. "to a line belong the attributes straight or curved," *Anal. post.* 1. 4, 73 b 19; "as in mathematics it is useful to know what is meant by the terms straight and curved," *De anima* 1. 1, 402 b 19). But from the class of "curved" lines Plato and Aristotle separate off the *περιφερής* or "circular" as a distinct species often similarly contrasted with straight. Aristotle seems to recognise broken lines forming an angle as one line: thus "a line, if it be bent (*κεκαμμένη*), but yet continuous, is called one" (*Metaph.* 1016 a 2); "the straight line is more one than the bent line" (*ibid.* 1016 a 12). Cf. Heron, Def. 12, "A broken line (*κεκλασμένη γραμμή*) so-called is a line which, when produced, does not meet *itself*."

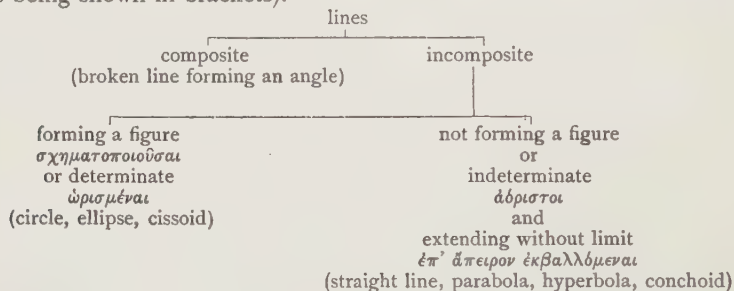
When Proclus says that both Plato and Aristotle divided lines into those which are "straight," "circular" (*περιφερής*) or "a mixture of the two," adding, as regards Plato, that he included in the last of these classes "those which are called helicoidal among plane (curves) and (curves) formed about solids, and such species of curved lines as arise from sections of solids" (p. 104, 1—5), he appears to be not quite exact. The reference as regards Plato seems to be to *Parmenides* 145 B: "At that rate it would seem that the one must have shape, either straight or round (*στρογγύλου*) or some combination of the two"; but this scarcely amounts to a formal classification of lines. As regards

Aristotle, Proclus seems to have in mind the passage (*De caelo* I. 2, 268 b 17) where it is stated that "all *motion* in space, which we call translation (*φορά*), is (in) a straight line, a circle, or a combination of the two; for the first two are the only simple (*motions*)."

For completeness it is desirable to add the substance of Proclus' account of the classification of lines, for which he quotes Geminus as his authority.

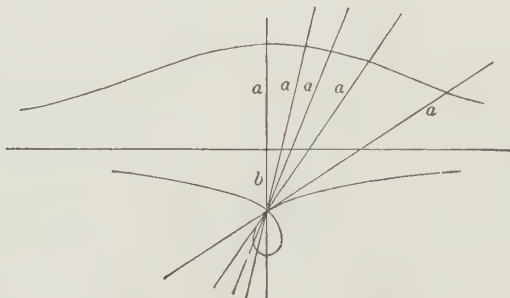
Geminus' first classification of lines.

This begins (p. 111, 1—9) with a division of lines into *composite* (σύνθετος) and *incomposite* (ἀσύνθετος). The only illustration given of the *composite* class is the "broken line which forms an angle" (ἡ κεκλασμένη καὶ γωνίαν ποιούσα); the subdivision of the *incomposite* class then follows (in the text as it stands the word "composite" is clearly an error for "incomposite"). The subdivisions of the *incomposite* class are repeated in a later passage (pp. 176, 27—177, 23) with some additional details. The following diagram reproduces the effect of both versions as far as possible (all the illustrations mentioned by Proclus being shown in brackets).



The additional details in the second version, which cannot easily be shown in the diagram, are as follows :

(1) Of the lines which extend without limit, some do not *form a figure* at all (viz. the straight line, the parabola and the hyperbola); but some first "come together and form a figure" (i.e. have a loop), "and, for the rest, extend without limit" (p. 177, 8).



As the only other curve, besides the parabola and the hyperbola, which has been mentioned as proceeding to infinity is the *conchoid* (of Nicomedes), we can hardly avoid the conclusion of Tannery¹ that the curve which has a loop and then proceeds to infinity is a variety of the *conchoid* itself. As is

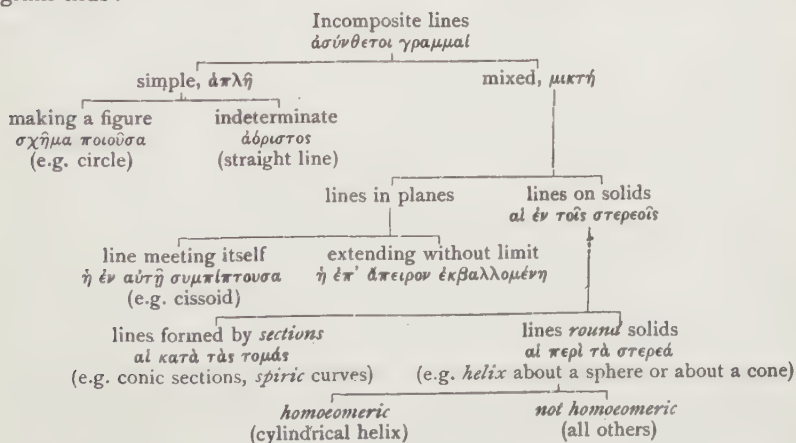
¹ Notes pour l'histoire des lignes et surfaces courbes dans l'antiquité in *Bulletin des sciences mathém. et astronom.* 2 sér. VIII. (1884), pp. 108—9 (*Mémoires scientifiques*, II. p. 23).

well known, the ordinary conchoid (which was used both for doubling the cube and for trisecting the angle) is obtained in this way. Suppose any number of rays passing through a fixed point (the *pole*) and intersecting a fixed straight line; and suppose that points are taken on the rays, beyond the fixed straight line, such that the portions of the rays intercepted between the fixed straight line and the point are equal to a constant *distance* (διάστημα), the locus of the points is a conchoid which has the fixed straight line for asymptote. If the "distance" a is measured from the intersection of the ray with the given straight line, not in the direction away from the pole, but towards the pole, we obtain three other curves according as a is less than, equal to, or greater than b , the distance of the pole from the fixed straight line, which is an asymptote in each case. The case in which $a > b$ gives a curve which forms a loop and then proceeds to infinity in the way Proclus describes. Now we know both from Eutocius (*Comm. on Archimedes*, ed. Heiberg, III. p. 98) and Proclus (p. 272, 3—7) that Nicomedes wrote on conchoids (in the plural), and Pappus (IV. p. 244. 18) says that besides the "first" (used as above stated) there were "the second, the third and the fourth which are useful for other theorems."

(2) Proclus next observes (p. 177, 9) that, of the lines which extend without limit, some are "*asymptotic*" (ἀσύμπτωτοι), namely "those which never meet, however they are produced," and some are "*sympnotic*," namely "those which will meet sometime"; and, of the "asymptotic" class, some are in one plane, and others not. Lastly, of the "asymptotic" lines in one plane, some preserve always the same distance from one another, while others continually "lessen the distance, like the hyperbola with reference to the straight line, and the conchoid with reference to the straight line."

Geminus' second classification.

This (from Proclus, pp. 111, 9—20 and 112, 16—18) can be shown in a diagram thus:



Notes on classes of "lines" and on particular curves.

We will now add the most interesting notes found in Proclus with reference to the above classifications or the particular curves mentioned.

1. Homoeomeric lines.

By this term (*ὁμοιομερεῖς*) are meant lines which are alike in all parts, so that in any one such curve any part can be made to coincide with any other part. Proclus observes that these lines are only three in number, two being "simple" and in a plane (the straight line and the circle), and the third "mixed," (subsisting) "about a solid," namely the cylindrical helix. The latter curve was also called the *cochlias* or *cochlion*, and its *homoeomeric* property was proved by Apollonius in his work *περὶ τοῦ κοχλίου* (Proclus, p. 105, 5). The fact that there are only three *homoeomeric* lines was proved by Geminus, "who proved, as a preliminary proposition, that, if from a point (*ἀπὸ τοῦ σημείου*, but on p. 251, 4 *ἀφ' ἐνὸς σημείου*) two straight lines be drawn to a homoeomeric line making equal angles with it, the straight lines are equal" (pp. 112, 1—113, 3, cf. p. 251, 2—19).

2. Mixed lines.

It might be supposed, says Proclus (p. 105, 11), that the cylindrical helix, being *homoeomeric*, like the straight line and the circle, must like them be *simple*. He replies that it is not simple, but *mixed*, because it is generated by *two unlike* motions. Two *like* motions, said Geminus, e.g. two motions at the same speed in the directions of two adjoining sides of a square, produce a *simple* line, namely a straight line (the diagonal); and again, if a straight line moves with its extremities upon the two sides of a right angle respectively, this same motion gives a *simple* curve (a circle) for the locus of the middle point of the straight line, and a *mixed* curve (an ellipse) for the locus of any other point on it (p. 106, 3—15).

Geminus also explained that the term "mixed," as applied to curves, and as applied to surfaces, respectively, is used in different senses. As applied to curves, "mixing" neither means simple "putting together" (*σύνθεσις*) nor "blending" (*κρᾶσις*). Thus the helix (or spiral) is a "mixed" line, but (1) it is not "mixed" in the sense of "putting together," as it would be if, say, part of it were straight and part circular, and (2) it is not mixed in the sense of "blending," because, if it is cut in any way, it does not present the appearance of any *simple* lines (of which it might be supposed to be compounded, as it were). The "mixing" in the case of lines is rather that in which the constituents are destroyed so far as their own character is concerned, and are replaced, as it were, by a *chemical* combination (*ἔστιν ἐν αὐτῇ συνεφθαρμένα τὰ ἄκρα καὶ συγκεχυμένα*). On the other hand "mixed" surfaces are mixed in the sense of a sort of "blending" (*κατὰ τινα κρᾶσιν*). For take a cone generated by a straight line passing through a fixed point and passing always through the circumference of a circle: if you cut this by a plane parallel to that of the circle, you obtain a circular section, and if you cut it by a plane through the vertex, you obtain a triangle, the "mixed" surface of the cone being thus cut into *simple* lines (pp. 117, 22—118, 23).

3. Spiric curves.

These curves, classed with conics as being sections of solids, were discovered by Perseus, according to an epigram of Perseus' own quoted by Proclus (p. 112, 1), which says that Perseus found "three lines upon (or, perhaps, in addition to) five sections" (*τρῆς γραμμὰς ἐπὶ πέντε τομαῖς*). Proclus throws some light on these in the following passages:

"Of the spiric sections, one is interlaced, resembling the horse-fetter (*ἵππου πέδη*); another is widened out in the middle and contracts on each

side (of the middle), a third is elongated and is narrower in the middle, broadening out on each side of it" (p. 112, 4—8).

"This is the case with the *spiric surface*; for it is conceived as generated by the revolution of a circle remaining at right angles [to a plane] and turning about a point which is not its centre [in other words, generated by the revolution of a circle about a straight line in its plane not passing through the centre]. Hence the *spire* takes three forms, for the centre [of rotation] is either on the circumference, or within it, or without it. And if the centre of rotation is on the circumference, we have the *continuous spire* (συνεχής), if within, the *interlaced* (ἐμπεπλεγμένη), and if without, the *open* (διεχής). And the spiric sections are three according to these three differences" (p. 119, 8—17).

"When the *hippope*, which is one of the spiric curves, forms an angle with itself, this angle also is contained by mixed lines" (p. 127, 1—3).

"Perseus showed for spirics what was their property (σύμπτωμα)" (p. 356, 12).

Thus the spiric surface was what we call a *tore*, or (when open) an *anchor-ring*. Heron (Def. 97) says it was called alternatively *spire* (σπείρα) or *ring* (κρίκος); he calls the variety in which "the circle cuts itself," not "interlaced," but "crossing-itself" (ἐπαλλάττουσα).

Tannery¹ has discussed these passages, as also did Schiaparelli². It is clear that Proclus' remark that the difference in the three curves which he mentions corresponds to the difference between the three surfaces is a slip, due perhaps to too hurried transcribing from Geminus: all three arise from plane sections of the *open* anchor-ring. If r is the radius of the revolving circle, a the distance of its centre from the axis of rotation, d the distance of the plane section (supposed to be parallel to the axis) from the axis, the three curves described in the first extract correspond to the following cases:

(1) $d = a - r$. In this case the curve is the *hippope*, of which the lemniscate of Bernoulli is a particular case, namely that in which $a = 2r$.

The name *hippope* was doubtless adopted for this one of Perseus' curves on the ground of its resemblance to the *hippope* of Eudoxus, which seems to have been the curve of intersection of a sphere with a cylinder touching it internally.

(2) $a + r > d > a$. Here the curve is an oval.

(3) $a > d > a - r$. The curve is now narrowest in the middle.

Tannery explains the "three lines upon (in addition to) five sections" thus. He points out that with the *open tore* there are two other sections corresponding to

(4) $d = a$: transition from (2) to (3).

(5) $a - r > d > 0$, in which case the section consists of two symmetrical ovals.

He then shows that the sections of the *closed* or *continuous tore*, corresponding to $a = r$, give curves corresponding to (2), (3) and (4) only. Instead of (1) and (5) we have only a section consisting of two equal circles touching one another.

On the other hand, the *third* spire (the *interlaced* variety) gives three *new* forms, which make a group of three in addition to the first group of *five* sections.

¹ Pour l'histoire des lignes et surfaces courbes dans l'antiquité in *Bulletin des sciences mathém. et astronom.* VIII. (1884), pp. 25—27 (*Mémoires scientifiques*, I. pp. 24—28).

² Die homocentrischen Sphären des Eudoxus, des Kallippus und des Aristoteles (*Abhandlungen zur Gesch. der Math.* I. Heft, 1877, pp. 149—152).

The difficulty which I see in this interpretation is the fact that, just after "three lines on five sections" are mentioned, Proclus describes three curves which were evidently the most important; but these three belong to three of the five sections of the open tore, and are not separate from them.

4. The cissoid.

This curve is assumed to be the same as that by means of which, according to Eutocius (*Comm. on Archimedes*, III. p. 66 sqq.), Diocles in his book *περί πύργων* (*On burning-glasses*) solved the problem of doubling the cube. It is the locus of points which he found by the following construction. Let AC , BD be diameters at right angles in a circle with centre O .

Let E , F be points on the quadrants BC , BA respectively such that the arcs BE , BF are equal.

Draw EG , FH perpendicular to CA . Join AE , and let P be its intersection with FH .

The cissoid is the locus of all the points P corresponding to different positions of E on the quadrant BC and of F at an equal distance from B along the arc BA .

A is the point on the curve corresponding to the position C for the point E , and B the point on the curve corresponding to the position of E in which it coincides with B .

It is easy to see that the curve extends in the direction AB beyond B , and that CK drawn perpendicular to CA is an asymptote. It may be regarded also as having a branch AD symmetrical with AB , and, beyond D , approaching KC produced as asymptote.

If OA , OD are coordinate axes, the equation of the curve is obviously

$$y^2(a+x) = (a-x)^3,$$

where a is the radius of the circle.

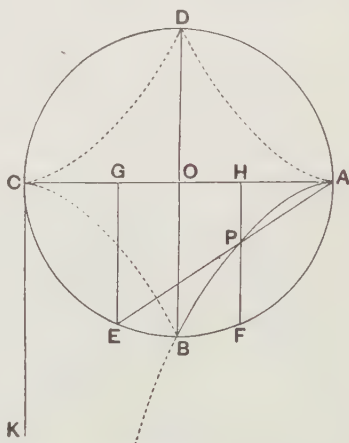
There is a cusp at A , and it agrees with this that Proclus should say (p. 126, 24) that "cissoidal lines converging to one point like the leaves of ivy—for this is the origin of their name—form an angle." He makes the slight correction (p. 128, 5) that it is not two *parts* of a curve, but *one* curve, which in this case makes an angle.

But what is surprising is that Proclus seems to have no idea of the curve passing outside the circle and having an asymptote, for he several times speaks of it as a *closed* curve (forming a figure and including an area): cf. p. 152, 7, "the plane (area) cut off by the cissoidal line has one bounding (line), but it has not in it a centre such that all (straight lines drawn to the curve) from it are equal." It would appear as if Proclus regarded the cissoid as formed by the *four* symmetrical cissoidal arcs shown in the figure.

Even more peculiar is Proclus' view of the

5. "Single-turn Spiral."

This is really the spiral of Archimedes traced by a point starting from the fixed extremity of a straight line and moving uniformly along it, while



simultaneously the straight line itself moves uniformly in a plane about its fixed extremity. In Archimedes the spiral has of course any number of turns, the straight line making the same number of complete revolutions. Yet Proclus, while giving the same account of the generation of the spiral (p. 180, 8—12), regards the *single-turn spiral* as actually *stopping short* at the point reached after one complete revolution of the straight line: "it is necessary to know that extending without limit is not a property of all lines; for it neither belongs to the circle nor to the cissoid, nor in general to lines which form figures; nor even to those which do not form figures. For even the single-turn spiral does not extend without limit—for it is constructed between two points—nor does any of the other lines so generated do so" (p. 187, 19—25). It is curious that Pappus (VIII. p. 1110 sqq.) uses the same term *μονόστροφος ἑλιξ* to denote one turn, not of the spiral, but of the *cylindrical helix*.

DEFINITION 3.

Γραμμῆς δὲ πέρατα σημεία.

The extremities of a line are points.

It being unscientific, as Aristotle said, to define a point as the "extremity of a line" (*πέρας γραμμῆς*), thereby explaining the prior by the posterior, Euclid defined a point differently; then, as it was necessary to connect a point with a line, he introduced this explanation after the definitions of both had been given. This compromise is no doubt his own idea; the same thing occurs with reference to a surface and a line as its extremity in Def. 6, and with reference to a solid and a surface as its extremity in XI. Def. 2.

We miss a statement of the facts, equally requiring to be known, that a "division" (*διαίρεσις*) of a line, no less than its "beginning" or "end," is a point (this is brought out by Aristotle: cf. *Metaph.* 1060 b 15), and that the *intersection* of two lines is also a point. If these additional explanations had been given, Proclus would have been spared the difficulty which he finds in the fact that some of the lines used in Euclid (namely infinite straight lines on the one hand, and circles on the other) have no "extremities." So also the ellipse, which Proclus calls by the old name *θυρεός* ("shield"). In the case of the circle and ellipse we can, he observes (p. 103, 7), take a portion bounded by points, and the definition applies to that portion. His rather far-fetched distinction between two aspects of a circle or ellipse as a *line* and as a *closed figure* (thus, while you are *describing* a circle, you have two extremities at any moment, but they disappear when it is finished) is an unnecessarily elaborate attempt to establish the literal universality of the "definition," which is really no more than an explanation that, if a line *has* extremities, those extremities are points.

DEFINITION 4.

Εὐθεῖα γραμμή ἐστίν, ἥτις ἐξ ἴσου τοῖς ἐφ' ἑαυτῆς σημείοις κείται.

A straight line is a line which lies evenly with the points on itself.

The only definition of a straight line authenticated as pre-Euclidean is that of Plato, who defined it as "*that of which the middle covers the ends*" (relatively, that is, to an eye placed at either end and looking along the straight line). It appears in the *Parmenides* 137 E: "straight is whatever has its middle in front of (i.e. so placed as to obstruct the view of) both its ends"

(εὐθύ γε οὐδ' ἂν τὸ μέσον ἀμφοῖν τοῖν ἐσχάτοι ἐπίπροσθεν ᾗ). Aristotle quotes it in equivalent terms (*Tōpica* VI. 11, 148 b 27), οὐδ' τὸ μέσον ἐπιπροσθὲι τοῖς πέρασιν; and, as he does not mention the name of its author, but states it in combination with the definition of a line as the extremity of a surface, we may assume that he used it as being well known. Proclus also quotes the definition as Plato's in almost identical terms, ἥς τὰ μέσα τοῖς ἄκροις ἐπιπροσθὲι (p. 109, 21). This definition is ingenious, but implicitly appeals to the sense of sight and involves the postulate that the line of sight is straight. (Cf. the Aristotelian *Problems* 31, 20, 959 a 39, where the question is why we can better observe straightness in a row, say, of letters with one eye than with two.) As regards the straightness of "visual rays," ὄψεις, cf. Euclid's own *Optics*, Def. 1, 2, assumed as *hypotheses*, in which he first speaks of the "straight lines" drawn from the eye, avoiding the word ὄψεις, and then says that the figure contained by the *visual rays* (ὄψεις) is a cone with its vertex in the eye.

As Aristotle mentions no definition of a straight line resembling Euclid's, but gives only Plato's definition and the other explaining it as the "extremity of a surface," the latter being evidently the current definition in contemporary textbooks, we may safely infer that Euclid's definition was a new departure of his own.

Proclus on Euclid's definition.

Coming now to the interpretation of Euclid's definition, εὐθεῖα γραμμὴ ἐστίν, ἥτις ἐξ ἴσου τοῖς ἐφ' ἑαυτῆς σημεῖοις κείται, we find any number of slightly different versions, but none that can be described as quite satisfactory; some authorities, e.g. Savile, have confessed that they could make nothing of it. It is natural to appeal to Proclus first; and we find that he does in fact give an interpretation which at first sight seems plausible. He says (p. 109, 8 sq.) that Euclid "shows by means of this that the straight line alone [of all lines] occupies a distance (κατέχειν διάστημα) equal to that between the points on it. For, as far as one of the points is distant from another, so great is the length (μέγεθος) of the straight line of which they are the extremities; and this is the meaning of lying ἐξ ἴσου to (or with) the points on it" [ἐξ ἴσου being thus, apparently, interpreted as "at" (or "over") "an equal distance"]. "But if you take two points on the circumference (of a circle) or any other line, the distance cut off between them along the line is greater than the interval separating them. And this is the case with every line except the straight line. Hence the ordinary remark, based on a common notion, that those who journey in a straight line only travel the necessary distance, while those who do not go straight travel more than the necessary distance." (Cf. Aristotle, *De caelo* 1. 4, 271 a 13, "we always call the distance of anything the straight line" drawn to it.) Thus Proclus would interpret somewhat in this way: "a straight line is that which represents extension equal with (the distances separating) the points on it." This explanation seems to be an attempt to graft on to Euclid's definition the *assumption* (it is a λαμβανόμενον, not a definition) of Archimedes (*On the sphere and cylinder* 1. *ad init.*) that "of all the lines which have the same extremities the straight line is least." For this purpose ἐξ ἴσου has apparently to be taken as meaning "at an equal distance," and again "lying at an equal distance" as equivalent to "extending over (or representing) an equal distance." This is difficult enough in itself, but is seen to be an impossible interpretation when applied to the similar definition of a plane by Euclid (Def. 7) as a surface "which lies evenly with the straight lines on itself." In that connexion Proclus tries to make the same words ἐξ ἴσου

κεῖται mean "extends over an equal *area* with." He says namely (p. 117, 2) that, "if two straight lines are set out" on the plane, the plane surface "occupies a space equal to that between the straight lines." But two straight lines do not determine by themselves any space at all; it would be necessary to have a *closed* figure with its boundaries in the plane before we could arrive at the equivalent of the other assumption of Archimedes that "of surfaces which have the same extremities, if those extremities are in a plane, the plane is the least [in area]." This seems to be an impossible sense for ἐξ ἴσου even on the assumption that it means "at an equal distance" in the present definition. The necessity therefore of interpreting ἐξ ἴσου similarly in both definitions makes it impossible to regard it as referring to *distance* or *length* at all. It should be added that Simplicius gave the same explanations as Proclus (an-Nairizi, p. 5).

The language and construction of the definition.

Let us now consider the actual wording and grammar of the phrase ἡ τις ἐξ ἴσου τοῖς ἐφ' αὐτῆς σημείοις κεῖται. As regards the expression ἐξ ἴσου we note that Plato and Aristotle (whose use of it seems typical) commonly have it in the sense of "on a footing of equality": cf. οἱ ἐξ ἴσου in Plato's *Laws* 777 D, 919 D; Aristotle, *Politics* 1259 b 5 ἐξ ἴσου εἶναι βούλεται τὴν φύσιν, "tend to be on an equality in nature," *Eth. Nic.* VIII. 12, 1161 a 8 ἐνταῦθα πάντες ἐξ ἴσου, "there all are on a footing of equality." Slightly different are the uses in Aristotle, *Eth. Nic.* X. 8, 1178 a 25 τῶν μὲν γὰρ ἀναγκαίων χρεια καὶ ἐξ ἴσου ἔστω, "both need the necessities of life to the same extent, let us say"; *Topics* IX. 15, 174 a 32 ἐξ ἴσου ποιοῦντα τὴν ἐρώτησιν, "asking the question indifferently" (i.e. without showing any expectation of one answer being given rather than another). The natural meaning would therefore appear to be "evenly placed" (or balanced), "in equal measure," "indifferently" or "without bias" one way or the other. Next, is the dative τοῖς ἐφ' αὐτῆς σημείοις constructed with ἐξ ἴσου or with κεῖται? In the first case the phrase must mean "that which lies *evenly with* (or in respect to) the points on it," in the second apparently "that which, in (or by) the points on it, lies (or is placed) evenly (or uniformly)." Max Simon takes the first construction to give the sense "die Gerade liegt in gleicher Weise wie ihre Punkte." If the last words mean "in the same way as (or in like manner as) its points," I cannot see that they tell us anything, although Simon attaches to the words the notion of *distance* (Abstand) like Proclus. The second construction he takes as giving "die Gerade liegt für (durch) ihre Punkte gleichmässig," "the straight line lies symmetrically for (or through) its points"; or, if κεῖται is taken as the passive of τίθημι, "die Gerade ist durch ihre Punkte gleichmässig gegeben worden," "the straight line is symmetrically determined by its points." He adds that the idea is here *direction*, and that both *direction* and *distance* (as between two different given points simply) would be to Euclid, as later to Bolzano (*Betrachtungen über einige Gegenstände der Elementargeometrie*, 1804, quoted by Schotten, *Inhalt und Methode des planimetrischen Unterrichts*, II. p. 16), primary irreducible notions.

While the language is thus seen to be hopelessly obscure, we can safely say that the sort of idea which Euclid wished to express was that of a line which presents the same shape at and relatively to all points on it, without any irregular or unsymmetrical feature distinguishing one part or side of it from another. Any such irregularity could, as Saccheri points out (Engel and Stäckel, *Die Theorie der Parallellinien von Euklid bis Gauss*, 1895, p. 109), be at once made perceptible by keeping the ends fixed and turning the line about

them right round; if any two positions were distinguishable, e.g. one being to the left or right relatively to another, "it would not lie in a uniform manner between its points."

A conjecture as to its origin and meaning.

The question arises, what was the origin of Euclid's definition, or, how was it suggested to him? It seems to me that the basis of it was really Plato's definition of a straight line as "that line the middle of which covers the ends." Euclid was a Platonist, and what more natural than that he should have adopted Plato's definition in substance, while regarding it as essential to change the form of words in order to make it independent of any implied appeal to vision, which, as a physical fact, could not properly find a place in a purely geometrical definition? I believe therefore that Euclid's definition is simply an attempt (albeit unsuccessful, from the nature of the case) to express, in terms to which a geometer could not object as not being part of geometrical subject-matter, the same thing as the Platonic definition.

The truth is that Euclid was attempting the impossible. As Pfeleiderer says (Scholia to Euclid), "It seems as though the notion of a *straight line*, owing to its simplicity, cannot be explained by any regular definition which does not introduce words already containing in themselves, by implication, the notion to be defined (such e.g. are direction, equality, uniformity or evenness of position, unswerving course), and as though it were impossible, if a person does not already know what the term *straight* here means, to teach it to him unless by putting before him in some way a picture or a drawing of it." This is accordingly done in such books as Veronese's *Elementi di geometria* (Part I., 1904, p. 10): "A stretched string, e.g. a plummet, a ray of light entering by a small hole into a dark room, are *rectilineal* objects. The image of them gives us the abstract idea of the limited line which is called a *rectilineal segment*."

Other definitions.

We will conclude this note with some other famous definitions of a straight line. The following are given by Proclus (p. 110, 18—23).

1. *A line stretched to the utmost, ἐπ' ἄκρον τεταμένη γραμμή.* This appears in Heron also, with the words "towards the ends" (ἐπὶ τὰ πέρατα) added. (Heron, Def. 4).

2. *Part of it cannot be in the assumed plane while part is in one higher up (ἐν μετεωροτέρῳ).* This is a *proposition* in Euclid (XI. 1).

3. *All its parts fit on all (other parts) alike, πάντα αὐτῆς τὰ μέρη πᾶσιν ὁμοίως ἐφαρμόζει.* Heron has this too (Def. 4), but instead of "alike" he says παντοίως, "in all ways," which is better as indicating that the applied part may be applied one way or the *reverse* way, with the same result.

4. *That line which, when its ends remain fixed, itself remains fixed, ἡ τῶν περάτων μενόντων καὶ αὐτὴ μένουσα.* Heron's addition to this, "*when it is, as it were, turned round in the same plane*" (ὡς ἐν τῷ αὐτῷ ἐπιπέδῳ στρεφόμενη), and his next variation, "and about the same ends having always the same position," show that the definition of a straight line as "that which does not change its position when it is turned about its extremities (or any two points in it) as poles" was no original discovery of Leibniz, or Saccheri, or Krafft, or Gauss, but goes back at least to the beginning of the Christian era. Gauss' form of this definition was: "The line in which lie all points that, during the revolution of a body (a part of space) about two fixed points, maintain their position unchanged is called a straight line." Schotten

(I. p. 315) maintains that the notion of a straight line and its property of being determined by two points are unconsciously assumed in this definition, which is therefore a logical "circle."

5. *That line which with one other of the same species cannot complete a figure*, ἡ μετὰ τῆς ὁμοειδοῦς μῆκος σχῆμα μὴ ἀποτελοῦσα. This is an obvious ὕστερον-πρότερον, since it assumes the notion of a figure.

Lastly Leibniz' definition should be mentioned: *A straight line is one which divides a plane into two halves identical in all but position*. Apart from the fact that this definition introduces the plane, it does not seem to have any advantages over the definition last but one referred to.

Legendre uses the Archimedean property of a straight line as *the shortest distance between two points*. Van Swinden observes (*Elemente der Geometrie*, 1834, p. 4), that to take this as the definition involves *assuming* the proposition that any two sides of a triangle are greater than the third and *proving* that straight lines which have two points in common coincide throughout their length (cf. Legendre, *Éléments de Géométrie* I. 3, 8).

The above definitions all illustrate the observation of Unger (*Die Geometrie des Euklid*, 1833): "*Straight* is a simple notion, and hence all definitions of it must fail.... But if the proper idea of a straight line has once been grasped, it will be recognised in all the various definitions usually given of it; all the definitions must therefore be regarded as *explanations*, and among them that one is the best from which further inferences can immediately be drawn as to the essence of the straight line."

DEFINITION 5.

Ἐπιφάνεια δέ ἐστιν, ὃ μῆκος καὶ πλάτος μόνον ἔχει.

A surface is that which has length and breadth only.

The word ἐπιφάνεια was used by Euclid and later writers to denote *surface* in general, while they appropriated the word ἐπίπεδον for *plane surface*, thus making ἐπίπεδον a *species* of the *genus* ἐπιφάνεια. A solitary use of ἐπιφάνεια by Euclid when a plane is meant (XI. Def. 11) is probably due to the fact that the particular definition came from an earlier textbook. Proclus (p. 116, 17) remarks that the older philosophers, including Plato and Aristotle, used the words ἐπιφάνεια and ἐπίπεδον indifferently for any kind of surface. Aristotle does indeed use both words for a surface, with perhaps a tendency to use ἐπιφάνεια more than ἐπίπεδον for a surface not plane. Cf. *Categories* 6, 5 a 1 sq., where both words are used in one sentence: "You can find a common boundary at which the parts fit together, a point in the case of a line, and a line in the case of a surface (ἐπιφάνεια); for the parts of the surface (ἐπίπεδον) do fit together at some common boundary. Similarly also in the case of a body you can find a common boundary, a line or a surface (ἐπιφάνεια), at which the parts of the body fit together." Plato however does not use ἐπιφάνεια at all in the sense of surface, but only ἐπίπεδον for both *surface* and *plane surface*. There is reason therefore for doubting the correctness of the notice in Diogenes Laertius, III. 24, that Plato "was the first philosopher to name, among extremities, the *plane surface*" (ἐπίπεδος ἐπιφάνεια).

ἐπιφάνεια of course means literally the feature of a body which is *apparent* to the eye (ἐπιφανής), namely the surface.

Aristotle tells us (*De sensu* 3, 439 a 31) that the Pythagoreans called a surface χροιά, which seems to have meant *skin* as well as *colour*. Aristotle explains the term with reference to colour (χρῶμα) as a thing inseparable from the extremity (πέρας) of a body.

Alternative definitions.

The definitions of a surface correspond to those of a line. As in Aristotle a line is a magnitude "extended" one way, or in one 'dimension'" (ἐφ' ἑν), "continuous one way" (ἐφ' ἐν συνεχές), or "divisible in one way" (μοναχῇ διαμερόν), so a surface is a magnitude extended or continuous *two ways* (ἐπὶ δύο), or divisible *in two ways* (διχῇ). As in Euclid a surface has "length and breadth" only, so in Aristotle "breadth" is characteristic of the surface and is once used as synonymous with it (*Metaph.* 1020 a 12), and again "lengths are made up of long and short, surfaces of broad and narrow, and solids (ὄγκοι) of deep and shallow" (*Metaph.* 1085 a 10).

Aristotle mentions the common remark that *a line by its motion produces a surface* (*De anima* 1. 4, 409 a 4). He also gives the *a posteriori* description of a surface as the "extremity of a solid" (*Topics* vi. 4, 141 b 22), and as "the section (τομή) or division (διαίρεσις) of a body" (*Metaph.* 1060 b 14).

Proclus remarks (p. 114, 20) that we get a notion of a surface when we measure areas and mark their boundaries in the sense of length and breadth; and we further get a sort of perception of it by looking at shadows, since these have no depth (for they do not penetrate the earth) but only have length and breadth.

Classification of surfaces.

Heron gives (Def. 74, p. 50, ed. Heiberg) two alternative divisions of surfaces into two classes, corresponding to Geminus' alternative divisions of lines, viz. into (1) *incomposite* and *composite* and (2) *simple* and *mixed*.

(1) *Incomposite* surfaces are "those which, when produced, fall into (or coalesce with) themselves" (ὅσαι ἐκβαλλόμεναι αὐταὶ καθ' ἑαυτῶν πίπτουσιν), i.e. are of continuous curvature, e.g. the sphere.

Composite surfaces are "those which, when produced, cut one another." Of composite surfaces, again, some are (a) made up of non-homogeneous (elements) (ἐξ ἀνομοιογενῶν) such as cones, cylinders and hemispheres, others (b) made up of homogeneous (elements), namely the rectilineal (or polyhedral) surfaces.

(2) Under the alternative division, *simple* surfaces are the plane and the spherical surfaces, but no others; the *mixed* class includes all other surfaces whatever and is therefore infinite in variety.

Heron specially mentions as belonging to the mixed class (a) the surface of cones, cylinders and the like, which are a mixture of plane and circular (μικτὰ ἐξ ἐπιπέδου καὶ περιφερείας) and (b) *spīric* surfaces, which are "a mixture of two circumferences" (by which he must mean a mixture of two circular elements, namely the generating circle and its circular motion about an axis in the same plane).

Proclus adds the remark that, curiously enough, *mixed* surfaces may arise by the revolution either of *simple* curves, e.g. in the case of the *spīre*, or of *mixed* curves, e.g. the "right-angled conoid" from a parabola, "another conoid" from the hyperbola, the "oblong" (ἐπίμηκες, in Archimedes παραμήκες) and "flat" (ἐπιπλατύ) spheroids from an ellipse according as it revolves about the major or minor axis respectively (pp. 119, 6—120, 2). The *homoeomeri*c surfaces, namely those any part of which will coincide with any other part, are *two* only (the plane and the spherical surface), not three as in the case of lines (p. 120, 7).

DEFINITION 6.

Ἐπιφανείας δὲ πέρατα γραμμαί.

The extremities of a surface are lines.

It being unscientific, as Aristotle says, to define a line as the extremity of a surface, Euclid avoids the error of defining the prior by means of the posterior in this way, and gives a different definition not open to this objection. Then, by way of compromise, and in order to show the connexion between a line and a surface, he adds the equivalent of the definition of a line previously current as an explanation.

As in the corresponding Def. 3 above, he omits to add what is made clear by Aristotle (*Metaph.* 1060 b 15) that a "division" (διαίρεσις) or "section" (τομή) of a solid or body is also a surface, or that the common boundary at which two parts of a solid fit together (*Categories* 6, 5 a 2) may be a surface.

Proclus discusses how the fact stated in Def. 6 can be said to be true of surfaces like that of the sphere "which is bounded (πεπερασται), it is true, but not by lines." His explanation (p. 116, 8—14) is that, "if we take the surface (of a sphere), so far as it is extended two ways (διχῇ διαστατή), we shall find that it is bounded by lines as to length and breadth; and if we consider the spherical surface as possessing a form of its own and invested with a fresh quality, we must regard it as having fitted end on to beginning and made the two ends (or extremities) one, being thus one potentially only, and not in actuality."

DEFINITION 7.

Ἐπίπεδος ἐπιφανεία ἐστίν, ἥτις ἐξ ἴσου ταῖς ἐφ' ἐαυτῆς εὐθείαις κείται.

A plane surface is a surface which lies evenly with the straight lines on itself.

The Greek follows exactly the definition of a straight line *mutatis mutandis*, i.e. with ταῖς...εὐθείαις for τοῖς...σημείοις. Proclus remarks that, in general, all the definitions of a straight line can be adapted to the plane surface by merely changing the *genus*. Thus, for instance, a plane surface is "a surface the middle of which covers the ends" (this being the adaptation of Plato's definition of a straight line). Whether Plato actually gave this as the definition of a plane surface or not, I believe that Euclid's definition of a plane surface as *lying evenly with the straight lines on itself* was intended simply to express the same idea without any implied appeal to vision (just as in the corresponding case of the definition of a straight line).

As already noted under Def. 4, Proclus tries to read into Euclid's definition the Archimedean assumption that "of surfaces which have the same extremities, if those extremities are in a plane, the plane is the least." But, as I have stated, his interpretation of the words seems impossible, although it is adopted by Simplicius also (see an-Nairizi).

Ancient alternatives.

The other ancient definitions recorded are as follows.

I. *The surface which is stretched to the utmost* (ἐπ' ἄκρον τεταμένη): a definition which Proclus describes as equivalent to Euclid's definition (on Proclus' own view of that definition). Cf. Heron, Def. 9, "a surface) which is right (and) stretched out" (ὀρθὴ οὖσα ἀποτεταμένη), words which he adds to Euclid's definition.

2. *The least surface among all those which have the same extremities.* Proclus is here (p. 117, 9) obviously quoting the Archimedean assumption.

3. *A surface all the parts of which have the property of fitting on (each other)* (Heron, Def. 9).

4. *A surface such that a straight line fits on all parts of it* (Proclus, p. 117, 8), or *such that the straight line fits on it all ways*, i.e. however placed (Proclus, p. 117, 20).

With this should be compared :

5. *“(A plane surface is) such that, if a straight line pass through two points on it, the line coincides wholly with it at every spot, all ways,”* i.e. however placed (one way or the reverse, no matter how), ἥς ἐπειδὴν δύο σημείων ἀψήται εὐθεία, καὶ ὅλη αὐτῇ κατὰ πάντα τόπον παντοίως ἐφαρμόζεται (Heron, Def. 9). This appears, with the words κατὰ πάντα τόπον παντοίως omitted, in Theon of Smyrna (p. 112, 5, ed. Hiller), so that it goes back at least as far as the 1st c. A.D. It is of course the same as the definition commonly attributed to Robert Simson, and very widely adopted as a substitute for Euclid's.

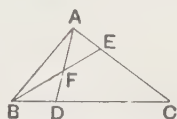
This same definition appears also in ἄν-Ναῖριζι (ed. Curtze, p. 10) who, after quoting Simplicius' explanation (on the same lines as Proclus') of the meaning of Euclid's definition, goes on to say that “others defined the plane surface as that in which it is possible to draw a straight line from any point to any other.”

Difficulties in ordinary definitions.

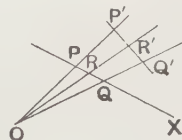
Gauss observed in a letter to Bessel that the definition of a plane surface as *a surface such that, if any two points in it be taken, the straight line joining them lies wholly in the surface* (which, for short, we will call “Simson's” definition) contains more than is necessary, in that a plane can be obtained by simply projecting a straight line lying in it from a point outside the line but also lying on the plane; in fact the definition includes a theorem, or postulate, as well. The same is true of Euclid's definition of a plane as the surface which “lies evenly with (all) the straight lines on itself,” because it is sufficient for a definition of a plane if the surface “lies evenly” with those lines only which pass through a fixed point on it and each of the several points of a straight line also lying in it but not passing through the point. But from Euclid's point of view it is immaterial whether a definition contains more than the necessary minimum *provided* that the *existence* of a thing possessing all the attributes contained in the definition is afterwards proved. This however is not done in regard to the plane. No proposition about the nature of a plane as such appears before Book XI., although its existence is presupposed in all the geometrical Books I.—IV. and VI.; nor in Book XI. is there any attempt to prove, e.g. by construction, the existence of a surface conforming to the definition. The explanation may be that the existence of the plane as defined was deliberately assumed from the beginning like that of points and lines, the existence of which, according to Aristotle, must be assumed as principles unproved, while the existence of everything else must be proved; and it may well be that Aristotle would have included plane surfaces with points and lines in this statement had it not been that he generally took his illustrations from *plane* geometry (excluding solid).

But, whatever definition of a plane is taken, the evolution of its essential properties is extraordinarily difficult. Crelle, who wrote an elaborate article *Zur Theorie der Ebene* (read in the Academie der Wissenschaften in 1834) of which account must be taken in any full history of the subject, observes that,

since the plane is the field, as it were, of almost all the rest of geometry, while a proper conception of it is necessary to enable Eucl. I. 1 to be understood, it might have been expected that the theory of the plane would have been the subject of at least the same amount of attention as, say, that of parallels. This however was far from being the case, perhaps because the subject of parallels (which, for the rest, presuppose the notion of a plane) is *much easier* than that of the plane. The nature of the difficulties as regards the plane have also been pointed out recently by Mr Frankland (*The First Book of Euclid's Elements*, Cambridge, 1905): it would appear that, whatever definition is taken, whether the simplest (as containing the minimum necessary to determine a plane) or the more complex, e.g. Simson's, some postulate has to be assumed in addition before the fundamental properties, or the truth of the other definitions, can be established. Crelle notes the same thing as regards Simson's definition, containing *more* than is necessary. Suppose a plane in which lies the triangle ABC . Let AD join the vertex A to any point D on BC , and BE the vertex B to any point E on CA . Then, according to the definition, AD lies wholly in the plane of the triangle; so does BE . But, if both AD and BE are to lie wholly in the one plane, AD , BE must intersect, say at F : if they did not, there would be two planes in question, not one. But the fact that the lines intersect and that, say, AD does not pass above or below BE , is by no means self-evident.



Mr Frankland points out the similar difficulty as regards the simpler definition of a plane as the surface generated by a straight line passing always through a fixed point and always intersecting a fixed straight line. Let OPP' , OQQ' drawn from O intersect the straight line X at P , Q respectively. Let R be any third point on X : then it needs to be proved that OR intersects $P'Q'$ in some point, say R' . Without some postulate, however, it is not easy to see how to prove this, or even to prove that $P'Q'$ intersects X .



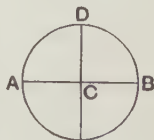
Crelle's essay. Definitions by Fourier, Deahna, Becker.

Crelle takes as the standard of a good definition that it shall be, not only as simple as possible, but also the best adapted for deducing, with the aid of the simplest possible principles, further properties belonging to the thing defined. He was much attracted by a very lucid definition, due, he says, to Fourier, according to which *a plane is formed by the aggregate of all the straight lines which, passing through one point on a straight line in space, are perpendicular to that straight line*. (This is really no more than an adaptation from Euclid's proposition XI. 5, to the effect that, if one of four concurrent straight lines be at right angles to each of the other three, those three are in one plane, which proposition is also used in Aristotle, *Meteorologica* III. 3, 373 a 13.) But Crelle confesses that he had not been able to deduce the necessary properties from this and had had to substitute the definition, already mentioned, of a plane as *the surface containing, throughout their whole length, all the straight lines passing through a fixed point and also intersecting a straight line in space*; and he only claims to have proved, after a long series of propositions, that the "Fourier"- or "perpendicular"-surface and the *plane* of the other definition just given are identical, after which the properties of the "Fourier"-surface can be used along with those of the plane. The advantage of the Fourier definition is that it leads easily, by means of the two propositions that

triangles are equal in all respects (1) when two sides and the included angle are respectively equal and (2) when all three sides are respectively equal, to the property expressed in Simson's definition. But Crelle uses to establish these two congruence-theorems a number of propositions about *equal angles*, *supplementary angles*, *right angles*, *greater* and *less angles*; and it is difficult to question the soundness of Schotten's criticism that these notions in themselves really presuppose that of a plane. The difficulty due to Fourier's use of the word "perpendicular," if that were all, could no doubt be got over. Thus Deahna in a dissertation (Marburg, 1837) constructed a plane as follows. Presupposing the notions of a straight line and a sphere, he observes that, if a sphere revolve about a diameter, all the points of its surface which move describe closed curves (circles). Each of these circles, during the revolution, moves along itself, and one of them divides the surface of the sphere into two congruent parts. The aggregate then of the lines joining the centre to the points of this circle forms the *plane*. Again, J. K. Becker (*Die Elemente der Geometrie*, 1877) pointed out that the revolution of a right angle about one side of it produces a conical surface which differs from all other conical surfaces generated by the revolution of other angles in the fact that *the particular cone coincides with the cone vertically opposite to it*: this characteristic might therefore be taken in order to get rid of the use of the *right angle*.

W. Bolyai and Lobachewsky.

Very similar to Deahna's equivalent for Fourier's definition is the device of W. Bolyai and Lobachewsky (described by Frischauf, *Elemente der absoluten Geometrie*, 1876). They worked upon a fundamental idea first suggested, apparently, by Leibniz. Briefly stated, their way of evolving a *plane* and a *straight line* was as follows. Conceive an infinite number of pairs of concentric spheres described about two fixed points in space, O , O' , as centres, and with equal radii, gradually increasing: these pairs of equal spherical surfaces intersect respectively in homogeneous curves (circles), and the "Inbegriff" or aggregate of these curves of intersection forms a *plane*. If A be a point on one of these circles (k say), suppose points M , M' to start simultaneously from A and to move in opposite directions *at the same speed* till they meet at B , say: B then is "opposite" to A , and A , B divide the circumference into two equal halves. If the points A , B be held fast and the whole system be turned about them until O takes the place of O' , and O' of O , the circle k will occupy the same position as before (though turned a different way). Two opposite points, P , Q say, of each of the other circles will remain stationary during the motion as well as A , B : the "Inbegriff" or aggregate of all such points which remain stationary forms a *straight line*. It is next observed that the *plane* as defined can be generated by the revolution of the straight line about OO' , and this suggests the following construction for a plane. Let a circle as one of the curves of intersection of the pairs of spherical surfaces be divided as before into two equal halves at A , B . Let the arc ADB be similarly bisected at D , and let C be the middle point of AB . This determines a straight line CD which is then *defined* as "perpendicular" to AB . The revolution of CD about AB generates a *plane*. The property stated in Simson's definition is then proved by means of the congruence-theorems proved in Eucl. I. 8 and I. 4. The first is taken as proved, practically by considerations of symmetry and homogeneity. If two spherical surfaces, not necessarily equal, with centres O , O' intersect, A and its "opposite" point B are taken as



before on the curve of intersection (a circle) and, relatively to OO' , the point A is taken to be convertible with B or any other point on the homogeneous curve. The second (that of Eucl. 1. 4) is established by simple application. Rausenberger objects to these proofs on the grounds that the first *assumes* that the two spherical surfaces intersect in one single curve, not in several, and that the second compares *angles*: a comparison which, he says, is possible only in a *plane*, so that a plane is really presupposed. Perhaps as regards the particular comparison of angles Rausenberger is hypercritical; but it is difficult to regard the supposed proof of the theorem of Eucl. 1. 8 as sufficiently rigorous (quite apart from the use of the uniform *motion* of points for the purpose of bisecting lines).

Simson's property is proved from the two congruence-theorems thus. Suppose that AB is "perpendicular" (as defined by Bolyai) to two generators CM , CN of a plane, or suppose CM , CN respectively to make with AB two angles congruent with one another. It is enough to prove that, if P be any point on the straight line MN , then CP , just as much as CM , CN respectively, makes with AB two angles congruent with one another and is therefore a generator. We prove successively the congruence of the following pairs of triangles:

ACM , BCM

ACN , BCN

AMN , BMN

AMP , BMP

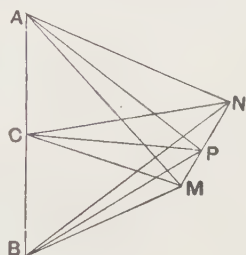
ACP , BCP ,

whence the angles ACP , BCP are congruent.

Other views.

Enriques and Amaldi (*Elementi di geometria*, Bologna, 1905), Veronese (in his *Elementi*) and Hilbert all assume as a *postulate* the property stated in Simson's definition. But G. Ingrams (*Elementi di geometria*, Bologna, 1904) proves it in the course of a remarkable series of closely argued propositions based upon a much less comprehensive postulate. He evolves the theory of the plane from that of a triangle, beginning with a triangle as a mere *three-side* (trilatero), i.e. a *frame*, as it were. His postulate relates to the *three-side* and is to the effect that each "(rectilinear) segment" joining a vertex to a point of the opposite side meets every segment similarly joining each of the other two vertices to the points of the sides opposite to them respectively, and, conversely, if a point be taken on a segment joining a vertex to a point of the opposite side, and if a straight line be drawn from another vertex to the point on the segment so taken, it will if produced meet the opposite side. A *triangle* is then defined as the figure formed by the aggregate of all the segments joining the respective vertices of a *three-side* to points on the opposite sides. After a series of propositions, Ingrams evolves a plane as the figure formed by the "half straight-lines" which project from an internal point of the triangle the points of the perimeter, and then, after two more theorems, proves that a plane is determined by any three of its points which are not in a straight line, and that a straight line which has two points in a plane has all its points in it.

The argument by which Bolyai and Lobachewsky evolved the plane is of course equivalent to the definition of a plane as the locus of all points equidistant from two fixed points in space.



Leibniz in a letter to Giordano defined a plane as *that surface which divides space into two congruent parts*. Adverting to Giordano's criticism that you could conceive of surfaces and lines which divided space or a plane into two congruent parts without being *plane* or *straight* respectively, Beez (*Über Euklidische und Nicht-Euklidische Geometrie*, 1888) pointed out that what was wanted to complete the definition was the further condition that the two congruent spaces could be *slid along each other* without the surfaces ceasing to coincide, and claimed priority for his completion of the definition in this way. But the idea of *all the parts* of a plane fitting exactly on *all other parts* is ancient, appearing, as we have seen, in Heron, Def. 9.

DEFINITIONS 8, 9.

8. Ἐπίπεδος δὲ γωνία ἐστὶν ἡ ἐν ἐπιπέδῳ δύο γραμμῶν ἀπτομένων ἀλλήλων καὶ μὴ ἐπ' εὐθείας κειμένων πρὸς ἀλλήλας τῶν γραμμῶν κλίσις.

9. Ὄταν δὲ αἱ περιέχουσai τὴν γωνίαν γραμμαὶ εὐθεῖαι ᾖσιν, εὐθύγραμμος καλεῖται ἡ γωνία.

8. A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

9. And when the lines containing the angle are straight, the angle is called rectilinear.

The phrase "not in a *straight line*" is strange, seeing that the definition purports to apply to angles formed by *curves* as well as straight lines. We should rather have expected *continuous* (συνεχής) with one another; and Heron takes this to be the meaning, since he at once adds an explanation as to what is meant by lines not being *continuous* (οὐ συνεχής). It looks as though Euclid really intended to define a *rectilinear* angle, but on second thoughts, as a concession to the then common recognition of curvilinear angles, altered "straight lines" into "lines" and separated the definition into two.

I think all our evidence suggests that Euclid's definition of an angle as *inclination* (κλίσις) was a new departure. The word does not occur in Aristotle; and we should gather from him that the idea generally associated with an angle in his time was rather *deflection* or *breaking* of lines (κλάσις): cf. his common use of κεκλάσθαι and other parts of the verb κλᾶν, and also his reference to *one bent line* forming an angle (τὴν κεκαμμένην καὶ ἔχουσαν γωνίαν, *Metaph.* 1016 a 13).

Proclus has a long and elaborate note on this definition, much of which (pp. 121, 12—126, 6) is apparently taken direct from a work by his master Syrianus (ὁ ἡμέτερος καθηγημῶν). Two criticisms contained in the note need occasion no difficulty. One of these asks how, if an angle be an inclination, one inclination can produce two angles. The other (p. 128, 2) is to the effect that the definition seems to exclude an angle formed by one and the same curve with itself, e.g. the complete *cissoïd* [at what we call the "cusp"] or the curve known as the *hippopede* (horse-fetter) [shaped like a lemniscate]. But such an "angle" as this belongs to higher geometry, which Euclid may well be excused for leaving out of account in any case.

Other ancient definitions: Apollonius, Plutarch, Carpus.

Proclus' note records other definitions of great interest. Apollonius defined an angle as *a contracting of a surface or a solid at one point under a broken line or surface* (συναγωγή ἐπιφανείας ἢ στερεοῦ πρὸς ἐνὶ σημείῳ ὑπὸ κεκλασμένη γραμμῇ ἢ ἐπιφανείᾳ), where again an angle is supposed to be formed by *one* broken line or surface. Still more interesting, perhaps, is the definition by "those who say that *the first distance under the point* (τὸ πρῶτον

διάστημα ὑπὸ τὸ σημεῖον) is the angle. Among these is Plutarch, who insists that Apollonius meant the same thing; for, he says, there must be *some* first distance under the breaking (or deflection) of the including lines or surfaces, though, the distance under the point being continuous, it is impossible to obtain the actual *first*, since every distance is divisible without limit" (ἐπ' ἄπειρον). There is some vagueness in the use of the word "distance" (διάστημα); thus it was objected that "if we anyhow separate off the *first*" (distance being apparently the word understood) "and draw a straight line *through it*, we get a *triangle* and not one angle." In spite of the objection, I cannot but see in the idea of Plutarch and the others the germ of a valuable conception in infinitesimals, an attempt (though partial and imperfect) to get at the *rate of divergence* between the lines at their point of meeting as a measure of the angle between them.

A third view of an angle was that of Carpus of Antioch, who said "that the angle was a *quantity* (ποσόν), namely a *distance* (διάστημα) between the lines or surfaces containing it. This means that it would be a distance (or divergence) in one *sense* (ἐφ' ἓν διεστώς), although the angle is not on that account a straight line. For it is not everything *extended in one sense* (τὸ ἐφ' ἓν διασταρόν) that is a line." This very phrase "extended one way" being held to define a *line*, it is natural that Carpus' idea should have been described as the greatest possible paradox (πάντων παραδοξότατον). The difficulty seems to have been caused by the want of a different technical term to express a new idea; for Carpus seems undoubtedly to have been anticipating the more modern idea of an angle as representing *divergence* rather than distance, and to have meant by ἐφ' ἓν in one sense (rotationally) as distinct from one way or in one dimension (linearly).

To what category does an angle belong?

There was much debate among philosophers as to the particular category (according to the Aristotelian scheme) in which an angle should be placed; is it, namely, a *quantum* (ποσόν), *quale* (ποιόν) or *relation* (πρός τι)?

1. Those who put it in the category of *quantity* argued from the fact that a plane angle is divided by a line and a solid angle by a surface. Since, then, it is a surface which is divided by a line, and a solid which is divided by a surface, they felt obliged to conclude that an angle is a surface or a solid, and therefore a magnitude. But homogeneous finite magnitudes, e.g. plane angles, must bear a ratio to one another, or one must be capable of being multiplied until it exceeds the other. This is, however, not the case with a rectilinear angle and the *horn-like* angle (κερατοειδής), by which latter is meant the "angle" between a circle and a tangent to it, since (Eucl. III. 16) the latter "angle" is less than any rectilinear angle whatever. The objection, it will be observed, assumes that the two sorts of angles are homogeneous. Plutarch and Carpus are classed among those who, in one way or other, placed an angle among *magnitudes*; and, as above noted, Plutarch claimed Apollonius as a supporter of his view, although the word *contraction* (of a surface or solid) used by the latter does not in itself suggest magnitude much more than Euclid's *inclination*. It was this last consideration which doubtless led "Aganis," the "friend" (socius) apparently of Simplicius, to substitute for Apollonius' wording "a quantity which has dimensions and the extremities of which arrive at one point" (an-Nairizi, p. 13).

2. Eudemus the Peripatetic, who wrote a whole work on the angle, maintained that it belonged to the category of *quality*. Aristotle had given as his fourth variety of *quality* "figure and the shape subsisting in each thing, and,

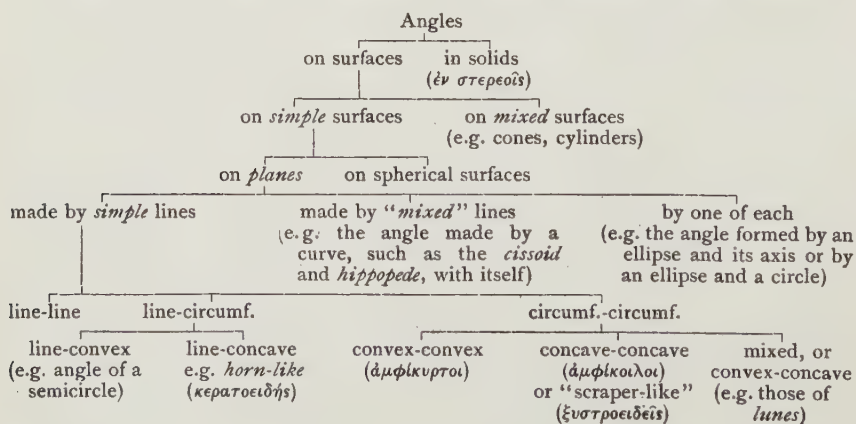
besides these, straightness, curvature, and the like" (*Categories* 8, 10 a 11). He says that each individual thing is spoken of as *quale* in respect of its form, and he instances a triangle and a square, using them again later on (*ibid.* 11 a 5) to show that it is not all qualities which are susceptible of *more* and *less*; again, in *Physics* 1. 5, 188 a 25 *angle, straight, circular* are called kinds of *figure*. Aristotle would no doubt have regarded *deflection* (κεκλᾶσθαι) as belonging to the same category with straightness and curvature (καμπυλότης). At all events, Eudemus took up an angle as having its origin in the *breaking* or *deflection* (κλάσις) of lines: deflection, he argued, was quality if straightness was, and that which has its origin in quality is itself quality. Objectors to this view argued thus. If an angle be a quality (ποιότης) like heat or cold, how can it be bisected, say? It can in fact be divided; and, if things of which divisibility is an essential attribute are varieties of *quantum* and not qualities, an angle cannot be a quality. Further, the *more* and the *less* are the appropriate attributes of quality, not the equal and the unequal; if therefore an angle were a quality, we should have to say of angles, not that one is greater and another smaller, but that one is more an angle and another less an angle, and that two angles are not unequal but *dissimilar* (ἀνόμοιοι). As a matter of fact, we are told by Simplicius, 538, 21, on Arist. *De caelo* that those who brought the angle under the category of *quale* did call equal angles *similar* angles; and Aristotle himself speaks of *similar* angles in this sense in *De caelo* 296 b 20, 311 b 34.

3. Euclid and all who called an angle an inclination are held by Syrianus to have classed it as a *relation* (πρός τι). Yet Euclid certainly regarded angles as magnitudes; this is clear both from the earliest propositions dealing specifically with angles, e.g. I. 9, 13, and also (though in another way) from his describing an angle in the very next definition and always as *contained* (περιεχομένη) by the two lines forming it (Simon, *Euclid*, p. 28).

Proclus (i.e. in this case Syrianus) adds that the truth lies between these three views. The angle partakes in fact of all those categories: it needs the *quantity* involved in magnitude, thereby becoming susceptible of equality, inequality and the like; it needs the *quality* given it by its *form*, and lastly the *relation* subsisting between the lines or planes bounding it.

Ancient classification of "angles."

An elaborate classification of angles given by Proclus (pp. 126, 7—127, 16) may safely be attributed to Geminus. In order to show it by a diagram it



will be necessary to make a convention about terms. Angles are to be understood under each class, "line-circumference" means an angle contained by a straight line and an arc of a circle, "line-convex" an angle contained by a straight line and a circular arc with convexity *outwards*, and so on in every case.

Definitions of angle classified.

As for the point, straight line, and plane, so for the *angle*, Schotten gives a valuable summary, classification and criticism of the different modern views up to date (*Inhalt und Methode des planimetrischen Unterrichts*, II., 1893, pp. 94—183); and for later developments represented by Veronese reference may be made to the third article (by Amaldi) in *Questioni riguardanti le matematiche elementari*, I. (Bologna, 1912).

With one or two exceptions, says Schotten, the definitions of an angle may be classed in three groups representing generally the following views:

1. *The angle is the difference of direction between two straight lines.* (With this group may be compared Euclid's definition of an angle as an inclination.)
2. *The angle is the quantity or amount (or the measure) of the rotation necessary to bring one of its sides from its own position to that of the other side without its moving out of the plane containing both.*
3. *The angle is the portion of a plane included between two straight lines in the plane which meet in a point (or two rays issuing from the point).*

It is remarkable however that nearly all of the text-books which give definitions different from those in group 2 add to them something pointing to a connexion between an angle and rotation: a striking indication that the essential nature of an angle is closely connected with rotation, and that a good definition must take account of that connexion.

The definitions in the first group must be admitted to be tautologous, or *circular*, inasmuch as they really presuppose some conception of an angle. *Direction* (as between two given points) may no doubt be regarded as a primary notion; and it may be defined as "the immediate relation of two points which the *ray* enables us to realise" (Schotten). But "a *direction* is no intensive magnitude, and therefore two directions cannot have any quantitative difference" (Bürklen). Nor is direction susceptible of differences such as those between qualities, e.g. colours. Direction is a *singular* entity: there cannot be different *sorts* or *degrees* of direction. If we speak of "a *different* direction," we use the word equivocally; what we mean is simply "another" direction. The fact is that these definitions of an angle as a difference of direction unconsciously appeal to something outside the notion of direction altogether, to some conception equivalent to that of the angle itself.

Recent Italian views.

The second group of definitions are (says Amaldi) based on the idea of the rotation of a straight line or ray in a plane about a point: an idea which, logically formulated, may lead to a convenient method of introducing the angle. But it must be made independent of *metric* conceptions, or of the conception of *congruence*, so as to bring out *first* the notion of an angle, and *afterwards* the notion of *equal* angles.

The third group of definitions satisfy the condition of not including metric conceptions; but they do not entirely correspond to our intuitive conception of an angle, to which we attribute the character of an entity in *one* dimension (as Veronese says) with respect to the *ray* as element, or an entity in *two*

dimensions with reference to *points* as elements, which may be called an *angular sector*. The defect is however easily remedied by considering the angle as "the aggregate of the rays issuing from the vertex and comprised in the angular sector."

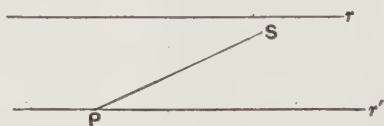
Proceeding to consider the principal methods of arriving at the logical formulation of the first superficial properties of the *plane* from which a definition of the angle may emerge, Amaldi distinguishes two points of view (1) the *genetic*, (2) the *actual*.

(1) From the first point of view we consider the *cluster of straight lines* or *rays* (the aggregate of all the straight lines in a plane passing through a point, or of all the rays with their extremities in that point) as generated by the movement of a straight line or ray in the plane, about a point. This leads to the *postulation of a closed order, or circular disposition*, of the straight lines or rays in a cluster. Next comes the connexion subsisting between the disposition of any two clusters whatever in one, plane, and so on.

(2) Starting from the point of view of the *actual*, we lay the foundation of the definition of an angle in the *division of the plane into two parts* (half-planes) *by the straight line*. Next, two straight lines (*a, b*) in the plane, intersecting at a point *O*, divide the plane into four *regions* which are called *angular sectors* (convex); and finally the angle (*ab*) or (*ba*) may be defined as the aggregate of the rays issuing from *O* and belonging to the angular sector which has *a* and *b* for sides.

Veronese's procedure (in his *Elementi*) is as follows. He begins with the first properties of the plane introduced by the following definition.

The figure given by all the straight lines joining the points of a straight line *r* to a point *P* outside it and by the parallel to *r* through *P* is called a *cluster of straight lines*, a *cluster of rays*, or a *plane*, according as we consider the *element* of the figure itself to be the *straight line*, the *ray* terminated at *P*, or a *point*.



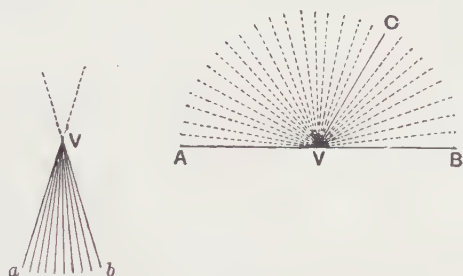
[It will be observed that this method of producing a plane involves using the *parallel* to *r*. This presents no difficulty to Veronese because he has previously defined parallels, without reference to the plane, by means of *reflex* or *opposite* figures, with respect to a point *O*: "two straight lines are called *parallel*, if one of them contains two points opposite to (or the reflex of) two points of the other with respect to the middle point of a common transversal (of the two lines)." He proves by means of a postulate that the parallel *r'* does belong to the plane *Pr*. Ingrams avoids the use of the parallel by defining a *plane* as "the figure formed by the half straight lines which project from an internal point of a triangle (i.e. a point on a line joining any vertex of a *three-side* to a point of the opposite side) the points of its perimeter," and then defining a *cluster* of rays as "the aggregate of the half straight lines in a plane starting from a given point of the plane and passing through the points of the perimeter of a triangle containing the point."]

Veronese goes on to the definition of an angle. "We call an angle a *part* of a cluster of rays, bounded by two rays (as the segment is a part of a straight line bounded by two points).

"An angle of the cluster, the bounding rays of which are opposite, is called a flat angle."

Then, after a postulate corresponding to postulates which he lays down for

a *rectilineal segment* and for a *straight line*, Veronese proves that *all flat angles* are equal to one another.



Hence he concludes that "the cluster of rays is a homogeneous linear system in which the element is the *ray* instead of the *point*. The cluster being a homogeneous linear system, all the propositions deduced from [Veronese's] Post. 1 for the straight line apply to it, e.g. that relative to the sum and difference of the segments: it is only necessary to substitute the ray for the point, and the angle for the segment."

DEFINITIONS 10, 11, 12.

10. "Όταν δὲ εὐθεία ἐπ' εὐθείαν σταθεῖσα τὰς ἐφεξῆς γωνίας ἴσας ἀλλήλαις ποιῇ, ὀρθὴ ἐκατέρα τῶν ἰσῶν γωνιῶν ἐστὶ, καὶ ἡ ἐφεστηκνία εὐθεῖα κάθετος καλεῖται, ἐφ' ἣν ἐφέστηκεν.

11. Ἀμβλεία γωνία ἐστὶν ἡ μείζων ὀρθῆς.

12. Ὀξεία δὲ ἡ ἐλάσσων ὀρθῆς.

10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.

11. An obtuse angle is an angle greater than a right angle.

12. An acute angle is an angle less than a right angle.

ἐφεξῆς is the regular term for *adjacent* angles, meaning literally "(next) in order." I do not find the term used in Aristotle of *angles*, but he explains its meaning in such passages as *Physics* vi. 1, 231 b 8: "those things are (next) in order which have nothing of the same kind (συγγενές) between them."

κάθετος, *perpendicular*, means literally *let fall*: the full expression is *perpendicular straight line*, as we see from the enunciation of Eucl. 1. 11, and the notion is that of a straight line let fall upon the surface of the earth, a *plumb-line*. Proclus (p. 283, 9) tells us that in ancient times the perpendicular was called *gnomon-wise* (κατὰ γνώμονα), because the gnomon (an upright stick) was set up at right angles to the horizon.

The three kinds of angles are among the things which according to the Platonic Socrates (*Republic* vi. 510 c) the geometer assumes and argues from, declining to give any account of them because they are obvious. Aristotle discusses the *priority* of the right angle in comparison with the acute (*Metaph.* 1084 b 7): in one way the right angle is prior, i.e. *in being defined* (ὅτι ὥρισται) and by its *notion* (τῷ λόγῳ), in another way the acute is prior, i.e. as being a *part*, and because the right angle is divided into acute angles; the acute angle is prior as *matter*, the right angle in respect of *form*; cf. also *Metaph.* 1035 b 6, "the notion of the right angle is not divided into

that of an acute angle, but the reverse; for, when defining an acute angle, you make use of the right angle." Proclus (p. 133, 15) observes that it is by the *perpendicular* that we measure the heights of figures, and that it is by reference to the right angle that we distinguish the other rectilinear angles, which are otherwise undistinguished the one from the other.

The Aristotelian *Problems* (16, 4, 913 b 36) contain an expression perhaps worth quoting. The question discussed is why things which fall on the ground and rebound make "similar" angles with the surface on both sides of the point of impact; and it is observed that "the right angle is the *limit* (*ὄρος*) of the opposite angles," where however "opposite" seems to mean, not "supplementary" (or acute and obtuse), but the equal angles made with the surface on opposite sides of the perpendicular.

Proclus, after his manner, remarks that the statement that an angle less than a right angle is acute is not true without qualification, for (1) the *horn-like* angle (between the circumference of a circle and a tangent) is less than a right angle, since it is less than an *acute* angle, but is not an acute angle, while (2) the "angle of a semicircle" (between the arc and a diameter) is also less than a right angle, but is not an acute angle.

The *existence* of the right angle is of course proved in I. 11.

DEFINITION 13.

"Ὅρος ἐστίν, ὃ τινός ἐστι πέρας.

A boundary is that which is an extremity of anything.

Aristotle also uses the words *ὄρος* and *πέρας* as synonymous. Cf. *De gen. animal.* II. 6, 745 a 6, 9, where in the expression "limit of magnitude" first one and then the other word is used.

Proclus (p. 136, 8) remarks that the word boundary is appropriate to the origin of geometry, which began from the measurement of areas of ground and involved the marking of boundaries.

DEFINITION 14.

Σχήμά ἐστι τὸ ὑπὸ τινὸς ἢ τινῶν ὁρῶν περιεχόμενον.

A figure is that which is contained by any boundary or boundaries.

Plato in the *Meno* observes that *roundness* (στρογγυλότης) or the *round* is a "figure," and that *the straight* and many other things are so too; he then inquires what there is common to all of them, in virtue of which we apply the term "figure" to them. His answer is (76 A): "with reference to every figure I say that *that in which the solid terminates* (τοῦτο, εἰς ὃ τὸ στερεὸν περαίνει) *is a figure*, or, to put it briefly, *a figure is an extremity of a solid.*" The first observation is similar to Aristotle's in the *Physics* I. 5, 188 a 25, where *angle*, *straight*, and *circular* are mentioned as genera of figure. In the *Categories* 8, 10 a 11, "figure" is placed with straightness and curvedness in the category of quality. Here however "figure" appears to mean *shape* (μορφή) rather than "figure" in our sense. Coming nearer to "figure" in our sense, Aristotle admits that figure is "*a sort of magnitude*" (*De anima* III. 1, 425 a 18), and he distinguishes *plane figures* of two kinds, in language not unlike Euclid's, as *contained* by straight and circular lines respectively: "every plane figure is either rectilinear or formed by circular lines (περιφερόγραμμα), and the rectilinear figure is contained by several lines, the circular by one line" (*De caelo* II. 4, 286 b 13). He is careful to explain that a plane is not a

figure, nor a figure a plane, but that a plane figure constitutes one notion and is a *species* of the *genus* figure (*Anal. post.* 11. 3, 90 b 37). Aristotle does not attempt to define figure in general, in fact he says it would be useless: "From this it is clear that there is one definition of soul in the same way as there is one definition of *figure*; for in the one case there is no figure except the triangle, quadrilateral, and so on, nor is there any soul other than those above mentioned. A definition might be constructed which should apply to all figures but not specially to any particular figure, and similarly with the species of soul referred to. [But such a general definition would serve no purpose.] Hence it is absurd here as elsewhere to seek a general definition which will not be properly a definition of anything in existence and will not be applicable to the particular irreducible species before us, to the neglect of the definition which is so applicable" (*De anima* 11. 3, 414 b 20-28).

Comparing Euclid's definition with the above, we observe that by introducing *boundary* (*ὄρος*) he at once excludes the *straight* which Aristotle classed as figure; he doubtless excluded *angle* also, as we may judge by (1) Heron's statement that "neither one nor two straight lines can complete a figure," (2) the alternative definition of a straight line as "that which cannot with another line of the same species form a figure," (3) Geminus' distinction between the line which *forms a figure* (*σχηματοποιούσα*) and the line which *extends indefinitely* (*ἐπ' ἀπειρον ἐκβαλλομένη*), which latter term includes a hyperbola and a parabola. Instead of calling figure an *extremity* as Plato did in the expression "extremity (or limit) of a solid," Euclid describes a figure as *that which has a boundary or boundaries*. And lastly, in spite of Aristotle's objection, he does attempt a general definition to cover all kinds of figure, solid and plane. It appears certain therefore that Euclid's definition is entirely his own.

Another view of a figure, recalling that of Plato in *Meno* 76 A, is attributed by Proclus (p. 143, 8) to Posidonius. The latter regarded the *figure* as the *confining extremity* or *limit* (*πέρας συγκλείον*), "separating the notion of figure from *quantity* (or magnitude) and making it the cause of *definition*, *limitation*, and *inclusion* (*τοῦ ὀρίσθαι καὶ πεπεράσθαι καὶ τῆς περιοχῆς*)...Posidonius thus seems to have in view only the boundary placed round from outside, Euclid the whole content, so that Euclid will speak of the circle as a figure in respect of its whole plane (surface) and of its inclusion (from) without, whereas Posidonius (makes it a figure) in respect of its circumference...Posidonius wished to explain the notion of figure as itself *limiting* and *confining* magnitude."

Proclus observes that a logical and refining critic might object to Euclid's definition as defining the genus from the species, since that which is enclosed by one boundary and that which is enclosed by several are both species of figure. The best answer to this seems to be supplied by the passage of Aristotle's *De anima* quoted above.

DEFINITIONS 15, 16.

15. Κύκλος ἐστὶ σχῆμα ἐπίπεδον ὑπὸ μιᾶς γραμμῆς περιεχόμενον [ἢ καλεῖται περιφέρεια], πρὸς ἣν ἀφ' ἐνὸς σημείου τῶν ἐντὸς τοῦ σχήματος κειμένων πᾶσαι αἱ προσπίπτουσαι εὐθεῖαι [πρὸς τὴν τοῦ κύκλου περιφέρειαν] ἴσαι ἀλλήλαις εἰσίν.

16. Κέντρον δὲ τοῦ κύκλου τὸ σημείον καλεῖται.

15. A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another;

16. And the point is called the centre of the circle.

The words ἡ καλεῖται περιφέρεια, "which is called the circumference," and πρὸς τὴν τοῦ κύκλου περιφέρειαν, "to the circumference of the circle," are bracketed by Heiberg because, although the mss. have them, they are omitted in other ancient sources, viz. Proclus, Taurus, Sextus Empiricus and Boethius, and Heron also omits the second gloss. The recently discovered papyrus Herculaneensis No. 1061 also quotes the definition without the words in question, confirming Heiberg's rejection of them (see Heiberg in *Hermes* xxxviii., 1903, p. 47). The words were doubtless added in view of the occurrence of the word "circumference" in Deff. 17, 18 immediately following, without any explanation. But no explanation was needed. Though the word περιφέρεια does not occur in Plato, Aristotle uses it several times (1) in the general sense of *contour* without any special mathematical signification, (2) mathematically, with reference to the rainbow and the circumference, as well as an arc, of a circle. Hence Euclid was perfectly justified in employing the word in Deff. 17, 18 and elsewhere, but leaving it undefined as being a word universally understood and not involving in itself any mathematical conception. It may be added that an-Nairizī had not the bracketed words in his text; for he comments on and tries to explain Euclid's omission to define the circumference.

The definition itself contained nothing new in substance. Plato (*Parmenides* 137 E) says: "Round is, I take it, that the extremes of which are every way equally distant from the middle" (στρογγύλον γέ πού ἐστι τοῦτο, οὗ ἂν τὰ ἔσχατα πανταχῇ ἀπὸ τοῦ μέσου ἴσον ἀπέχῃ). In Aristotle we find the following expressions: "the circular (περιφερόγραμμα) plane figure bounded by one line" (*De caelo* II. 4, 286 b 13—16); "the plane equal (i.e. extending equally all ways) from the middle" (ἐπίπεδον τὸ ἐκ τοῦ μέσου ἴσον), meaning a circle (*Rhetoric* III. 6, 1407 b 27); he also contrasts with the circle "any other figure which has not the lines from the middle equal, as for example an egg-shaped figure" (*De caelo* II. 4, 287 a 19). The word "centre" (κέντρον) was also regularly used: cf. Proclus' quotation from the "oracles" (λόγια), "the centre from which all (lines extending) as far as the rim are equal."

The definition as it stands has no *genetic* character. It says nothing as to the existence or non-existence of the thing defined or as to the method of constructing it. It simply explains what is meant by the word "circle," and is a provisional definition which cannot be used until the existence of circles is proved or assumed. Generally, in such a case, existence is proved by actual construction; but here the possibility of constructing the circle as defined, and consequently its existence, are *postulated* (Postulate 3). A *genetic* definition might state that a circle is the figure described when a straight line, always remaining in one plane, moves about one extremity as a fixed point until it returns to its first position (so Heron, Def. 27).

Simplicius indeed, who points out that the distance between the feet of a pair of compasses is a straight line from the centre to the circumference, will have it that Euclid intended by this definition to show how to construct a circle by the revolution of a straight line about one end as centre; and an-Nairizī points to this as the explanation (1) of Euclid's definition of a circle as a *plane figure*, meaning the whole surface bounded by the circumference, and not the circumference itself, and (2) of his omission to mention the "circumference," since with this construction the circumference is not drawn separately as a *line*. But it is not necessary to suppose that Euclid himself did more than follow the traditional view; for the same conception of the circle as a *plane figure* appears, as we have seen, in Aristotle. While, however,

Euclid is generally careful to say the "*circumference* of a circle" when he means the circumference, or an arc, only, there are cases where "circle" means "circumference of a circle," e.g. in III. 10: "A circle does not cut a circle in more points than two."

Heron, Proclus and Simplicius are all careful to point out that the centre is not the only point which is equidistant from all points of the circumference. The centre is the only point *in the plane of the circle* ("lying within the figure," as Euclid says) of which this is true; any point not in the same plane which is equidistant from all points of the circumference is a *pole*. If you set up a "gnomon" (an upright stick) at the centre of a circle (i.e. a line through the centre perpendicular to the plane of the circle), its upper extremity is a pole (Proclus, p. 153, 3); the perpendicular is the locus of all such poles.

DEFINITION 17.

Διάμετρος δὲ τοῦ κύκλου ἐστὶν εὐθεῖα τις διὰ τοῦ κέντρου ἡγμένη καὶ περατομένη ἐφ' ἑκάτερα τὰ μέρη ὑπὸ τῆς τοῦ κύκλου περιφερείας, ἣτις καὶ δίχα τέμνει τὸν κύκλον.

A diameter of the circle is any straight line drawn through the centre and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.

The last words, literally "which (straight line) also bisects the circle," are omitted by Simson and the editors who followed him. But they are necessary even though they do not "belong to the definition" but only express a property of the diameter as defined. For, without this explanation, Euclid would not have been justified in describing as a *semi-circle* a portion of a circle bounded by a diameter and the circumference cut off by it.

Simplicius observes that the *diameter* is so called because it passes *through* the whole surface of a circle as if *measuring* it, and also because it divides the circle into two equal parts. He might however have added that, in general, it is a line passing through a figure where it is *widest*, as well as dividing it equally: thus in Aristotle τὰ κατὰ διάμετρον κείμενα, "things diametrically situated" in space, are at their maximum distance apart. *Diameter* was the regular word in Euclid and elsewhere for the diameter of a *square*, and also of a parallelogram; *diagonal* (διαγώνιος) was a later term, defined by Heron (Def. 67) as the straight line drawn from an angle to an angle.

Proclus (p. 157, 10) says that Thales was the first to prove that a circle is bisected by its diameter; but we are not told how he proved it. Proclus gives as the *reason* of the property "the undeviating course of the straight line through the centre" (a simple appeal to symmetry), but adds that, if it is desired to prove it mathematically, it is only necessary to imagine the diameter drawn and one part of the circle applied to the other; it is then clear that they must coincide, for, if they did not, and one fell inside or outside the other, the straight lines from the centre to the circumference would not all be equal: which is absurd.

Saccheri's proof is worth quoting. It depends on three "Lemmas" immediately preceding, (1) that two straight lines cannot enclose a space, (2) that two straight lines cannot have one and the same segment common, (3) that, if two straight lines meet at a point, they do not touch, but cut one another, at it.

"Let *MDHNKM* be a circle, *A* its centre, *MN* a diameter. Suppose

the portion $MNKM$ of the circle turned about the fixed points M , N , so that it ultimately comes near to or coincides with the remaining portion $MNHD$.

"Then (i) the whole diameter MAN , with all its points, clearly remains in the same position, since otherwise two straight lines would enclose a space (contrary to the first Lemma).

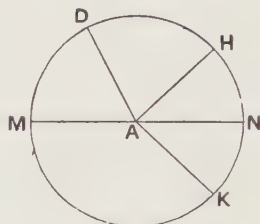
"(ii) Clearly no point K of the circumference NKM falls within or outside the surface enclosed by the diameter MAN and the other part, $NHDM$, of the circumference, since otherwise, contrary to the nature of the circle, a radius as AK would be less or greater than another radius as AH .

"(iii) Any radius MA can clearly be rectilinearly produced only along a single other radius AN , since otherwise (contrary to the second Lemma) two lines assumed straight, e.g. MAN , MAH , would have one and the same common segment.

"(iv) All diameters of the circle obviously cut one another in the centre (Lemma 3 preceding), and they bisect one another there, by the general properties of the circle.

"From all this it is manifest that the diameter MAN divides its circle and the circumference of it just exactly into two equal parts, and the same may be generally asserted for every diameter whatsoever of the same circle; which was to be proved."

Simson observes that the property is easily deduced from III. 31 and 24; for it follows from III. 31 that the two parts of the circle are "similar segments" of a circle (segments containing equal angles, III. Def. 11), and from III. 24 that they are equal to one another.



DEFINITION 18.

Ἡμικύκλιον δέ ἐστι τὸ περιεχόμενον σχῆμα ὑπὸ τε τῆς διαμέτρου καὶ τῆς ἀπολαμβανομένης ὑπ' αὐτῆς περιφερείας. κέντρον δὲ τοῦ ἡμικυκλίου τὸ αὐτό, ὃ καὶ τοῦ κύκλου ἐστίν.

A semicircle is the figure contained by the diameter and the circumference cut off by it. And the centre of the semicircle is the same as that of the circle.

The last words, "And the centre of the semicircle is the same as that of the circle," are added from Proclus to the definition as it appears in the MSS. Scarburgh remarks that a semicircle has no centre, properly speaking, and thinks that the words are not Euclid's, but only a note by Proclus. I am however inclined to think that they are genuine, if only because of the very futility of an observation added by Proclus. He explains, namely, that the semicircle is the only plane figure that has its centre on its perimeter (!), "so that you may conclude that the centre has three positions, since it may be within the figure, as in the case of a circle, or on the perimeter, as with the semicircle, or outside, as with some conic lines (the single-branch hyperbola presumably)!"

Proclus and Simplicius point out that, in the order adopted by Euclid for these definitions of figures, the first figure taken is that bounded by *one* line (the circle), then follows that bounded by *two* lines (the semicircle), then the triangle, bounded by *three* lines, and so on. Proclus, as usual, distinguishes

different kinds of figures bounded by two lines (pp. 159, 14-160, 9). Thus they may be formed

(1) by circumference and circumference, e.g. (*a*) those forming angles, as a *lune* (τὸ μηνοειδές) and the figure included by two arcs with convexities outward, and (*b*) the *angle-less* (ἄγωνιον), as the figure included between two concentric circles (the *corona*);

(2) by circumference and straight line, e.g. the semicircle or segments of circles (ἀψίδες is a name given to those less than a semicircle);

(3) by "mixed" line and "mixed" line, e.g. two ellipses cutting one another;

(4) by "mixed" line and circumference, e.g. intersecting ellipse and circle;

(5) by "mixed" line and straight line, e.g. half an ellipse.

Following Def. 18 in the MSS. is a definition of a *segment of a circle* which was obviously interpolated from III. Def. 6. Proclus, Martianus Capella and Boethius do not give it in this place, and it is therefore properly omitted.

DEFINITIONS 19, 20, 21.

19. Σχήματα εὐθύγραμμά ἐστι τὰ ὑπὸ εὐθειῶν περιεχόμενα, τρίπλευρα μὲν τὰ ὑπὸ τριῶν, τετράπλευρα δὲ τὰ ὑπὸ τεσσάρων, πολὺπλευρα δὲ τὰ ὑπὸ πλείονων ἢ τεσσάρων εὐθειῶν περιεχόμενα.

20. Τῶν δὲ τριπλευρῶν σχημάτων ἰσόπλευρον μὲν τρίγωνόν ἐστι τὸ τὰς τρεῖς ἴσας ἔχον πλευράς, ἰσοσκελὲς δὲ τὸ τὰς δύο μόνας ἴσας ἔχον πλευράς, σκαληνὸν δὲ τὸ τὰς τρεῖς ἀνίσους ἔχον πλευράς.

21. Ἐτι δὲ τῶν τριπλευρῶν σχημάτων ὀρθογώνιον μὲν τρίγωνόν ἐστι τὸ ἔχον ὀρθὴν γωνίαν, ἀμβλυγώνιον δὲ τὸ ἔχον ἀμβλείαν γωνίαν, ὀξυγώνιον δὲ τὸ τὰς τρεῖς ὀξείας ἔχον γωνίας.

19. Rectilinear figures are those which are contained by straight lines, trilateral figures being those contained by three, quadrilateral those contained by four, and multilateral those contained by more than four straight lines.

20. Of trilateral figures, an equilateral triangle is that which has its three sides equal, an isosceles triangle that which has two of its sides alone equal, and a scalene triangle that which has its three sides unequal.

21. Further, of trilateral figures, a right-angled triangle is that which has a right angle, an obtuse-angled triangle that which has an obtuse angle, and an acute-angled triangle that which has its three angles acute.

19.

The latter part of this definition, distinguishing *three-sided*, *four-sided* and *many-sided* figures, is probably due to Euclid himself, since the words τρίπλευρον, τετράπλευρον and πολὺπλευρον do not appear in Plato or Aristotle (only in one passage of the *Mechanics* and of the *Problems* respectively does even τετράπλευρον, quadrilateral, occur). By his use of τετράπλευρον, quadrilateral, Euclid seems practically to have put an end to any ambiguity in the use by mathematicians of the word τετράγωνον, literally "four-angled (figure)," and to have got it restricted to the *square*. Cf. note on Def. 22.

20.

Isosceles (ἰσοσκελής, with equal legs) is used by Plato as well as Aristotle. *Scalene* (σκαληνός, with the variant σκαληνής) is used by Aristotle of a triangle with no two sides equal: cf. also Tim. Loc. 98 B. Plato, *Euthyphro* 12 D.

applies the term "scalene" to an *odd* number in contrast to "isosceles" used of an even number. Proclus (p. 168, 24) seems to connect it with *σκάζω*, to *limp*; others make it akin to *σκολιός*, *crooked*, *aslant*. Apollonius uses the same word "scalene" of an *oblique* circular cone.

Triangles are classified, first with reference to their sides, and then with reference to their angles. Proclus points out that seven distinct species of triangles emerge: (1) the *equilateral* triangle, (2) three species of *isosceles* triangles, the right-angled, the obtuse-angled and the acute-angled, (3) the same three varieties of *scalene* triangles.

Proclus gives an odd reason for the dual classification according to sides and angles, namely that Euclid was mindful of the fact that it is not every *triangle* that is *trilateral* also. He explains this statement by reference (p. 165, 22) to a figure which some called *barb-like* (*ἀκιδοειδής*) while Zenodorus called it *hollow-angled* (*κοιλογώνιος*). Proclus mentions it again in his note on I. 22 (p. 328, 21 sqq.) as one of the paradoxes of geometry, observing that it is seen in the figure of that proposition. This "triangle" is merely a *quadrilateral* with a re-entrant angle; and the idea that it has only three angles is due to the non-recognition of the fourth angle (which is greater than two right angles) as being an angle at all. Since Proclus speaks of the *four-sided triangle* as "one of the paradoxes in geometry," it is perhaps not safe to assume that the misconception underlying the expression existed in the mind of Proclus alone; but there does not seem to be any evidence that Zenodorus called the figure in question a triangle (cf. Pappus, ed. Hultsch, pp. 1154, 1206).



DEFINITION 22.

Τῶν δὲ τετραπλεύρων σχημάτων τετράγωνον μὲν ἐστίν, ὃ ἰσοπλευρόν τε ἐστὶ καὶ ὀρθογώνιον, ἐτερόμηκες δέ, ὃ ὀρθογώνιον μὲν, οὐκ ἰσοπλευρον δέ, ῥόμβος δέ, ὃ ἰσοπλευρον μὲν, οὐκ ὀρθογώνιον δέ, ῥομβοειδὲς δὲ τὸ τὰς ἀπεναντίον πλευρὰς τε καὶ γωνίας ἰσὰς ἀλλήλαις ἔχον, ὃ οὔτε ἰσοπλευρόν ἐστίν οὔτε ὀρθογώνιον· τὰ δὲ παρὰ ταῦτα τετράπλευρα τραπέζια καλεῖσθω.

Of *quadrilateral figures*, a square is that which is both *equilateral* and *right-angled*; an oblong that which is *right-angled* but not *equilateral*; a rhombus that which is *equilateral* but not *right-angled*; and a rhomboid that which has its opposite sides and angles equal to one another but is neither *equilateral* nor *right-angled*. And let quadrilaterals other than these be called *trapezia*.

τετράγωνον was already a *square* with the Pythagoreans (cf. Aristotle, *Metaph.* 986 a 26), and it is so most commonly in Aristotle; but in *De anima* II. 3, 414 b 31 it seems to be a quadrilateral, and in *Metaph.* 1054 b 2, "equal and equiangular *τετράγωνα*," it cannot be anything else but quadrilateral if "equiangular" is to have any sense. Though, by introducing *τετράπλευρον* for any quadrilateral, Euclid enabled ambiguity to be avoided, there seem to be traces of the older vague use of *τετράγωνον* in much later writers. Thus Heron (Def. 100) speaks of a cube as "contained by six equilateral and *equiangular* *τετράγωνα*" and Proclus (p. 166, 10) adds to his remark about the "four-sided triangle" that "you might have *τετράγωνα* with more than the four sides," where *τετράγωνα* can hardly mean squares.

ἐτερόμηκες, *oblong* (with sides of *different length*), is also a Pythagorean term.

The word *right-angled* (*ὀρθογώνιον*) as here applied to quadrilaterals must mean *rectangular* (i.e., practically, having all its angles right angles); for, although it is tempting to take the word in the same sense for a

square as for a triangle (i.e. "having *one* right angle"), this will not do in the case of the oblong, which, unless it were stated that *three* of its angles are right angles, would not be sufficiently defined.

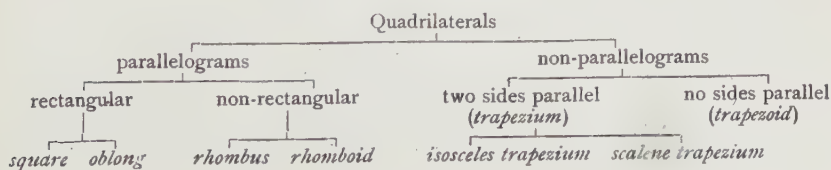
If it be objected, as it was by Todhunter for example, that the definition of a square assumes more than is necessary, since it is sufficient that, being equilateral, it should have one right angle, the answer is that, as in other cases, the superfluity does not matter from Euclid's point of view; on the contrary, the more of the essential attributes of a thing that could be included in its definition the better, provided that the existence of the thing defined and its possession of all those attributes is proved before the definition is, actually used; and Euclid does this in the case of the square by construction in I. 46, making no use of the definition before that proposition.

The word *rhombus* (ῥόμβος) is apparently derived from ῥέμβω, to *turn round and round*, and meant among other things a *spinning-top*. Archimedes uses the term *solid rhombus* to denote a solid figure made up of two right cones with a common circular base and vertices turned in opposite directions. We can of course easily imagine this solid generated by *spinning*; and, if the cones were equal, the section through the common axis would be a *plane rhombus*, which would also be the *apparent* form of the spinning solid to the eye. The difficulty in the way of supposing the plane figure to have been named after the solid figure is that in Archimedes the cones forming the solid are not necessarily equal. It is however possible that the solid to which the name was originally given was made up of two equal cones, that the plane rhombus then received its name from that solid, and that Archimedes, in taking up the old name again, extended its signification (cf. J. H. T. Müller, *Beiträge zur Terminologie der griechischen Mathematiker*, 1860, p. 20). Proclus, while he speaks of a rhombus as being like a shaken, i.e. deformed, square, and of a rhomboid as an oblong that has been moved, tries to explain the rhombus by reference to the appearance of a *spinning* square (τετραγωνον ῥομβοῦμενον).

It is true that the definition of a rhomboid says more than is necessary in describing it as having its opposite sides *and* angles equal to one another. The answer to the objection is the same as the answer to the similar objection to the definition of a square.

Euclid makes no use in the *Elements* of the *oblong*, the *rhombus* and the *rhomboid*. The explanation of his inclusion of definitions of these figures is no doubt that they were taken from earlier text-books. From the words "*let* quadrilaterals other than these *be called* trapezia" we may perhaps infer that *trapezium* was a new name or a new application of an old name.

As Euclid has not yet defined parallel lines and does not anywhere define a *parallelogram*, he is not in a position to make the more elaborate classification of quadrilaterals attributed by Proclus to Posidonius and appearing also in Heron's Definitions. It may be shown by the following diagram, distinguishing seven species of quadrilaterals.



It will be observed that, while Euclid in the above definition classes as *trapezia* all quadrilaterals other than squares, oblongs, rhombi, and rhomboids, the word is in this classification restricted to quadrilaterals having two sides (only) parallel, and *trapezoid* is used to denote the rest. Euclid appears to have used *trapezium* in the restricted sense of a quadrilateral with two sides parallel in his book *περὶ διαμέσεων* (on divisions of figures). Archimedes uses it in the same sense, but in one place describes it more precisely as a trapezium with its two sides parallel.

DEFINITION 23.

Παράλληλοι εἰσιν εὐθεῖαι, αἵτινες ἐν τῷ αὐτῷ ἐπιπέδῳ οὔσαι καὶ ἐκβαλλόμεναι εἰς ἄπειρον ἐφ' ἑκάτερα τὰ μέρη ἐπὶ μηδέτερα συμπίπτουσιν ἀλλήλαις.

Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Παράλληλος (alongside one another) written in one word does not appear in Plato; but with Aristotle it was already a familiar term.

εἰς ἄπειρον cannot be translated "to infinity" because these words might seem to suggest a *region* or *place* infinitely distant, whereas εἰς ἄπειρον, which seems to be used indifferently with ἐπ' ἄπειρον, is adverbial, meaning "without limit," i.e. "indefinitely." Thus the expression is used of a magnitude being "indefinitely divisible," or of a series of terms extending without limit.

In both directions, ἐφ' ἑκάτερα τὰ μέρη, literally "towards both the parts" where "parts" must be used in the sense of "regions" (cf Thuc. II. 96).

It is clear that with Aristotle the general notion of parallels was that of straight lines which do not meet, as in Euclid: thus Aristotle discusses the question whether to think that parallels do meet should be called a geometrical or an ungeometrical error (*Anal. post.* I. 12, 77 b 22), and (more interesting still in relation to Euclid) he observes that there is nothing surprising in different hypotheses leading to the same error, as one might conclude that parallels meet by starting from the assumption, either (a) that the interior (angle) is greater than the exterior, or (b) that the angles of a triangle make up more than two right angles (*Anal. prior.* II. 17, 66 a 11).

Another definition is attributed by Proclus to Posidonius, who said that "parallel lines are those which, (being) in one plane, neither converge nor diverge, but have all the perpendiculars equal which are drawn from the points of one line to the other, while such (straight lines) as make the perpendiculars less and less continually do converge to one another; for the perpendicular is enough to define (ὁρίζειν δύναται) the heights of areas and the distances between lines. For this reason, when the perpendiculars are equal, the distances between the straight lines are equal, but when they become greater and less, the interval is lessened, and the straight lines converge to one another in the direction in which the less perpendiculars are" (Proclus, p. 176, 6—17).

Posidonius' definition, with the explanation as to distances between straight lines, their convergence and divergence, amounts to the definition quoted by Simplicius (an-Nairizi, p. 25, ed. Curtze) which described straight lines as parallel if, when they are produced indefinitely both ways, the distance between them, or the perpendicular drawn from either of them to the other, is always equal and not different. To the objection that it should be proved that the distance between two parallel lines is the perpendicular to them Simplicius

replies that the definition will do equally well if all mention of the *perpendicular* be omitted and it be merely stated that the *distance* remains equal, although "for *proving* the matter in question it is necessary to say that one straight line is perpendicular to both" (an-Nairizi, ed. Besthorn-Heiberg, p. 9). He then quotes the definition of "the philosopher Aganis": "*Parallel straight lines are straight lines, situated in the same plane, the distance between which, if they are produced indefinitely in both directions at the same time, is everywhere the same.*" (This definition forms the basis of the attempt of "Aganis" to prove the Postulate of Parallels.) On the definition Simplicius remarks that the words "situated in the same plane" are perhaps unnecessary, since, if the distance between the lines is everywhere the same, and one does not incline at all towards the other, they must for that reason be in the same plane. He adds that the "distance" referred to in the definition is the shortest line which joins things disjoined. Thus, between point and point, the distance is the straight line joining them; between a point and a straight line or between a point and a plane it is the perpendicular drawn from the point to the line or plane; "as regards the distance between two lines, that distance is, if the lines are parallel, one and the same, equal to itself at all places on the lines, it is the *shortest* distance and, at all places on the lines, perpendicular to both" (*ibid.* p. 10).

The same idea occurs in a quotation by Proclus (p. 177, 11) from Geminus. As part of a classification of lines which do not meet he observes: "Of lines which do not meet, some are in one plane with one another, others not. Of those which meet and are in one plane, *some are always the same distance from one another*, others lessen the distance continually, as the hyperbola (approaches) the straight line, and the conchoid the straight line (i.e. the asymptote in each case). For these, while the distance is being continually lessened, are continually (in the position of) not meeting, though they converge to one another; they never converge entirely, and this is the most paradoxical theorem in geometry, since it shows that the convergence of some lines is non-convergent. But of lines which are always an equal distance apart, those which are straight and never make the (distance) between them smaller, and which are in one plane, are parallel."

Thus the *equidistance*-theory of parallels (to which we shall return) is very fully represented in antiquity. I seem also to see traces in Greek writers of a conception equivalent to the vicious *direction*-theory which has been adopted in so many modern text-books. Aristotle has an interesting, though obscure, allusion in *Anal. prior.* II. 16, 65 a 4 to a *petitio principii* committed by "those who think that they draw parallels" (or "establish the theory of parallels," which is a possible translation of τὰς παραλλήλους γράφειν): "for they unconsciously assume such things as it is not possible to demonstrate if parallels do not exist." It is clear from this that there was a vicious circle in the then current theory of parallels; something which depended for its truth on the properties of parallels was assumed in the actual proof of those properties, e.g. that the three angles of a triangle make up two right angles. This is not the case in Euclid, and the passage makes it clear that it was Euclid himself who got rid of the *petitio principii* in earlier text-books by formulating and premising before 1. 29 the famous Postulate 5, which must ever be regarded as among the most epoch-making achievements in the domain of geometry. But one of the commentators on Aristotle, Philoponus, has a note on the above passage purporting to give the specific character of the *petitio principii* alluded to; and it is here that a *direction*-theory of parallels may be hinted at,

whether Philoponus is or is not right in supposing that this was what Aristotle had in mind. Philoponus says: "The same thing is done by those who draw parallels, namely begging the original question; for they will have it that it is possible to draw parallel straight lines from the meridian circle, and they assume a point, so to say, falling on the plane of that circle and thus they draw the straight lines. And what was sought is thereby assumed; for he who does not admit the genesis of the parallels will not admit the point referred to either." What is meant is, I think, somewhat as follows. Given a straight line and a point through which a parallel to it is to be drawn, we are to suppose the given straight line placed in the plane of the meridian. Then we are told to draw through the given point another straight line in the plane of the meridian (strictly speaking it should be drawn in a plane parallel to the plane of the meridian, but the idea is that, compared with the size of the meridian circle, the distance between the point and the straight line is negligible); and this, as I read Philoponus, is supposed to be equivalent to assuming a very distant point in the meridian plane and joining the given point to it. But obviously no ruler would stretch to such a point, and the objector would say that we cannot really direct a straight line to the assumed distant point except by drawing it, without more ado, *parallel* to the given straight line. And herein is the *petitio principii*. I am confirmed in seeing in Philoponus an allusion to a *direction*-theory by a remark of Schotten on a similar reference to the meridian plane supposed to be used by advocates of that theory. Schotten is arguing that direction is not in itself a conception such that you can predicate *one* direction of *two* different lines. "If any one should reply that nevertheless many lines can be conceived *which all have the direction from north to south*," he replies that this represents only a nominal, not a real, identity of direction.

Coming now to modern times, we may classify under three groups practically all the different definitions that have been given of parallels (Schotten, *op. cit.* II. p. 188 sqq.).

(1) *Parallel straight lines have no point common*, under which general conception the following varieties of statement may be included:

- (a) *they do not cut one another,*
- (b) *they meet at infinity, or*
- (c) *they have a common point at infinity.*

(2) *Parallel straight lines have the same, or like, direction or directions*, under which class of definitions must be included all those which introduce transversals and say that the parallels *make equal angles with a transversal*.

(3) *Parallel straight lines have the distance between them constant*; with which group we may connect the attempt to explain a parallel as *the geometrical locus of all points which are equidistant from a straight line*.

But the three points of view have a good deal in common; some of them lead easily to the others. Thus the idea of the lines having no point common led to the notion of their having a common point at infinity, through the influence of modern geometry seeking to embrace different cases under one conception; and then again the idea of the lines having a common point at infinity might suggest their having the same direction. The "non-secant" idea would also naturally lead to that of equidistance (3), since our observation shows that it is things which come nearer to one another that tend to meet, and hence, if lines are not to meet, the obvious thing is to see that they shall not come nearer, i.e. shall remain the same distance apart.

We will now take the three groups in order.

(1) The first observation of Schotten is that the varieties of this group which regard parallels as (a) meeting at infinity or (b) having a common point at infinity (first mentioned apparently by Kepler, 1604, as a "façon de parler" and then used by Desargues, 1639) are at least unsuitable definitions for elementary text-books. How do we know that the lines cut or meet at infinity? We are not entitled to assume either that they do or that they do not, because "infinity" is outside our field of observation and we cannot verify either. As Gauss says (letter to Schumacher), "Finite man cannot claim to be able to regard the infinite as something to be grasped by means of ordinary methods of observation." Steiner, in speaking of the rays passing through a point and successive points of a straight line, observes that as the point of intersection gets further away the ray moves continually in one and the same direction ("nach einer und derselben Richtung hin"); only in one position, that in which it is parallel to the straight line, "there is *no real cutting*" between the ray and the straight line; what we have to say is that the ray is "*directed towards the infinitely distant point on the straight line.*" It is true that higher geometry has to assume that the lines do meet at infinity: whether such lines exist in nature or not does not matter (just as we deal with "straight lines" although there is no such thing as a straight line). But if two lines do not cut at any finite distance, may not the same thing be true at infinity also? Are lines conceivable which would not cut even at infinity but always remain at the same distance from one another even there? Take the case of a line of railway. Must the two rails meet at infinity so that a train could not stand on them there (whether we could *see* it or not makes no difference)? It seems best therefore to leave to higher geometry the conception of infinitely distant points on a line and of two straight lines meeting at infinity, like *imaginary* points of intersection, and, for the purposes of elementary geometry, to rely on the plain distinction between "parallel" and "cutting" which average human intelligence can readily grasp. This is the method adopted by Euclid in his definition, which of course belongs to the group (1) of definitions regarding parallels as non-secant.

It is significant, I think, that such authorities as Ingrami (*Elementi di geometria*, 1904) and Enriques and Amaldi (*Elementi di geometria*, 1905), after all the discussion of principles that has taken place of late years, give definitions of parallels equivalent to Euclid's: "those straight lines in a plane which have not any point in common are called parallels." Hilbert adopts the same point of view. Veronese, it is true, takes a different line. In his great work *Fondamenti di geometria*, 1891, he had taken a ray to be parallel to another when a point at infinity on the second is situated on the first; but he appears to have come to the conclusion that this definition was unsuitable for his *Elementi*. He avoids however giving the Euclidean definition of parallels as "straight lines in a plane which, though produced indefinitely, never meet," because "no one has ever seen two straight lines of this sort," and because the postulate generally used in connexion with this definition is not evident in the way that, in the field of our experience, it is evident that only one straight line can pass through two points. Hence he gives a different definition, for which he claims the advantage that it is independent of the plane. It is based on a definition of figures "opposite to one another with respect to a point" (or *reflex* figures). "Two figures are opposite to one another with respect to a point O , e.g. the figures $ABC \dots$ and $A'B'C' \dots$, if to every point of the one there corresponds one sole point of the other, and if the segments

OA , OB , OC , ... joining the points of one figure to O are respectively equal and opposite to the segments OA' , OB' , OC' , ... joining to O the corresponding points of the second": then, a *transversal* of two straight lines being any segment having as its extremities one point of one line and one point of the other, "*two straight lines are called parallel if one of them contains two points opposite to two points of the other with respect to the middle point of a common transversal.*" It is true, as Veronese says, that the parallels so defined and the parallels of Euclid are in substance the same; but it can hardly be said that the definition gives as good an idea of the essential nature of parallels as does Euclid's. Veronese has to *prove*, of course, that his parallels have no point in common, and his "Postulate of Parallels" can hardly be called more evident than Euclid's: "If two straight lines are parallel, they are figures opposite to one another with respect to the middle points of all their transversal segments."

(2) The *direction*-theory.

The fallacy of this theory has nowhere been more completely exposed than by C. L. Dodgson (*Euclid and his modern Rivals*, 1879). According to Killing (*Einführung in die Grundlagen der Geometrie*, I. p. 5) it would appear to have originated with no less a person than Leibniz. In the text-books which employ this method the notion of *direction* appears to be regarded as a primary, not a derivative notion, since no definition is given. But we ought at least to know how the same direction or like directions can be recognised when two different straight lines are in question. But no answer to this question is forthcoming. The fact is that the whole idea as applied to non-coincident straight lines is derived from knowledge of the properties of *parallels*; it is a case of explaining a thing by itself. The idea of parallels being in the *same* direction perhaps arose from the conception of an angle as a *difference* of direction (the hollowness of which has already been exposed); sameness of direction for parallels follows from the same "difference of direction" which both exhibit relatively to a third line. But this is not enough. As Gauss said (*Werke*, IV. p. 365), "If it [identity of direction] is recognised by the equality of the angles formed with *one* third straight line, we do not yet know without an antecedent proof whether this same equality will also be found in the angles formed with a *fourth* straight line" (and any number of other transversals); and in order to make this theory of parallels valid, so far from getting rid of axioms such as Euclid's, you would have to assume as an axiom what is much less axiomatic, namely that "straight lines which make equal corresponding angles with a certain transversal do so with *any* transversal" (Dodgson, p. 101).

(3) In modern times the conception of parallels as *equidistant* straight lines was practically adopted by Clavius (the editor of Euclid, born at Bamberg, 1537) and (according to Saccheri) by Borelli (*Euclides restitutus*, 1658) although they do not seem to have *defined* parallels in this way. Saccheri points out that, before such a definition can be used, it has to be *proved* that "the geometrical locus of points equidistant from a straight line is a straight line." To do him justice, Clavius saw this and tried to prove it: he makes out that the locus is a straight line according to the definition of Euclid, because "it lies evenly with respect to all the points on it"; but there is a confusion here, because such "evenness" as the locus has is with respect to the straight line from which its points are equidistant, and there is nothing to show that it possesses this property with respect to itself. In fact the theorem cannot be proved without a postulate.

POSTULATE 1.

Ἡτιθέσθω ἀπὸ παντὸς σημείου ἐπὶ πᾶν σημεῖον εὐθεῖαν γραμμὴν ἀγαγεῖν.

✓ *Let the following be postulated: to draw a straight line from any point to any point.*

From any point to any point. In general statements of this kind the Greeks did not say, as we do, "*any* point," "*any* triangle" etc., but "*every* point," "*every* triangle" and the like. Thus the words are here literally "from every point to every point." Similarly the first words of Postulate 3 are "with *every* centre and distance," and the enunciation, e.g., of 1. 18 is "In *every* triangle the greater side subtends the greater angle."

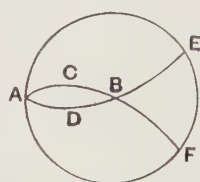
It will be remembered that, according to Aristotle, the geometer must in general assume *what* a thing is, or its definition, but must prove *that* it is, i.e. the *existence* of the thing corresponding to the definition: only in the case of the two most primary things, points and lines, does he assume, without proof, both the definition and the existence of the thing defined. Euclid has indeed no separate assumption affirming the existence of *points* such as we find nowadays in text-books like those of Veronese, Ingrami, Enriques, "there exist distinct points" or "there exist an infinite number of points." But, as regards the only lines dealt with in the *Elements*, straight lines and circles, existence is asserted in Postulates 1 and 3 respectively. Postulate 1 however does much more than (1) postulate the existence of straight lines. It is (2) an answer to a possible objector who should say that you cannot, with the imperfect instruments at your disposal, draw a mathematical straight line at all, and consequently (in the words of Aristotle, *Anal. post.* 1. 10, 76 b 41) that the geometer uses false hypotheses, since he calls a line a foot long when it is not or straight when it is not straight. It would seem (if Gherard's translation is right) that an-Nairizi saw that one purpose of the Postulate was to refute this criticism: "the utility of the first three postulates is (to ensure) that the weakness of our equipment shall not prevent (scientific) demonstration" (ed. Curtze, p. 30). The fact is, as Aristotle says, that the geometer's demonstration is not concerned with the particular imperfect straight line which he has drawn, but with the ideal straight line of which it is the imperfect representation. Simplicius too indicates that the object of the Postulate is rather to enable the drawing of a mathematical straight line to be *imagined* than to assert that it can actually be realised in practice: "he would be a rash person who, taking things as they actually are, should postulate the drawing of a straight line from Aries to Libra."

There is still something more that must be inferred from the Postulate combined with the definition of a straight line, namely (3) that the straight line joining two points is *unique*: in other words that, *if two straight lines* ("rectilineal segments," as Veronese would call them) *have the same extremities, they must coincide throughout their length*. The omission of Euclid to state this in so many words, though he assumes it in 1. 4, is no doubt answerable for the interpolation in the text of the equivalent assumption that *two straight lines cannot enclose a space*, which has constantly appeared in mss. and editions of Euclid, either among Axioms or Postulates. That Postulate 1 included it, by conscious implication, is even clear from Proclus' words in his note on 1. 4 (p. 239, 16): "therefore two straight lines do not enclose a space, and it was with knowledge of this fact that the writer of the *Elements* said in the first of his Postulates, *to draw a straight line from any point to any point*, implying that it is *one* straight line which would always join the two points, not *two*."

Proclus attempts in the same note (p. 239) to *prove* that two straight lines cannot enclose a space, using as his basis the definition of the diameter of a circle and the theorem, stated in it, that any diameter divides the circle into two equal parts.

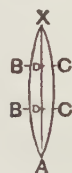
Suppose, he says, ACB , ADB to be two straight lines enclosing a space. Produce them (beyond B) indefinitely. With centre B and distance AB describe a circle, cutting the lines so produced in F , E respectively.

Then, since $ACBF$, $ADBE$ are both diameters cutting off semi-circles, the arcs AE , AF are equal: which is impossible. Therefore etc.



It will be observed, however, that the straight lines produced are assumed to meet the circle given in two *different* points E , F , whereas, for anything we know, E , F might coincide and the straight lines have *three* common points. The proof is therefore delusive.

Saccheri gives a different proof. From Euclid's definition of a straight line as that which lies evenly with its points he infers that, when such a line is turned about its two extremities, which remain fixed, all the points on it must remain throughout in the same position, and cannot take up different positions as the revolution proceeds. "In this view of the straight line the truth of the assertion that two straight lines do not enclose a space is obviously involved. In fact, if two lines are given which enclose a space, and of which the two points A and X are the common extremities, it is easily shown that neither, or else only one, of the two lines is straight."



It is however better to assume as a *postulate* the fact, inseparably connected with the idea of a straight line, that *there exists only one straight line containing two given points*, or, *if two straight lines have two points in common, they coincide throughout*.

POSTULATE 2.

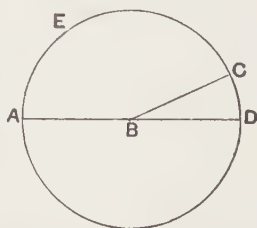
Καὶ πεπερασμένην εὐθείαν κατὰ τὸ συνεχὲς ἐπ' εὐθείας ἐκβαλεῖν.

To produce a finite straight line continuously in a straight line.

I translate *πεπερασμένην* by *finite*, because that is the received equivalent, and because any alternative word such as *limited*, *terminated*, if applied to a straight line, would equally fail to express what modern Italian geometers aptly call a *rectilineal segment*, that is, a straight line having *two* extremities.

Just as Post. 1 asserting the possibility of drawing a straight line from any one point to another must be held to declare at the same time that the straight line so drawn is unique, so Post. 2 maintaining the possibility of producing a finite straight line (a "rectilineal segment") continuously in a straight line must also be held to assert that the straight line can only be produced *in one way* at either end, or that the produced part in either direction is *unique*; in other words, that *two straight lines cannot have a common segment*. This latter assumption is not expressly appealed to by Euclid until XI. 1. But it is needed at the very beginning of Book I. Proclus (p. 214, 18) says that Zeno of Sidon, an Epicurean, maintained that the very first proposition I. 1 requires it to be admitted that "two straight lines cannot have the same segments"; otherwise AC , BC might meet before they arrive at C and have the rest of their length common, in which case the actual triangle formed by them and AB would not be equilateral. The assumption that two straight lines cannot have a common segment is certainly necessary in I. 4, where one side of one triangle is placed on that side of the other

triangle which is equal to it, and it is inferred that the two coincide throughout their length: this would by no means follow if two straight lines could have a common segment. Proclus (p. 215, 24), while observing that Post. 2 clearly indicates that the produced portion must be *one*, attempts to prove it, but unsuccessfully. Both he and Simplicius practically use the same argument. Suppose, says Proclus, that the straight lines AC , AD have AB as a common segment. With centre B and radius BA describe a circle (Post. 3) meeting AC , AD in C , D . Then, since ABC is a straight line through the centre, AEC is a semi-circle. Similarly, ABD being a straight line through the centre, AED is a semi-circle. Therefore AEC is equal to AED : which is impossible.



Proclus observes that Zeno would object to this proof as really depending on the assumption that "two circumferences (of circles) cannot have one portion common"; for this, he would say, is assumed in the common proof by superposition of the fact that a circle is bisected by a diameter, since that proof takes it for granted that, if one part of the circumference cut off by the diameter, when applied to the other, does not coincide with it, it must necessarily fall either *entirely* outside or *entirely* inside it, whereas there is nothing to prevent their coinciding, not altogether, but in part only; and, until you really prove the bisection of a circle by its diameter, the above proof is not valid. Posidonius is represented as having derided Zeno for not seeing that the proof of the bisection of a circle by its diameter goes on just as well if the circumferences fail to coincide *in part* only. But the true objection to the proof above given is that the proof of the bisection of a circle by any diameter *itself* assumes that two straight lines cannot have a common segment; for, if we wish to draw the diameter of a circle which has its extremity at a given point of the circumference we have to join the latter point to the centre (Post. 1) and then to *produce* the straight line so drawn till it meets the circle again (Post. 2), and it is necessary for the proof that the produced part shall be *unique*.

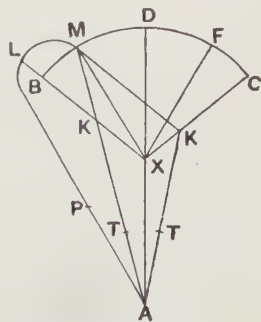
Saccheri adopted the proper order when he gave, first the proposition that two straight lines cannot have a common segment, and after that the proposition that any diameter of a circle bisects the circle and its circumference.

Saccheri's proof of the former is very interesting as showing the thoroughness of his method, if not at the end entirely convincing. It is in five stages which I shall indicate shortly, giving the full argument of the first only.

Suppose, if possible, that AX is a common segment of both the straight lines AXB , AXC , in one plane, produced beyond X . Then describe about X as centre, with radius XB or XC , the arc BMC , and draw through X to any point on it the straight line XM .

(i) I maintain that, with the assumption made, the line AXM is also a straight line which is drawn from the point A to the point X and produced beyond X .

For, if this line were not straight, we could draw another straight line AM which for its part would be straight. This straight line will either (a) cut one of the two straight lines XB , XC in a certain point K or (b) enclose one of them, for instance XB , in the area bounded by AX , XM and $APLM$.



But the first alternative (a) obviously contradicts the foregoing lemma [that two straight lines cannot enclose a space], since in that case the two lines AXK , ATK , which by hypothesis are straight, would enclose a space.

The second possibility (b) is at once seen to involve a similar absurdity. For the straight line XB must, when produced beyond B , ultimately meet $APLM$ in a point L . Consequently the two lines AXB , APL , which by hypothesis are straight, would again enclose a space. If however we were to assume that the straight line XB produced beyond B will ultimately meet either the straight line XM or the straight line XA in another point, we should in the same way arrive at a contradiction.

From this it obviously follows that, on the assumption made, the line AXM is itself the straight line which was drawn from the point A to the point M ; and that is what was maintained.

The remaining stages are in substance these.

(ii) *If the straight line AXB , regarded as rigid, revolves about AX as axis, it cannot assume two more positions in the same plane, so that, for example, in one position XB should coincide with XC , and in the other with XM .*

[This is proved by considerations of symmetry. AXB cannot be altogether "similar or equal to" AXC , if viewed from the same side (left or right) of both: otherwise they would coincide, which by hypothesis they do not. But there is nothing to prevent AXB viewed from one side (say the left) being "similar or equal to" AXC viewed from the other side (i.e. the right), so that AXB can, without any change, be brought into the position AXC .

AXB cannot however take the position of the other straight line AXM as well. If they were like on one side, they would coincide; if they were like on opposite sides, AXM , AXC would be like on the same side and therefore coincide.]

(iii) The other positions of AXB during the revolution must be above or below the original plane.

(iv) It is next maintained that *there is a point D on the arc BC such that, if XD is drawn, AXD is not only a straight line but is such that viewed from the left side it is exactly "similar or equal" to what it is when viewed from the right side.*

[First, it is proved that points M , F can be found on the arc, corresponding in the same way as B , C do, but nearer together, and of course AXM , AXF are both straight lines.

Secondly, similar corresponding points can be found still nearer together, and so on continually, until either (a) we come to one point D such that AXD is exactly like itself when the right and left sides are compared, or (b) there are two ultimate points of this sort M , F , so that both AXM , AXF have this property.

Thirdly, (b) is ruled out by reference to the definition of a straight line.

Hence (a) only is true, and there is only one point D such as described.]

(v) Lastly, Saccheri concludes that the straight line AXD so determined "is alone a straight line, and the immediate prolongation from A beyond X to D ," relying again on the definition of a straight line as "lying evenly."

Simson deduced the proposition that *two straight lines cannot have a common segment* as a corollary from I. 11; but his argument is a complete *petitio principii*, as shown by Todhunter in his note on that proposition.

Proclus (p. 217, 10) records an ancient proof also based on the proposition I. 11. Zeno, he says, propounded this proof and then criticised it.

Suppose that two straight lines AC , AD have a common segment AB , and let BE be drawn at right angles to AC .

Then the angle EBC is right.

If then the angle EBD is also right, the two angles will be equal: which is impossible.

If the angle EBD is not right, draw BF at right angles to AD ; therefore the angle FBA is right.

But the angle EBA is right.

Therefore the angles EBA , FBA are equal: which is impossible.



Zeno objected to this, says Proclus, because it assumed the later proposition I. 11 for its proof. Posidonius said that there was no trace of such a proof to be found in the text-books of Elements, and that it was only invented by Zeno for the purpose of slandering contemporary geometers. Posidonius maintains further that even this proof has something to be said for it. There must be some straight line at right angles to each of the two straight lines AC , AD (the very definition of right angles assumes this): "*suppose then it happens to be the straight line we have set up.*" Here then we have an ancient instance of a defence of *hypothetical construction*, but in such apologetic terms ("it is possible to say something even for this proof") that we may conclude that in general it would not have been accepted by geometers of that time as a legitimate means of proving a proposition.

Todhunter proposed to deduce that *two straight lines cannot have a common segment* from I. 13. But this will not serve either, since, as before mentioned, the assumption is really required for I. 4.

It is best to make it a postulate.

POSTULATE 3.

Καὶ παντὶ κέντρῳ καὶ διαστήματι κύκλον γράφεσθαι.

To describe a circle with any centre and distance.

In this case Euclid's text has the passive of the verb: "a circle can be drawn"; Proclus however has the active (*γράφαι*) as Euclid has in the first two Postulates.

Distance, διαστήματι. This word, meaning "distance" quite generally (cf. Arist. *Metaph.* 1055 a 9 "it is between extremities that distance is greatest," *ibid.* 1056 a 36 "things which have something between them, that is, a certain distance"), and also "distance" in the sense of "dimension" (as in "space has three dimensions, length, breadth and depth," Arist. *Physics* IV. 1, 209 a 4), was the regular word used for describing a circle with a certain *radius*, the idea being that each point of the circumference was at that *distance* from the centre (cf. Arist. *Meteorologica* III. 5, 376 b 8: "if a circle be drawn...with distance ΜΠ"). The Greeks had no word corresponding to *radius*: if they had to express it, they said "(straight lines) drawn from the centre" (*αἱ ἐκ τοῦ κέντρου*, Eucl. III. Def. 1 and Prop. 26; *Meteorologica* II. 5, 362 b 1 has the full phrase *αἱ ἐκ τοῦ κέντρου ἀγόμεναι γραμμαί*).

Mr Frankland observes that it would be remarkable if, unlike Postulates 1 and 2, this Postulate implied *merely* what it says, that a circle can be drawn with any centre and distance. We may regard it, if we please, as helping to the complete delineation of the Space which Euclid's geometry is to investigate formally. The Postulate has the effect of removing any restriction upon the size of the circle. It may (1) be indefinitely small, and this implies that space is *continuous*, not discrete, with an irreducible minimum distance between

contiguous points in it. (2) The circle may be indefinitely large, which implies the fundamental hypothesis of *infinitude* of space. This last assumed characteristic of space is essential to the proof of I. 16, a theorem not universally valid in a space which is unbounded in extent but finite in size. It would however be unsafe to suppose that Euclid foresaw the use to which his Postulate might thus be put, or formulated it with such an intention.

POSTULATE 4.

Καὶ πάσας τὰς ὀρθὰς γωνίας ἴσας ἀλλήλαις εἶναι.

That all right angles are equal to one another.

While this Postulate asserts the essential truth that a right angle is a *determinate magnitude* so that it really serves as an invariable standard by which other (acute and obtuse) angles may be measured, much more than this is implied, as will easily be seen from the following consideration. If the statement is to be *proved*, it can only be proved by the method of applying one pair of right angles to another and so arguing their equality. But this method would not be valid unless on the assumption of the *invariability of figures*, which would therefore have to be asserted as an antecedent postulate. Euclid preferred to assert as a postulate, directly, the fact that all right angles are equal; and hence his postulate must be taken as equivalent to the principle of *invariability of figures* or its equivalent, the *homogeneity of space*.

According to Proclus, Geminus held that this Postulate should not be classed as a postulate but as an axiom, since it does not, like the first three Postulates, assert the possibility of some *construction* but expresses an essential property of right angles. Proclus further observes (p. 188, 8) that it is not a postulate in Aristotle's sense either. (In this I think he is wrong, as explained above.) Proclus himself, while regarding the assumption as axiomatic ("the equality of right angles suggests itself even by virtue of our common notions"), is prepared with a proof, if such is asked for.

Let ABC , DEF be two right angles.

If they are not equal, one of them must be the greater, say ABC .

Then, if we apply DE to AB , EF will fall within ABC , as BG .

Produce CB to H . Then, since ABC is a right angle, so is ABH , and the two angles are equal (a right angle being by definition equal to its adjacent angle).

Therefore the angle ABH is *greater* than the angle ABG .

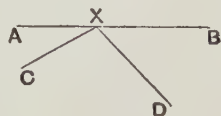
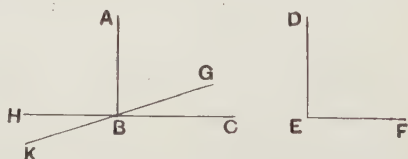
Producing GB to K , we have similarly the two angles ABK , ABG both right and equal to one another; whence the angle ABH is *less* than the angle ABG .

But it is also greater: which is impossible.

Therefore etc.

A defect in this proof is the assumption that CB , GB can each be produced only in one way, and that BK falls outside the angle ABH .

Saccheri's proof is more careful in that he premises a third lemma in addition to those asserting (1) that two straight lines cannot enclose a space and (2) that two straight lines cannot have a common segment. The third lemma is: *If two straight lines AB , CXD meet one another at an intermediate point X , they do not touch at that point, but cut one another.*



Suppose now that DA standing on BAC makes the two angles DAB , DAC equal, so that each is a right angle by the definition.

Similarly, let LH form with the straight line FHM the right angles LHF , LHM .

Let DA , HL be equal; and suppose the whole of the second figure so laid upon the first that the point H falls on A , and L on D .

Then the straight line FHM will (by the third lemma) not touch the straight line BC at A ; it will either (a) coincide exactly with BC , or

(b) cut it so that one of its extremities, as F , will fall above $[BC]$ and the other, M , below it.

If the alternative (a) is true, we have already proved the exact equality of all rectilinear right angles.

Under alternative (b) we prove that the angle LHF , being equal to the angle DAF , is less than the angle DAB or DAC , and *a fortiori* less than the angle DAM or LHM : which is contrary to the hypothesis.

[Hence (a) is the only possible alternative, so that all right angles are equal.]

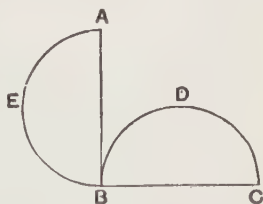
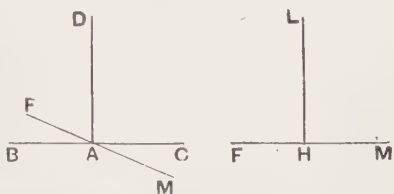
Saccheri adds that it makes no difference if the angle DAF diverges *infinitely little* from the angle DAB . This would equally lead to a conclusion contradicting the hypothesis.

It will be observed that Saccheri speaks of "the exact equality of all rectilinear right angles." He may have had in mind the remark of Pappus, quoted by Proclus (p. 189, 11), that the converse of this postulate, namely that an angle which is equal to a right angle is also right, is not necessarily true, unless the former angle is *rectilinear*. Suppose two equal straight lines BA , BC at right angles to one another, and semi-circles described on BA , BC respectively as AEB , BDC in the figure. Then, since the semi-circles are equal, they coincide if applied to one another. Hence the "angles" EBA , DBC are equal. Add to each the "angle" ABD ; and it follows that the *lunular* angle EBD is equal to the right angle ABC . (Similarly, if BA , BC be inclined at an acute or obtuse angle, instead of at a right angle, we find a *lunular* angle equal to an acute or obtuse angle.) This is one of the curiosities which Greek commentators delighted in.

Veronese, Ingrami, and Enriques and Amaldi deduce the fact that *all right angles are equal* from the equivalent fact that *all flat angles are equal*, which is either itself assumed as a postulate or immediately deduced from some other postulate.

Hilbert takes quite a different line. He considers that Euclid did wrong in placing Post. 4 among "axioms." He himself, after his Group III. of Axioms containing six relating to congruence, proves several theorems about the congruence of triangles and angles, and then deduces our Postulate.

As to the *raison d'être* and the place of Post. 4 one thing is quite certain. It was essential from Euclid's point of view that it should come before Post. 5, since the condition in the latter that a certain pair of angles are together less than two right angles would be useless unless it were first made clear that right angles are angles of determinate and invariable magnitude.



POSTULATE 5.

Καὶ ἐὰν εἰς δύο εὐθείας εὐθεία ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθῶν ἐλάσσονας ποιῇ, ἐκβαλλομένης τὰς δύο εὐθείας ἐπ' ἄπειρον συμπέπτειν, ἐφ' ἃ μέρη εἰσὶν αἱ τῶν δύο ὀρθῶν ἐλάσσονες.

That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Although Aristotle gives a clear idea of what he understood by a *postulate*, he does not give any instances from geometry; still less has he any allusion recalling the particular postulates found in Euclid. We naturally infer that the formulation of these postulates was Euclid's own work. There is a more positive indication of the originality of Postulate 5, since in the passage (*Anal. prior.* II. 16, 65 a 4) quoted above in the note on the definition of parallels he alludes to some *petitio principii* involved in the theory of parallels current in his time. This reproach was removed by Euclid when he laid down this epoch-making Postulate. When we consider the countless successive attempts made through more than twenty centuries to prove the Postulate, many of them by geometers of ability, we cannot but admire the genius of the man who concluded that such a hypothesis, which he found necessary to the validity of his whole system of geometry, was really indemonstrable.

From the very beginning, as we know from Proclus, the Postulate was attacked as such, and attempts were made to prove it as a theorem or to get rid of it by adopting some other definition of parallels; while in modern times the literature of the subject is enormous. Riccardi (*Saggio di una bibliografia Euclidea*, Part IV., Bologna, 1890) has twenty quarto pages of titles of monographs relating to Post. 5 between the dates 1607 and 1887. Max Simon (*Ueber die Entwicklung der Elementar-geometrie im XIX. Jahrhundert*, 1906) notes that he has seen three new attempts, as late as 1891 (a century after Gauss laid the foundation of non-Euclidean geometry), to prove the theory of parallels independently of the Postulate. Max Simon himself (pp. 53—61) gives a large number of references to books or articles on the subject and refers to the copious information, as to contents as well as names, contained in Schotten's *Inhalt und Methode des planimetrischen Unterrichts*, II. pp. 183—332.

This note will include some account of or allusion to a few of the most noteworthy attempts to prove the Postulate. Only those of ancient times, as being less generally accessible, will be described at any length; shorter references must suffice in the case of the modern geometers who have made the most important contributions to the discussion of the Postulate and have thereby, in particular, contributed most towards the foundation of the non-Euclidean geometries, and here I shall make use principally of the valuable Article 8, *Sulla teoria delle parallele e sulle geometrie non-euclidee* (by Roberto Bonola), in *Questioni riguardanti le matematiche elementari*, I. pp. 247—363.

Proclus (p. 191, 21 sqq.) states very clearly the nature of the first objections taken to the Postulate.

"This ought even to be struck out of the Postulates altogether; for it is a theorem involving many difficulties, which Ptolemy, in a certain book, set himself to solve, and it requires for the demonstration of it a number of definitions as well as theorems. And the converse of it is actually proved by Euclid himself as a theorem. It may be that some would be

deceived and would think it proper to place even the assumption in question among the postulates as affording, in the lessening of the two right angles, ground for an instantaneous belief that the straight lines converge and meet. To such as these Geminus correctly replied that we have learned from the very pioneers of this science not to have any regard to mere plausible imaginings when it is a question of the reasonings to be included in our geometrical doctrine. For Aristotle says that it is as justifiable to ask scientific proofs of a rhetorician as to accept mere plausibilities from a geometer; and Simmias is made by Plato to say that he recognises as quacks those who fashion for themselves proofs from probabilities. So in this case the fact that, when the right angles are lessened, the straight lines converge is true and necessary; but the statement that, since they converge more and more as they are produced, they will sometime meet is plausible but not necessary, in the absence of some argument showing that this is true in the case of straight lines. For the fact that some lines exist which approach indefinitely, but yet remain non-secant (*ἀσύνευκτοι*), although it seems improbable and paradoxical, is nevertheless true and fully ascertained with regard to other species of lines. May not then the same thing be possible in the case of straight lines which happens in the case of the lines referred to? Indeed, until the statement in the Postulate is clinched by proof, the facts shown in the case of other lines may direct our imagination the opposite way. And, though the controversial arguments against the meeting of the straight lines should contain much that is surprising, is there not all the more reason why we should expel from our body of doctrine this merely plausible and unreasoned (hypothesis)?

"It is then clear from this that we must seek a proof of the present theorem, and that it is alien to the special character of postulates. But how it should be proved, and by what sort of arguments the objections taken to it should be removed, we must explain at the point where the writer of the Elements is actually about to recall it and use it as obvious. It will be necessary at that stage to show that its obvious character does not appear independently of proof, but is turned by proof into matter of knowledge."

Before passing to the attempts of Ptolemy and Proclus to prove the Postulate, I should note here that Simplicius says (in an-Nairizi, ed. Besthorn-Heiberg, p. 119, ed. Curtze, p. 65) that this Postulate is by no means manifest, but requires proof, and accordingly "Abthiniathus" and Diodorus had already proved it by means of many different propositions, while Ptolemy also had explained and proved it, using for the purpose Eucl. I. 13, 15 and 16 (or 18). The Diodorus here mentioned may be the author of the *Analemma* on which Pappus wrote a commentary. It is difficult even to frame a conjecture as to who "Abthiniathus" is. In one place in the Arabic text the name appears to be written "Anthisathus" (H. Suter in *Zeitschrift für Math. und Physik*, xxxviii., hist. litt. Abth. p. 194). It has occurred to me whether he might be Peithon, a friend of Serenus of Antinoëia (Antinoupolis) who was long known as Serenus of Antissa. Serenus says (*De sectione cylindri*, ed. Heiberg, p. 96): "Peithon the geometer, explaining parallels in a work of his, was not satisfied with what Euclid said, but showed their nature more cleverly by an example; for he says that parallel straight lines are such a thing as we see on walls or on the ground in the shadows of pillars which are made when either a torch or a lamp is burning behind them. And, although this has only been matter of merriment to every one, I at least must not deride it, for the respect I have for the author, who is my friend." If Peithon was known as "of Antinoëia" or "of Antissa," the two forms of the mysterious name might perhaps be an attempt at an equivalent; but this is no more than a guess.

Simplicius adds in full and word for word the attempt of his "friend" or his "master Aganis" to prove the Postulate.

Proclus returns to the subject (p. 365, 5) in his note on Eucl. I. 29. He says that before his time a certain number of geometers had classed as a theorem this Euclidean postulate and thought it matter for proof, and he then proceeds to give an account of Ptolemy's argument.

Noteworthy attempts to prove the Postulate.

Ptolemy.

We learn from Proclus (p. 365, 7—11) that Ptolemy wrote a book on the proposition that "straight lines drawn from angles less than two right angles meet if produced," and that he used in his "proof" many of the theorems in Euclid preceding I. 29. Proclus excuses himself from reproducing the early part of Ptolemy's argument, only mentioning as one of the propositions proved in it the theorem of Eucl. I. 28 that, if two straight lines meeting a transversal make the two interior angles on the same side equal to two right angles, the straight lines do not meet, however far produced.

I. From Proclus' note on I. 28 (p. 362, 14 sq.) we know that Ptolemy proved this somewhat as follows.

Suppose that there are two straight lines AB , CD , and that $EFGH$, meeting them, makes the angles BFG , FGD equal to two right angles. I say that AB , CD are parallel, that is, they are non-secant.

For, if possible, let FB , GD meet at K .

Now, since the angles BFG , FGD are equal to two right angles, while the four angles AFG , BFG , FGD , FGC are together equal to four right angles,

the angles AFG , FGC are equal to two right angles.

"If therefore FB , GD , when the interior angles are equal to two right angles, meet at K , the straight lines FA , GC will also meet if produced; for the angles AFG , CGF are also equal to two right angles.

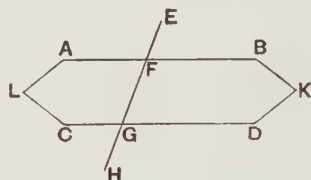
"Therefore the straight lines will either meet in both directions or in neither direction, if the two pairs of interior angles are both equal to two right angles.

"Let, then, FA , GC meet at L .

"Therefore the straight lines $LABK$, $LCDK$ enclose a space: which is impossible.

"Therefore it is not possible for two straight lines to meet when the interior angles are equal to two right angles. Therefore they are parallel."

[The argument in the words italicised would be clearer if it had been shown that the two interior angles on one side of EH are *severally* equal to the two interior angles on the other, namely BFG to CGF and FGD to AFG ; whence, assuming FB , GD to meet in K , we can take the triangle KFG and place it (e.g. by rotating it in the plane about O the middle point of FG) so that FG falls where GF is in the figure and GD falls on FA , in which case FB must also fall on GC ; hence, since FB , GD meet at K , GC and FA must meet at a corresponding point L . Or, as Mr Frankland does, we may substitute for FG a straight line MN through O the middle point of FG drawn perpendicular to one of the parallels, say AB . Then, since the two triangles OMF , ONG have two angles equal respectively, namely FOM to



GON (i. 15) and OFM to OGN , and one side OF equal to one side OG , the triangles are congruent, the angle ONG is a right angle, and MN is perpendicular to both AB and CD . Then, by the same method of application, MA , NC are shown to form with MN a triangle $MALCN$ congruent with the triangle $NDKBM$, and MA , NC meet at a point L corresponding to K . Thus the two straight lines would meet at the *two* points K , L . This is what happens under the Riemann hypothesis, where the axiom that two straight lines cannot enclose a space does not hold, but all straight lines meeting in one point have another point common also, and e.g. in the particular figure just used K , L are points common to all perpendiculars to MN . If we suppose that K , L are not distinct points, but *one* point, the axiom that two straight lines cannot enclose a space is *not* contradicted.]

II. Ptolemy now tries to prove i. 29 without using our Postulate, and then deduces the Postulate from it (Proclus, pp. 365, 14—367, 27).

The argument to prove i. 29 is as follows.

The straight line which cuts the parallels must make the sum of the interior angles on the same side equal to, greater than, or less than, two right angles.

"Let AB , CD be parallel, and let FG meet them. I say (1) that FG does not make the interior angles on the same side greater than two right angles.



"For, if the angles AFG , CGF are greater than two right angles, the remaining angles BFG , DGF are less than two right angles.

"But the same two angles are also greater than two right angles; for AF , CG are no more parallel than FB , GD , so that, if the straight line falling on AF , CG makes the interior angles greater than two right angles, the straight line falling on FB , GD will also make the interior angles greater than two right angles.

"But the same angles are also less than two right angles; for the four angles AFG , CGF , BFG , DGF are equal to four right angles: which is impossible.

"Similarly (2) we can show that the straight line falling on the parallels does not make the interior angles on the same side less than two right angles.

"But (3), if it makes them neither greater nor less than two right angles, it can only make the interior angles on the same side *equal* to two right angles."

III. Ptolemy deduces Post. 5 thus:

Suppose that the straight lines making angles with a transversal less than two right angles do not meet on the side on which those angles are.

Then, *a fortiori*, they will not meet on the other side on which are the angles *greater* than two right angles.

Hence the straight lines will not meet in either direction; they are therefore *parallel*.

But, if so, the angles made by them with the transversal are equal to two right angles, by the preceding proposition (= i. 29).

Therefore the same angles will be both equal to and less than two right angles:

which is impossible.

Hence the straight lines will meet.

IV. Ptolemy lastly enforces his conclusion that the straight lines will meet on the side on which are the angles less than two right angles by recurring to the *a fortiori* step in the foregoing proof.

Let the angles AFG , CGF in the accompanying figure be together less than two right angles.

Therefore the angles BFG , DGF are greater than two right angles.

We have proved that the straight lines are not non-secant.

If they meet, they must meet either towards A , C , or towards B , D .

(1) Suppose they meet towards B , D , at K .

Then, since the angles AFG , CGF are less than two right angles, and the angles AFG , GFB are equal to two right angles, take away the common angle AFG , and

the angle CGF is less than the angle BFG ;

that is, the exterior angle of the triangle KFG is less than the interior and opposite angle BFG ;

which is impossible.

Therefore AB , CD do not meet towards B , D .

(2) But they do meet, and therefore they must meet in one direction or the other:

therefore they meet towards A , B , that is, on the side where are the angles less than two right angles.

The flaw in Ptolemy's argument is of course in the part of his proof of 1. 29 which I have italicised. As Proclus says, he is not entitled to assume that, if AB , CD are parallel, whatever is true of the interior angles on one side of FG (i.e. that they are together equal to, greater than, or less than, two right angles) is necessarily true at the same time of the interior angles on the other side. Ptolemy justifies this by saying that FA , GC are no more parallel in one direction than FB , GD are in the other: which is equivalent to the assumption that *through any point only one parallel can be drawn to a given straight line*. That is, he assumes an equivalent of the very Postulate he is endeavouring to prove.

Proclus.

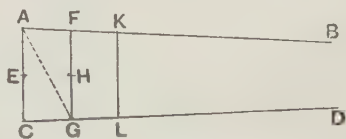
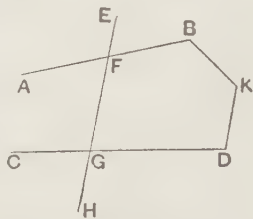
Before passing to his own attempt at a proof, Proclus (p. 368, 26 sqq.) examines an ingenious argument (recalling somewhat the famous one about Achilles and the tortoise) which appeared to show that it was *impossible* for the lines described in the Postulate to meet.

Let AB , CD make with AC the angles BAC , ACD together less than two right angles.

Bisect AC at E and along AB , CD respectively measure AF , CG so that each is equal to AE .

Bisect FG at H and mark off FK , GL each equal to FH ; and so on.

Then AF , CG will not meet at any point on FG ; for, if that were the case, two sides of a triangle would be together equal to the third: which is impossible.



Similarly, AB , CD will not meet at any point on KL ; and "proceeding like this indefinitely, joining the non-coincident points, bisecting the lines so drawn, and cutting off from the straight lines portions equal to the half of these, they say they thereby prove that the straight lines AB , CD will not meet anywhere."

It is not surprising that Proclus does not succeed in exposing the fallacy here (the fact being that the process will indeed be endless, and yet the straight lines will intersect within a finite distance). But Proclus' criticism contains nevertheless something of value. He says that the argument will prove too much, since we have only to join AG in order to see that straight lines making *some* angles which are together less than two right angles do in fact meet, namely AG , CG . "Therefore it is not possible to assert, without some definite limitation, that the straight lines produced from angles less than two right angles do not meet. On the contrary, it is manifest that *some* straight lines, when produced from angles less than two right angles, do meet, although the argument seems to require it to be proved that this property belongs to *all* such straight lines. For one might say that, the lessening of the two right angles being subject to no limitation, *with such and such an amount of lessening the straight lines remain non-secant, but with an amount of lessening in excess of this they meet* (p. 371, 2—10)."

[Here then we have the germ of such an idea as that worked out by Lobachewsky, namely that the straight lines issuing from a point in a plane can be divided with reference to a straight line lying in that plane into two classes, "secant" and "non-secant," and that we may define as *parallel* the two straight lines which divide the secant from the non-secant class.]

Proclus goes on (p. 371, 10) to base his own argument upon "an axiom such as Aristotle too used in arguing that the universe is finite. For, *if from one point two straight lines forming an angle be produced indefinitely, the distance (διάστασις, Arist. διάστημα) between the said straight lines produced indefinitely will exceed any finite magnitude.* Aristotle at all events showed that, if the straight lines drawn from the centre to the circumference are infinite, the interval between them is infinite. For, if it is finite, it is impossible to increase the distance, so that the straight lines (the radii) are not infinite. Hence the straight lines, when produced indefinitely, will be at a distance from one another greater than any assumed finite magnitude."

This is a fair representation of Aristotle's argument in *De caelo* 1. 5, 271 b 28, although of course it is not a proof of what Proclus assumes as an axiom.

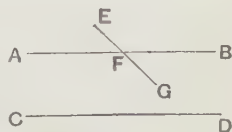
This being premised, Proclus proceeds (p. 371, 24):

I. "I say that, *if any straight line cuts one of two parallels, it will cut the other also.*

"For let AB , CD be parallel, and let EFG cut AB ; I say that it will cut CD also.

"For, since BF , FG are two straight lines from one point F , they have, when produced indefinitely, a distance greater than any magnitude, so that it will also be greater than the interval between the parallels. Whenever therefore they are at a distance from one another greater than the distance between the parallels, FG will cut CD .

"Therefore etc."



II. "Having proved this, we shall prove, as a deduction from it, the theorem in question.

"For let AB , CD be two straight lines, and let EF falling on them make the angles BEF , DFE less than two right angles.

"I say that the straight lines will meet on that side on which are the angles less than two right angles.

"For, since the angles BEF , DFE are less than two right angles, let the angle HEB be equal to the excess of two right angles (over them), and let HE be produced to K .

"Since then EF falls on KH , CD and makes the two interior angles HEF , DFE equal to two right angles,

the straight lines HK , CD are parallel.

"And AB cuts KH ; therefore it will also cut CD , by what was before shown.

"Therefore AB , CD will meet on that side on which are the angles less than two right angles.

"Hence the theorem is proved."

Clavius criticised this proof on the ground that the axiom from which it starts, taken from Aristotle, itself requires proof. He points out that, just as you cannot assume that two lines which continually approach one another will meet (witness the hyperbola and its asymptote), so you cannot assume that two lines which continually diverge will ultimately be so far apart that a perpendicular from a point on one let fall on the other will be greater than any assigned distance; and he refers to the *conchoid* of Nicomedes, which continually approaches its asymptote, and therefore continually gets farther away from the tangent at the vertex; yet the perpendicular from any point on the curve to that tangent will always be less than the distance between the tangent and the asymptote. Saccheri supports the objection.

Proclus' first proposition is open to the objection that it assumes that two "parallels" (in the Euclidean sense) or, as we may say, two straight lines which have a common perpendicular, are (not necessarily equidistant, but) so related that, when they are produced indefinitely, the perpendicular from a point of one upon the other remains finite.

This last assumption is incorrect on the hyperbolic hypothesis; the "axiom" taken from Aristotle does not hold on the elliptic hypothesis.

Naşiraddin at-Tūsî.

The Persian-born editor of Euclid, whose date is 1201—1274, has three lemmas leading up to the final proposition. Their content is substantially as follows, the first lemma being apparently assumed as evident.

I. (a) If AB , CD be two straight lines such that successive perpendiculars, as EF , GH , KL , from points on AB to CD always make with AB unequal angles, which are always acute on the side towards B and always obtuse on the side towards A , then the lines AB , CD , so long as they do not cut, approach continually nearer in the direction of the acute angles and diverge continually in the direction of the obtuse angles; and the perpendiculars diminish towards B , D , and increase towards A , C .

(b) Conversely, if the perpendiculars so drawn continually become shorter in the direction of B , D , and longer in the



direction of A, C , the straight lines AB, CD approach continually nearer in the direction of B, D and diverge continually in the other direction; also each perpendicular will make with AB two angles one of which is acute and the other is obtuse, and all the acute angles will lie in the direction towards B, D , and the obtuse angles in the opposite direction.

[Saccheri points out that even the first part (*a*) requires proof. As regards the converse (*b*) he asks, why should not the successive acute angles made by the perpendiculars with AB , while remaining acute, become greater and greater as the perpendiculars become smaller until we arrive at last at a perpendicular which is a common perpendicular to *both* lines? If that happens, all the author's efforts are in vain. And, if you are to assume the truth of the statement in the lemma without proof, would it not, as Wallis said, be as easy to assume as axiomatic the statement in Post. 5 without more ado?]

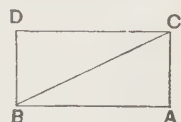
II. If AC, BD be drawn from the extremities of AB at right angles to it and on the same side, and if AC, BD be made equal to one another and CD be joined, each of the angles ACD, BDC will be right, and CD will be equal to AB .

The first part of this lemma is proved by *reductio ad absurdum* from the preceding lemma. If, e.g., the angle ACD is not right, it must either be acute or obtuse.

Suppose it is acute; then, by lemma 1, AC is greater than BD , which is contrary to the hypothesis. And so on.

The angles ACD, BDC being proved to be right angles, it is easy to prove that AB, CD are equal.

[It is of course assumed in this "proof" that, if the angle ACD is acute, the angle BDC is obtuse, and *vice versa*.]



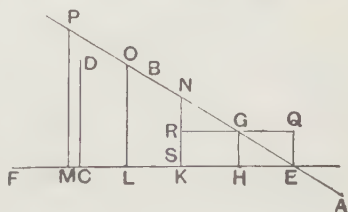
III. In any triangle the three angles are together equal to two right angles. This is proved for a right-angled triangle by means of the foregoing lemma, the four angles of the quadrilateral $ABCD$ of that lemma being all right angles. The proposition is then true for any triangle, since any triangle can be divided into two right-angled triangles

IV. Here we have the final "proof" of Post. 5. Three cases are distinguished, but it is enough to show the case where one of the interior angles is right and the other acute.

Suppose AB, CD to be two straight lines met by FCE making the angle ECD a right angle and the angle CEB an acute angle.

Take any point G on EB , and draw GH perpendicular to EC .

Since the angle CEG is acute, the perpendicular GH will fall on the side of E towards D , and will either coincide with CD or not coincide with it. In the former case the proposition is proved.



If GH does not coincide with CD but falls on the side of it towards F , CD , being within the triangle formed by the perpendicular and by CE, EG , must cut EG . [An axiom is here used, namely that, if CD be produced far enough, it must pass *outside* the triangle and therefore cut *some* side, which must be EB , since it cannot be the perpendicular (i. 27), or CE .]

Lastly, let GH fall on the side of CD towards E .

Thus CP and AP meet at P ; and, as by the definition of similar figures the angles of the triangles PCA , πCa are respectively equal, the angle PCA being equal to the angle πCa and the angle PAC to the angle πaC or BAC , it follows that CP , AP lie on CD , AB produced respectively.

Hence AB , CD meet on the side on which are the angles less than two right angles.

[The whole gist of this proof lies in the assumed postulate as to the existence of similar figures; and, as Saccheri points out, this is equivalent to unconditionally assuming the "hypothesis of the right angle," and consequently Euclid's Postulate 5.]

Gerolamo Saccheri.

The book *Euclides ab omni naevo vindicatus* (1733) by Gerolamo Saccheri (1667—1733), a Jesuit, and professor at the University of Pavia, is now accessible (1) edited in German by Engel and Stäckel, *Die Theorie der Parallellinien von Euklid bis auf Gauss*, 1895, pp. 41—136, and (2) in an Italian version, abridged but annotated, *L'Euclide emendato del P. Gerolamo Saccheri*, by G. Boccardini (Hoepli, Milan, 1904). It is of much greater importance than all the earlier attempts to prove Post. 5 because Saccheri was the first to contemplate the possibility of hypotheses other than that of Euclid, and to work out a number of consequences of those hypotheses. He was therefore a true precursor of Legendre and of Lobachewsky, as Beltrami called him (1889), and, it might be added, of Riemann also. For, as Veronese observes (*Fondamenti di geometria*, p. 570), Saccheri obtained a glimpse of the theory of parallels in all its generality, while Legendre, Lobachewsky and G. Bolyai excluded *a priori*, without knowing it, the "hypothesis of the obtuse angle," or the Riemann hypothesis. Saccheri, however, was the victim of the preconceived notion of his time that the sole possible geometry was the Euclidean, and he presents the curious spectacle of a man laboriously erecting a structure upon new foundations for the very purpose of demolishing it afterwards; he sought for contradictions in the heart of the systems which he constructed, in order to prove thereby the falsity of his hypotheses.

For the purpose of formulating his hypotheses he takes a plane quadrilateral $ABDC$, two opposite sides of which, AC , BD , are equal and perpendicular to a third AB . Then the angles at C and D are easily proved to be equal. On the Euclidean hypothesis they are both right angles; but apart from this hypothesis they might be both obtuse or both acute. To the three possibilities, which Saccheri distinguishes by the names (1) *the hypothesis of the right angle*, (2) *the hypothesis of the obtuse angle* and (3) *the hypothesis of the acute angle* respectively, there corresponds a certain group of theorems; and Saccheri's point of view is that the Postulate will be completely proved if the consequences which follow from the last two hypotheses comprise results inconsistent with one another.

Among the most important of his propositions are the following:

(1) *If the hypothesis of the right angle, or of the obtuse angle, or of the acute angle is proved true in a single case, it is true in every other case.* (Props. v., vi., vii.)

(2) *According as the hypothesis of the right angle, the obtuse angle, or the acute angle is true, the sum of the three angles of a triangle is equal to, greater than, or less than two right angles.* (Prop. ix.)



(3) *From the existence of a single triangle in which the sum of the angles is equal to, greater than, or less than two right angles the truth of the hypothesis of the right angle, obtuse angle, or acute angle respectively follows.* (Prop. xv.)

These propositions involve the following: *If in a single triangle the sum of the angles is equal to, greater than, or less than two right angles, then any triangle has the sum of its angles equal to, greater than, or less than two right angles respectively*, which was proved about a century later by Legendre for the two cases only where the sum is *equal to* or *less than* two right angles.

The proofs are not free from imperfections, as when, in the proofs of Prop. xii. and the part of Prop. xiii. relating to the hypothesis of the *obtuse angle*, Saccheri uses Eucl. I. 18 depending on I. 16, a proposition which is only valid on the assumption that *straight lines are infinite in length*; for this assumption itself does not hold under the hypothesis of the obtuse angle (the Riemann hypothesis).

The hypothesis of the acute angle takes Saccheri much longer to dispose of, and this part of the book is less satisfactory; but it contains the following propositions afterwards established anew by Lobachewsky and Bolyai, viz.:

(4) *Two straight lines in a plane (even on the hypothesis of the acute angle) either have a common perpendicular, or must, if produced in one and the same direction, either intersect once at a finite distance or at least continually approach one another.* (Prop. xxiii.)

(5) *In a cluster of rays issuing from a point there exist always (on the hypothesis of the acute angle) two determinate straight lines which separate the straight lines which intersect a fixed straight line from those which do not intersect it, ending with and including the straight line which has a common perpendicular with the fixed straight line.* (Props. xxx., xxxi., xxxii.)

Lambert.

A dissertation by G. S. Klügel, *Conatuum præcipuorum theoriæ parallelarum demonstrandi recensio* (1763), contained an examination of some thirty "demonstrations" of Post. 5 and is remarkable for its conclusion expressing, apparently for the first time, *doubt as to its demonstrability* and observing that the certainty which we have in us of the truth of the Euclidean hypothesis is not the result of a series of rigorous deductions but rather of experimental observations. It also had the greater merit that it called the attention of Johann Heinrich Lambert (1728—1777) to the theory of parallels. His *Theory of Parallels* was written in 1766 and published after his death by G. Bernoulli and C. F. Hindenburg; it is reproduced by Engel and Stäckel (*op. cit.* pp. 152—208).

The third part of Lambert's tract is devoted to the discussion of the same three hypotheses as Saccheri's, the hypothesis of the *right angle* being for Lambert the *first*, that of the *obtuse angle* the *second*, and that of the *acute angle* the *third*, hypothesis; and, with reference to a quadrilateral with *three right angles* from which Lambert starts (that is, one of the halves into which the median divides Saccheri's quadrilateral), the three hypotheses are the assumptions that the fourth angle is a right angle, an obtuse angle, or an acute angle respectively.

Lambert goes much further than Saccheri in the deduction of new propositions from the *second* and *third* hypotheses. The most remarkable is the following.

The area of a plane triangle, under the second and third hypotheses, is proportional to the difference between the sum of the three angles and two right angles.

Thus the numerical expression for the area of a triangle is, under the *third* hypothesis

$$\Delta = k(\pi - A - B - C) \dots\dots\dots(1),$$

and under the *second* hypothesis

$$\Delta = k(A + B + C - \pi) \dots\dots\dots(2),$$

where k is a positive constant.

A remarkable observation is appended (§ 82): "In connexion with this it seems to be remarkable that the *second* hypothesis holds if *spherical* instead of plane triangles are taken, because in the former also the sum of the angles is greater than two right angles, and the excess is proportional to the area of the triangle.

"It appears still more remarkable that what I here assert of spherical triangles can be proved independently of the difficulty of parallels."

This discovery that the *second* hypothesis is realised on the surface of a sphere is important in view of the development, later, of the Riemann hypothesis (1854).

Still more remarkable is the following prophetic sentence: "*I am almost inclined to draw the conclusion that the third hypothesis arises with an imaginary spherical surface*" (cf. Lobachewsky's *Géométrie imaginaire*, 1837).

No doubt Lambert was confirmed in this by the fact that, in the formula (2) above, which, for $k = r^2$, represents the area of a spherical triangle, if $r\sqrt{-1}$ is substituted for r , and $r^2 = k$, we obtain the formula (1).

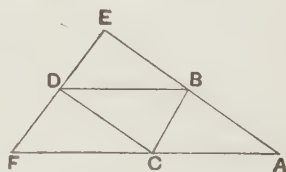
Legendre.

No account of our present subject would be complete without a full reference to what is of permanent value in the investigations of Adrien Marie Legendre (1752—1833) relating to the theory of parallels, which extended over the space of a generation. His different attempts to prove the Euclidean hypothesis appeared in the successive editions of his *Éléments de Géométrie* from the first (1794) to the twelfth (1823), which last may be said to contain his last word on the subject. Later, in 1833, he published, in the *Mémoires de l'Académie Royale des Sciences*, xii. p. 367 sqq., a collection of his different proofs under the title *Réflexions sur différentes manières de démontrer la théorie des parallèles*. His exposition brought out clearly, as Saccheri had done, and kept steadily in view, the essential connexion between the theory of parallels and the sum of the angles of a triangle. In the first edition of the *Éléments* the proposition that *the sum of the angles of a triangle is equal to two right angles* was proved analytically on the basis of the assumption that the choice of a *unit of length* does not affect the correctness of the proposition to be proved, which is of course equivalent to Wallis' assumption of *the existence of similar figures*. A similar analytical proof is given in the notes to the twelfth edition. In his second edition Legendre proved Postulate 5 by means of the assumption that, *given three points not in a straight line, there exists a circle passing through all three*. In the third edition (1800) he gave the proposition that *the sum of the angles of a triangle is not greater than two right angles*; this proof, which was geometrical, was replaced later by another, the best known, depending on a construction like that of Euclid I. 16, the continued application of which enables any number of successive triangles to be evolved in which, while the sum of the angles in each remains always equal to the sum of the angles of the original triangle, one of the angles increases and the sum of the other two diminishes continually. But Legendre found the proof of the equally necessary proposition that the sum of the angles of a triangle is

not less than two right angles to present great difficulties. He first observed that, as in the case of spherical triangles (in which the sum of the angles is greater than two right angles) the excess of the sum of the angles over two right angles is proportional to the area of the triangle, so in the case of rectilinear triangles, if the sum of the angles is less than two right angles by a certain *deficit*, the *deficit* will be proportional to the area of the triangle. Hence if, starting from a given triangle, we could construct another triangle in which the original triangle is contained at least m times, the *deficit* of this new triangle will be equal to at least m times that of the original triangle, so that the sum of the angles of the greater triangle will diminish progressively as m increases, until it becomes zero or negative: which is absurd. The whole difficulty was thus reduced to that of the construction of a triangle containing the given triangle at least twice; but the solution of even this simple problem requires it to be assumed (or proved) that *through a given point within a given angle less than two-thirds of a right angle we can always draw a straight line which shall meet both sides of the angle*. This is however really equivalent to Euclid's Postulate. The proof in the course of which the necessity for the assumption appeared is as follows.

It is required to prove that the sum of the angles of a triangle cannot be less than two right angles.

Suppose A is the least of the three angles of a triangle ABC . Apply to the opposite side BC a triangle DBC , equal to the triangle ACB , and such that the angle DBC is equal to the angle ACB , and the angle DCB to the angle ABC ; and draw any straight line through D cutting AB , AC produced in E , F .



If now the sum of the angles of the triangle ABC is less than two right angles, being equal to $2R - \delta$ say, the sum of the angles of the triangle DBC , equal to the triangle ABC , is also $2R - \delta$.

Since the sum of the three angles of the remaining triangles DEB , FDC respectively cannot at all events be *greater* than two right angles [for Legendre's proofs of this see below], the sum of the twelve angles of the four triangles in the figure *cannot be greater* than

$$4R + (2R - \delta) + (2R - \delta), \text{ i.e. } 8R - 2\delta.$$

Now the sum of the three angles at each of the points B , C , D is $2R$.

Subtracting these nine angles, we have the result that the three angles of the triangle AEF *cannot be greater* than $2R - 2\delta$.

Hence, if the sum of the angles of the triangle ABC is less than two right angles by δ , the sum of the angles of the larger triangle AEF is less than two right angles by *at least* 2δ .

We can continue the construction, making a still larger triangle from AEF , and so on.

But, however small δ is, we can arrive at a multiple $2^n\delta$ which shall exceed any given angle and therefore $2R$ itself; so that the sum of the three angles of a triangle sufficiently large would be zero or even less than zero: which is absurd.

Therefore etc.

The difficulty caused by the necessity of making the above-mentioned assumption made Legendre abandon, in his ninth edition, the method of the

editions from the third to the eighth and return to Euclid's method pure and simple.

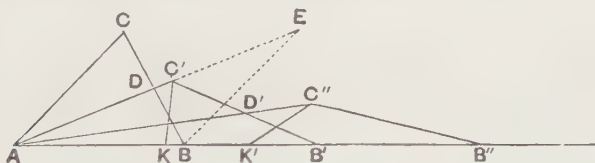
But again, in the twelfth, he returned to the plan of constructing any number of successive triangles such that the sum of the three angles in all of them remains equal to the sum of the three angles of the original triangle, but two of the angles of the new triangles become smaller and smaller, while the third becomes larger and larger; and this time he claims to prove in one proposition that the sum of the three angles of the original triangle is *equal* to two right angles by continuing the construction of new triangles *indefinitely* and compressing the two smaller angles of the ultimate triangle into nothing, while the third angle is made to become a *flat* angle at the same time. The construction and attempted proof are as follows.

Let ABC be the given triangle; let AB be the greatest side and BC the least; therefore C is the greatest angle and A the least.

From A draw AD to the middle point of BC , and produce AD to C' , making AC' equal to AB .

Produce AB to B' , making AB' equal to twice AD .

The triangle $AB'C'$ is then such that the sum of its three angles is equal to the sum of the three angles of the triangle ABC .



For take AK along AB equal to AD , and join $C'K$.

Then the triangles ABD , $AC'K$ have two sides and the included angles respectively equal, and are therefore equal in all respects; and $C'K$ is equal to BD or DC .

Next, in the triangles $B'C'K$, ACD , the angles $B'KC'$, ADC are equal, being respectively supplementary to the equal angles AKC' , ADB ; and the two sides about the equal angles are respectively equal;

therefore the triangles $B'C'K$, ACD are equal in all respects.

Thus the angle $AC'B'$ is the sum of two angles respectively equal to the angles B , C of the original triangle; and the angle A in the original triangle is the sum of two angles respectively equal to the angles at A and B' in the triangle $AB'C'$.

It follows that the sum of the three angles of the new triangle $AB'C'$ is equal to the sum of the angles of the triangle ABC .

Moreover, the side AC' , being equal to AB , and therefore greater than AC , is greater than $B'C'$ which is equal to AC .

Hence the angle $C'AB'$ is less than the angle $AB'C'$; so that the angle $C'AB'$ is less than $\frac{1}{2}A$, where A denotes the angle CAB of the original triangle.

[It will be observed that the triangle $AB'C'$ is really the same triangle as the triangle AEB obtained by the construction of Eucl. 1. 16, but differently placed so that the longest side lies along AB .]

By taking the middle point D' of the side $B'C'$ and repeating the same construction, we obtain a triangle $AB''C''$ such that (1) the sum of its three angles is equal to the sum of the three angles of ABC ; (2) the sum of the

two angles $C''AB''$, $AB''C''$ is equal to the angle $C'AB'$ in the preceding triangle, and is therefore less than $\frac{1}{2}A$, and (3) the angle $C''AB''$ is less than half the angle $C'AB'$, and therefore less than $\frac{1}{4}A$.

Continuing in this way, we shall obtain a triangle Abc such that the sum of two angles, those at A and b , is less than $\frac{1}{2^n}A$, and the angle at c is greater than the corresponding angle in the preceding triangle.

If, Legendre argues, the construction be continued indefinitely so that $\frac{1}{2^n}A$ becomes smaller than any assigned angle, the point c ultimately lies on Ab , and the sum of the three angles of the triangle (which is equal to the sum of the three angles of the original triangle) becomes identical with the angle at c , which is then a *flat* angle, and therefore equal to two right angles.

This proof was however shown to be unsound (in respect of the final inference) by J. P. W. Stein in Gergonne's *Annales de Mathématiques* xv., 1824, pp. 77—79.

We will now reproduce shortly the substance of the theorems of Legendre which are of the most permanent value as not depending on a particular hypothesis as regards parallels.

I. *The sum of the three angles of a triangle cannot be greater than two right angles.*

This Legendre proved in two ways.

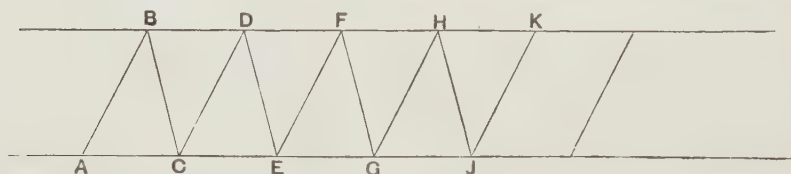
(1) *First proof* (in the third edition of the *Éléments*).

Let ABC be the given triangle, and ACJ a straight line.

Make CE equal to AC , the angle DCE equal to the angle BAC , and DC equal to AB . Join DE .

Then the triangle DCE is equal to the triangle BAC in all respects.

If then the sum of the three angles of the triangle ABC is greater than



$2R$, the said sum must be greater than the sum of the angles BCA , BCD , DCE , which sum is *equal* to $2R$.

Subtracting the equal angles on both sides, we have the result that

the angle ABC is *greater* than the angle BCD .

But the two sides AB , BC of the triangle ABC are respectively equal to the two sides DC , CB of the triangle BCD .

Therefore the base AC is *greater* than the base BD (Eucl. I. 24).

Next, make the triangle FEG (by the same construction) equal in all respects to the triangle BAC or DCE ; and we prove in the same way that CE (or AC) is *greater* than DF .

And, at the same time, BD is equal to DF , because the angles BCD , DEF are equal.

Continuing the construction of further triangles, however small the difference between AC and BD is, we shall ultimately reach some multiple

of this difference, represented in the figure by (say) the difference between the straight line AJ and the composite line $BDFHK$, which will be greater than any assigned length, and greater therefore than the sum of AB and JK .

Hence, on the assumption that the sum of the angles of the triangle ABC is greater than $2R$, the broken line $ABDFHKJ$ may be less than the straight line AJ : which is impossible.

Therefore etc.

(2) *Proof substituted later.*

If possible, let $2R + \alpha$ be the sum of the three angles of the triangle ABC , of which α is not greater than either of the others.

Bisect BC at H , and produce AH to D , making HD equal to AH ; join BD .

Then the triangles AHC , DHB are equal in all respects (I. 4); and the angles CAH , ACH are respectively equal to the angles BDH , DBH .

It follows that the sum of the angles of the triangle ABD is equal to the sum of the angles of the original triangle, i.e. to $2R + \alpha$.

And one of the angles DAB , ADB is either equal to or less than half the angle CAB .

Continuing the same construction with the triangle ADB , we find a third triangle in which the sum of the angles is still $2R + \alpha$, while one of them is equal to or less than $(\angle CAB)/4$.

Proceeding in this way, we arrive at a triangle in which the sum of the angles is $2R + \alpha$, and one of them is not greater than $(\angle CAB)/2^n$.

And, if n is sufficiently large, this will be less than α ; in which case we should have a triangle in which two angles are together greater than two right angles: which is absurd.

Therefore α is equal to or less than zero.

(It will be noted that in both these proofs, as in Eucl. I. 16; it is taken for granted that a straight line is infinite in length and does not return into itself, which is not true under the Riemann hypothesis.)

II. On the assumption that the sum of the angles of a triangle is less than two right angles, if a triangle is made up of two others, the "deficit" of the former is equal to the sum of the "deficits" of the others.

In fact, if the sums of the angles of the component triangles are $2R - \alpha$, $2R - \beta$ respectively, the sum of the angles of the whole triangle is

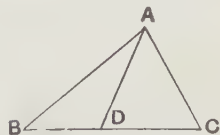
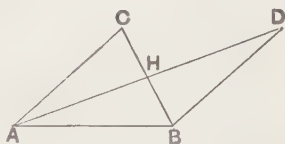
$$(2R - \alpha) + (2R - \beta) - 2R = 2R - (\alpha + \beta).$$

III. If the sum of the three angles of a triangle is equal to two right angles, the same is true of all triangles obtained by subdividing it by straight lines drawn from a vertex to meet the opposite side.

Since the sum of the angles of the triangle ABC is equal to $2R$, if the sum of the angles of the triangle ABD were $2R - \alpha$, it would follow that the sum of the angles of the triangle ADC must be $2R + \alpha$, which is absurd (by I. above).

IV. If in a triangle the sum of the three angles is equal to two right angles, a quadrilateral can always be constructed with four right angles and four equal sides exceeding in length any assigned rectilineal segment.

Let ABC be a triangle in which the sum of the angles is equal to two



right angles. We can assume ABC to be an *isosceles right-angled* triangle because we can reduce the case to this by making subdivisions of ABC by straight lines through vertices (as in Prop. III. above).

Taking two equal triangles of this kind and placing their hypotenuses together, we obtain a quadrilateral with four right angles and four equal sides.

Putting four of these quadrilaterals together, we obtain a new quadrilateral of the same kind but with its sides double of those of the first quadrilateral.

After n such operations we have a quadrilateral with four right angles and four equal sides, each being equal to 2^n times the side AB .

The diagonal of this quadrilateral divides it into two equal isosceles right-angled triangles in each of which the sum of the angles is equal to two right angles.

Consequently, from the existence of *one* triangle in which the sum of the three angles is equal to two right angles it follows that there exists an isosceles right-angled triangle with sides greater than any assigned rectilinear segment and such that the sum of its three angles is also equal to two right angles.

V. *If the sum of the three angles of one triangle is equal to two right angles, the sum of the three angles of any other triangle is also equal to two right angles.*

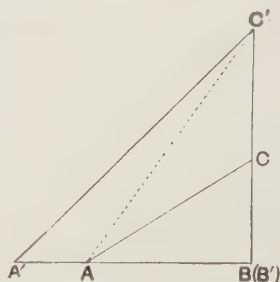
It is enough to prove this for a *right-angled* triangle, since any triangle can be divided into two right-angled triangles.

Let ABC be any right-angled triangle.

If then the sum of the angles of any one triangle is equal to two right angles, we can construct (by the preceding Prop.) an isosceles right-angled triangle with the same property and with its perpendicular sides greater than those of ABC .

Let $A'B'C'$ be such a triangle, and let it be applied to ABC , as in the figure.

Applying then Prop. III. above, we deduce first that the sum of the three angles of the triangle $A'B'C'$ is equal to two right angles, and next, for the same reason, that the sum of the three angles of the original triangle ABC is equal to two right angles.



VI. *If in any one triangle the sum of the three angles is less than two right angles, the sum of the three angles of any other triangle is also less than two right angles.*

This follows from the preceding theorem.

(It will be observed that the last two theorems are included among those of Saccheri, which contain however in addition the corresponding theorem touching the case where the sum of the angles is *greater* than two right angles.)

We come now to the bearing of these propositions upon Euclid's Postulate 5; and the next theorem is

VII. *If the sum of the three angles of a triangle is equal to two right angles, through any point in a plane there can only be drawn one parallel to a given straight line.*

For the proof of this we require the following

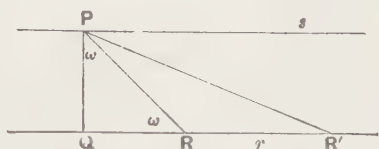
LEMMA. *It is always possible, through a point P , to draw a straight line which shall make, with a given straight line (r), an angle less than any assigned angle.*

Let Q be the foot of the perpendicular from P upon r .

Let a segment QR be taken on r , on either side of Q , such that QR is equal to PQ .

Join PR , and mark off the segment RR' equal to PR ; join PR' .

If ω represents the angle QPR or the angle QRP , each of the equal angles RPR' , $RR'P$ is not greater than $\omega/2$.



Continuing the construction, we obtain, after the requisite number of operations, a triangle $PR_{n-1}R_n$ in which each of the equal angles is equal to or less than $\omega/2^n$.

Hence we shall arrive at a straight line PR_n which, starting from P and meeting r , makes with r an angle as small as we please.

To return now to the Proposition. Draw from P the straight line s perpendicular to PQ .

Then any straight line drawn from P which meets r in R will form equal angles with r and s , since, by hypothesis, the sum of the angles of the triangle PQR is equal to two right angles.

And since, by the Lemma, it is always possible to draw through P straight lines which form with r angles as small as we please, it follows that all the straight lines through P , except s , will meet r . Hence s is the only parallel to r that can be drawn through P .

The history of the attempts to prove Postulate 5 or something equivalent has now been brought down to the parting of the ways. The further developments on lines independent of the Postulate, beginning with Schweikart (1780—1857), Taurinus (1794—1874), Gauss (1777—1855), Lobachewsky (1793—1856), J. Bolyai (1802—1860), Riemann (1826—1866), belong to the history of non-Euclidean geometry, which is outside the scope of this work. I may refer the reader to the full article *Sulla teoria delle parallele e sulle geometrie non-euclidee* by R. Bonola in *Questioni riguardanti le matematiche elementari*, 1., of which I have made considerable use in the above, to the same author's *La geometria non-euclidea*, Bologna, 1906, to the first volume of Killing's *Einführung in die Grundlagen der Geometrie*, Paderborn, 1893, to P. Mansion's *Premiers principes de métalogéométrie*, and P. Barbarin's *La géométrie non-Euclidienne*, Paris, 1902, to the historical summary in Veronese's *Fondamenti di geometria*, 1891, p. 565 sqq., and (for original sources) to Engel and Stäckel, *Die Theorie der Parallellinien von Euklid bis auf Gauss*, 1895, and *Urkunden zur Geschichte der nicht-Euklidischen Geometrie*, 1. (Lobachewsky), 1899, and 11. (Wolfgang und Johann Bolyai). I will only add that it was Gauss who first expressed a conviction that the Postulate could never be proved; he indicated this in reviews in the *Göttingische gelehrte Anzeigen*, 20 Apr. 1816 and 28 Oct. 1822, and affirmed it in a letter to Bessel of 27 January, 1829. The actual indemonstrability of the Postulate was proved by Beltrami (1868) and by Hoüel (*Note sur l'impossibilité de démontrer par une construction plane le principe de la théorie des parallèles dit Postulat d'Euclide* in Battaglini's *Giornale di matematica*, viii., 1870, pp. 84—89).

Alternatives for Postulate 5.

It may be convenient to collect here a few of the more noteworthy substitutes which have from time to time been formally suggested or tacitly assumed.

(1) *Through a given point only one parallel can be drawn to a given straight line or, Two straight lines which intersect one another cannot both be parallel to one and the same straight line.*

This is commonly known as "Playfair's Axiom," but it was of course not a new discovery. It is distinctly stated in Proclus' note to Eucl. I. 31.

(1 a) *If a straight line intersect one of two parallels, it will intersect the other also* (Proclus).

(1 b) *Straight lines parallel to the same straight line are parallel to one another.*

The forms (1 a) and (1 b) are exactly equivalent to (1).

(2) *There exist straight lines everywhere equidistant from one another* (Posidonius and Geminus); with which may be compared Proclus' tacit assumption that *Parallels remain, throughout their length, at a finite distance from one another.*

(3) *There exists a triangle in which the sum of the three angles is equal to two right angles* (Legendre).

(4) *Given any figure, there exists a figure similar to it of any size we please* (Wallis, Carnot, Laplace).

Saccheri points out that it is not necessary to assume so much, and that it is enough to postulate that *there exist two unequal triangles with equal angles.*

(5) *Through any point within an angle less than two-thirds of a right angle a straight line can always be drawn which meets both sides of the angle* (Legendre).

With this may be compared the similar axiom of Lorenz (*Grundriss der reinen und angewandten Mathematik*, 1791): *Every straight line through a point within an angle must meet one of the sides of the angle.*

(6) *Given any three points not in a straight line, there exists a circle passing through them* (Legendre, W. Bolyai).

(7) *"If I could prove that a rectilinear triangle is possible the content of which is greater than any given area, I am in a position to prove perfectly rigorously the whole of geometry"* (Gauss, in a letter to W. Bolyai, 1799).

Cf. the proposition of Legendre numbered iv. above, and the axiom of Worpitzky: *There exists no triangle in which every angle is as small as we please.*

(8) *If in a quadrilateral three angles are right angles, the fourth angle is a right angle also* (Clairaut, 1741).

(9) *If two straight lines are parallel, they are figures opposite to (or the reflex of) one another with respect to the middle points of all their transversal segments* (Veronese, *Elementi*, 1904).

Or, *Two parallel straight lines intercept, on every transversal which passes through the middle point of a segment included between them, another segment the middle point of which is the middle point of the first* (Ingrami, *Elementi*, 1904).

Veronese and Ingrami deduce immediately Playfair's Axiom.

AXIOMS OR COMMON NOTIONS.

In a paper *Sur l'authenticité des axiomes d'Euclide* in the *Bulletin des sciences math. et astron.* 1884, p. 162 sq. (*Mémoires scientifiques*, II., pp. 48—63), Paul Tannery maintained that the *Common Notions* (including the first three) were not in Euclid's work but were interpolated later. The following are his main arguments. (1) If Euclid had set about distinguishing between indemonstrable principles (*a*) common to all demonstrative sciences and (*b*) peculiar to geometry, he would, says Tannery, certainly not have placed the common principles second and the special principles (the Postulates) first. (2) If the *Common Notions* are Euclid's, this designation of them must be his too; for he must have used *some* name to distinguish them from the Postulates and, if he had used another name, such as *Axioms*, it is impossible to imagine why that name was changed afterwards for a less suitable one. The word *ἐννοια* (*notion*), says Tannery, never signified a notion in the sense of a *proposition*, but a notion of some *object*; nor is it found in any technical sense in Plato and Aristotle. (3) Tannery's own view was that the formulation of the *Common Notions* dates from the time of Apollonius, and that it was inspired by his work relating to the *Elements* (we know from Proclus that Apollonius tried to prove the *Common Notions*). This idea, Tannery thought, was confirmed by a "fortunate coincidence" furnished by the occurrence of the word *ἐννοια* (*notion*) in a quotation by Proclus (p. 100, 6): "we shall agree with Apollonius when he says that we have a *notion* (*ἐννοίαν*) of a line when we order the lengths, only, of roads or walls to be measured."

In reply to argument (1) that it is an unnatural order to place the purely geometrical Postulates first, and the *Common Notions*, which are not peculiar to geometry, last, it may be pointed out that it would surely have been a still more awkward arrangement to give the Definitions first and then to separate from them, by the interposition of the *Common Notions*, the Postulates, which are so closely connected with the Definitions in that they proceed to postulate the *existence* of certain of the things defined, namely straight lines and circles.

(2) Though it is true that *ἐννοια* in Plato and Aristotle is generally a notion of an *object*, not of a *fact* or proposition, there are instances in Aristotle where it does mean a notion of a fact: thus in the *Eth. Nic.* IX. II, 1171a 32 he speaks of "the notion (or consciousness) *that friends sympathise*" (*ἡ ἐννοια τοῦ συναλγεῖν τοὺς φίλους*) and again, b 14, of "the *notion* (or consciousness) *that they are pleased* at his good fortune." It is true that Plato and Aristotle do not use the word in a technical sense; but neither was there apparently in Aristotle's time any fixed technical term for what we call "axioms," since he speaks of them variously as "the so-called axioms in mathematics," "the so-called common axioms," "the common (things)" (*τὰ κοινά*), and even "the common *opinions*" (*κοινὰ δόξαι*). I see therefore no reason why Euclid should not himself have given a technical sense to "Common Notions," which is at least a distinct improvement upon "common opinions."

(3) The use of *ἐννοια* in Proclus' quotation from Apollonius seems to me to be an unfortunate, rather than a fortunate, coincidence from Tannery's point of view, for it is there used precisely in the old sense of the notion of an *object* (in that case a line).

No doubt it is difficult to feel certain that Euclid did himself use the term *Common Notions*, seeing that Proclus' commentary generally speaks of *Axioms*. But even Proclus (p. 194, 8), after explaining the meaning of the word "axiom," first as used by the Stoics, and secondly as used by "Aristotle and

the geometers," goes on to say: "For in their view (that of Aristotle and the geometers) *axiom* and *common notion* are the same thing." This, as it seems to me, may be a sort of apology for using the word "axiom" exclusively in what has gone before, as if Proclus had suddenly bethought himself that he had described both Aristotle and the geometers as using the one term "axiom," whereas he should have said that Aristotle spoke of "axioms," while "the geometers" (in fact Euclid), though meaning the same thing, called them *Common Notions*. It may be for a like reason that in another passage (p. 76, 16), after quoting Aristotle's view of an "axiom," as distinct from a postulate and a hypothesis, he proceeds: "For it is not by virtue of a *common notion* that, without being taught, we preconceive the circle to be such and such a figure." If this view of the two passages just quoted is correct, it would strengthen rather than weaken the case for the genuineness of *Common Notions* as the Euclidean term.

Again, it is clear from Aristotle's allusions to the "common axioms in mathematics" that more than one axiom of this kind had a place in the text-books of his day; and as he constantly quotes the particular axiom that, *if equals be taken from equals, the remainders are equal*, which is Euclid's *Common Notion* 3, it would seem that at least the first three *Common Notions* were adopted by Euclid from earlier text-books. It is, besides, scarcely credible that, if the *Common Notions* which Apollonius tried to prove had not been introduced earlier (e.g. by Euclid), they would then have been interpolated as axioms and not as propositions to be proved. The line taken by Apollonius is much better explained on the assumption that he was directly attacking axioms which he found already admitted into the *Elements*.

Proclus, who recognised the five *Common Notions* given in the text, warns us, not only against the error of unnecessarily multiplying the axioms, but against the contrary error of reducing their number unduly (p. 196, 15), "as Heron does in enunciating three only; for it is also an axiom that *the whole is greater than the part*, and indeed the geometer employs this in many places for his demonstrations, and again that *things which coincide are equal*."

Thus Heron recognised the first three of the *Common Notions*; and this fact, together with Aristotle's allusions to "common axioms" (in the plural), and in particular to our *Common Notion* 3, may satisfy us that at least the first three *Common Notions* were contained in the *Elements* as they left Euclid's hands.

COMMON NOTION I.

Τὰ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα.

Things which are equal to the same thing are also equal to one another.

Aristotle throughout emphasises the fact that axioms are self-evident truths, which it is impossible to demonstrate. If, he says, any one should attempt to prove them, it could only be through ignorance. Aristotle therefore would undoubtedly have agreed in Proclus' strictures on Apollonius for attempting to prove the axioms. Proclus gives (p. 194, 25), as a specimen of these attempted proofs by Apollonius, that of the first of the *Common Notions*. "Let *A* be equal to *B*, and the latter to *C*; I say that *A* is also equal to *C*. For, since *A* is equal to *B*, it occupies the same space with it; and since *B* is equal to *C*, it occupies the same space with it.

A	B	C
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Therefore *A* also occupies the same space with *C*."

Proclus rightly remarks (p. 194, 22) that "the middle term is no more

intelligible (better known, γνωριμώτερον) than the conclusion, if it is not actually more disputable." Again (p. 195, 6), the proof assumes two things, (1) that things which "occupy the same space" (τόπος) are equal to one another, and (2) that things which occupy the same space with one and the same thing occupy the same space with one another; which is to explain the obvious by something much more obscure, for space is an entity more unknown to us than the things which exist in space.

Aristotle would also have objected to the proof that it is partial and not general (καθόλου), since it refers only to things which can be supposed to occupy a space (or take up room), whereas the axiom is, as Proclus says (p. 196, 1), true of numbers, speeds, and periods of time as well, though of course each science uses axioms in relation to its own subject-matter only.

COMMON NOTIONS 2, 3.

2. Καὶ ἐὰν ἴσοις ἴσα προστεθῇ, τὰ ὅλα ἐστὶν ἴσα.
3. Καὶ ἐὰν ἀπὸ ἴσων ἴσα ἀφαιρεθῇ, τὰ καταλειπόμενά ἐστιν ἴσα.
2. *If equals be added to equals, the wholes are equal.*
3. *If equals be subtracted from equals, the remainders are equal.*

These two Common Notions are recognised by Heron and Proclus as genuine. The latter is the axiom which is so favourite an illustration with Aristotle.

Following them in the MSS. and editions there came four others of the same type as 1—3. Three of these are given by Heiberg in brackets; the fourth he omits altogether.

The three are:

- (a) *If equals be added to unequals, the wholes are unequal.*
- (b) *Things which are double of the same thing are equal to one another.*
- (c) *Things which are halves of the same thing are equal to one another.*

The fourth, which was placed between (a) and (b), was:

- (d) *If equals be subtracted from unequals, the remainders are unequal.*

Proclus, in observing that axioms ought not to be multiplied, indicates that all should be rejected which follow from the five admitted by him and appearing in the text above (p. 155). He mentions the second of those just quoted (b) as one of those to be excluded, since it follows from *Common Notion 1*. Proclus does not mention (a), (c) or (d); an-Nairizi gives (a), (d), (b) and (c), in that order, as Euclid's, adding a note of Simplicius that "three axioms (sententiae acceptae) only are extant in the ancient manuscripts, but the number was increased in the more recent."

(a) stands self-condemned because "unequal" tells us nothing. It is easy to see what is wanted if we refer to 1. 17, where the same angle is added to a *greater* and a *less*, and it is inferred that the first sum is greater than the second. So far however as the wording of (a) is concerned, the addition of equal to *greater* and *less* might be supposed to produce *less* and *greater* respectively. If therefore such an axiom were given at all, it should be divided into two. Heiberg conjectures that this axiom may have been taken from the commentary of Pappus, who had the axiom about equals added to unequals quoted below (e); if so, it can only be an unskilful adaptation of some remark of Pappus, for his axiom (e) has some point, whereas (a) is useless.

As regards (b), I agree with Tannery in seeing no sufficient reason why, if

we reject it (as we certainly must), the words in 1. 47 "But things which are double of equals are equal to one another" should be condemned as an interpolation. If they were interpolated, we should have expected to find the same interpolation in 1. 42, where the axiom is *tacitly* assumed. I think it quite possible that Euclid may have inserted such words in one case and left them out in another, without necessarily implying either that he was quoting a formal *Common Notion* of his own or that he had *not* included among his Common Notions the particular fact stated as obvious.

The corresponding axiom (c) about the *halves* of equals can hardly be genuine if (b) is not, and Proclus does not mention it. Tannery acutely observes however that, when Heiberg, in 1. 37, 38, brackets words stating that "the halves of equal things are equal to one another" on the ground that axiom (c) was interpolated (although before Theon's time), and explains that Euclid used *Common Notion* 3 in making his inference, he is clearly mistaken. For, while axiom (b) is an obvious inference from *Common Notion* 2, axiom (c) is not an inference from *Common Notion* 3. Tannery says, in a note, that (c) would have to be established by *reductio ad absurdum* with the help of axiom (b), that is to say, of *Common Notion* 2. But, as the hypothesis in the *reductio ad absurdum* would be that one of the halves is *greater* than the other, and it would therefore be necessary to prove that the one whole is *greater* than the other, while axiom (b) or *Common Notion* 2 only refers to *equals*, a little argument would be necessary in addition to the reference to *Common Notion* 2. I think Euclid would not have gone through this process in order to prove (c), but would have assumed it as equally obvious with (b).

Proclus (pp. 197, 6—198, 5) definitely rejects two other axioms of the above kind given by Pappus, observing that, as they follow from the genuine axioms, they are rightly omitted in most copies, although Pappus said that they were "on record" with the others (συναναγράφεσθαι):

(e) *If unequals be added to equals, the difference between the wholes is equal to the difference between the added parts*; and

(f) *If equals be added to unequals, the difference between the wholes is equal to the difference between the original unequals.*

Proclus and Simplicius (in an-Nairizi) give proofs of both. The proof of the former, as given by Simplicius, is as follows:

Let AB , CD be equal magnitudes; and let EB , FD be added to them respectively, EB being greater than FD .

I say that AE exceeds CF by the same difference as that by which BE exceeds DF .

Cut off from BE the magnitude BG equal to DF .

Then, since AE exceeds AG by GE , and AG is equal to CF and BG to DF ,

AE exceeds CF by the same difference as that by which BE exceeds DF .



COMMON NOTION 4.

Καὶ τὰ ἐφαρμόζοντα ἐπ' ἀλλήλα ἴσα ἀλλήλοις ἐστίν.

Things which coincide with one another are equal to one another.

The word *ἐφαρμόζειν*, as a geometrical term, has a different meaning according as it is used in the active or in the passive. In the passive, *ἐφαρμόζεσθαι*, it means "to be *applied* to" without any implication that the applied figure will exactly fit, or coincide with, the figure to which it is applied; on the other hand the active *ἐφαρμόζειν* is used intransitively and means "to

fit exactly," "to coincide with." In Euclid and Archimedes ἐφαρμόζειν is constructed with ἐπί and the accusative, in Pappus with the dative.

On *Common Notion 4* Tannery observes that it is incontestably geometrical in character, and should therefore have been excluded from the *Common Notions*; again, it is difficult to see why it is not accompanied by its converse, at all events for straight lines (and, it might be added, angles also), which Euclid makes use of in 1. 4. As it is, says Tannery, we have here a definition of geometrical equality more or less sufficient, but not a real axiom.

It is true that Proclus seems to recognise this *Common Notion* and the next as proper axioms in the passage (p. 196, 15—21) where he says that we should not cut down the axioms to the minimum, as Heron does in giving only three axioms; but the statement seems to rest, not upon authority, but upon an assumption that Euclid would state explicitly at the beginning all axioms subsequently used and not reducible to others unquestionably included. Now in 1. 4 this *Common Notion* is not quoted; it is simply inferred that "the base BC will coincide with EF , and will be equal to it." The position is therefore the same as it is in regard to the statement in the same proposition that, "if... the base BC does not coincide with EF , two straight lines will enclose a space: which is impossible"; and, if we do not admit that Euclid had the axiom that "two straight lines cannot enclose a space," neither need we infer that he had *Common Notion 4*. I am therefore inclined to think that the latter is more likely than not to be an interpolation.

It seems clear that the *Common Notion*, as here formulated, is intended to assert that superposition is a legitimate way of proving the equality of two figures which have the necessary parts respectively equal, or, in other words, to serve as an axiom of congruence.

The phraseology of the propositions, e.g. 1. 4 and 1. 8, in which Euclid employs the method indicated, leaves no room for doubt that he regarded one figure as actually moved and placed upon the other. Thus in 1. 4 he says, "The triangle ABC being applied (*ἐφαρμοζομένον*) to the triangle DEF , and the point A being placed (*τιθεμένον*) upon the point D , and the straight line AB on DE , the point B will also coincide with E because AB is equal to DE "; and in 1. 8, "If the sides BA , AC do not coincide with ED , DF , but fall beside them (take a different position, *παράλλασουσιν*), then" etc. At the same time, it is clear that Euclid disliked the method and avoided it wherever he could, e.g. in 1. 26, where he proves the equality of two triangles which have two angles respectively equal to two angles and one side of the one equal to the corresponding side of the other. It looks as though he found the method handed down by tradition (we can hardly suppose that, if Thales proved that the diameter of a circle divides it into two equal parts, he would do so by any other method than that of superposition), and followed it, in the few cases where he does so, only because he had not been able to see his way to a satisfactory substitute. But seeing how much of the *Elements* depends on 1. 4, directly or indirectly, the method can hardly be regarded as being, in Euclid, of only subordinate importance; on the contrary, it is fundamental. Nor, as a matter of fact, do we find in the ancient geometers any expression of doubt as to the legitimacy of the method. Archimedes uses it to prove that any spheroidal figure cut by a plane through the centre is divided into two equal parts in respect of both its surface and its volume; he also postulates in *Equilibrium of Planes* 1. that "when equal and similar plane figures coincide if applied to one another, their centres of gravity coincide also."

Killing: (*Einführung in die Grundlagen der Geometrie*, II. pp. 4, 5)

contrasts the attitude of the Greek geometers with that of the philosophers, who, he says, appear to have agreed in banishing motion from geometry altogether. In support of this he refers to the view frequently expressed by Aristotle that mathematics has to do with *immovable* objects (*ἀκίνητα*), and that only where astronomy is admitted as part of mathematical science is motion mentioned as a subject for mathematics. Cf. *Metaph.* 989 b 32 "For mathematical objects are among things which exist apart from motion, except such as relate to astronomy"; *Metaph.* 1064 a 30 "Physics deals with things which have in themselves the principle of motion; mathematics is a theoretical science and one concerned with things which are *stationary* (*μένοντα*) but not separable" (sc. from matter); in *Physics* II. 2, 193 b 34 he speaks of the subjects of mathematics as "in thought separable from motion."

But I doubt whether in Aristotle's use of the words "immovable," "without motion" etc. as applied to the subjects of mathematics there is any implication such as Killing supposes. We arrive at mathematical concepts by abstraction from material objects; and just as we, in thought, eliminate the matter, so according to Aristotle we eliminate the attributes of matter as such, e.g. qualitative change and *motion*. It does not appear to me that the use of "immovable" in the passages referred to means more than this. I do not think that Aristotle would have regarded it as illegitimate to *move* a geometrical figure from one position to another; and I infer this from a passage in *De caelo* III. 1 where he is criticising "those who make up every body that has an origin by putting together *planes*, and resolve it again into *planes*." The reference must be to the *Timaeus* (54 B sqq.) where Plato evolves the four elements in this way. He begins with a right-angled triangle in which the hypotenuse is double of the smaller side; six of these put together in the proper way produce one equilateral triangle. Making solid angles with (a) three, (b) four, and (c) five of these equilateral triangles respectively, and taking the requisite number of these solid angles, namely four of (a), six of (b) and twelve of (c) respectively, and putting them together so as to form regular solids, he obtains (α) a tetrahedron, (β) an octahedron, (γ) an icosahedron respectively. For the fourth element (earth), four isosceles right-angled triangles are first put together so as to form a square, and then six of these squares are put together to form a cube. Now, says Aristotle (299 b 23), "it is absurd that planes should only admit of being put together so as to touch in a *line*; for just as a line and a line are put together in both ways, lengthwise and breadthwise, so must a plane and a plane. A line can be combined with a line in the sense of being a line *superposed*, and not *added*"; the inference being that a *plane* can be superposed on a *plane*. Now this is precisely the sort of motion in question here; and Aristotle, so far from denying its permissibility, seems to blame Plato for not using it. Cf. also *Physics* v. 4, 228 b 25, where Aristotle speaks of "the spiral or other magnitude in which any part will not coincide with any other part," and where superposition is obviously contemplated.

Motion without deformation.

It is well known that Helmholtz maintained that geometry requires us to assume the actual existence of rigid bodies and their free mobility in space, whence he inferred that geometry is dependent on mechanics.

Veronese exposed the fallacy in this (*Fondamenti di geometria*, pp. xxxv—xxxvi, 239—240 note, 615—7), his argument being as follows. Since geometry is concerned with empty space, which is immovable, it would be at least strange if it was necessary to have recourse to the real motion of bodies for a definition,

and for the proof of the properties, of immovable space. We must distinguish the intuitive principle of motion in itself from that of motion *without deformation*. Every point of a figure which moves is transferred to another point in space. "Without deformation" means that the mutual relations between the points of the figure do not change, but the relations between them and other figures do change (for if they did not, the figure could not move). Now consider what we mean by saying that, when the figure A has moved from the position A_1 to the position A_2 , the relations between the points of A in the position A_2 are unaltered from what they were in the position A_1 , are the same in fact as if A had not moved but remained at A_1 . We can only say that, judging of the figure (or the body with its physical qualities eliminated) by the impressions it produces in us during its movement, the impressions produced in us in the two different positions (which are in time distinct) *are equal*. In fact, we are making use of the notion of *equality* between two distinct figures. Thus, if we say that two bodies are equal when they can be superposed by means of *movement without deformation*, we are committing a *petitio principii*. The notion of the equality of spaces is really prior to that of rigid bodies or of motion without deformation. Helmholtz supported his view by reference to the process of measurement in which the measure must be, at least approximately, a rigid body, but the existence of a rigid body as a standard to measure by, and the question how we discover two equal spaces to be equal, are matters of no concern to the geometer. The method of superposition, depending on motion without deformation, is only of use as a *practical* test; it has nothing to do with the *theory* of geometry.

Compare an acute observation of Schopenhauer (*Die Welt als Wille*, 2 ed. 1844, II. p. 130) which was a criticism in advance of Helmholtz' theory: "I am surprised that, instead of the eleventh axiom [the Parallel-Postulate], the eighth is not rather attacked: 'Figures which coincide (sich decken) are equal to one another.' For *coincidence* (das Sichdecken) is either mere tautology, or something entirely empirical, which belongs, not to pure intuition (Anschauung), but to external sensuous experience. It presupposes in fact the mobility of figures; but that which is movable in space is matter and nothing else. Thus this appeal to coincidence means leaving pure space, the sole element of geometry, in order to pass over to the material and empirical."

Mr Bertrand Russell observes (*Encyclopaedia Britannica*, Suppl. Vol. 4, 1902, Art. "Geometry, non-Euclidean") that the apparent use of motion here is deceptive; what in geometry is called a motion is merely the transference of our attention from one figure to another. Actual superposition, which is nominally employed by Euclid, is not required; all that is required is the transference of our attention from the original figure to a new one defined by the position of some of its elements and by certain properties which it shares with the original figure.

If the method of superposition is given up as a means of defining theoretically the equality of two figures, some other definition of equality is necessary. But such a definition can be evolved out of *empirical* or *practical* observation of the result of superposing two material representations of figures. This is done by Veronese (*Elementi di geometria*, 1904) and Ingrami (*Elementi di geometria*, 1904). Ingrami says, namely (p. 66):

"If a sheet of paper be folded double, and a triangle be drawn upon it and then cut out, we obtain two triangles *superposed* which we in practice call *equal*. If points $A, B, C, D \dots$ be marked on one of the triangles, then, when we place this triangle upon the other (so as to coincide with it), we see

that *each* of the particular points taken on the first is superposed on one particular point of the second in such a way that the segments $AB, AC, AD, BC, BD, CD, \dots$ are respectively superposed on as many segments in the second triangle and are therefore equal to them respectively. In this way we justify the following

“Definition of equality.

“Any two figures whatever will be called *equal* when to the points of one the points of the other can be made to correspond *univocally* [i.e. every *one* point in one to *one distinct* point in the other and *vice versa*] in such a way that the segments which join the points, two and two, in one figure are respectively equal to the segments which join, two and two, the corresponding points in the other.”

Ingrami has of course previously postulated as known the signification of the phrase *equal (rectilineal) segments*, of which we get a *practical* notion when we can place one upon the other or can place a third movable segment successively on both.

New systems of Congruence-Postulates.

In the fourth Article of *Questioni riguardanti le matematiche elementari*, I., pp. 93—122, a review is given of three different systems: (1) that of Pasch in *Vorlesungen über neuere Geometrie*, 1882, p. 101 sqq., (2) that of Veronese according to the *Fondamenti di geometria*, 1891, and the *Elementi* taken together, (3) that of Hilbert (see *Grundlagen der Geometrie*, 1903, pp. 7—15).

These systems differ in the particular conceptions taken by the three authors as primary. (1) Pasch considers as primary the notion of *congruence* or *equality* between *any figures which are made up of a finite number of points only*. The definitions of congruent *segments* and of congruent *angles* have to be *deduced* in the way shown on pp. 102—103 of the Article referred to, after which Eucl. I. 4 follows immediately, and Eucl. I. 26 (1) and I. 8 by a method recalling that in Eucl. I. 7, 8.

(2) Veronese takes as primary the conception of congruence between *segments* (rectilineal). The transition to congruent *angles*, and thence to *triangles* is made by means of the following postulate:

“Let AB, AC and $A'B', A'C'$ be two pairs of straight lines intersecting at A, A' , and let there be determined upon them the congruent segments $AB, A'B'$ and the congruent segments $AC, A'C'$;

then, if $BC, B'C'$ are congruent, the two *pairs of straight lines* are congruent.”

(3) Hilbert takes as primary the notions of congruence between *both segments and angles*.

It is shown in the Article referred to that, from the theoretical standpoint, Veronese's system is an advance upon that of Pasch, since the idea of congruence between *segments* is more simple than that of congruence between *any figures*; but, didactically, the development of the theory is more complicated when we start from Veronese's system than when we start from that of Pasch.

The system of Hilbert offers advantages over both the others from the point of view of the teaching of geometry, and I shall therefore give a short account of his system only, following the Article above quoted.

Hilbert's system.

The following are substantially the Postulates laid down.

(1) *If one segment is congruent with another, the second is also congruent with the first.*

(2) *If an angle is congruent with another angle, the second angle is also congruent with the first.*

(3) *Two segments congruent with a third are congruent with one another.*

(4) *Two angles congruent with a third are congruent with one another.*

(5) *Any segment AB is congruent with itself, independently of its sense.*

This we may express symbolically thus :

$$AB \equiv AB \equiv BA.$$

(6) *Any angle (ab) is congruent with itself, independently of its sense.*

This we may express symbolically thus :

$$(ab) \equiv (ab) \equiv (ba).$$

(7) *On any straight line r , starting from any one of its points A' , and on each side of it respectively, there exists one and only one segment congruent with a segment AB belonging to the straight line r .*

(8) *Given a ray a , issuing from a point O, in any plane which contains it and on each of the two sides of it, there exists one and only one ray b issuing from O such that the angle (ab) is congruent with a given angle (a'b').*

(9) *If AB, BC are two consecutive segments of the same straight line r (segments, that is, having an extremity and no other point common), and $A'B'$, $B'C'$ two consecutive segments on another straight line r' , and if $AB \equiv A'B'$, $BC \equiv B'C'$, then*

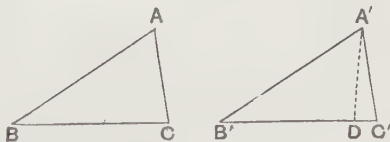
$$AC \equiv A'C'.$$

(10) *If (ab), (bc) are two consecutive angles in the same plane π (angles, that is, having the vertex and one side common), and (a'b'), (b'c') two consecutive angles in another plane π' , and if (ab) $\equiv$ (a'b'), (bc) $\equiv$ (b'c'), then*

$$(ac) \equiv (a'c').$$

(11) *If two triangles have two sides and the included angles respectively congruent, they have also their third sides congruent as well as the angles opposite to the congruent sides respectively.*

As a matter of fact, Hilbert's postulate corresponding to (11) does not assert the equality of the third sides in each, but only the equality of the two remaining angles in one triangle to the two remaining angles in the other respectively. He proves the equality of the third sides (thereby completing the theorem of Eucl. I. 4) by *reductio ad absurdum* thus. Let ABC , $A'B'C'$ be the two triangles which have the sides AB , AC respectively congruent with the sides $A'B'$, $A'C'$ and the included angle at A congruent with the included angle at A' .



Then, by Hilbert's own postulate, the angles ABC , $A'B'C'$ are congruent, as also the angles ACB , $A'C'B'$.

If BC is not congruent with $B'C'$, let D be taken on $B'C'$ such that BC , $B'D$ are congruent. and join $A'D$.

Then the two triangles ABC , $A'B'D$ have two sides and the included angles congruent respectively; therefore, by the same postulate, the angles BAC , $B'A'D$ are congruent.

But the angles BAC , $B'A'C'$ are congruent; therefore, by (4) above, the angles $B'A'C'$, $B'A'D$ are congruent: which is impossible, since it contradicts (8) above.

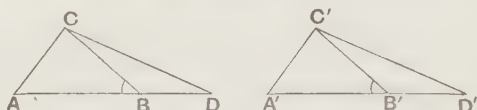
Hence BC , $B'C'$ cannot but be congruent.

Eucl. I. 4 is thus proved; but it seems to be as well to include all of that theorem in the postulate, as is done in (11) above, since the two parts of it are equally suggested by empirical observation of the result of one superposition.

A proof similar to that just given immediately establishes Eucl. I. 26 (1), and Hilbert next proves that

If two angles ABC , $A'B'C'$ are congruent with one another, their supplementary angles CBD , $C'B'D'$ are also congruent with one another.

We choose A , D on one of the straight lines forming the first angle, and A' , D' on one of those forming the second angle, and again C , C' on the other



straight lines forming the angles, so that $A'B'$ is congruent with AB , $C'B'$ with CB , and $D'B'$ with DB .

The triangles ABC , $A'B'C'$ are congruent, by (11) above; and AC is congruent with $A'C'$, and the angle CAB with the angle $C'A'B'$.

Thus, AD , $A'D'$ being congruent, by (9), the triangles CAD , $C'A'D'$ are also congruent, by (11);

whence CD is congruent with $C'D'$, and the angle ADC with the angle $A'D'C'$.

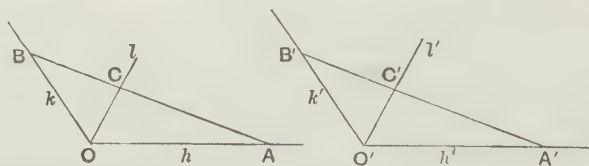
Lastly, by (11), the triangles CDB , $C'D'B'$ are congruent, and the angles CBD , $C'B'D'$ are thus congruent.

Hilbert's next proposition is that

Given that the angle (h, k) in the plane α is congruent with the angle (h', k') in the plane α' , and that l is a half-ray in the plane α starting from the vertex of the angle (h, k) and lying within that angle, there always exists a half-ray l' in the second plane α' , starting from the vertex of the angle (h', k') and lying within that angle, such that

$$(h, l) \equiv (h', l'), \text{ and } (k, l) \equiv (k', l').$$

If O , O' are the vertices, we choose points A , B on h , k , and points A' , B' on h' , k' respectively, such that OA , $O'A'$ are congruent and also OB , $O'B'$.



The triangles OAB , $O'A'B'$ are then congruent; and, if l meets AB in C , we can determine C' on $A'B'$ such that $A'C'$ is congruent with AC .

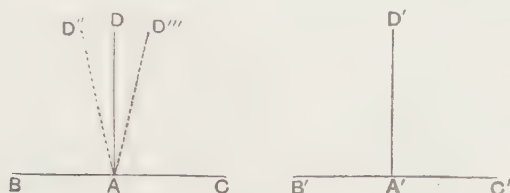
Then l' drawn from O' through C' is the half-ray required.

The congruence of the angles (h, l) , (h', l') follows from (11) directly, and that of (k, l) and (k', l') follows in the same way after we have inferred by means of (9) that, AB, AC being respectively congruent with $A'B', A'C'$, the difference BC is congruent with the difference $B'C'$.

It is by means of the two propositions just given that Hilbert proves that

All right angles are congruent with one another.

Let the angle BAD be congruent with its adjacent angle CAD , and likewise the angle $B'A'D'$ congruent with its adjacent angle $C'A'D'$. All four angles are then right angles.



If the angle $B'A'D'$ is not congruent with the angle BAD , let the angle with AB for one side and congruent with the angle $B'A'D'$ be the angle BAD'' , so that AD'' falls either within the angle BAD or within the angle DAC . Suppose the former.

By the last proposition but one (about adjacent angles), the angles $B'A'D', BAD''$ being congruent, the angles $C'A'D', CAD''$ are congruent.

Hence, by the hypothesis and postulate (4) above, the angles BAD'', CAD'' are also congruent.

And, since the angles BAD, CAD are congruent, we can find within the angle CAD a half-ray CAD''' such that the angles BAD'', CAD''' are congruent, and likewise the angles DAD'', DAD''' (by the last proposition).

But the angles BAD'', CAD'' were congruent (see above); and it follows, by (4), that the angles CAD'', CAD''' are congruent: which is impossible, since it contradicts postulate (8).

Therefore etc.

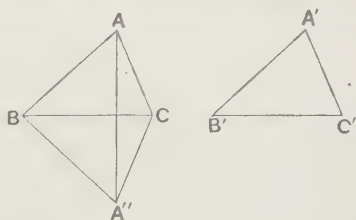
Euclid 1. 5 follows directly by applying the postulate (11) above to ABC, ACB as distinct triangles.

Postulates (9), (10) above give in substance the proposition that "the sums or differences of segments, or of angles, respectively equal, are equal."

Lastly, Hilbert proves Eucl. 1. 8 by means of the theorem of Eucl. 1. 5 and the proposition just stated as applied to angles.

$ABC, A'B'C'$ being the given triangles with three sides respectively congruent, we suppose an angle CBA'' to be determined, on the side of BC opposite to A , congruent with the angle $A'B'C'$, and we make BA'' equal to $A'B'$.

The proof is obvious, being equivalent to the alternative proof often given in our text-books for Eucl. 1. 8.



COMMON NOTION 5.

καὶ τὸ ὅλον τοῦ μέρους μείζον [ἐστίν].

The whole is greater than the part.

Proclus includes this "axiom" on the same ground as the preceding one. I think however there is force in the objection which Tannery takes to it, namely that it replaces a *different* expression in Eucl. I. 6, where it is stated that "the triangle *DBC* will be equal to the triangle *ACB*, the less to the greater: which is absurd." The axiom appears to be an abstraction or generalisation substituted for an immediate inference from a geometrical figure, but it takes the form of a sort of definition of whole and part. The probabilities seem to be against its being genuine, notwithstanding Proclus' approval of it.

Clavius added the axiom that *the whole is the equal to the sum of its parts*.

OTHER AXIOMS INTRODUCED AFTER EUCLID'S TIME.

[9] *Two straight lines do not enclose (or contain) a space.*

Proclus (p. 196, 21) mentions this in illustration of the undue multiplication of axioms, and he points out, as an objection to it, that it belongs to the subject matter of geometry, whereas axioms are of a general character, and not peculiar to any one science. The real objection to the axiom is that it is unnecessary, since the fact which it states is included in the meaning of Postulate 1. It was no doubt taken from the passage in I. 4, "if...the base *BC* does not coincide with the base *EF*, two straight lines will enclose a space: which is impossible"; and we must certainly regard it as an interpolation, notwithstanding that two of the best mss. have it after Postulate 5, and one gives it as *Common Notion* 9.

Pappus added some others which Proclus objects to (p. 198, 5) because they are either anticipated in the definitions or follow from them.

(g) *All the parts of a plane, or of a straight line, coincide with one another.*

(h) *A point divides a line, a line a surface, and a surface a solid; on which Proclus remarks that everything is divided by the same things as those by which it is bounded.*

An-Nairizi (ed. Besthorn-Heiberg, p. 31, ed. Curtze, p. 38) in his version of this axiom, which he also attributes to Pappus, omits the reference to solids, but mentions planes as a particular case of surfaces.

"(a) *A surface cuts a surface in a line;*

(β) *If two surfaces which cut one another are plane, they cut one another in a straight line;*

(γ) *A line cuts a line in a point (this last we need in the first proposition)."*

(k) *Magnitudes are susceptible of the infinite (or unlimited) both by way of addition and by way of successive diminution, but in both cases potentially only (τὸ ἄπειρον ἐν τοῖς μεγέθεσιν ἐστίν καὶ τῇ προσθέσει καὶ τῇ ἐπικαθαιρέσει, δυνάμει δὲ ἐκάτερον).*

An-Nairizi's version of this refers to straight lines and plane surfaces only: "as regards the straight line and the plane surface, in consequence of their evenness, it is possible to produce them indefinitely.

This "axiom" of Pappus, as quoted by Proclus, seems to be taken directly from the discussion of τὸ ἄπειρον in Aristotle, *Physics* III. 5—8, even to the wording, for, while Aristotle uses the term *division* (διαίρεσις) most frequently as the antithesis of *addition* (σύνθεσις), he occasionally speaks of *subtraction* (ἀφαίρεσις) and *diminution* (καθαιρέσις). Hankel (*Zur Geschichte der Mathematik im Alterthum und Mittelalter*, 1874, pp. 119—120) gave an admirable

summary of Aristotle's views on this subject; and they are stated in greater detail in Görland, *Aristoteles und die Mathematik*, Marburg, 1899, pp. 157—183. The infinite or unlimited (*ἄπειρον*) only exists potentially (*δυνάμει*), not in actuality (*ἐνεργείᾳ*). The infinite is so in virtue of its endlessly changing into something else, like day or the Olympic Games (*Phys.* III. 6, 206 a 15—25). The infinite is manifested in different forms in time, in Man, and in the division of magnitudes. For, in general, the infinite consists in something new being continually taken, that something being itself always finite but always different. Therefore the infinite must not be regarded as a particular thing (*τόδε τι*), as man, house, but as being always in course of becoming or decay, and, though finite at any moment, always different from moment to moment. But there is the distinction between the forms above referred to that, whereas in the case of magnitudes what is once taken remains, in the case of time and Man it passes or is destroyed but the succession is unbroken. The case of addition is in a sense the same as that of division; in the finite magnitude the former takes place in the converse way to the latter; for, as we see the finite magnitude divided *ad infinitum*, so we shall find that addition gives a sum tending to a definite limit. I mean that, in the case of a finite magnitude, you may take a definite fraction of it and add to it (continually) in the same ratio; if now the successive added terms do not include one and the same magnitude whatever it is [i.e. if the successive terms diminish in geometrical progression], you will not come to the end of the finite magnitude, but, if the ratio is increased so that each term does include one and the same magnitude whatever it is, you will come to the end of the finite magnitude, for every finite magnitude is exhausted by continually taking from it any definite fraction whatever. Thus in no other sense does the infinite exist, but only in the sense just mentioned, that is, potentially and by way of diminution (206 a 25—b 13). And in this sense you may have potentially infinite addition, the process being, as we say, in a manner, the same as with division *ad infinitum*: for in the case of addition you will always be able to find something outside the total for the time being, but the total will never exceed every definite (or assigned) magnitude in the way that, in the direction of division, the result will pass every definite magnitude, that is, by becoming smaller than it. The infinite therefore cannot exist even potentially in the sense of exceeding every finite magnitude as the result of successive addition (206 b 16—22). It follows that the correct view of the infinite is the opposite of that commonly held: it is not that which has nothing outside it, but that which always has something outside it (206 b 33—207 a 1).

Contrasting the case of number and magnitude, Aristotle points out that (1) in number there is a limit in the direction of smallness, namely unity, but none in the other direction: a number may exceed any assigned number however great; but (2) with magnitude the contrary is the case: you can find a magnitude smaller than any assigned magnitude, but in the other direction there is no such thing as an infinite magnitude (207 b 1—5). The latter assertion he justified by the following argument. However large a thing can be potentially, it can be as large actually. But there is no magnitude perceptible to sense that is infinite. Therefore excess over every assigned magnitude is an impossibility; otherwise there would be something larger than the universe (*οὐρανός*) (207 b 17—21).

Aristotle is aware that it is essentially of physical magnitudes that he is speaking. He had observed in an earlier passage (*Phys.* I.1. 5, 204 a 34) that it is perhaps a more general inquiry that would be necessary to determine

whether the infinite is possible in mathematics, and in the domain of thought and of things which have no magnitude; but he excuses himself from entering upon this inquiry on the ground that his subject is physics and sensible objects. He returns however to the bearing of his conclusions on mathematics in III. 7, 207 b 27: "my argument does not even rob mathematicians of their study, although it denies the existence of the infinite in the sense of actual existence as something increased to such an extent that it cannot be gone through (*ἀδιεξήγητον*); for, as it is, they do not even need the infinite or use it, but only require that the finite (straight line) shall be as long as they please; and another magnitude of any size whatever can be cut in the same ratio as the greatest magnitude. Hence it will make no difference to them for the purpose of demonstration."

Lastly, if it should be urged that the infinite exists in *thought*, Aristotle replies that this does not involve its existence in *fact*. A thing is not greater than a certain size because it is conceived to be so, but because it *is*; and magnitude is not infinite in virtue of increase in thought (208 a 16—22).

Hankel and Görland do not quote the passage about an infinite series of magnitudes (206 b 3—13) included in the above paraphrase; but I have thought that mathematicians would be interested in the distinct expression of Aristotle's view that the existence of an infinite series the terms of which are *magnitudes* is impossible unless it is convergent, and (with reference to Riemann's developments) in the statement that it does not matter to geometry if the straight line is not infinite in length, provided that it is as long as we please.

Aristotle's denial of even the potential existence of a sum of magnitudes which shall exceed every definite magnitude was, as he himself implies, in conflict with the lemma or assumption used by Eudoxus (as we infer from Archimedes) to prove the theorem about the volume of a pyramid. The lemma is thus stated by Archimedes (*Quadrature of a parabola*, preface): "The excess by which the greater of two unequal areas exceeds the less can, if it be continually added to itself, be made to exceed any assigned finite area." We can therefore well understand why, a century later, Archimedes felt it necessary to justify his own use of the lemma as he does in the same preface: "The earlier geometers too have used this lemma: for it is by its help that they have proved that circles have to one another the duplicate ratio of their diameters, that spheres have to one another the triplicate ratio of their diameters, and so on. And, in the result, each of the said theorems has been accepted no less than those proved without the aid of this lemma."

Principle of continuity.

The use of actual construction as a method of proving the existence of figures having certain properties is one of the characteristics of the *Elements*. Now constructions are effected by means of straight lines and circles drawn in accordance with Postulates 1—3; the essence of them is that such straight lines and circles determine by their intersections other points in addition to those given, and these points again are used to determine new lines, and so on. This being so, the *existence* of such points of intersection must be postulated or proved in the same way as that of the lines which determine them. Yet there is no postulate of this character expressed in Euclid except Post. 5. This postulate asserts that two straight lines meet if they satisfy a certain condition. The condition is of the nature of a *διορισμός* (*discrimination*, or condition of possibility) in a problem; and, if the existence of the point of

intersection were not granted, the solutions of problems in which the points of intersection of straight lines are used would not in general furnish the required proofs of the existence of the figures to be constructed.

But, equally with the intersections of straight lines, the intersections of circle with straight line, and of circle with circle, are used in constructions. Hence, in addition to Postulate 5, we require postulates asserting the actual existence of points of intersection of circle with straight line and of circle with circle. In the very first proposition the vertex of the required equilateral triangle is determined as one of the intersections of two circles, and we need therefore to be assured that the circles will intersect. Euclid seems to assume it as obvious, although it is not so; and he makes a similar assumption in 1. 22. It is true that in the latter case Euclid adds to the enunciation that two of the given straight lines must be together greater than the third; but there is nothing to show that, if this condition is satisfied, the construction is always possible. In 1. 12, in order to be sure that the circle with a given centre will intersect a given straight line, Euclid makes the circle pass through a point on the side of the line opposite to that where the centre is. It appears therefore as if, in this case, he based his inference in some way upon the definition of a circle combined with the fact that the point within it called the centre is on one side of the straight line and one point of the circumference on the other, and, in the case of two intersecting circles, upon similar considerations. But not even in Book III., where there are several propositions about the relative positions of two circles, do we find any discussion of the conditions under which two circles have two, one, or no point common.

The deficiency can only be made good by the *Principle of Continuity*.

Killing (*Einführung in die Grundlagen der Geometrie*, II. p. 43) gives the following forms as sufficient for most purposes.

(a) Suppose a line belongs entirely to a figure which is divided into two parts; then, if the line has at least one point common with each part, it must also meet the boundary between the parts; or

(b) If a point moves in a figure which is divided into two parts, and if it belongs at the beginning of the motion to one part and at the end of the motion to the other part, it must during the motion arrive at the boundary between the two parts.

In the *Questioni riguardanti le matematiche elementari*, I., Art. 5, pp. 123—143, the principle of continuity is discussed with special reference to the Postulate of Dedekind, and it is shown, first, how the Postulate may be led up to and, secondly, how it may be applied for the purposes of elementary geometry.

Suppose that in a segment AB of a straight line a point C determines two segments AC , CB . If we consider the point C as belonging to only one of the two segments AC , CB , we have a division of the segment AB into two parts with the following properties.

1. Every point of the segment AB belongs to one of the two parts.
2. The point A belongs to one of the two parts (which we will call the *first*) and the point B to the other; the point C may belong indifferently to one or the other of the two parts according as we choose to premise.
3. Every point of the first part precedes every point of the second in the order AB of the segment.

(For generality we may also suppose the case in which the point C falls at A or at B . Considering C , in these cases respectively, as belonging to the first or second part, we still have a division into parts which have the properties above enunciated, one part being then a single point A or B .)

Now, considering carefully the inverse of the above proposition, we see that it agrees with the idea which we have of the continuity of the straight line. Consequently we are induced to admit as a *postulate* the following.

If a segment of a straight line AB is divided into two parts so that

- (1) *every point of the segment AB belongs to one of the parts,*
- (2) *the extremity A belongs to the first part and B to the second, and*
- (3) *any point whatever of the first part precedes any point whatever of the second part, in the order AB of the segment,*

there exists a point C of the segment AB (which may belong either to one part or to the other) such that every point of AB that precedes C belongs to the first part, and every point of AB that follows C belongs to the second part in the division originally assumed.

(If one of the two parts consists of the single point A or B , the point C is the said extremity A or B of the segment.)

This is the Postulate of Dedekind, which was enunciated by Dedekind himself in the following slightly different form (*Stetigkeit und irrationale Zahlen*, 1872, new edition 1905, p. 11).

"If all points of a straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, there exists one and only one point which produces this division of all the points into two classes, this division of the straight line into two parts."

The above enunciation may be said to correspond to the intuitive notion which we have that, if in a segment of a straight line two points start from the ends and describe the segment in opposite senses, they meet in a point. The point of meeting might be regarded as belonging to both parts, but for the present purpose we must regard it as belonging to one only and subtracted from the other part.

Application of Dedekind's postulate to angles.

If we consider an angle less than two right angles bounded by two rays a, b , and draw the straight line connecting A , a point on a , with B , a point on b , we see that all points on the finite segment AB correspond univocally to all the rays of the angle, the point corresponding to any ray being the point in which the ray cuts the segment AB ; and if a ray be supposed to move about the vertex of the angle from the position a to the position b , the corresponding points of the segment AB are seen to follow in the same order as the corresponding rays of the angle (ab).

Consequently, if the angle (ab) is divided into two parts so that

- (1) *each ray of the angle (ab) belongs to one of the two parts,*
- (2) *the outside ray a belongs to the first part and the ray b to the second,*
- (3) *any ray whatever of the first part precedes any ray whatever of the second part,*

the corresponding points of the segment AB determine two parts of the segments such that

- (1) *every point of the segment AB belongs to one of the two parts,*
- (2) *the extremity A belongs to the first part and B to the second,*
- (3) *any point whatever of the first part precedes any point whatever of the second.*

But in that case there exists a point C of AB (which may belong to one or the other of the two parts) such that every point of AB that precedes C belongs to the first part and every point of AB that follows C belongs to the second part.

Thus exactly the same thing holds of c , the ray corresponding to C , with reference to the division of the angle (ab) into two parts.

It is not difficult to extend this to an angle (ab) which is either *flat* or greater than two right angles; this is done (Vitali, *op. cit.* pp. 126—127) by supposing the angle to be divided into two, (ad), (db), each less than two right angles, and considering the three cases in which

- (1) the ray d is such that all the rays that precede it belong to the first part and those which follow it to the second part,
- (2) the ray d is followed by some rays of the first part,
- (3) the ray d is preceded by some rays of the second part.

Application to circular arcs.

If we consider an arc AB of a circle with centre O , the points of the arc correspond univocally, and in the same order, to the rays from the point O passing through those points respectively, and the same argument by which we passed from a segment of a straight line to an angle can be used to make the transition from an angle to an arc.

Intersections of a straight line with a circle.

It is possible to use the Postulate of Dedekind to prove that

If a straight line has one point inside and one point outside a circle, it has two points common with the circle.

For this purpose it is necessary to assume (1) the proposition with reference to the perpendicular and obliques drawn from a given point to a given straight line, namely that of all straight lines drawn from a given point to a given straight line the perpendicular is the shortest, and of the rest (the obliques) that is the longer which has the longer projection upon the straight line, while those are equal the projections of which are equal, so that for any given length of projection there are two equal obliques and two only, one on each side of the perpendicular, and (2) the proposition that any side of a triangle is less than the sum of the other two.

Consider the circle (C) with centre O , and a straight line (r) with one point A inside and one point B outside the circle.

By the definition of the circle, if R is the radius,

$$OA < R, OB > R.$$

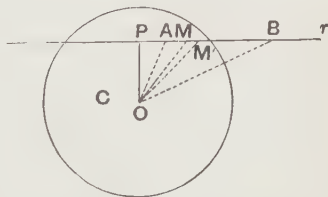
Draw OP perpendicular to the straight line r .

Then $OP < OA$, so that OP is always less than R , and P is therefore within the circle C .

Now let us fix our attention on the finite segment AB of the straight line r . It can be divided into two parts, (1) that containing all the points H for which $OH < R$ (i.e. points inside C), and (2) that containing all the points K for which $OK \geq R$ (points outside C or on the circumference of C).

Thus, remembering that, of two obliques from a given point to a given straight line, that is greater the projection of which is greater, we can assert that all the points of the segment PB which *precede* a point inside C are inside C , and those which *follow* a point on the circumference of C or outside C are outside C .

Hence, by the Postulate of Dedekind, there exists on the segment PB a



point M such that all the points which precede it belong to the first part and those which follow it to the second part.

I say that M is common to the straight line r and the circle C , or

$$OM = R.$$

For suppose, e.g., that $OM < R$.

There will then exist a segment (or length) σ less than the difference between R and OM .

Consider the point M' , one of those which *follow* M , such that MM' is equal to σ .

Then, because any side of a triangle is less than the sum of the other two,

$$OM' < OM + MM'.$$

But

$$OM + MM' = OM + \sigma < R,$$

whence

$$OM' < R,$$

which is absurd.

A similar absurdity would follow if we suppose that $OM > R$.

Therefore OM must be equal to R .

It is immediately obvious that, corresponding to the point M on the segment PB which is common to r and C , there is another point on r which has the same property, namely that which is symmetrical to M with respect to P .

And the proposition is proved.

Intersections of two circles.

We can likewise use the Postulate of Dedekind to prove that

If in a given plane a circle C has one point X inside and one point Y outside another circle C' , the two circles intersect in two points.

We must first prove the following

Lemma.

If O , O' are the centres of two circles C , C' , and R , R' their radii respectively, the straight line OO' meets the circle C in two points A , B , one of which is inside C' and the other outside it.

Now one of these points must fall (1) on the prolongation of $O'O$ beyond O or (2) on OO' itself or (3) on the prolongation of OO' beyond O' .

(1) First, suppose A to lie on $O'O$ produced.

Then $AO' = AO + OO' = R + OO' \dots\dots(a).$

But, in the triangle $OO'Y$,

$$O'Y < OY + OO',$$

and, since $O'Y > R'$, $OY = R$,

$$R' < R + OO'.$$

It follows from (a) that $AO' > R'$; and A therefore lies *outside* C' .

(2) Secondly, suppose A to lie on OO' .

Then $OO' = OA + AO' = R + AO' \dots(\beta).$

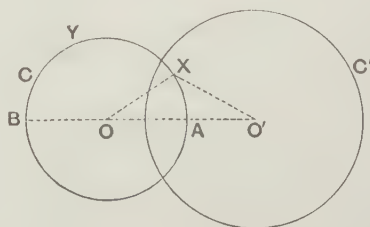
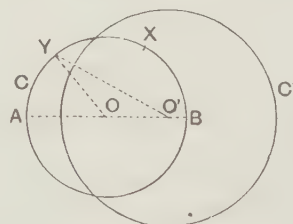
From the triangle $OO'X$ we have

$$OO' < OX + O'X,$$

and, since $OX = R$, $O'X < R'$, it follows that

$$OO' < R + R',$$

whence, by (β), $AO' < R'$, so that A lies *inside* C' .



(3) Thirdly, suppose A to lie on OO' produced.

Then $R = OA = OO' + O'A \dots\dots(\gamma)$.

And, in the triangle $OO'X$,

$$OX < OO' + O'X,$$

that is $R < OO' + O'X$,

whence, by (γ) ,

$$OO' + O'A < OO' + O'X,$$

or $O'A < O'X$,

and A lies *inside* C' .

It is to be observed that one of the two points A, B is in the position of case (1) and the other in the position of either case (2) or case (3): whence we must conclude that one of the two points A, B is *inside* and the other *outside* the circle C' .

Proof of theorem.

The circle C is divided by the points A, B into two semicircles. Consider one of them, and suppose it to be described by a point moving from A to B .

Take two separate points P, Q on it and, to fix our ideas, suppose that P precedes Q .

Comparing the triangles $OO'P, OO'Q$, we observe that one side OO' is common, OP is equal to OQ , and the angle POO' is less than the angle QOO' .

Therefore $O'P < O'Q$.

Now, considering the semicircle $APQB$ as divided into two parts, so that the points of the first part are inside the circle C' , and those of the second part on the circumference of C' or outside it, we have the conditions necessary for the applicability of the Postulate of Dedekind (which is true for arcs of circles as for straight lines); whence *there exists a point M separating the two parts*.

I say that $O'M = R'$.

For, if not, suppose $O'M < R'$.

If then σ signifies the difference between R' and $O'M$, suppose a point M' , which *follows* M , taken on the semicircle such that the chord MM' is not greater than σ (for a way of doing this see below).

Then, in the triangle $O'MM'$,

$$O'M' < O'M + MM' < O'M + \sigma,$$

and therefore

$$O'M' < R'.$$

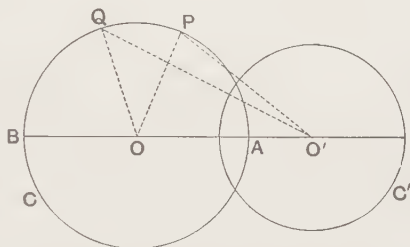
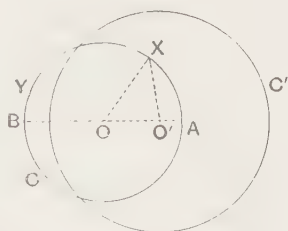
It follows that M' , a point on the arc MB , is inside the circle C' : which is absurd.

Similarly it may be proved that $O'M$ is not greater than R .

Hence $O'M = R$.

[To find a point M' such that the chord MM' is not greater than σ , we may proceed thus.

Draw from M a straight line MP distinct from OM , and cut off MP on it equal to $\sigma/2$.

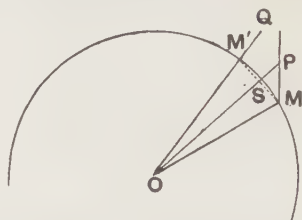


Join OP , and draw another radius OQ such that the angle POQ is equal to the angle MOP .

The intersection, M' , of OQ with the circle satisfies the required condition.

For MM' meets OP at right angles in S .

Therefore, in the right-angled triangle MSP , MS is not greater than MP (it is less, unless MP coincides with MS , when it is equal).



Therefore MS is not greater than $\sigma/2$, so that MM' is not greater than σ .]

BOOK I. PROPOSITIONS.

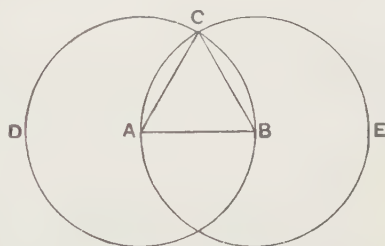
PROPOSITION I.

On a given finite straight line to construct an equilateral triangle.

Let AB be the given finite straight line.

Thus it is required to construct an equilateral triangle on the straight line AB .

With centre A and distance AB let the circle BCD be described ; [Post. 3]
 10 again, with centre B and distance BA let the circle ACE be described ; [Post. 3]



and from the point C , in which the circles cut one another, to the points A , B let the straight lines CA , CB be joined.

[Post. 1]

15 Now, since the point A is the centre of the circle CDB ,
 AC is equal to AB . [Def. 15]

Again, since the point B is the centre of the circle CAE ,
 BC is equal to BA . [Def. 15]

But CA was also proved equal to AB ;

20 therefore each of the straight lines CA , CB is equal to AB .

And things which are equal to the same thing are also equal to one another ; [C. N. 1]

therefore CA is also equal to CB .

Therefore the three straight lines CA , AB , BC are
 25 equal to one another.

Therefore the triangle ABC is equilateral; and it has been constructed on the given finite straight line AB .
(Being) what it was required to do.

1. On a given finite straight line. The Greek usage differs from ours in that the definite article is employed in such a phrase as this where we have the indefinite. *ἐπὶ τῆς δοθείσης εὐθείας πεπερασμένης*, "on the given finite straight line," i.e. the finite straight line which we choose to take.

3. Let AB be the given finite straight line. To be strictly literal we should have to translate in the reverse order "let the given finite straight line be the (straight line) AB "; but this order is inconvenient in other cases where there is more than one datum, e.g. in the setting-out of 1. 2, "let the given point be A , and the given straight line BC ," the awkwardness arising from the omission of the verb in the second clause. Hence I have, for clearness' sake, adopted the other order throughout the book.

8. let the circle BCD be described. Two things are here to be noted, (1) the elegant and practically universal use of the perfect passive imperative in constructions, *γεγράφθω* meaning of course "let it *have been* described" or "suppose it described," (2) the impossibility of expressing shortly in a translation the force of the words in their original order. *κύκλος γεγράφθω ὁ ΒΓΔ* means literally "let a circle have been described, the (circle, namely, which I denote by) BCD ." Similarly we have lower down "let straight lines, (namely) the (straight lines) CA, CB , be joined," *ἐπεζεύχθωσαν εὐθεῖαι αὐτὰς ΓΑ, ΒΒ*. There seems to be no practicable alternative, in English, but to translate as I have done in the text.

13. from the point C ... Euclid is careful to adhere to the phraseology of Postulate 1 except that he speaks of "joining" (*ἐπεζεύχθωσαν*) instead of "drawing" (*γράφειν*). He does not allow himself to use the shortened expression "let the straight line FC be joined" (without mention of the points F, C) until 1. 5.

20. each of the straight lines CA, CB , *ἐκαστέρα τῶν ΓΑ, ΒΒ* and 24. the three straight lines CA, AB, BC , *αἱ τρεῖς αὐτὰς ΓΑ, ΑΒ, ΒΓ*. I have, here and in all similar expressions, inserted the words "straight lines" which are not in the Greek. The possession of the inflected definite article enables the Greek to omit the words, but this is not possible in English, and it would scarcely be English to write "each of CA, CB " or "the three CA, AB, BC ."

It is a commonplace that Euclid has no right to assume, without promising some postulate, that the two circles *will* meet in a point C . To supply what is wanted we must invoke the Principle of Continuity (see note thereon above, p. 235). It is sufficient for the purpose of this proposition and of 1. 22, where there is a similar tacit assumption, to use the form of postulate suggested by Killing. "*If a line [in this case e.g. the circumference ACE] belongs entirely to a figure [in this case a plane] which is divided into two parts [namely the part enclosed within the circumference of the circle BCD and the part outside that circle], and if the line has at least one point common with each part, it must also meet the boundary between the parts [i.e. the circumference ACE must meet the circumference BCD]."*

Zeno's remark that the problem is not solved unless it is taken for granted that two straight lines cannot have a common segment has already been mentioned (note on Post. 2, p. 196). Thus, if AC, BC meet at F before reaching C , and have the part FC common, the triangle obtained, namely FAB , will not be equilateral, but FA, FB will each be less than AB . But Post. 2 has already laid it down that two straight lines cannot have a common segment.

Proclus devotes considerable space to this part of Zeno's criticism, but satisfies himself with the bare mention of the other part, to the effect that it is also necessary to assume that two *circumferences* (with different centres) cannot have a common part. That is, for anything we know, there may be any number of points C common to the two circumferences ACE, BCD . It is not until III. 10 that it is proved that two circles cannot intersect in more

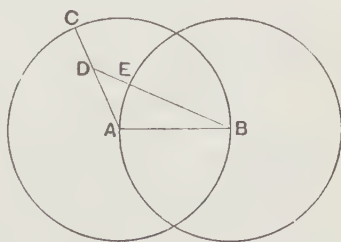
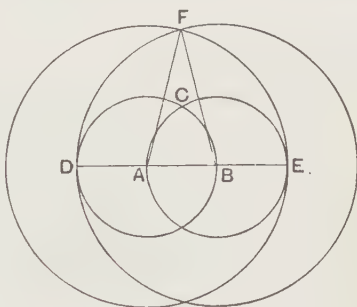
points than two, so that we are not entitled to assume it here. The most we can say is that it is enough for the purpose of this proposition if *one* equilateral triangle can be found with the given base; that the construction only gives *two* such triangles has to be left over to be proved subsequently. And indeed we have not long to wait; for I. 7 clearly shows that on either side of the base AB only *one* equilateral triangle can be described. Thus I. 7 gives us the *number of solutions* of which the present problem is susceptible, and it supplies the same want in I. 22 where a triangle has to be described with three sides of given length; that is, I. 7 furnishes us, in both cases, with one of the essential parts of a complete *διορισμός*, which includes not only the determination of the conditions of possibility but also the number of solutions (*ποσαχώς ἐγγχωρεῖ*, Proclus, p. 202, 5). This view of I. 7 as supplying an equivalent for III. 10 absolutely needed in I. 1 and I. 22 should serve to correct the idea so common among writers of text-books that I. 7 is merely of use as a lemma to Euclid's proof of I. 8, and therefore may be left out if an alternative proof of that proposition is adopted.

Agreeably to his notion that it is from I. 1 that we must satisfy ourselves that isosceles and scalene triangles actually exist, as well as equilateral triangles, Proclus shows how to draw, first a particular isosceles triangle, and then a scalene triangle, by means of the figure of the proposition. To make an isosceles triangle he produces AB in both directions to meet the respective circles in D , E , and then describes circles with A , B as centres and AE , BD as radii respectively. The result is an isosceles triangle with each of two sides double of the third side. To make an isosceles triangle in which the equal sides are not so related to the third side but have any given length would require the use of I. 3; and there is no object in treating the question at all in advance of I. 22. An easier way of satisfying ourselves of the existence of some isosceles triangles would surely be to conceive any two radii of a circle drawn and their extremities joined.

There is more point in Proclus' construction of a *scalene* triangle. Suppose AC to be a radius of one of the two circles, and D a point on AC lying in that portion of the circle with centre A which is outside the circle with centre B . Then, joining BD , as in the figure, we have a triangle which obviously has all its sides unequal, that is, a *scalene* triangle.

The above two constructions appear in an-Nairizi's commentary under the name of Heron; Proclus does not mention his source.

In addition to the above construction for a scalene triangle (producing a triangle in which the "given" side is greater than one and less than the other of the two remaining sides), Heron has two others showing the other two possible cases, in which the "given" side is (1) less than, (2) greater than, either of the other two sides.



PROPOSITION 2.

To place at a given point (as an extremity) a straight line equal to a given straight line.

Let A be the given point, and BC the given straight line.

Thus it is required to place at the point A (as an extremity)
5 a straight line equal to the given straight line BC .

From the point A to the point B
let the straight line AB be joined ;

[Post. 1]

and on it let the equilateral triangle
10 DAB be constructed. [I. 1]

Let the straight lines AE , BF be
produced in a straight line with DA ,
 DB ;

[Post. 2]

with centre B and distance BC let the
15 circle CGH be described ;

[Post. 3]

and again, with centre D and distance DG let the circle GKL
be described.

[Post. 3]

Then, since the point B is the centre of the circle CGH ,
 BC is equal to BG .

20 Again, since the point D is the centre of the circle GKL ,
 DL is equal to DG .

And in these DA is equal to DB ;

therefore the remainder AL is equal to the remainder
 BG .

[C.N. 3]

25 But BC was also proved equal to BG ;

therefore each of the straight lines AL , BC is equal
to BG .

And things which are equal to the same thing are also
equal to one another ;

[C.N. 1]

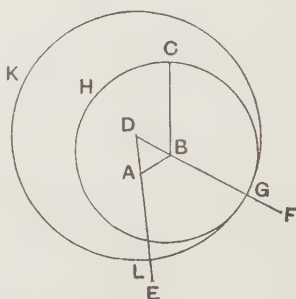
30 therefore AL is also equal to BC .

Therefore at the given point A the straight line AL is
placed equal to the given straight line BC .

(Being) what it was required to do.

1. (as an extremity). I have inserted these words because "to place a straight line at a given point" (*πρὸς τῷ δοθέντι σημείῳ*) is not quite clear enough, at least in English.

11. Let the straight lines AE , BF be produced.... It will be observed that in this first application of Postulate 2, and again in I. 5, Euclid speaks of the *continuation* of the straight line as that which is produced in such cases, *ἐκβεβλήσθωσαν* and *προσεκβεβλήσθωσαν* meaning little more than *drawing* straight lines "in a straight line with" the given straight lines. The first place in which Euclid uses phraseology exactly corresponding to ours when



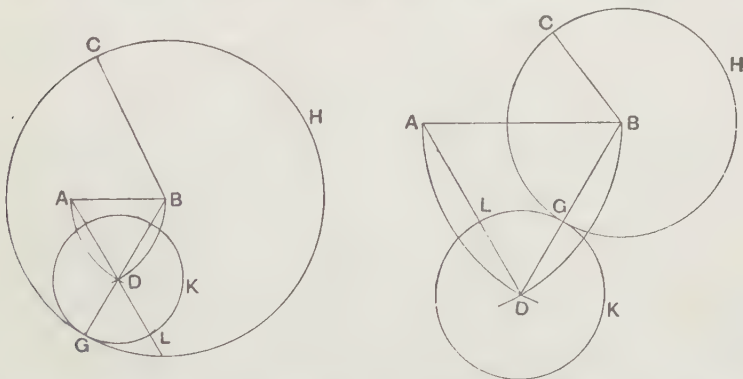
speaking of a straight line being produced is in I. 16: "let one side of it, BC , be produced to D " (προσεκβεβλήσθω αὐτοῦ μία πλευρά ἢ $BΓ$ ἐπὶ τὸ $Δ$).

23. the remainder AL ...the remainder BG . The Greek expressions are λοιπὴ ἢ $ΑΛ$ and λοιπὴ τῇ $BΗ$, and the literal translation would be " AL (or BG) remaining," but the shade of meaning conveyed by the position of the definite article can hardly be expressed in English.

This proposition gives Proclus an opportunity, such as the Greek commentators revelled in, of distinguishing a multitude of *cases*. After explaining that those theorems and problems are said to have *cases* which have the same force, though admitting of a number of different figures, and preserve the same method of demonstration while admitting variations of position, and that cases reveal themselves in the *construction*, he proceeds to distinguish the cases in this problem arising from the different positions which the given point may occupy relatively to the given straight line. It may be (he says) either (1) outside the line or (2) on the line, and, if (1), it may be either (a) on the line produced or (b) situated obliquely with regard to it; if (2), it may be either (a) one of the extremities of the line or (b) an intermediate point on it. It will be seen that Proclus' anxiety to subdivide leads him to give a "case," (2) (a), which is useless, since in that "case" we are given what we are required to find, and there is really no problem to solve. As Savile says, "qui quaerit ad β punctum ponere rectam aequalem τῇ $\beta\gamma$ rectae, quaerit quod datum est, quod nemo faceret nisi forte insaniat."

Proclus gives the construction for (2) (b) following Euclid's way of taking G as the point in which the circle with centre B intersects DB produced, and then proceeds to "cases," of which there are still more, which result from the different ways of drawing the equilateral triangle and of producing its sides.

This last class of "cases" he subdivides into three according as AB is (1) equal to, (2) greater than or (3) less than BC . Here again "case" (1) serves no purpose, since, if AB is equal to BC , the problem is already solved. But Proclus' figures for the other two cases are worth giving, because in one of them the point G is on BD produced beyond D , and in the other it lies on BD itself and there is no need to produce any side of the equilateral triangle.



A glance at these figures will show that, if they were used in the proposition, each of them would require a slight modification in the wording (1) of the construction, since BD is in one case produced beyond D instead of B and in the other case not produced at all, (2) of the proof, since BG , instead of being the difference between DG and DB , is in one case the sum of DG and DB and in the other the difference between DB and DG .

Modern editors generally seem to classify the cases according to the possible variations in the construction rather than according to differences in the data. Thus Lardner, Potts, and Todhunter distinguish eight cases due to the three possible alternatives, (1) that the given point may be joined to either end of the given straight line, (2) that the equilateral triangle may then be described on either side of the joining line, and (3) that the side of the equilateral triangle which is produced may be produced in either direction. (But it should have been observed that, where AB is greater than BC , the third alternative is between producing DB and not producing it at all.) Potts adds that, when the given point lies either on the line or on the line produced, the distinction which arises from joining the two ends of the line with the given point no longer exists, and there are only four cases of the problem (I think he should rather have said *solutions*).

To distinguish a number of cases in this way was foreign to the really classical manner. Thus, as we shall see, Euclid's method is to give one case only, for choice the most difficult, leaving the reader to supply the rest for himself. Where there was a real distinction between cases, sufficient to necessitate a substantial difference in the proof, the practice was to give separate *enunciations* and proofs altogether, as we may see, e.g., from the *Conics* and the *De sectione rationis* of Apollonius.

Proclus alludes, in conclusion, to the error of those who proposed to solve 1. 2 by describing a circle with the given point as centre and with a distance equal to BC , which, as he says, is a *petitio principii*. De Morgan puts the matter very clearly (*Supplementary Remarks on the first six Books of Euclid's Elements* in the *Companion to the Almanac*, 1849, p. 6). We should "insist," he says, "here upon the restrictions imposed by the first three postulates, which do not allow a circle to be drawn with a compass-carried distance; suppose the compasses to close of themselves the moment they cease to touch the paper. These two propositions [1. 2, 3] extend the power of construction to what it would have been if *all* the usual power of the compasses had been assumed; they are mysterious to all who do not see that postulate iii does not ask for *every use of the compasses*."

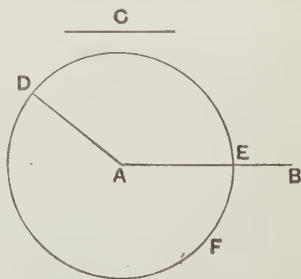
PROPOSITION 3.

Given two unequal straight lines, to cut off from the greater a straight line equal to the less.

Let AB , C be the two given unequal straight lines, and let AB be the greater of them.

Thus it is required to cut off from AB the greater a straight line equal to C the less.

At the point A let AD be placed equal to the straight line C ; [1. 2] and with centre A and distance AD let the circle DEF be described.



[Post. 3]

Now, since the point A is the centre of the circle DEF ,
 AE is equal to AD . [Def. 15]

But C is also equal to AD .

Therefore each of the straight lines AE , C is equal to
 AD ; so that AE is also equal to C . [C.N. 1]

Therefore, given the two straight lines AB , C , from AB
the greater AE has been cut off equal to C the less.

(Being) what it was required to do.

Proclus contrives to make a number of "cases" out of this proposition also, and gives as many as eight figures. But he only produces this variety by practically incorporating the construction of the preceding proposition, instead of assuming it as we are entitled to do. If Prop. 2 is assumed, there is really only one "case" of the present proposition, for Potts' distinction between two cases according to the particular extremity of the straight line from which the given length has to be cut off scarcely seems to be worth making.

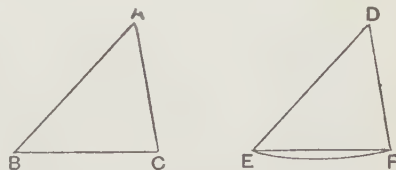
PROPOSITION 4.

If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend.

Let ABC , DEF be two triangles having the two sides
 AB , AC equal to the two sides DE , DF respectively, namely
 AB to DE and AC to DF , and the angle BAC equal to the
10 angle EDF .

I say that the base BC is also equal to the base EF , the
triangle ABC will be equal to the triangle DEF , and the
remaining angles will be equal to the remaining angles
respectively, namely those which the equal sides subtend, that
15 is, the angle ABC to the angle DEF , and the angle ACB
to the angle DFE .

For, if the triangle ABC be
applied to the triangle DEF ,
and if the point A be placed
20 on the point D
and the straight line AB
on DE ,



then the point B will also coincide with E , because AB is
equal to DE .

25 Again, AB coinciding with DE ,
the straight line AC will also coincide with DF , because the
angle BAC is equal to the angle EDF ;

hence the point C will also coincide with the point F ,
because AC is again equal to DF .

30 But B also coincided with E ;
hence the base BC will coincide with the base EF .

[For if, when B coincides with E and C with F , the base
 BC does not coincide with the base EF , two straight lines
will enclose a space: which is impossible.

35 Therefore the base BC will coincide with
 EF] and will be equal to it. [C.N. 4]

Thus the whole triangle ABC will coincide with the
whole triangle DEF ,

and will be equal to it.

40 And the remaining angles will also coincide with the
remaining angles and will be equal to them,

the angle ABC to the angle DEF ,

and the angle ACB to the angle DFE .

Therefore etc.

45 (Being) what it was required to prove.

1—3. It is a fact that Euclid's enunciations not infrequently leave something to be desired in point of clearness and precision. Here he speaks of the triangles having "the angle equal to the angle, namely the angle contained by the equal straight lines" (*τὴν γωνίαν τῇ γωνίᾳ ἴσην ἔχῃ τὴν ὑπὸ τῶν ἴσων εὐθειῶν περιεχομένην*), only one of the two angles being described in the latter expression (in the accusative), and a similar expression in the dative being left to be understood of the other angle. It is curious too that, after mentioning two "sides," he speaks of the angles contained by the equal "straight lines," not "sides." It may be that he wished to adhere scrupulously, at the outset, to the phraseology of the definitions, where the angle is the inclination to one another of two lines or straight lines. Similarly in the enunciation of 1. 5 he speaks of producing the equal "straight lines" as if to keep strictly to the wording of Postulate 2.

2. respectively. I agree with Mr H. M. Taylor (*Euclid*, p. ix) that it is best to abandon the traditional translation of "each to each," which would naturally seem to imply that all the four magnitudes are equal rather than (as the Greek *ἐκάτερα ἐκάτερα* does) that one is equal to one and the other to the other.

3. the base. Here we have the word *base* used for the first time in the *Elements*. Proclus explains it (p. 236, 12—15) as meaning (1), when no side of a triangle has been mentioned before, the side "which is on a level with the sight" (*τὴν πρὸς τῇ ὀψει κεμένην*), and (2), when two sides have already been mentioned, the third side. Proclus thus avoids the mistake made by some modern editors who explain the term exclusively with reference to the case where two sides have been mentioned before. That this is an error is proved (1) by the occurrence of the term in the enunciations of 1. 37 etc. about triangles on the same base and equal bases, (2) by the application of the same term to the bases of parallelograms in 1. 35 etc. The truth is that the use of the term must have been suggested by the practice of drawing the particular side horizontally, as it were, and the rest of the figure above it. The *base* of a figure was therefore spoken of, primarily, in the same sense as the base of anything

else, e.g. of a pedestal or column; but when, as in I. 5, two triangles were compared occupying other than the normal positions which gave rise to the name, and when two sides had been previously mentioned, the base was, as Proclus says, necessarily the third side.

6. **subtend.** *ὑποτείνειν ὑπὸ*, "to stretch under," with accusative.

9. **the angle BAC.** The full Greek expression would be *ἡ ὑπὸ τῶν BA, AC περιεχομένη γωνία*, "the angle contained by the (straight lines) *BA, AC*." But it was a common practice of Greek geometers, e.g. of Archimedes and Apollonius (and Euclid too in Books x.—xiii.), to use the abbreviation *αὶ BAC* for *αὶ BA, AC*, "the (straight lines) *BA, AC*." Thus, on *περιεχομένη* being dropped, the expression would become first *ἡ ὑπὸ τῶν BAC γωνία*, then *ἡ ὑπὸ BAC γωνία*, and finally *ἡ ὑπὸ BAC*, without *γωνία*, as we regularly find it in Euclid.

17. **if the triangle be applied to...** 23. **coincide.** The difference between the technical use of the passive *ἐφαρμόζεσθαι* "to be applied (to)," and of the active *ἐφαρμόζειν* "to coincide (with)" has been noticed above (note on *Common Notion 4*, pp. 224—5).

32. [For if, when *B* coincides... 36. coincide with *EF*]. Heiberg (*Paralipomena zu Euklid* in *Hermes*, xxxviii., 1903, p. 56) has pointed out, as a conclusive reason for regarding these words as an early interpolation, that the text of an-Nairizi (*Codex Leidensis* 399, I, ed. Besthorn-Heiberg, p. 55) does not give the words in this place but after the conclusion Q.E.D., which shows that they constitute a *scholium* only. They were doubtless added by some commentator who thought it necessary to explain the immediate inference that, since *B* coincides with *E* and *C* with *F*, the straight line *BC* coincides with the straight line *EF*, an inference which really follows from the definition of a straight line and Post. 1; and no doubt the Postulate that "Two straight lines cannot enclose a space" (afterwards placed among the *Common Notions*) was interpolated at the same time.

44. **Therefore etc.** Where (as here) Euclid's *conclusion* merely repeats the enunciation word for word, I shall avoid the repetition and write "Therefore etc." simply.

In the note on *Common Notion 4* I have already mentioned that Euclid obviously used the method of superposition with reluctance, and I have given, after Veronese for the most part, the reason for holding that that method is not admissible as a *theoretical* means of proving equality, although it may be of use as a *practical* test, and may thus furnish an empirical basis on which to found a postulate. Mr Bertrand Russell observes (*Principles of Mathematics* I. p. 405) that Euclid would have done better to assume I. 4 as an axiom, as is practically done by Hilbert (*Grundlagen der Geometrie*, p. 9). It may be that Euclid himself was as well aware of the objections to the method as are his modern critics; but at all events those objections were stated, with almost equal clearness, as early as the middle of the 16th century. Peletarius (Jacques Peletier) has a long note on this proposition (*In Euclidis Elementa geometrica demonstrationum libri sex*, 1557), in which he observes that, if superposition of lines and figures could be assumed as a method of proof, the whole of geometry would be full of such proofs, that it could equally well have been used in I. 2, 3 (thus in I. 2 we could simply have supposed the line taken up and placed at the point), and that in short it is obvious how far removed the method is from the dignity of geometry. The theorem, he adds, is obvious in itself and does not require proof; although it is introduced as a theorem, it would seem that Euclid intended it rather as a *definition* than a theorem, "for I cannot think that two angles are equal unless I have a conception of what equality of angles is." Why then did Euclid include the proposition among theorems, instead of placing it among the axioms? Peletarius makes the best excuse he can, but concludes thus: "Huius itaque propositionis veritatem non aliunde quam a communi iudicio petemus; cogitabimusque figuras figuris superponere, Mechanicum quippiam esse: intelligere verò, id demum esse Mathematicum."

Expressed in terms of the modern systems of Congruence-Axioms referred to in the note on *Common Notion 4*, what Euclid really assumes amounts to the following:

(1) On the line *DE*, there is a point *E*, on either side of *D*, such that *AB* is equal to *DE*.

(2) On either side of the ray DE there is a ray DF such that the angle EDF is equal to the angle BAC .

It now follows that on DF there is a point F such that DF is equal to AC .

And lastly (3), we require an axiom from which to infer that the two remaining angles of the triangles are respectively equal and that the bases are equal.

I have shown above (pp. 229—230) that Hilbert has an axiom stating the equality of the remaining angles simply, but proves the equality of the bases.

Another alternative is that of Pasch (*Vorlesungen über neuere Geometrie*, p. 109) who has the following "Grundsatz":

If two figures AB and FGH are given (FGH not being contained in a straight length), and AB , FG are congruent, and if a plane surface be laid through A and B , we can specify in this plane surface, produced if necessary, two points C , D , neither more nor less, such that the figures ABC and ABD are congruent with the figure FGH , and the straight line CD has with the straight line AB or with AB produced one point common.

I pass to two points of detail in Euclid's proof:

(1) The inference that, since B coincides with E , and C with F , the bases of the triangles are wholly coincident rests, as expressly stated, on the impossibility of two straight lines enclosing a space, and therefore presents no difficulty.

But (2) most editors seem to have failed to observe that at the very beginning of the proof a much more serious assumption is made without any explanation whatever, namely that, if A be placed on D , and AB on DE , the point B will coincide with E , because AB is equal to DE . That is, the converse of *Common Notion 4* is assumed for straight lines. Proclus merely observes, with regard to the converse of this *Common Notion*, that it is only true in the case of things "of the same form" ($\delta\mu\omicron\sigma\epsilon\iota\delta\eta$), which he explains as meaning straight lines, arcs of one and the same circle, and angles "contained by lines similar and similarly situated" (p. 241, 3—8).

Savile however saw the difficulty and grappled with it in his note on the *Common Notion*. After stating that all straight lines with two points common are congruent between them (for otherwise two straight lines would enclose a space), he argues thus. Let there be two straight lines AB , DE , and let A be placed on D , and AB on DE . Then B will coincide with E . For, if not, let B fall somewhere short of E or beyond E ; and in either case it will follow that the less is equal to the greater, which is impossible.

Savile seems to assume (and so apparently does Lardner who gives the same proof) that, if the straight lines be "applied," B will fall somewhere on DE or DE produced. But the ground for this assumption should surely be stated; and it seems to me that it is necessary to use, not *Postulate 1* alone, nor *Postulate 2* alone, but both, for this purpose (in other words to assume, not only that *two straight lines cannot enclose a space*, but also that *two straight lines cannot have a common segment*). For the only safe course is to place A upon D and then turn AB about D until *some* point on AB intermediate between A and B coincides with *some* point on DE . In this position AB and DE have two points common. Then *Postulate 1* enables us to infer that the straight lines coincide *between* the two common points, and *Postulate 2* that they coincide beyond the second common point towards B and E . Thus the straight lines coincide throughout so far as *both* extend; and Savile's argument then proves that B coincides with E .

PROPOSITION 5.

In isosceles triangles the angles at the base are equal to one another, and, if the equal straight lines be produced further, the angles under the base will be equal to one another.

Let ABC be an isosceles triangle having the side AB equal to the side AC ;

and let the straight lines BD , CE be produced further in a straight line with AB , AC . [Post. 2]

I say that the angle ABC is equal to the angle ACB , and the angle CBD to the angle BCE .

Let a point F be taken at random on BD ;

from AE the greater let AG be cut off equal to AF the less; [I. 3]

and let the straight lines FC , GB be joined. [Post. 1]

Then, since AF is equal to AG and AB to AC ,

the two sides FA , AC are equal to the two sides GA , AB , respectively;

and they contain a common angle, the angle FAG .

Therefore the base FC is equal to the base GB ,

and the triangle AFC is equal to the triangle AGB ,

and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend,

that is, the angle ACF to the angle ABG ,

and the angle AFC to the angle AGB . [I. 4]

And, since the whole AF is equal to the whole AG ,

and in these AB is equal to AC ,

the remainder BF is equal to the remainder CG .

But FC was also proved equal to GB ;

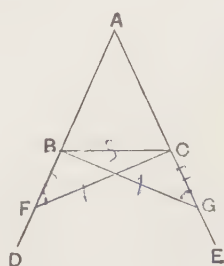
therefore the two sides BF , FC are equal to the two sides CG , GB respectively;

and the angle BFC is equal to the angle CGB ,

while the base BC is common to them;

therefore the triangle BFC is also equal to the triangle CGB ,

and the remaining angles will be equal to the remaining



angles respectively, namely those which the equal sides subtend;

therefore the angle FBC is equal to the angle GCB ,
and the angle BCF to the angle CBG .

40 Accordingly, since the whole angle ABG was proved equal to the angle ACF ,

and in these the angle CBG is equal to the angle BCF ,
the remaining angle ABC is equal to the remaining angle ACB ;

45 and they are at the base of the triangle ABC .

But the angle FBC was also proved equal to the angle GCB ;
and they are under the base.

Therefore etc.

Q. E. D.

2. the equal straight lines (meaning the equal *sides*). Cf. note on the similar expression in Prop. 4, lines 2, 3.

10. Let a point F be taken at random on BD , $\epsilon\lambda\lambda\eta\phi\theta\omega$ ἐπὶ τῆς BD τυχόν σημείον τὸ Z , where $\tau\upsilon\chi\acute{o}\nu$ σημείον means "a chance point."

17. the two sides FA , AC are equal to the two sides GA , AB respectively, $\delta\acute{o}\nu\alpha\iota$ ZA , AF δυοὶ ταῖς HA , AB ἴσαι εἰσὶν ἑκάτερα ἑκατέρῃ. Here, and in numberless later passages, I have inserted the word "sides" for the reason given in the note on I. 1, line 20. It would have been permissible to supply either "straight lines" or "sides"; but on the whole "sides" seems to be more in accordance with the phraseology of I. 4.

33. the base BC is common to them, i.e., apparently, common to the *angles*, as the $\alpha\upsilon\tau\acute{\omega}\nu$ in $\beta\acute{\alpha}\sigma\epsilon\iota\varsigma$ $\alpha\upsilon\tau\acute{\omega}\nu$ κοινὴ can only refer to $\gamma\omega\gamma\iota\alpha$ and $\gamma\omega\gamma\iota\alpha$ preceding. Simson wrote "and the base BC is common to the two triangles BFC , CGB "; Todhunter left out these words as being of no use and tending to perplex a beginner. But Euclid evidently chose to quote the conclusion of I. 4 exactly; the first phrase of that conclusion is that the bases (of the two triangles) are equal, and, as the equal bases are here the *same* base, Euclid naturally substitutes the word "common" for "equal."

48. As "(Being) what it was required to prove" (or "do") is somewhat long, I shall henceforth write the time-honoured "Q. E. D." and "Q. E. F." for $\delta\pi\epsilon\rho$ $\epsilon\delta\epsilon\iota$ $\delta\epsilon\acute{\iota}\xi\alpha\iota$ and $\delta\pi\epsilon\rho$ $\epsilon\delta\epsilon\iota$ $\pi\omicron\upsilon\eta\sigma\alpha\iota$.

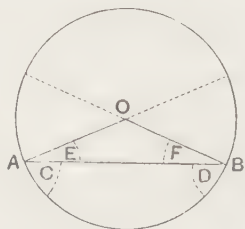
According to Proclus (p. 250, 20) the discoverer of the fact that in any isosceles triangle the angles at the base are equal was Thales, who however is said to have spoken of the angles as being *similar*, and not as being *equal*. (Cf. Arist. *De caelo* IV. 4, 311 b 34 πρὸς ὁμοίας γωνίας φαίνεται φερόμενον where *equal* angles are meant.)

A pre-Euclidean proof of I. 5.

One of the most interesting of the passages in Aristotle indicating differences between Euclid's proofs and those with which Aristotle was familiar, in other words, those of the text-books immediately preceding Euclid's, has reference to the theorem of I. 5. The passage (*Anal. Prior.* I. 24, 41 b 13—22) is so important that I must quote it in full. Aristotle is illustrating the fact that in any syllogism one of the propositions must be affirmative and *universal* (καθόλου). "This," he says, "is better shown in the case of geometrical propositions" (ἐν τοῖς διαγράμμασιν), e.g. the proposition that *the angles at the base of an isosceles triangle are equal*.

"For let A , B be drawn [i.e. joined] to the centre.

"If, then, we assumed (1) that the angle AC [i.e. $A + C$] is equal to the angle BD [i.e. $B + D$] without asserting generally that *the angles of semicircles are equal*, and again (2) that the angle C is equal to the angle D without making the further assumption that *the two angles of all segments are equal*, and if we then inferred, lastly, that, since the whole angles are equal, and equal angles are subtracted from them, the angles which remain, namely E, F , are equal. we should commit a *petitio principii*, unless we assumed [generally] that, *when equals are subtracted from equals, the remainders are equal*."



The language is noteworthy in some respects.

(1) A, B are said to be *drawn* ($\eta\gamma\gamma\mu\acute{\epsilon}\nu\alpha\iota$) to the centre (of the circle of which the two equal sides are radii) as if A, B were not the angular points but the sides or the radii themselves. (There is a parallel for this in Eucl. iv. 4.)

(2) "The angle AC " is the angle which is the sum of A and C , and A means here the angle at A of the *isosceles triangle* shown in the figure, and afterwards spoken of by Aristotle as E , while C is the "mixed" angle between AB and the circumference of the smaller segment cut off by it.

(3) The "angle of a semicircle" (i.e. the "angle" between the diameter and the circumference, at the extremity of the diameter) and the "angle of a segment" appear in Euclid III. 16 and III. Def. 7 respectively, obviously as survivals from earlier text-books.

But the most significant facts to be gathered from the extract are that in the text-books which preceded Euclid's "mixed" angles played a much more important part than they do with Euclid, and, in particular, that at least two propositions concerning such angles appeared quite at the beginning, namely the propositions that *the (mixed) angles of semicircles are equal* and that *the (mixed) angles of any segment of a circle are equal*. The wording of the first of the two propositions is vague, but it does not necessarily mean more than that the two (mixed) angles in *one* semicircle are equal, and I know of no evidence going to show that it asserts that the angle of any one semicircle is equal to the angle of any other semicircle (of different size). It is quoted in the same form, "because the angles of semicircles are equal," in the Latin translation from the Arabic of Heron's *Catoptrica*, Prop. 9 (Heron, Vol. II., Teubner, p. 334), but it is only inferred that the different radii of *one* circle make equal "angles" with the circumference; and in the similar proposition of the Pseudo-Euclidean *Catoptrica* (Euclid, Vol. VII., p. 294) angles of the same sort in *one* circle are said to be equal "because they are (angles) of a semicircle." Therefore the first of the two propositions may be only a particular case of the second.

But it is remarkable enough that the second proposition (that *the two "angles of" any segment of a circle are equal*) should, in earlier text-books, have been placed before the theorem of Eucl. I. 5. We can hardly suppose it to have been proved otherwise than by the superposition of the semicircles into which the circle is divided by the diameter which bisects at right angles the base of the segment; and no doubt the proof would be closely connected with that of Thales' other proposition that any diameter of a circle bisects it, which must also (as Proclus indicates) have been proved by superposing one of the two parts upon the other.

It is a natural inference from the passage of Aristotle that Euclid's proof of

I. 5 was his own, and it would thus appear that his innovations as regards order of propositions and methods of proof began at the very threshold of the subject.

Proof without producing the sides.

In this proof, given by Proclus (pp. 248, 22—249, 19), D and E are taken on AB , AC , instead of on AB , AC *produced*, so that AD , AE are equal. The method of proof is of course exactly like Euclid's, but it does not establish the equality of the angles beyond the base as well.

Pappus' proof.

Proclus (pp. 249, 20—250, 12) says that Pappus proved the theorem in a still shorter manner without the help of any construction whatever.

This very interesting proof is given as follows :

"Let ABC be an isosceles triangle, and AB equal to AC .

Let us conceive this one triangle as two triangles, and let us argue in this way.

Since AB is equal to AC , and AC to AB ,

the two sides AB , AC are equal to the two sides AC , AB .

And the angle BAC is equal to the angle CAB , for it is the same.

Therefore all the corresponding parts (in the triangles) are equal, namely

BC to BC ,

the triangle ABC to the triangle ABC (i.e. ACB),

the angle ABC to the angle ACB ,

and the angle ACB to the angle ABC ,

(for these are the angles subtended by the equal sides AB , AC).

Therefore in isosceles triangles the angles at the base are equal."

This will no doubt be recognised as the foundation of the alternative proof frequently given by modern editors, though they do not refer to Pappus. But they state the proof in a different form, the common method being to suppose the triangle to be taken up, turned over, and placed again upon *itself*, after which the same considerations of congruence as those used by Euclid in I. 4 are used over again. There is the obvious difficulty that it supposes the triangle to be taken up and at the same time to remain where it is. (Cf. Dodgson's humorous remark upon this, *Euclid and his modern Rivals*, p. 47.) Whatever we may say in justification of the proceeding (e.g. that the triangle may be supposed to leave a *trace*), it is really equivalent to assuming the construction (hypothetical, if you will) of another triangle equal in all respects to the given triangle; and such an assumption is not in accordance with Euclid's principles and practice.

It seems to me that the form given to the proof by Pappus himself is by far the best, for the reasons (1) that it assumes no construction of a second triangle, real or hypothetical, (2) that it avoids the distinct awkwardness involved by a proof which, instead of merely quoting and applying the *result* of a previous proposition, repeats, with reference to a new set of data, the *process* by which that result was established. If it is asked how we are to realise Pappus' idea of *two* triangles, surely we may answer that we keep to one triangle and merely view it in two aspects. If it were a question of helping a beginner to understand this, we might say that one triangle is the triangle



looked at in front and that the other triangle is the *same* triangle looked at from *behind*; but even this is not really necessary.

Pappus' proof, of course, does not include the proof of the second part of the proposition about the angles under the base, and we should still have to establish this much in the same way as Euclid does.

Purpose of the second part of the theorem.

An interesting question arises as to the reason for Euclid's insertion of the second part, to which, it will be observed, the converse proposition 1. 6 has nothing corresponding. As a matter of fact, it is not necessary for any subsequent demonstration that is to be found in the original text of Euclid, but only for the interpolated second case of 1. 7; and it was perhaps not unnatural that the undoubted genuineness of the second part of 1. 5 convinced many editors that the second case of 1. 7 must necessarily be Euclid's also. Proclus' explanation, which must apparently be the right one, is that the second part of 1. 5 was inserted for the purpose of fore-arming the learner against a possible *objection* (*ἐνστάσις*), as it was technically called, which might be raised to 1. 7 as given in the text, with one case only. The *objection* would, as we have seen, take the specific ground that, as demonstrated, the theorem was not conclusive, since it did not cover all possible cases. From this point of view, the second part of 1. 5 is useful not only for 1. 7 but, according to Proclus, for 1. 9 also. Simson does not seem to have grasped Proclus' meaning, for he says: "And Proclus acknowledges, that the second part of Prop. 5 was added upon account of Prop. 7 but gives a ridiculous reason for it, 'that it might afford an answer to objections made against the 7th,' as if the case of the 7th which is left out were, as he expressly makes it, an objection against the proposition itself."

PROPOSITION 6.

If in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another.

Let ABC be a triangle having the angle ABC equal to the angle ACB ;

I say that the side AB is also equal to the side AC .

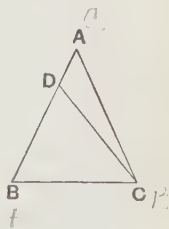
For, if AB is unequal to AC , one of them is greater.

Let AB be greater; and from AB the greater let DB be cut off equal to AC the less;

let DC be joined.

Then, since DB is equal to AC ,
and BC is common,

the two sides DB , BC are equal to the two sides AC , CB respectively;



and the angle DBC is equal to the angle ACB ;

therefore the base DC is equal to the base AB ,
and the triangle DBC will be equal to the triangle ACB ,
the less to the greater :

which is absurd.

Therefore AB is not unequal to AC ;
it is therefore equal to it.

Therefore etc.

Q. E. D.

Euclid assumes that, because D is between A and B , the triangle DBC is less than the triangle ABC . Some postulate is necessary to justify this tacit assumption; considering an angle less than two right angles, say the angle ACB in the figure of the proposition, as a cluster of rays issuing from C and bounded by the rays CA , CB , and joining AB (where A , B are any two points on CA , CB respectively), we see that to each successive ray taken in the direction from CA to CB there corresponds one point on AB in which the said ray intersects AB , and that all the points on AB taken in order from A to B correspond univocally to all the rays taken in order from CA to CB , each point namely to the ray intersecting AB in the point.

We have here used, for the first time in the *Elements*, the method of *reductio ad absurdum*, as to which I would refer to the section above (pp. 136, 140) dealing with this among other technical terms.

This proposition also, being the *converse* of the preceding proposition, brings us to the subject of

Geometrical Conversion.

This must of course be distinguished from the *logical* conversion of a proposition. Thus, from the proposition that all isosceles triangles have the angles opposite to the equal sides equal, *logical* conversion would only enable us to conclude that *some* triangles with two angles equal are isosceles. Thus I. 6 is the geometrical, but not the logical, converse of I. 5. On the other hand, as De Morgan points out (*Companion to the Almanac*, 1849, p. 7), I. 6 is a purely *logical* deduction from I. 5 and I. 18 taken together, as is I. 19 also. For the general argument see the note on I. 19. For the present proposition it is enough to state the matter thus. Let X denote the class of triangles which have the two sides other than the base equal, Y the class of triangles which have the base angles equal; then we may call non- X the class of triangles having the sides other than the base unequal, non- Y the class of triangles having the base angles unequal.

Thus we have

All X is Y , [I. 5]

- All non- X is non- Y ; [I. 18]

and it is a purely logical deduction that

All Y is X . [I. 6]

According to Proclus (p. 252, 5 sqq.) two forms of *geometrical conversion* were distinguished.

(1) The leading form ($\pi\rho\omicron\gamma\gamma\omicron\upsilon\mu\acute{\epsilon}\nu\eta$), the conversion *par excellence* ($\eta\ \kappa\upsilon\rho\iota\omega\varsigma$

ἀντιστροφή), is the complete or simple conversion in which the hypothesis and the conclusion of a theorem change places exactly, the conclusion of the theorem being the hypothesis of the converse theorem, which again establishes, as its conclusion, the hypothesis of the original theorem. The relation between the first part of 1. 5 and 1. 6 is of this character. In the former the hypothesis is that two sides of a triangle are equal and the conclusion is that the angles at the base are equal, while the converse (1. 6) starts from the hypothesis that two angles are equal and proves that the sides subtending them are equal.

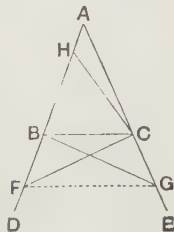
(2) The other form of conversion, which we may call *partial*, is seen in cases where a theorem starts from two or more hypotheses combined into one enunciation and leads to a certain conclusion, after which the converse theorem takes this conclusion in substitution for one of the hypotheses of the original theorem and from the said conclusion along with the rest of the original hypotheses obtains, as its conclusion, the omitted hypothesis of the original theorem. 1. 8 is in this sense a converse proposition to 1. 4; for 1. 4 takes as hypotheses (1) that two sides in two triangles are respectively equal, (2) that the included angles are equal, and proves (3) that the bases are equal, while 1. 8 takes (1) and (3) as hypotheses and proves (2) as its conclusion. It is clear that a conversion of the *leading* type must be unique, while there may be many *partial* conversions of a theorem according to the number of hypotheses from which it starts.

Further, of convertible theorems, those which took as their hypothesis the *genus* and proved a *property* were distinguished as the *leading* theorems (προηγούμενα), while those which started from the property as hypothesis and described, as the conclusion, the genus possessing that property were the *converse* theorems. 1. 5 is thus the leading theorem and 1. 6 its converse, since the genus is in this case taken to be the isosceles triangle.

Converse of second part of I. 5.

Why, asks Proclus, did not Euclid convert the *second* part of 1. 5 as well? He suggests, properly enough, two reasons: (1) that the second part of 1. 5 itself is not wanted for any proof occurring in the original text, but is only put in to enable *objections* to the existing form of later propositions to be met, whereas the converse is not even wanted for this purpose; (2) that the converse could be deduced from 1. 6, if wanted, at any time after we have passed 1. 13, which can be used to prove that, if the angles formed by producing two sides of a triangle beyond the base are equal, the base angles themselves are equal.

Proclus adds a proof of the converse of the second part of 1. 5, i.e. of the proposition that, if the angles formed by producing two sides of a triangle beyond the base are equal, the triangle is isosceles; but it runs to some length and then only effects a reduction to the theorem of 1. 6 as we have it. As the result of this should hardly be assumed, a better proof would be an independent one *adapting* Euclid's own method in 1. 6. Thus, with the construction of 1. 5, we first prove by means of 1. 4 that the triangles *BFC*, *CGB* are equal in all respects, and therefore that *FC* is equal to *GB*, and the angle *BFC* equal to the angle *CGB*. Then we have to prove that *AF*, *AG* are equal. If they are not, let *AF* be the greater, and from *FA* cut off *FH* equal to *GA*. Join *CH*.



Then we have, in the two triangles HFC , AGB ,
 two sides HF , FC equal to two sides AG , GB
 and the angle HFC equal to the angle AGB .

Therefore (I. 4) the triangles HFC , AGB are equal. But the triangles BFC , CGB are also equal.

Therefore (if we take away these equals respectively) the triangles HBC , ACB are equal: which is impossible.

Therefore AF , AG are not unequal.

Hence AF is equal to AG and, if we subtract the equals BF , CG respectively, AB is equal to AC .

This proof is found in the commentary of an-Nairizi (ed. Besthorn-Heiberg, p. 61; ed. Curtze, p. 50).

Alternative proofs of I. 6.

Todhunter points out that I. 6, not being wanted till II. 4, could be postponed till later and proved by means of I. 26. Bisect the angle BAC by a straight line meeting the base at D . Then the triangles ABD , ACD are equal in all respects.

Another method depending on I. 26 is given by an-Nairizi after that proposition.

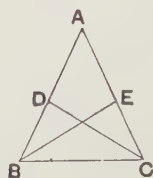
Measure equal lengths BD , CE along the sides BA , CA .
 Join BE , CD .

Then [I. 4] the triangles DBC , ECB are equal in all respects;

therefore EB , DC are equal, and the angles BEC , CDB are equal.

The supplements of the latter angles are equal [I. 13], and hence the triangles ABE , ACD have two angles equal respectively and the side BE equal to the side CD .

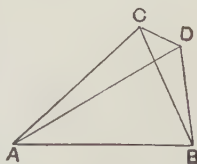
Therefore [I. 26] AB is equal to AC .



PROPOSITION 7.

Given two straight lines constructed on a straight line (from its extremities) and meeting in a point, there cannot be constructed on the same straight line (from its extremities), and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each to that which has the same extremity with it.

For, if possible, given two straight lines AC , CB constructed on the straight line AB and meeting at the point C , let two other straight lines AD , DB be constructed on the same straight line AB , on the same side of it, meeting in another point D and equal to the former two respectively, namely each to that which has the same extremity with it, so that CA is equal to DA which has the same extremity A with it, and



CB to DB which has the same extremity B with it; and let CD be joined.

Then, since AC is equal to AD ,

the angle ACD is also equal to the angle ADC ; [I. 5]

20 therefore the angle ADC is greater than the angle DCB ; therefore the angle CDB is much greater than the angle DCB .

Again, since CB is equal to DB ,

the angle CDB is also equal to the angle DCB .

25 But it was also proved much greater than it:

which is impossible.

Therefore etc.

Q. E. D.

1--6. In an English translation of the enunciation of this proposition it is absolutely necessary, in order to make it intelligible, to insert some words which are not in the Greek. The reason is partly that the Greek enunciation is itself very elliptical, and partly that some words used in it conveyed more meaning than the corresponding words in English do. Particularly is this the case with οὐ συσταθήσονται ἐπὶ "there shall not be constructed upon," since συνίστασθαι is the regular word for constructing a triangle in particular. Thus a Greek would easily understand συσταθήσονται ἐπὶ as meaning the construction of two lines forming a triangle on a given straight line as base; whereas, to "construct two straight lines on a straight line" is not in English sufficiently definite unless we explain that they are drawn from the ends of the straight line to meet at a point. I have had the less hesitation in putting in the words "from its extremities" because they are actually used by Euclid in the somewhat similar enunciation of I. 21.

How impossible a literal translation into English is, if it is to convey the meaning of the enunciation intelligibly, will be clear from the following attempt to render literally: "On the same straight line there shall not be constructed two other straight lines equal, each to each, to the same two straight lines, (terminating) at different points on the same side, having the same extremities as the original straight lines" (ἐπὶ τῆς αὐτῆς εὐθείας δύο ταῖς αὐταῖς εὐθείαις ἄλλαι δύο εὐθείαι ἴσαι ἑκάτερα ἑκάτερα οὐ συσταθήσονται πρὸς ἄλλω καὶ ἄλλω σημείῳ ἐπὶ τὰ αὐτὰ μέρη τὰ αὐτὰ πέρατα ἔχουσαι ταῖς ἐξ ἀρχῆς εὐθείαις).

The reason why Euclid allowed himself to use, in this enunciation, language apparently so obscure is no doubt that the phraseology was traditional and therefore, vague as it was, had a conventional meaning which the contemporary geometer well understood. This is proved, I think, by the occurrence in Aristotle (*Meteorologica* III. 5, 376 a 2 sqq.) of the very same, evidently technical, expressions. Aristotle is there alluding to the theorem given by Eutocius from Apollonius' *Plane Loci* to the effect that, if H, K be two fixed points and M such a variable point that the ratio of MH to MK is a given ratio (not one of equality), the locus of M is a circle. (For an account of this theorem see note on VI. 3 below.) Now Aristotle says "The lines drawn up from H, K in this ratio cannot be constructed to two different points of the semicircle A " (αὶ οὖν ἀπὸ τῶν HK ἀναγόμεναι γραμμαὶ ἐν τούτῳ τῷ λόγῳ οὐ συσταθήσονται τοῦ ἐφ' ᾧ A ἡμικυκλίου πρὸς ἄλλο καὶ ἄλλο σημείον).

If a paraphrase is allowed instead of a translation adhering as closely as possible to the original, Simson's is the best that could be found, since the fact that the straight lines form triangles on the same base is really conveyed in the Greek. Simson's enunciation is, *Upon the same base, and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and likewise those which are terminated at the other extremity.* Th. Taylor (the translator of Proclus) attacks Simson's alteration as "indiscreet" and as detracting from the beauty and accuracy of Euclid's enunciation which are enlarged upon by Proclus in his commentary. Yet, when Taylor says "Whatever difficulty learners may find in conceiving this proposition abstractedly is easily removed by its exposition in the figure," he really gives his case away. The fact is that Taylor, always enthusiastic over his author, was nettled by Simson's slighting remarks on Proclus' comments on the proposition. Simson had said, with reference to Proclus' explanation of the bearing of the second part of I. 5 on I. 7, that it was not "worth while

to relate his trifles at full length," to which Taylor retorts "But Mr Simson was no philosopher; and therefore the greatest part of these Commentaries must be considered by him as trifles, from the want of a philosophic genius to comprehend their meaning, and a taste superior to that of a *mere mathematician*, to discover their beauty and elegance."

20. It would be natural to insert here the step "but the angle ACD is greater than the angle BCD . [*C. N. 5.*]"

21. **much greater**, literally "greater by much" (πολλῶ μείζων). Simson and those who follow him translate: "*much more then* is the angle BDC greater than the angle BCD ," but the Greek for this would have to be $\text{πολλῶ (or πολὺ) μᾶλλον ἔστι... μείζων}$. πολλῶ μᾶλλον , however, though used by Apollonius, is not, apparently, found in Euclid or Archimedes.

Just as in 1. 6 we need a Postulate to justify theoretically the statement that CD falls within the angle ACB , so that the triangle DBC is less than the triangle ABC , so here we need Postulates which shall satisfy us as to the relative positions of CA , CB , CD on the one hand and of DC , DA , DB on the other, in order that we may be able to infer that the angle BDC is greater than the angle ADC , and the angle ACD greater than the angle BCD .

De Morgan (*op. cit.* p. 7) observes that 1. 7 would be made easy to beginners if they were first familiarised, as a common notion, with "if two magnitudes be equal, any magnitude greater than the one is greater than any magnitude less than the other." I doubt however whether a beginner would follow this easily; perhaps it would be more easily apprehended in the form "if any magnitude A is greater than a magnitude B , the magnitude A is greater than any magnitude equal to B , and (*a fortiori*) greater than any magnitude less than B ."

It has been mentioned already (note on 1. 5) that the second case of 1. 7 given by Simson and in our text-books generally is not in the original text (the omission being in accordance with Euclid's general practice of giving only one case, and that the most difficult, and leaving the others to be worked out by the reader for himself). The second case is given by Proclus as the answer to a possible *objection* to Euclid's proposition, which should assert that the proposition is not proved to be universally true, since the proof given does not cover all possible cases. Here the objector is supposed to contend that what Euclid declares to be impossible may still be possible if one pair of lines lie wholly within the other pair of lines; and the second part of 1. 5 enables the objection to be refuted.

If possible, let AD , DB be entirely within the triangle formed by AC , CB with AB , and let AC be equal to AD and BC to BD .

Join CD , and produce AC , AD to E and F .

Then, since AC is equal to AD ,

the triangle ACD is isosceles,

and the angles ECD , FDC under the base are equal.

But the angle ECD is greater than the angle BCD ; therefore the angle FDC is also greater than the angle BCD .

Therefore the angle BDC is greater by far than the angle BCD .

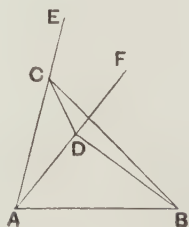
Again, since DB is equal to CB ,

the angles at the base of the triangle BDC are equal,

that is, the angle BDC is equal to the angle BCD .

Therefore the same angle BDC is both greater than and equal to the angle BCD ; which is impossible.

The case in which D falls on AC or BC does not require proof.



[1. 5]

I have already referred (note on I. 1) to the mistake made by those editors who regard I. 7 as being of no use except to prove I. 8. What I. 7 proves is that if, in addition to the base of a triangle, the length of the side terminating at each extremity of the base is given, only one triangle satisfying these conditions can be constructed on one and the same side of the given base. Hence not only does I. 7 enable us to prove I. 8, but it supplements I. 1 and I. 22 by showing that the constructions of those propositions give one triangle only on one and the same side of the base. But for I. 7 this could not be proved except by anticipating III. 10, of which therefore I. 7 is the equivalent for Book I. purposes. Dodgson (*Euclid and his modern Rivals*, pp. 194—5) puts it in another way. "It [I. 7] shows that, of all plane figures that can be made by hinging rods together, the *three-sided* ones (and these only) are *rigid* (which is another way of stating the fact that there cannot be *two* such figures on the same base). This is analogous to the fact, in relation to solids contained by plane surfaces hinged together, that *any* such solid is rigid, there being no maximum number of sides. And there is a close analogy between I. 7, 8 and III. 23, 24. These analogies give to geometry much of its beauty, and I think that they ought not to be lost sight of." It will therefore be apparent how ill-advised are those editors who eliminate I. 7 altogether and rely on Philo's proof for I. 8.

Proclus, it may be added, gives (pp. 268, 19—269, 10) another explanation of the retention of I. 7, notwithstanding that it was apparently only required for I. 8. It was said that astronomers used it to prove that three successive eclipses could not occur at equal intervals of time, i.e. that the third could not follow the second at the same interval as the second followed the first; and it was argued that Euclid had an eye to this astronomical application of the proposition. But, as we have seen, there are other grounds for retaining the proposition which are quite sufficient of themselves.

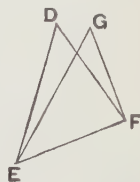
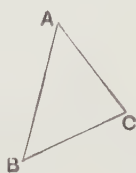
PROPOSITION 8.

If two triangles have the two sides equal to two sides respectively, and have also the base equal to the base, they will also have the angles equal which are contained by the equal straight lines.

5 Let ABC , DEF be two triangles having the two sides AB , AC equal to the two sides DE , DF respectively, namely AB to DE , and AC to DF ; and let them have the base BC equal
10 to the base EF ;

I say that the angle BAC is also equal to the angle EDF .

For, if the triangle ABC be applied to the triangle DEF , and if the point B be placed on
15 the point E and the straight line BC on EF ,
the point C will also coincide with F ,
because BC is equal to EF .



Then, BC coinciding with EF ,

BA , AC will also coincide with ED , DF ;

20 for, if the base BC coincides with the base EF , and the sides BA , AC do not coincide with ED , DF but fall beside them as EG , GF ,

then, given two straight lines constructed on a straight line (from its extremities) and meeting in a point, there will
25 have been constructed on the same straight line (from its extremities), and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each to that which has the same extremity with it.

30 But they cannot be so constructed. [I. 7]

Therefore it is not possible that, if the base BC be applied to the base EF , the sides BA , AC should not coincide with ED , DF ;

they will therefore coincide,

35 so that the angle BAC will also coincide with the angle EDF , and will be equal to it.

If therefore etc.

Q. E. D.

19. **BA, AC.** The text has here " BA , CA ."

21. fall beside them. The Greek has the future, $\piαραλλάξουσι$. $\piαραλλάττω$ means "to pass by without touching," "to miss" or "to deviate."

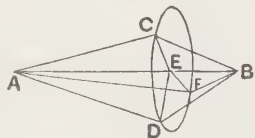
As pointed out above (p. 257) I. 8 is a *partial* converse of I. 4.

It is to be observed that in I. 8 Euclid is satisfied with proving the equality of the vertical angles and does not, as in I. 4, add that the triangles are equal, and the remaining angles are equal respectively. The reason is no doubt (as pointed out by Proclus and by Savile after him) that, when once the vertical angles are proved equal, the rest follows from I. 4, and there is no object in proving again what has been proved already.

Aristotle has an allusion to the theorem of this proposition in *Meteorologica* III. 3, 373 a 5—16. He is speaking of the rainbow and observes that, if equal rays be reflected from one and the same point to one and the same point, the points at which reflection takes place are on the circumference of a circle. "For let the broken lines ACB , AFB , ADB be all reflected from the point A to the point B (in such a way that) AC , AF , AD are all equal to one another, and the lines (terminating) at B , i.e. CB , FB , DB , are likewise all equal; and let AEB be joined. It follows that *the triangles are equal*; for they are upon the equal (base) AEB ."

Heiberg (*Mathematisches zu Aristoteles*, p. 18) thinks that the form of the conclusion quoted is an indication that in the corresponding proposition to Eucl. I. 8, as it lay before Aristotle, it was maintained that the *triangles* were equal, and not only the angles, and "we see here therefore, in a clear example, how the stones of the ancient fabric were recut for the rigid structure of his

Elements." I do not, however, think that this inference from Aristotle's language as to the form of the pre-Euclidean proposition is safe. Thus if we, nowadays, were arguing from the data in the passage of Aristotle, we should doubtless infer directly that the triangles are equal in all respects, quoting 1. 8 alone. Besides, Aristotle's language is rather careless, as the next sentences of the same passage show. "Let perpendiculars," he says, "be drawn to AEB from the angles, CE from C , FE from F and DE from D . These, then, are equal; for they are all in equal triangles, and in one plane; for all of them are perpendicular to AEB , and they meet at one point E . Therefore the (line) drawn (through C, F, D) will be a circle, and its centre (will be) E ." Aristotle should obviously have proved that the three perpendiculars *will* meet at one point E on AEB before he spoke of drawing the perpendiculars CE, FE, DE . This of course follows from their being "in equal triangles" (by means of Eucl. 1. 26); and then, from the fact that the perpendiculars meet at one point on AB , it can be inferred that all three are in one plane.



Philo's proof of I. 8.

This alternative proof avoids the use of 1. 7, and it is elegant; but it is inconvenient in one respect, since three cases have to be distinguished. Proclus gives the proof in the following order (pp. 266, 15—268, 14).

Let ABC, DEF be two triangles having the sides AB, AC equal to the sides DE, DF respectively, and the base BC equal to the base EF .

Let the triangle ABC be applied to the triangle DEF , so that B is placed on E and BC on EF , but so that A falls on the opposite side of EF from D , taking the position G . Then C will coincide with F , since BC is equal to EF .

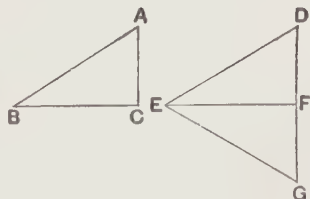
Now FG will either be in a straight line with DF , or make an angle with it, and in the latter case the angle will either be *interior* ($\kappa\alpha\tau\alpha\ \tau\omicron\ \epsilon\nu\tau\omicron\varsigma$) to the figure or *exterior* ($\kappa\alpha\tau\alpha\ \tau\omicron\ \epsilon\kappa\tau\omicron\varsigma$).

I. Let FG be in a straight line with DF .

Then, since DE is equal to EG , and DFG is a straight line,

DEG is an isosceles triangle, and the angle at D is equal to the angle at G .

[I. 5].



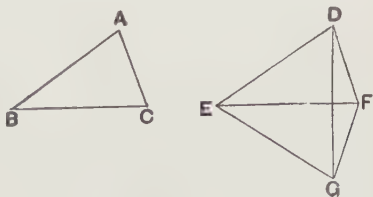
II. Let DF, FG form an angle *interior* to the figure.

Let DG be joined.

Then, since DE, EG are equal,

the angle EDG is equal to the angle EGD .

Again, since DF is equal to FG , the angle FDG is equal to the angle FGD .



Therefore, by addition, the whole angle EDF is equal to the whole angle EGF .

III. Let DF , FG form an angle *exterior* to the figure.

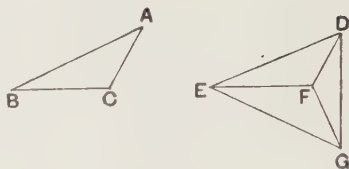
Let DG be joined.

The proof proceeds as in the last case, except that subtraction takes the place of addition, and

the remaining angle EDF is equal to the remaining angle EGF .

Therefore in all three cases the angle EDF is equal to the angle EGF , that is, to the angle BAC .

It will be observed that, in accordance with the practice of the Greek geometers in not recognising as an "angle" any angle not less than two right angles, the re-entrant angle of the quadrilateral $DEGF$ is ignored and the angle DFG is said to be *outside* the figure.



PROPOSITION 9.

To bisect a given rectilineal angle.

Let the angle BAC be the given rectilineal angle.

Thus it is required to bisect it.

Let a point D be taken at random on AB ;

let AE be cut off from AC equal to AD ; [I. 3]

let DE be joined, and on DE let the equilateral triangle DEF be constructed;

let AF be joined.



I say that the angle BAC has been bisected by the straight line AF .

For, since AD is equal to AE ,
and AF is common,

the two sides DA , AF are equal to the two sides EA , AF respectively.

And the base DF is equal to the base EF ;

therefore the angle DAF is equal to the angle EAF .

[I. 8]

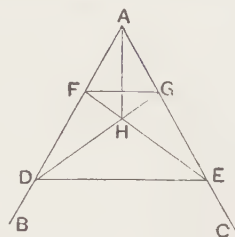
Therefore the given rectilineal angle BAC has been bisected by the straight line AF .

Q. E. F.

It will be observed from the translation of this proposition that Euclid does not say, in his description of the construction, that the equilateral triangle should be constructed on the side of DE opposite to A ; he leaves this to be inferred from his figure. There is no particular value in Proclus' explanation as to how we should proceed in case any one should assert that he could not recognise the existence of any space below DE . He supposes, then, the equilateral triangle described on the side of DE towards A , and hence has to consider three cases according as the vertex of the equilateral triangle falls on A , above A or below it. The second and third cases do not differ

substantially from Euclid's. In the first case, where ADE is the equilateral triangle constructed on DE , take any point F on AD , and from AE cut off AG equal to AF . Join DG , EF meeting in H ; and join AH . Then AH is the bisector required.

Proclus also answers the possible objection that might be raised to Euclid's proof on the ground that it assumes that, if the equilateral triangle be described on the side of DE opposite to A , its vertex F will lie within the angle BAC . The objector is supposed to argue that this is not necessary, but that F might fall either on one of the lines forming the angle or outside it altogether. The two cases are disposed of thus.



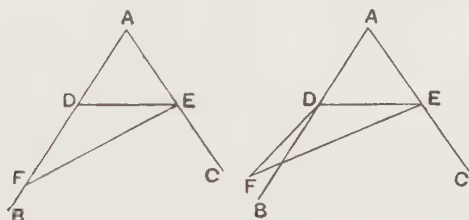
Suppose F to fall as shown in the two figures below respectively.

Then, since FD is equal to FE ,

the angle FDE is equal to the angle FED .

Therefore the angle CED is greater than the angle FDE ; and, in the second figure, *a fortiori*, the angle CED is greater than the angle BDE .

But, since ADE is an isosceles triangle, and the equal sides are produced,



the angles under the base are equal,

i.e., the angle CED is equal to the angle BDE .

But the angle CED was proved greater: which is impossible.

Here then is the second case in which, in Proclus' view, the second part of I. 5 is useful for refuting objections.

On this proposition Proclus takes occasion (p. 271, 15—19) to emphasize the fact that the given angle must be *rectilineal*, since the bisection of any sort of angle (including angles made by curves with one another or with straight lines) is not matter for an elementary treatise, besides which it is questionable whether such bisection is always possible. "Thus it is difficult to say whether it is possible to bisect the so-called *horn-like angle*" (formed by the circumference of a circle and a tangent to it).

Trisection of an angle.

Further it is here that Proclus gives us his valuable historical note about the *trisection* of any acute angle, which (as well as the division of an angle in any given ratio) requires resort to other curves than circles, i.e. curves of the species which, after Geminus, he calls "mixed." "This," he says (p. 272, 1—12), "is shown by those who have set themselves the task of trisecting such a given rectilineal angle. For Nicomedes trisected any rectilineal angle by means of the *conchoidal* lines, the origin, order, and properties of which he has handed down to us, being himself the discoverer of their peculiarity. Others have done the same thing by means of the *quadratrices* of Hippias and Nicomedes, thereby again using 'mixed' curves. But others, starting from the Archimedean spirals, cut a given rectilineal angle in a given ratio."

The property of the curve is that, if F is any point, the arc BED is to the arc ED as AB is to FH .

In other words, if ϕ is the angle FAD , ρ the radius vector AF and a the side of the square,

$$(\rho \sin \phi)/a = \phi/\frac{1}{2}\pi.$$

Now the angle EAD can not only be *trisected* but *divided in any given ratio* by means of the quadratrix (Pappus iv. p. 286).

For let FH be divided at K in the given ratio.

Draw KL parallel to AD , meeting the curve in L ; join AL and produce it to meet the circle in N .

Then the angles EAN , NAD are in the ratio of FK to KH , as is easily proved.

(c) Use of the *spiral of Archimedes*.

The trisection of an angle, or the division of an angle in any ratio, by means of the *spiral of Archimedes* is of course an equally simple matter. Suppose any angle included between the two radii vectores OA and OB of the spiral, and let it be required to cut the angle AOB in a given ratio. Since the radius vector increases proportionally with the angle described by the vector which generates the curve (reckoned from the original position of the vector coinciding with the initial line to the particular position assumed), we have only to take the radius vector OB (the greater of the two OA , OB), mark off OC along it equal to OA , cut CB in the given ratio (at D say), and then draw the circle with centre O and radius OD cutting the spiral in E . Then OE will divide the angle AOB in the required manner.

PROPOSITION 10.

To bisect a given finite straight line.

Let AB be the given finite straight line.

Thus it is required to bisect the finite straight line AB .

Let the equilateral triangle ABC be constructed on it, [1. 1]

and let the angle ACB be bisected by the straight line CD ; [1. 9]

I say that the straight line AB has been bisected at the point D .

For, since AC is equal to CB , and CD is common,

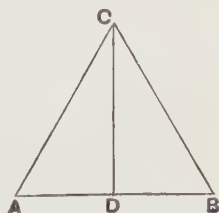
the two sides AC , CD are equal to the two sides BC , CD respectively;

and the angle ACD is equal to the angle BCD ;

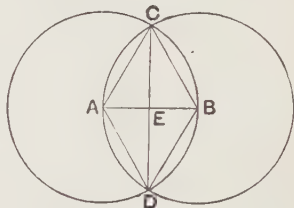
therefore the base AD is equal to the base BD . [1. 4]

Therefore the given finite straight line AB has been bisected at D .

Q. E. F.



Apollonius, we are told (Proclus, pp. 279, 16—280, 4), bisected a straight line AB by a construction like that of I. 1. With centres A , B , and radii AB , BA respectively, two circles are described, intersecting in C , D . Joining CD , AC , CB , AD , DB , Apollonius proves in two steps that CD bisects AB .



- (1) Since, in the triangles ACD , BCD ,
two sides AC , CD are equal to two sides
 BC , CD ,
and the bases AD , BD are equal,
the angle ACD is equal to the angle
 BCD .

[I. 8]

- (2) The latter angles being equal, and AC being equal to CB , while CE is common,

the equality of AE , EB follows by I. 4.

The objection to this proof is that, instead of *assuming* the bisection of the angle ACB , as already effected by I. 9, Apollonius goes a step further back and embodies a construction for bisecting the angle. That is, he unnecessarily does over again what has been done before, which is open to objection from a theoretical point of view.

Proclus (pp. 277, 25—279, 4) warns us against being moved by this proposition to conclude that geometers assumed, as a preliminary hypothesis, that a line is not made up of indivisible parts (ἐξ ἀμερῶν). This might be argued thus. If a line is made up of indivisibles, there must be in a finite line either an odd or an even number of them. If the number were odd, it would be necessary in order to bisect the line to bisect an indivisible (the odd one). In that case therefore it would not be possible to bisect a straight line, if it is a magnitude made up of indivisibles. But, if it is not so made up, the straight line can be divided *ad infinitum* or without limit (ἐπ' ἀπειρον διαίρεται). Hence it was argued (φασίν), says Proclus, that the divisibility of magnitudes without limit was admitted and assumed as a geometrical principle. To this he replies, following Geminus, that geometers did indeed assume, by way of a common notion, that a continuous magnitude, i.e. a magnitude consisting of parts connected together (συνημμένον), is divisible (διαπετόν). But *infinite* divisibility was not assumed by them; it was *proved* by means of the first principles applicable to the case. "For when," he says, "they prove that the incommensurable exists among magnitudes, and that it is not all things that are commensurable with one another, what else will any one say that they prove but that every magnitude can be divided for ever, and that we shall never arrive at the indivisible, that is, the least common measure of the magnitudes? This then is matter of demonstration, whereas it is an *axiom* that everything continuous is divisible, so that a finite continuous line is divisible. The writer of the *Elements* bisects a finite straight line, starting from the latter notion, and not from any assumption that it is divisible without limit." Proclus adds that the proposition may also serve to refute Xenocrates' theory of indivisible lines (ἄτομοι γραμμαί). The argument given by Proclus to disprove the existence of indivisible lines is substantially that used by Aristotle as regards magnitudes generally (cf. *Physics* VI. 1, 231 a 21 sqq. and especially VI. 2, 233 b 15—32).

PROPOSITION 11.

To draw a straight line at right angles to a given straight line from a given point on it.

Let AB be the given straight line, and C the given point on it.

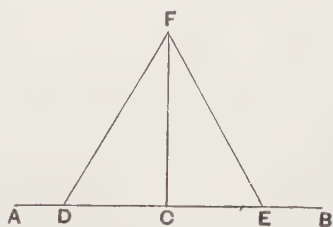
5 Thus it is required to draw from the point C a straight line at right angles to the straight line AB .

Let a point D be taken at random on AC ;

10 let CE be made equal to CD ; [I. 3]

on DE let the equilateral triangle FDE be constructed, [I. 1]

and let FC be joined;



I say that the straight line FC has been drawn at right
15 angles to the given straight line AB from C the given point on it.

For, since DC is equal to CE ,
and CF is common,

the two sides DC , CF are equal to the two sides EC ,
20 CF respectively;

and the base DF is equal to the base FE ;

therefore the angle DCF is equal to the angle ECF ;

[I. 8]

and they are adjacent angles.

But, when a straight line set up on a straight line makes
25 the adjacent angles equal to one another, each of the equal angles is right; [Def. 10]

therefore each of the angles DCF , FCE is right.

Therefore the straight line CF has been drawn at right
angles to the given straight line AB from the given point
30 C on it.

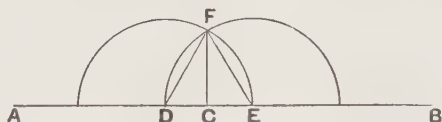
Q. E. F.

10. let CE be made equal to CD . The verb is *κείσθω* which, as well as the other parts of *κείμαι*, is constantly used for the passive of *τίθημι* "to place"; and the latter word is constantly used in the sense of *making*, e.g., one straight line equal to another straight line.

De Mörgan remarks that this proposition, which is "to bisect the angle made by a straight line and its continuation" [i.e. a *flat* angle], should be a particular case of 1. 9, the constructions being the same. This is certainly

worth noting, though I doubt the advantage of rearranging the propositions in consequence.

Apollonius gave a construction for this proposition (see Proclus, p. 282, 8) differing from Euclid's in much the same way as his construction for bisecting a straight line differed from that of I. 10. Instead of assuming an equilateral triangle drawn without repeating the process of I. 1, Apollonius takes D and E equidistant from C as in Euclid, and then draws circles in the manner of



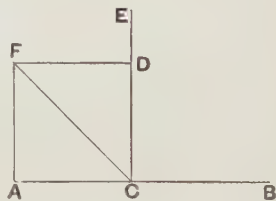
I. 1 meeting at F . This necessitates proving again that DF is equal to FE ; whereas Euclid's assumption of the construction of I. 1 in the words "let the equilateral triangle FDE be constructed" enables him to dispense with the drawing of circles and with the proof that DF is equal to FE at the same time. While however the substitution of Apollonius' constructions for I. 10 and I. 11 would show faulty arrangement in a theoretical treatise like Euclid's, they are entirely suitable for what we call *practical* geometry, and such may have been Apollonius' object in these constructions and in his alternative for I. 23.

Proclus gives a construction for drawing a straight line at right angles to another straight line but from *one end* of it, instead of from an intermediate point on it, it being supposed (for the sake of argument) that we are not permitted to *produce* the straight line. In the commentary of an-Nairizi (ed. Besthorn-Heiberg, pp. 73—4; ed. Curtze, pp. 54—5) this construction is attributed to Heron.

Let it be required to draw from A a straight line at right angles to AB .

On AB take any point C , and in the manner of the proposition draw CE at right angles to AB .

From CE cut off CD equal to AC , bisect the angle ACE by the straight line CF , [I. 9] and draw DF at right angles to CE meeting CF in F . Join FA .



Then the angle FAC will be a right angle.

For, since, in the triangles ACF , DCF , the two sides AC , CF are equal to the two sides DC , CF respectively, and the included angles ACF , DCF are equal,

the triangles are equal in all respects.

[I. 4]

Therefore the angle at A is equal to the angle at D , and is accordingly a right angle.

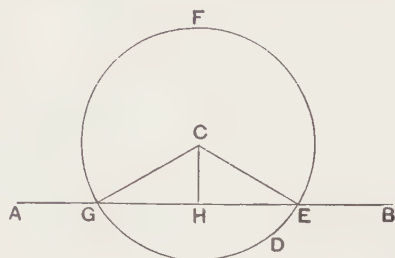
PROPOSITION 12.

To a given infinite straight line, from a given point which is not on it, to draw a perpendicular straight line.

Let AB be the given infinite straight line, and C the given point which is not on it;

5 thus it is required to draw to the given infinite straight line AB , from the given point C which is not on it, a perpendicular straight line.

For let a point D be taken
10 at random on the other side of the straight line AB , and with centre C and distance CD let the circle EFG be described ;



[Post. 3]

let the straight line EG

15

be bisected at H ,

[I. 10]

and let the straight lines CG , CH , CE be joined.

[Post. 1]

I say that CH has been drawn perpendicular to the given infinite straight line AB from the given point C which is not on it.

20 For, since GH is equal to HE ,

and HC is common,

the two sides GH , HC are equal to the two sides EH , HC respectively ;

and the base CG is equal to the base CE ;

25

therefore the angle CHG is equal to the angle EHC .

[I. 8]

And they are adjacent angles.

But, when a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is
30 called a perpendicular to that on which it stands. [Def. 10]

Therefore CH has been drawn perpendicular to the given infinite straight line AB from the given point C which is not on it.

Q. E. F.

2. a perpendicular straight line, *κάθετον εὐθείαν γραμμὴν*. This is the full expression for a perpendicular, *κάθετος* meaning *let fall* or *let down*, so that the expression corresponds to our *plumb-line*. *ἡ κάθετος* is however constantly used alone for a perpendicular, *γραμμὴ* being understood.

10. on the other side of the straight line AB , literally "towards the other parts of the straight line AB ," ἐπὶ τὰ ἑτέρα μέρη τῆς AB . Cf. "on the same side" (ἐπὶ τὰ αὐτὰ μέρη) in Post. 5 and "in both directions" (ἐφ' ἑκάτερα τὰ μέρη) in Def. 23.

"This problem," says Proclus (p. 283, 7--10), "was first investigated by Oenopides [5th cent. B.C.], who thought it useful for astronomy. He however calls the perpendicular, in the archaic manner, (a line drawn)

gnomon-wise (κατὰ γνῶμονα), because the gnomon is also at right angles to the horizon." In this earlier sense the *gnomon* was a staff placed in a vertical position for the purpose of casting shadows and so serving as a means of measuring time (Cantor, *Geschichte der Mathematik*, 13, p. 161). The later meanings of the word as used in Eucl. Book II. and elsewhere will be explained in the note on Book II. Def. 2.

Proclus says that two kinds of perpendicular were distinguished, the "plane" (ἐπίπεδος) and the "solid" (στερεά), the former being the perpendicular dropped on a line in a plane and the latter the perpendicular dropped on a plane. The term "solid perpendicular" is sufficiently curious, but it may perhaps be compared with the Greek term "solid locus" applied to a conic section, apparently on the ground that it has its origin in the section of a solid, namely a cone.

Attention is called by most editors to the assumption in this proposition that, if only D be taken on the side of AB remote from C , the circle described with CD as radius must necessarily cut AB in two points. To satisfy us of this we need, as in 1. 1, some postulate of continuity, e.g. something like that suggested by Killing (see note on the Principle of Continuity above, p. 235): "If a point [here the point describing the circle] moves in a figure which is divided into two parts [by the straight line], and if it belongs at the beginning of the motion to one part and at another stage of the motion to the other part, it must during the motion cut the boundary between the two parts," and this of course applies to the motion in *two* directions from D .

But the editors have not, as a rule, noticed a possible *objection* to the Euclidean statement of this problem which is much more difficult to dispose of at this stage, i.e. without employing any proposition later than this in Euclid's order. How do we know, says the supposed critic, that the circle does not cut AB in *three* or more points, in which case there would be not *one* perpendicular but *three* or more? Proclus (pp. 286, 12—289, 6) tries to refute this objection, and it is interesting to follow his argument, though it will easily be seen to be inconclusive. He takes in order three possible suppositions.

1. May not the circle meet AB in a third point K between the middle point of GE and either extremity of it, taking the form drawn in the figure appended?

Suppose this possible. Bisect GE in H . Join CH , and produce it to meet the circle in L . Join CG , CK , CE .

Then, since CG is equal to CE , and CH is common, while the base GH is equal to the base HE ,

the angles CHG , CHE are equal and, since they are adjacent, they are both right.

Again, since CG is equal to CE ,

the angles at G and E are equal.

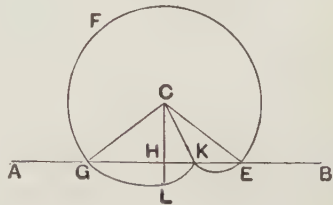
Lastly, since CK is equal to CG and also to CE , the angles CGK , CKG are equal, as also are the angles CKE , CEK .

Since the angles CGK , CEK are equal, it follows that

the angles CKG , CKE are equal and therefore both right.

Therefore the angle CKH is equal to the angle CHK ,

and CH is equal to CK .



But CK is equal to CL , by the definition of the circle; therefore CH is equal to CL : which is impossible.

Thus Proclus; but why should not the circle meet AB in H as well as K ?

2. May not the circle meet AB in H the middle point of GE and take the form shown in the second figure?

In that case, says Proclus, join CG , CH , CE as before. Then bisect HE at K , join CK and produce it to meet the circumference at L .

Now, since HK is equal to KE , CK is common, and the base CH is equal to the base CE ,

the angles at K are equal and therefore both right angles.

Therefore the angle CHK is equal to the angle CKH , whence CK is equal to CH and therefore to CL : which is impossible.

So Proclus; but why should not the circle meet AB in K as well as H ?

3. May not the circle meet AB in *two* points besides G , E and pass, between those two points, to the side of AB towards C , as in the next figure?

Here again, by the same method, Proclus proves that, K , L being the other two points in which the circle cuts AB ,

CK is equal to CH ,

and, since the circle cuts CH in M ,

CM is equal to CK and therefore to CH : which is impossible.

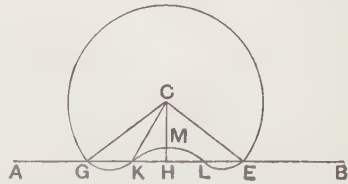
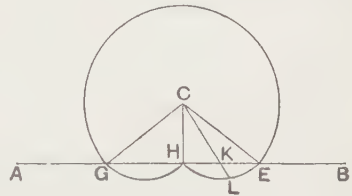
But, again, why should the circle not cut AB in the point H as well?

In fact, Proclus' cases are not mutually exclusive, and his method of proof only enables us to show that, if the circle meets AB in one more point besides G , E , it must meet it in more points still. We can always find a new point of intersection by bisecting the distance separating any two points of intersection, and so, applying the method *ad infinitum*, we should have to conclude ultimately that the circle with radius CH (or CG) coincides with AB . It would follow that a circle with centre C and radius greater than CH would not meet AB at all. Also, since all straight lines from C to points on AB would be equal in length, there would be an infinite number of perpendiculars from C on AB .

Is this under any circumstances possible? It is not possible in Euclidean space, but it is possible, under the Riemann hypothesis (where a straight line is a "closed series" and returns on itself), in the case where C is the pole of the straight line AB .

It is natural therefore that, for a proof that in Euclidean space there is only one perpendicular from a point to a straight line, we have to wait until I. 16, the precise proposition which under the Riemann hypothesis is only valid with a certain restriction and not universally. There is no difficulty involved by waiting until I. 16, since I. 12 is not used before that proposition is reached; and we are only in the same position as when, in order to satisfy ourselves of the number of possible solutions of I. 1, we have to wait till I. 7.

But if we wish, after all, to prove the truth of the assumption *without* recourse to any later proposition than I. 12, we can do so by means of this same invaluable I. 7.



If the circle intersects AB as before in G, E , let H be the middle point of GE , and suppose, if possible, that the circle also intersects AB in any other point K on AH .

From H , on the side of AB opposite to C , draw HL at right angles to AB , and make HL equal to HC .

Join CG, LG, CK, LK .

Now, in the triangles CHG, LHG , CH is equal to LH , and HG is common.

Also the angles CHG, LHG , being both right, are equal.

Therefore the base CG is equal to the base LG .

Similarly we prove that CK is equal to LK .

But, by hypothesis, since K is on the circle,

CK is equal to CG .

Therefore CG, CK, LG, LK are all equal.

Now the next proposition, I. 13, will tell us that CH, HL are in a straight line; but we will not assume this. Join CL .

Then on the same base CL and on the same side of it we have two pairs of straight lines drawn from C, L to G and K such that CG is equal to CK and LG to LK .

But this is impossible [I. 7].

Therefore the circle cannot cut BA or BA produced in any point other than G on that side of CL on which G is.

Similarly it cannot cut AB or AB produced at any point other than E on the other side of CL .

The only possibility left therefore is that the circle might cut AB in the same point as that in which CL cuts it. But this is shown to be impossible by an adaptation of the proof of I. 7.

For the assumption is that there may be some point M on CL such that CM is equal to CG and LM to LG .

If possible, let this be the case, and produce CG to N .

Then, since CM is equal to CG , the angle NGM is equal to the angle GML [I. 5, part 2].

Therefore the angle GML is greater than the angle MGL .

Again, since LG is equal to LM , the angle GML is equal to the angle MGL .

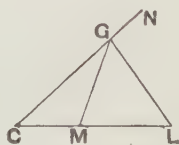
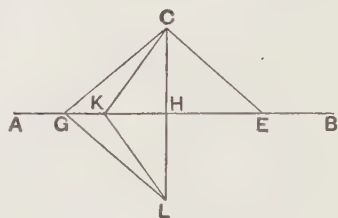
But it was also greater: which is impossible.

Hence the circle in the original figure cannot cut AB in the point in which CL cuts it.

Therefore the circle cannot cut AB in any point whatever except G and E .

[This proof of course does not prove that CK is less than CG , but only that it is not equal to it. The proposition that, of the obliques drawn from C to AB , that is less the foot of which is nearer to H can only be proved later. The proof by I. 7 also fails, under the Riemann hypothesis, if C, L are the poles of the straight line AB , since the broken lines CGL, CKL etc. become equal straight lines, all perpendicular to AB .]

Proclus rightly adds (p. 289, 18 sqq.) that it is not necessary to take D on the side of AB away from A if an objector "says that there is no space on



that side." If it is not desired to trespass on that side of AB , we can take D anywhere on AB and describe the arc of a circle between D and the point where it meets AB again, drawing the arc on the side of AB on which C is. If it should happen that the selected point D is such that the circle only meets AB in *one* point (D itself), we have only to describe the circle with CD as radius, then, if E be a point on this circle, take F a point further from C than E is, and describe with CF as radius the circular arc meeting AB in two points.

PROPOSITION 13.

If a straight line set up on a straight line make angles, it will make either two right angles or angles equal to two right angles.

For let any straight line AB set up on the straight line
5 CD make the angles CBA , ABD ;

I say that the angles CBA , ABD
are either two right angles or equal to
two right angles.

Now, if the angle CBA is equal to
10 the angle ABD ,

they are two right angles. [Def. 10]

But, if not, let BE be drawn from the point B at right
angles to CD ; [I. 11]

therefore the angles CBE , EBD are two right angles.

15 Then, since the angle CBE is equal to the two angles
 CBA , ABE ,

let the angle EBD be added to each;

therefore the angles CBE , EBD are equal to the three
angles CBA , ABE , EBD . [C. N. 2]

20 Again, since the angle DBA is equal to the two angles
 DBE , EBA ,

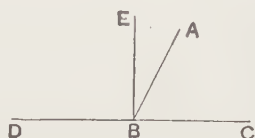
let the angle ABC be added to each;

therefore the angles DBA , ABC are equal to the three
angles DBE , EBA , ABC . [C. N. 2]

25 But the angles CBE , EBD were also proved equal to
the same three angles;

and things which are equal to the same thing are also
equal to one another; [C. N. 1]

therefore the angles CBE , EBD are also equal to the
30 angles DBA , ABC .



But the angles CBE , EBD are two right angles ;

therefore the angles DBA , ABC are also equal to two right angles.

Therefore etc.

Q. E. D.

17. let the angle EBD be added to each, literally "let the angle EBD be added (so as to be) common," *κοινὴ προσκελισθω ἡ ὑπὸ EBD* . Similarly *κοινὴ ἀφηρήσθω* is used of subtracting a straight line or angle from each of two others. "Let the common angle EBD be added" is clearly an inaccurate translation, for the angle is not common before it is added, i.e. the *κοινὴ* is proleptic. "Let the common angle be *subtracted*" as a translation of *κοινὴ ἀφηρήσθω* would be less unsatisfactory, it is true, but, as it is desirable to use corresponding words when translating the two expressions, it seems hopeless to attempt to keep the word "common," and I have therefore said "to each" and "from each" simply.

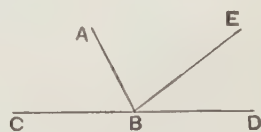
PROPOSITION 14.

If with any straight line, and at a point on it, two straight lines not lying on the same side make the adjacent angles equal to two right angles, the two straight lines will be in a straight line with one another.

5 For with any straight line AB , and at the point B on it, let the two straight lines BC , BD not lying on the same side make the adjacent angles ABC , ABD equal to two right angles ;

I say that BD is in a straight line with CB .

10 For, if BD is not in a straight line with BC , let BE be in a straight line with CB .



Then, since the straight line AB stands on the straight line CBE ,

15 the angles ABC , ABE are equal to two right angles.

[I. 13]

But the angles ABC , ABD are also equal to two right angles ;

therefore the angles CBA , ABE are equal to the angles CBA , ABD .

[Post. 4 and C. N. 1]

Let the angle CBA be subtracted from each ;

20 therefore the remaining angle ABE is equal to the remaining angle ABD ,

[C. N. 3]

the less to the greater : which is impossible.

Therefore BE is not in a straight line with CB .

Similarly we can prove that neither is any other straight
25 line except BD .

Therefore CB is in a straight line with BD .
Therefore etc.

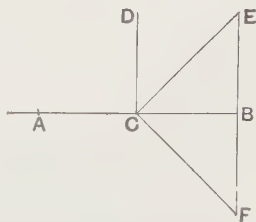
Q. E. D.

1. If with any straight line.... There is no greater difficulty in translating the works of the Greek geometers than that of accurately giving the force of prepositions. $\pi\rho\acute{o}s$, for instance, is used in all sorts of expressions with various shades of meaning. The present enunciation begins 'Εάν $\pi\rho\acute{o}s$ τινι εὐθείᾳ καὶ τῷ $\pi\rho\acute{o}s$ αὐτῇ σημείω, and it is really necessary in this one sentence to translate $\pi\rho\acute{o}s$ by three different words, *with*, *at*, and *on*. The first $\pi\rho\acute{o}s$ must be translated by *with* because two straight lines "make" an angle *with* one another. On the other hand, where the similar expression $\pi\rho\acute{o}s$ τῇ δοθείσῃ εὐθείᾳ occurs in I. 23, but it is a question of "constructing" an angle ($\sigma\upsilon\sigma\tau\eta\sigma\alpha\sigma\theta\alpha\iota$), we have to say "to construct *on* a given straight line." *Against* would perhaps be the English word coming nearest to expressing all these meanings of $\pi\rho\acute{o}s$, but it would be intolerable as a translation.

17. Todhunter points out that for the inference in this line Post. 4, that all right angles are equal, is necessary as well as the Common Notion that things which are equal to the same thing (or rather, here, to *equal things*) are equal. A similar remark applies to steps in the proofs of I. 15 and I. 28.

24. **we can prove.** The Greek expresses this by the future of the verb, $\delta\epsilon\lambda\acute{o}\mu\epsilon\nu$, "we shall prove," which however would perhaps be misleading in English.

Proclus observes (p. 297) that two straight lines on the *same* side of another straight line and meeting it in one and the same point may make with one and the same portion of the straight line terminated at the point two angles which are together equal to two right angles, in which case however the two straight lines would not be in a straight line with one another. And he quotes from Porphyry a construction for two such straight lines in the particular case where they form with the given straight line angles equal respectively to half a right angle and one and a half right angles. There is no particular value in the construction, which will be gathered from the annexed figure where CE , CF are drawn at the prescribed inclinations to CD .



PROPOSITION 15.

If two straight lines cut one another, they make the vertical angles equal to one another.

For let the straight lines AB , CD cut one another at the point E ;

5 I say that the angle AEC is equal to the angle DEB ,

and the angle CEB to the angle AED .



For, since the straight line AE stands
10 on the straight line CD , making the angles CEA , AED ,
the angles CEA , AED are equal to two right angles.

[I. 13]

Again, since the straight line DE stands on the straight line AB , making the angles AED , DEB ,
the angles AED , DEB are equal to two right angles.

[I. 13]

15 But the angles CEA , AED were also proved equal to two right angles ;

therefore the angles CEA , AED are equal to the angles AED , DEB . [Post. 4 and C. N. 1]

Let the angle AED be subtracted from each ;

20 therefore the remaining angle CEA is equal to the remaining angle DEB . [C. N. 3]

Similarly it can be proved that the angles CEB , DEA are also equal.

Therefore etc.

Q. E. D.

25 [PORISM. From this it is manifest that, if two straight lines cut one another, they will make the angles at the point of section equal to four right angles.]

1. the vertical angles. The difference between *adjacent* angles (*αἱ ἐφεξῆς γωνίαι*) and *vertical* angles (*αἱ κατὰ κορυφὴν γωνίαι*) is thus explained by Proclus (p. 298, 14—24). The first term describes the angles made by two straight lines when one only is divided by the other, i.e. when one straight line meets another at a point which is not either of its extremities, but is not itself produced beyond the point of meeting. When the first straight line *is* produced, so that the lines cross at the point, they make two pairs of *vertical* angles (which are more clearly described as *vertically opposite* angles), and which are so called because their convergence is from opposite directions to one point (the intersection of the lines) as vertex (*κορυφή*).

26. at the point of section, literally “at the section,” *πρὸς τῇ τομῇ*.

This theorem, according to Eudemus, was first discovered by Thales, but found its scientific demonstration in Euclid (Proclus, p. 299, 3—6).

Proclus gives a converse theorem which may be stated thus. *If a straight line is met at one and the same point intermediate in its length by two other straight lines on different sides of it and such as to make the vertical angles equal, the latter straight lines are in a straight line with one another.* The proof need not be given, since it is almost self-evident, whether (1) it is direct, by means of I. 13, 14, or (2) indirect, by *reductio ad absurdum* depending on I. 15.

The balance of ms. authority seems to be against the genuineness of this *Porism*, but Proclus and Psellus both have it. The word is not here used, as it is in the title of Euclid's lost *Porisms*, to signify a particular class of independent propositions which Proclus describes as being in some sort intermediate between theorems and problems (requiring us, not to bring a thing into existence, but to *find* something which we know to exist). *Porism* has here (and wherever the term is used in the *Elements*) its second meaning ; it is what we call a corollary, i.e. an incidental result springing from the proof of a theorem or the solution of a problem, a result not directly sought but appearing as it were by chance without any additional labour, and constituting, as Proclus says, a sort of *windfall* (*ἐρμαιον*) and *bonus* (*κέρδος*). These *Porisms* appear in both the

geometrical and arithmetical Books of the *Elements*, and may either result from theorems or problems. Here the Porism is geometrical, and springs out of a theorem; VII. 2 affords an instance of an arithmetical Porism. As an instance of a Porism to a problem Proclus cites "that which is found in the second Book" (*τὸ ἐν τῷ δευτέρῳ βιβλίῳ κείμενον*); but as to this see notes on II. 4 and IV. 15.

The present Porism, says Proclus, formed the basis of "that paradoxical theorem which proves that only the following three (regular) polygons can fill up the whole space surrounding one point, the equilateral triangle, the square, and the equilateral and equiangular hexagon." We can in fact place round a point in this manner six equilateral triangles, three regular hexagons, or four squares. "But only the angles of these regular figures, to the number specified, can make up four right angles: a theorem due to the Pythagoreans."

Proclus further adds that it results from the Porism that, if any number of straight lines intersect one another at one point, the sum of all the angles so formed will still be equal to four right angles. This is of course what is generally given in the text-books as Corollary 2.

PROPOSITION 16.

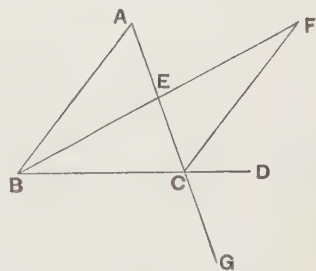
In any triangle, if one of the sides be produced, the exterior angle is greater than either of the interior and opposite angles.

Let ABC be a triangle, and let one side of it BC be produced to D ;

5 I say that the exterior angle ACD is greater than either of the interior and opposite angles CBA , BAC .

Let AC be bisected at E [I. 10],
and let BE be joined and produced
10 in a straight line to F ;

let EF be made equal to BE [I. 3],
let FC be joined [Post. 1], and let AC
be drawn through to G [Post. 2].



Then, since AE is equal to EC ,
15 and BE to EF ,

the two sides AE , EB are equal to the two sides CE , EF respectively;

and the angle AEB is equal to the angle FEC ,

for they are vertical angles.

[I. 15]

20 Therefore the base AB is equal to the base FC ,

and the triangle ABE is equal to the triangle CFE ,

and the remaining angles are equal to the remaining angles respectively, namely those which the equal sides subtend; [I. 4]

therefore the angle BAE is equal to the angle ECF .

25 But the angle ECD is greater than the angle ECF ;
[C. N. 5]

therefore the angle ACD is greater than the angle BAE .

Similarly also, if BC be bisected, the angle BCG , that is, the angle ACD [I. 15], can be proved greater than the angle ABC as well.

Therefore etc.

Q. E. D.

1. the exterior angle, literally "the outside angle," ἡ ἐκτὸς γωνία.
2. the interior and opposite angles, τῶν ἐντὸς καὶ ἀπεναντίων γωνιῶν.
12. let AC be drawn through to G . The word is διήχθω, a variation on the more usual ἐκβεβλήσθω, "let it be produced."
21. CFE , in the text " FEC ."

As is well known, this proposition is not universally true under the Riemann hypothesis of a space endless in extent but not infinite in size. On this hypothesis a straight line is a "closed series" and returns on itself; and two straight lines which have one point of intersection have another point of intersection also, which bisects the whole length of the straight line measured from the first point on it to the same point again; thus the axiom of Euclidean geometry that two straight lines do not enclose a space does not hold. If 4Δ denotes the finite length of a straight line measured from any point once round to the same point again, 2Δ is the distance between the two intersections of two straight lines which meet. Two points A, B do not determine one sole straight line unless the distance between them is different from 2Δ . In order that there may only be one perpendicular from a point C to a straight line AB , C must not be one of the two "poles" of the straight line.

Now, in order that the proof of the present proposition may be universally valid, it is necessary that CF should always fall within the angle ACD so that the angle ACF may be less than the angle ACD . But this will not always be so on the Riemann hypothesis. For, (1) if BE is equal to Δ , so that BF is equal to 2Δ , F will be the second point in which BE and BD intersect; i.e. F will lie on CD , and the angle ACF will be equal to the angle ACD . In this case the exterior angle ACD will be equal to the interior angle BAC . (2) If BE is greater than Δ and less than 2Δ , so that BF is greater than 2Δ and less than 4Δ , the angle ACF will be greater than the angle ACD , and therefore the angle ACD will be less than the interior angle BAC . Thus, e.g., in the particular case of a right-angled triangle, the angles other than the right angle may be (1) both acute, (2) one acute and one obtuse, or (3) both obtuse according as the perpendicular sides are (1) both less than Δ , (2) one less and the other greater than Δ , (3) both greater than Δ .

Proclus tells us (p. 307, 1—12) that some combined this theorem with the next in one enunciation thus: *In any triangle, if one side be produced, the exterior angle of the triangle is greater than either of the interior and opposite angles, and any two of the interior angles are less than two right angles, the combination having been suggested by the similar enunciation of Euclid I. 32, In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.*

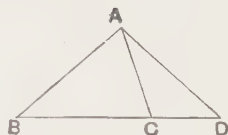
The present proposition enables Proclus to prove what he did not succeed in establishing conclusively in his note on I. 12, namely that *from one point there cannot be drawn to the same straight line three straight lines equal in length.*

For, if possible, let AB , AC , AD be all equal, B , C , D being in a straight line.

Then, since AB , AC are equal, the angles ABC , ACB are equal.

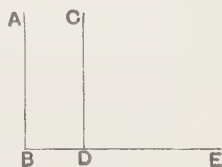
Similarly, since AB , AD are equal, the angles ABD , ADB are equal.

Therefore the angle ACB is equal to the angle ADC , i.e. the exterior angle to the interior and opposite angle: which is impossible.



Proclus next (p. 308, 14 sqq.) undertakes to prove by means of I. 16 that, *if a straight line falling on two straight lines make the exterior angle equal to the interior and opposite angle, the two straight lines will not form a triangle or meet*, for in that case the same angle would be both greater and equal.

The proof is really equivalent to that of Eucl. I. 27. If BE falls on the two straight lines AB , CD in such a way that the angle CDE is equal to the interior and opposite angle ABD , AB and CD cannot form a triangle or meet. For, if they did, then (by I. 16) the angle CDE would be *greater* than the angle ABD , while by the hypothesis it is at the same time *equal* to it.



Hence, says Proclus, in order that BA , DC may form a triangle it is necessary for them to *approach* one another in the sense of being turned round one pair of corresponding extremities, e.g. B , D , so that the other extremities A , C come nearer. This may be brought about in one of three ways: (1) AB may remain fixed and CD be turned about D so that the angle CDE increases; (2) CD may remain fixed and AB be turned about B so that the angle ABD becomes smaller; (3) both AB and CD may move so as to make the angle ABD smaller and the angle CDE larger at the same time. The *reason*, then, of the straight lines AB , CD coming to form a triangle or to meet is (says Proclus) *the movement of the straight lines*.

Though he does not mention it here, Proclus does in another passage (p. 371, 2—10, quoted on p. 207 above) hint at the possibility that, while I. 16 may remain universally true, either of the straight lines BA , DC (or both together) may be turned through any angle not greater than a certain finite angle and yet may not meet (the Bolyai-Lobachewsky hypothesis).

PROPOSITION 17.

In any triangle two angles taken together in any manner are less than two right angles.

Let ABC be a triangle;

I say that two angles of the triangle ABC taken together in any manner are less than two right angles.

For let BC be produced to D .

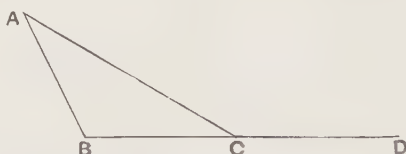
[Post. 2]

Then, since the angle ACD is an exterior angle of the triangle ABC ,

it is greater than the interior and opposite angle ABC .

[I. 16]

Let the angle ACB be added to each ;
therefore the angles ACD , ACB are greater than the angles
 ABC , BCA .



But the angles ACD , ACB are equal to two right angles.

[I. 13]

Therefore the angles ABC , BCA are less than two right angles.

Similarly we can prove that the angles BAC , ACB are also less than two right angles, and so are the angles CAB , ABC as well.

Therefore etc.

Q. E. D.

- 1. taken together in any manner, πάντη μεταλαμβάνομεναι, i.e. any pair added together.

As in his note on the previous proposition, Proclus tries to state the *cause* of the property. He takes the case of two straight lines forming right angles with a transversal and observes that it is the *convergence of the straight lines towards one another* (σύνεσις τῶν εὐθειῶν), the *lessening* of the two right angles, which produces the triangle. He will not have it that the fact of the exterior angle being greater than the interior and opposite angle is the *cause* of the property, for the odd reason that "it is not necessary that a side should be produced, or that there should be any exterior angle constructed...and how can what is not necessary be the cause of what is necessary?" (p. 311, 17—21).

Agreeably to this view, Proclus then sets himself to prove the theorem without producing a side of the triangle.

Let ABC be a triangle. Take any point D on BC , and join AD .

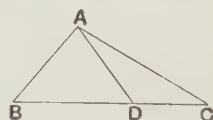
Then the exterior angle ADC of the triangle ABD is greater than the interior and opposite angle ABD .

Similarly the exterior angle ADB of the triangle ADC is greater than the interior and opposite angle ACD .

Therefore, by addition, the angles ADB , ADC are together greater than the angles ABC , ACB .

But the angles ADB , ADC are equal to two right angles ; therefore the angles ABC , ACB are less than two right angles.

Lastly, Proclus proves (what is obvious from this proposition) that *there cannot be more than one perpendicular to a straight line from a point without it*. For, if this were possible, two of such perpendiculars would form a triangle in which two angles would be right angles: which is impossible, since any two angles of a triangle are together less than two right angles.



PROPOSITION 18.

In any triangle the greater side subtends the greater angle.

For let ABC be a triangle having the side AC greater than AB ;

I say that the angle ABC is also greater than the angle BCA .

For, since AC is greater than AB , let AD be made equal to AB [I. 3], and let BD be joined.

Then, since the angle ADB is an exterior angle of the triangle BCD ,

it is greater than the interior and opposite angle DCB . [I. 16]

But the angle ADB is equal to the angle ABD ,

since the side AB is equal to AD ;

therefore the angle ABD is also greater than the angle ACB ;

therefore the angle ABC is much greater than the angle ACB .

Therefore etc.

Q. E. D.

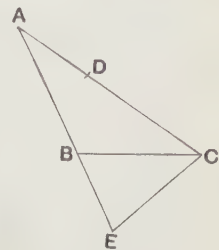
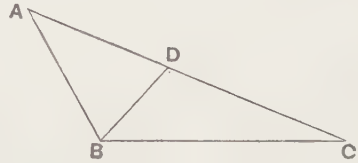
In the enunciation of this proposition we have *ὑποτείνειν* ("subtend") used with the simple accusative instead of the more usual *ὑπὸ* with accusative. The latter construction is used in the enunciation of I. 19, which otherwise only differs from that of I. 18 in the order of the words. The point to remember in order to distinguish the two is that the *datum* comes first and the *quæsitum* second, the *datum* being in this proposition the greater side and in the next the greater angle. Thus the enunciations are (I. 18) *παντὸς τριγώνου ἡ μείζων πλευρὰ τὴν μείζονα γωνίαν ὑποτείνει* and (I. 19) *παντὸς τριγώνου ὑπὸ τὴν μείζονα γωνίαν ἡ μείζων πλευρὰ ὑποτείνει*. In order to keep the proper order in English we must use the passive of the verb in I. 19. Aristotle quotes the result of I. 19, using the exact wording, *ὑπὸ γὰρ τὴν μείζω γωνίαν ὑποτείνεται* (*Meteorologica* III. 5, 376 a 12).

"In order to assist the student in remembering which of these two propositions [I. 18, 19] is demonstrated directly and which indirectly, it may be observed that the order is similar to that in I. 5 and I. 6" (Todhunter).

An alternative proof of I. 18 given by Porphyry (see Proclus; pp. 315, 11—316, 13) is interesting. It starts by supposing a length equal to AB cut off from the other end of AC ; that is, CD and not AD is made equal to AB .

Produce AB to E so that BE is equal to AD , and join EC .

Then, since AB is equal to CD , and BE to AD ,
 AE is equal to AC .



Therefore the angle AEC is equal to the angle ACE .

Now the angle ABC is greater than the angle AEC , [I. 16]

and therefore greater than the angle ACE .

Hence, *a fortiori*, the angle ABC is greater than the angle ACB .

PROPOSITION 19.

In any triangle the greater angle is subtended by the greater side.

Let ABC be a triangle having the angle ABC greater than the angle BCA ;

I say that the side AC is also greater than the side AB .

For, if not, AC is either equal to AB or less.

Now AC is not equal to AB ;

for then the angle ABC would also have been equal to the angle ACB ; [I. 5]

but it is not;

therefore AC is not equal to AB .

Neither is AC less than AB ,

for then the angle ABC would also have been less than the angle ACB ; [I. 18]

but it is not;

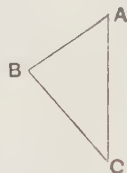
therefore AC is not less than AB .

And it was proved that it is not equal either.

Therefore AC is greater than AB .

Therefore etc.

Q. E. D.



This proposition, like I. 6, can be proved by merely *logical* deduction from I. 5 and I. 18 taken together, as pointed out by De Morgan. The general form of the argument used by De Morgan is given in his *Formal Logic* (1847), p. 25, thus:

Hypothesis. Let there be any number of propositions or assertions—three for instance, X , Y and Z —of which it is the property that one or the other must be true, *and one only*. Let there be three other propositions P , Q and R of which it is also the property that one, and one only, must be true. Let it be a connexion of those assertions that:

when X is true, P is true,

when Y is true, Q is true,

when Z is true, R is true.

Consequence: then it follows that,

when P is true, X is true,

when Q is true, Y is true,

when R is true, Z is true."

To apply this to the case before us, let us denote the sides of the triangle ABC by a, b, c , and the angles opposite to these sides by A, B, C respectively, and suppose that a is the base.

Then we have the three propositions,

- | | |
|--|--------|
| when b is equal to c , B is equal to C , | [1. 5] |
| when b is greater than c , B is greater than C , | } |
| when b is less than c , B is less than C , | |

and it follows *logically* that,

- | | |
|--|--------|
| when B is equal to C , b is equal to c , | [1. 6] |
| when B is greater than C , b is greater than c , | } |
| when B is less than C , b is less than c . | |

Reductio ad absurdum by exhaustion.

Here, says Proclus (p. 318, 16—23), Euclid proves the impossibility “by means of *division*” (ἐκ διαμέσεως). This means simply the separation of different hypotheses, each of which is inconsistent with the truth of the theorem to be proved, and which therefore must be successively shown to be impossible. If a straight line is not greater than a straight line, it must be either equal to it or less; thus in a *reductio ad absurdum* intended to prove such a theorem as 1. 19 it is necessary to dispose successively of *two* hypotheses inconsistent with the truth of the theorem.

Alternative (direct) proof.

Proclus gives a direct proof (pp. 319—321) which an-Nairīzī also has and attributes to Heron. It requires a lemma and is consequently open to the slight objection of separating a theorem from its converse. But the lemma and proof are worth giving.

Lemma.

If an angle of a triangle be bisected and the straight line bisecting it meet the base and divide it into unequal parts, the sides containing the angle will be unequal, and the greater will be that which meets the greater segment of the base, and the less that which meets the less.

Let AD , the bisector of the angle A of the triangle ABC , meet BC in D , making CD greater than BD .

I say that AC is greater than AB .

Produce AD to E so that DE is equal to AD . And, since DC is greater than BD , cut off DF equal to BD .

Join EF and produce it to G .

Then, since the two sides AD, DB are equal to the two sides ED, DF , and the vertical angles at D are equal,

AB is equal to EF ,

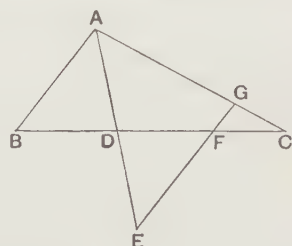
and the angle DEF to the angle BAD ,

i.e. to the angle DAG (by hypothesis).

Therefore AG is equal to EG ,

and therefore greater than EF , or AB .

Hence, *a fortiori*, AC is greater than AB .



[1. 6]

Proof of I. 19.

Let ABC be a triangle in which the angle ABC is greater than the angle ACB .

Bisect BC at D , join AD , and produce it to E so that DE is equal to AD . Join BE .

Then the two sides BD , DE are equal to the two sides CD , DA , and the vertical angles at D are equal; therefore BE is equal to AC ,

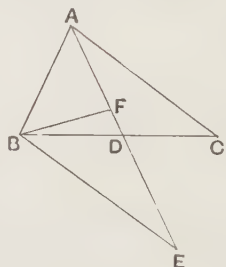
and the angle DBE to the angle at C .

But the angle at C is less than the angle ABC ; therefore the angle DBE is less than the angle ABD .

Hence, if BF bisect the angle ABE , BF meets AE between A and D . Therefore EF is greater than FA .

It follows, by the lemma, that BE is greater than BA ,

that is, AC is greater than AB .

**PROPOSITION 20.**

In any triangle two sides taken together in any manner are greater than the remaining one.

For let ABC be a triangle;

I say that in the triangle ABC two sides taken together in any manner are greater than the remaining one, namely

BA , AC greater than BC ,

AB , BC greater than AC ,

BC , CA greater than AB .

For let BA be drawn through to the point D , let DA be made equal to CA , and let DC be joined.

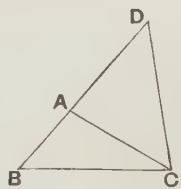
Then, since DA is equal to AC , the angle ADC is also equal to the angle ACD ;

therefore the angle BCD is greater than the angle ADC .

And, since DCB is a triangle having the angle BCD greater than the angle BDC ,

and the greater angle is subtended by the greater side,

therefore DB is greater than BC .



But DA is equal to AC ;

therefore BA , AC are greater than BC .

Similarly we can prove that AB , BC are also greater than CA , and BC , CA than AB .

Therefore etc.

Q. E. D.

It was the habit of the Epicureans, says Proclus (p. 322), to ridicule this theorem as being evident even to an ass and requiring no proof, and their allegation that the theorem was "known" (*γνώριμον*) even to an ass was based on the fact that, if fodder is placed at one angular point and the ass at another, he does not, in order to get to his food, traverse the two sides of the triangle but only the one side separating them (an argument which makes Savile exclaim that its authors were "*digni ipsi, qui cum Asino foenum essent,*" p. 78). Proclus replies truly that a mere perception of the truth of the theorem is a different thing from a scientific proof of it and a knowledge of the reason *why* it is true. Moreover, as Simson says, the number of axioms should not be increased without necessity.

Alternative Proofs.

Heron and Porphyry, we are told (Proclus, pp. 323—6), proved this theorem in different ways as follows, without producing one of the sides.

First proof.

Let ABC be the triangle, and let it be required to prove that the sides BA , AC are greater than BC .

Bisect the angle BAC by AD meeting BC in D .

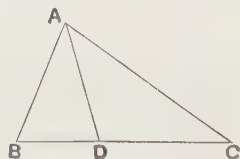
Then, in the triangle ABD ,
the exterior angle ADC is greater than the
interior and opposite angle BAD , [I. 16]

that is, greater than the angle DAC .

Therefore the side AC is greater than the side
 CD . [I. 19]

Similarly we can prove that AB is greater than BD .

Hence, by addition, BA , AC are greater than BC .



Second proof.

This, like the first proof, is direct. There are several cases to be considered.

(1) If the triangle is *equilateral*, the truth of the proposition is obvious.

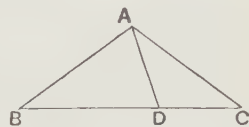
(2) If the triangle is *isosceles*, the proposition needs no proof in the case
(a) where each of the equal sides is greater than the base.

(b) If the base is greater than either of the other sides, we have to prove that the sum of the two equal sides is greater than the base. Let BC be the base in such a triangle.

Cut off from BC a length BD equal to AB , and join AD .

Then, in the triangle ADB , the exterior angle ADC is greater than the interior and opposite angle BAD . [I. 16]

Similarly, in the triangle ADC , the exterior angle ADB is greater than the interior and opposite angle CAD .



By addition, the two angles BDA , ADC are together greater than the two angles BAD , DAC (or the whole angle BAC).

Subtracting the equal angles BDA , BAD , we have the angle ADC greater than the angle CAD .

It follows that AC is greater than CD ; [I. 19]

and, adding the equals AB , BD respectively, we have BA , AC together greater than BC .

(3) If the triangle be *scalene*, we can arrange the sides in order of length. Suppose BC is the greatest, AB the intermediate and AC the least side. Then it is obvious that AB , BC are together greater than AC , and BC , CA together greater than AB .

It only remains therefore to prove that CA , AB are together greater than BC .

We cut off from BC a length BD equal to the adjacent side, join AD , and proceed exactly as in the above case of the isosceles triangle.

Third proof.

This proof is by *reductio ad absurdum*.

Suppose that BC is the greatest side and, as before, we have to prove that BA , AC are greater than BC .

If they are not, they must be either equal to or less than BC .

(1) Suppose BA , AC are together equal to BC .

From BC cut off BD equal to BA , and join AD .

It follows from the hypothesis that DC is equal to AC .

Then, since BA is equal to BD , the angle BDA is equal to the angle BAD .

Similarly, since AC is equal to CD , the angle CDA is equal to the angle CAD .

By addition, the angles BDA , ADC are together equal to the whole angle BAC .

That is, the angle BAC is equal to two right angles : which is impossible.

(2) Suppose BA , AC are together less than BC .

From BC cut off BD equal to BA , and from CB cut off CE equal to CA . Join AD , AE .

In this case, we prove in the same way that the angle BDA is equal to the angle BAD , and the angle CEA to the angle CAE .

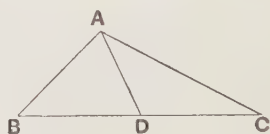
By addition, the sum of the angles BDA , AEC is equal to the sum of the angles BAD , CAE .

Now, by I. 16, the angle BDA is greater than the angle DAC , and therefore, *a fortiori*, greater than the angle EAC .

Similarly the angle AEC is greater than the angle BAD .

Hence the sum of the angles BDA , AEC is greater than the sum of the angles BAD , EAC .

But the former sum was also equal to the latter : which is impossible.



PROPOSITION 21.

If on one of the sides of a triangle, from its extremities, there be constructed two straight lines meeting within the triangle, the straight lines so constructed will be less than the remaining two sides of the triangle, but will contain a greater angle.

On BC , one of the sides of the triangle ABC , from its extremities B, C , let the two straight lines BD, DC be constructed meeting within the triangle;

I say that BD, DC are less than the remaining two sides of the triangle BA, AC , but contain an angle BDC greater than the angle BAC .

For let BD be drawn through, to E .

Then, since in any triangle two sides are greater than the remaining one,

[I. 20]

therefore, in the triangle ABE , the two sides AB, AE are greater than BE .

Let EC be added to each;

therefore BA, AC are greater than BE, EC .

Again, since, in the triangle CED ,

the two sides CE, ED are greater than CD , let DB be added to each;

therefore CE, EB are greater than CD, DB .

But BA, AC were proved greater than BE, EC ;

therefore BA, AC are much greater than BD, DC .

Again, since in any triangle the exterior angle is greater than the interior and opposite angle, [I. 16] therefore, in the triangle CDE ,

the exterior angle BDC is greater than the angle CED .

For the same reason, moreover, in the triangle ABE also, the exterior angle CEB is greater than the angle BAC .

But the angle BDC was proved greater than the angle CEB ; therefore the angle BDC is much greater than the angle BAC .

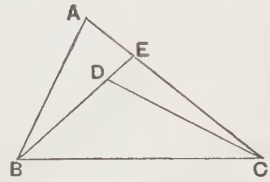
Therefore etc.

Q. E. D.

2. be constructed...meeting within the triangle. The word "meeting" is not in the Greek, where the words are *ἐντὸς συνισθῶν*. *συνιστᾶσθαι* is the word used of constructing two straight lines to a point (cf. I. 7) or so as to form a triangle, but it is necessary in English to indicate that they meet.

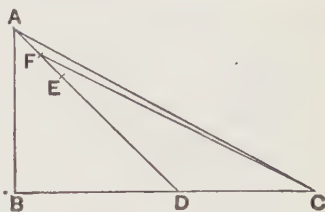
3. the straight lines so constructed. Observe the elegant brevity of the Greek *αὐτὰς συνισθῆναι*.

H. E.



The editors generally call attention to the fact that the lines drawn within the triangle in this proposition must be drawn, as the enunciation says, from the *ends* of the side; otherwise it is not necessary that their sum should be less than that of the remaining sides of the triangle. Proclus (p. 327, 12 sqq.) gives a simple illustration.

Let ABC be a right-angled triangle. Take any point D on BC , join DA , and cut off from it DE equal to AB . Bisect AE at F , and join FC .



Then shall CF , FD be together greater than CA , AB .

For CF , FE are equal to CF , FA ,
and therefore greater than CA .

Add the equals ED , AB respectively;

therefore CF , FD are together greater than CA , AB .

Pappus gives the same proposition as that just proved, but follows it up by a number of others more elaborate in character, selected apparently from "the so-called paradoxes" of one Erycinus (Pappus, III. p. 106 sqq.). Thus he proves the following:

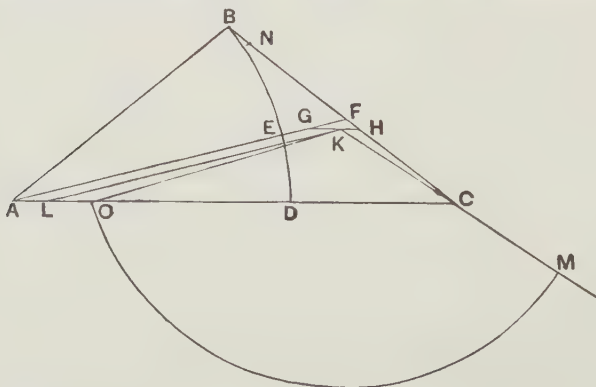
1. In any triangle, except an equilateral triangle or an isosceles triangle with base less than one of the other sides, it is possible to construct on the base and within the triangle two straight lines the sum of which is equal to the sum of the other two sides of the triangle.

2. In any triangle in which it is possible to construct two straight lines on the base which are equal to the sum of the other two sides of the triangle it is also possible to construct two others the sum of which is *greater* than that sum.

3. Under the same conditions, if the base is greater than either of the other two sides, two straight lines can be constructed in the manner described which are *respectively* greater than the other two sides of the triangle; and the lines may be constructed so as to be respectively *equal* to the two sides, if one of those two sides is less than the other and each of them less than the base.

4. The lines may be so constructed that their sum will bear to the sum of the two sides of the triangle any ratio less than $2 : 1$.

As a specimen of the proofs we will give that of the proposition which has



been numbered (1) for the case where the triangle is isosceles (Pappus, III. pp. 108—110).

Let ABC be an isosceles triangle in which the base AC is greater than either of the equal sides AB, BC .

With centre A and radius AB describe a circle meeting AC in D .

Draw any radius AEF such that it meets BC in a point F outside the circle.

Take any point G on EF , and through it draw GH parallel to AC . Take any point K on GH , and draw KL parallel to FA meeting AC in L .

From BC cut off BN equal to EG .

Thus AG , or LK , is equal to the sum of AB, BN , and CN is less than LK .

Now GF, FH are together greater than GH ,

and CH, HK together greater than CK .

Therefore, by addition,

CF, FG, HK are together greater than CK, HG .

Subtracting HK from each side, we see that

CF, FG are together greater than CK, KG ;

therefore, if we add AG to each,

AF, FC are together greater than AG, GK, KC .

And AB, BC are together greater than AF, FC .

[I. 21]

Therefore AB, BC are together greater than AG, GK, KC .

But, by construction, AB, BN are together equal to AG ;

therefore, by subtraction, NC is greater than GK, KC ,

and *a fortiori* greater than KC .

Take on KC produced a point M such that KM is equal to NC ;

with centre K and radius KM describe a circle meeting CL in O , and join KO .

Then shall LK, KO be equal to AB, BC .

For, by construction, LK is equal to the sum of AB, BN , and KO is equal to NC ;

therefore LK, KO are together equal to AB, BC .

It is after I. 21 that (as remarked by De Morgan) the important proposition about the perpendicular and obliques drawn from a point to a straight line of unlimited length is best introduced:

Of all straight lines that can be drawn to a given straight line of unlimited length from a given point without it:

(a) *the perpendicular is the shortest;*

(b) *of the obliques, that is the greater the foot of which is further from the perpendicular;*

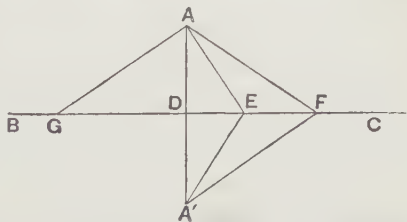
(c) *given one oblique, only one other can be found of the same length, namely that the foot of which is equally distant with the foot of the given one from the perpendicular, but on the other side of it.*

Let A be the given point, BC the given straight line; let AD be the perpendicular from A on BC , and AE, AF any two obliques of which AF makes the greater angle with AD .

Produce AD to A' , making $A'D$ equal to AD , and join $A'E, A'F$.

Then the triangles $ADE, A'DE$ are equal in all respects; and so are the triangles $ADF, A'DF$.

Now (I) in the triangle AEA' the two sides AE, EA' are greater than AA' [I. 20], that is, twice AE is greater than twice AD .



Therefore AE is greater than AD .

- (2) Since $AE, A'E$ are drawn to E , a point within the triangle $AF A'$,
 AF, FA' are together greater than AE, EA' , [I. 21]
 or twice AF is greater than twice AE .

Therefore AF is greater than AE .

- (3) Along DB measure off DG equal to DF , and join AG .

The triangles AGD, AFD are then equal in all respects, so that the angles GAD, FAD are equal, and AG is equal to AF .

PROPOSITION 22.

Out of three straight lines, which are equal to three given straight lines, to construct a triangle: thus it is necessary that two of the straight lines taken together in any manner should be greater than the remaining one. [I. 20]

Let the three given straight lines be A, B, C , and of these let two taken together in any manner be greater than the remaining one,
 namely

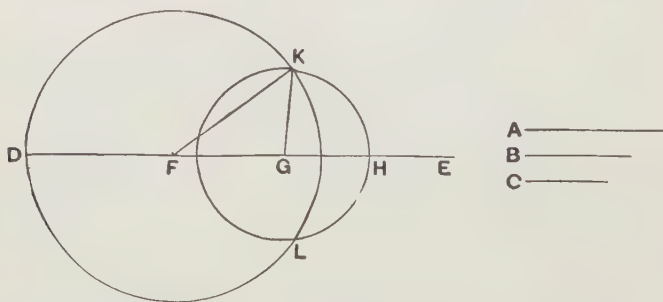
A, B greater than C ,

A, C greater than B ,

and

B, C greater than A ;

thus it is required to construct a triangle out of straight lines equal to A, B, C .



Let there be set out a straight line DE , terminated at D but of infinite length in the direction of E ,
 and let DF be made equal to A , FG equal to B , and GH equal to C . [I. 3]

With centre F and distance FD let the circle DKL be described;
 again, with centre G and distance GH let the circle KLH be described;
 and let KF, KG be joined;

I say that the triangle KFG has been constructed out of three straight lines equal to A, B, C .

For, since the point F is the centre of the circle DKL ,
 FD is equal to FK .

But FD is equal to A ;

therefore KF is also equal to A .

Again, since the point G is the centre of the circle LKH ,
 GH is equal to GK .

But GH is equal to C ;

therefore KG is also equal to C .

And FG is also equal to B ;

therefore the three straight lines KF , FG , GK are equal to
the three straight lines A , B , C .

Therefore out of the three straight lines KF , FG , GK ,
which are equal to the three given straight lines A , B , C , the
triangle KFG has been constructed.

Q. E. F.

2—4. This is the first case in the *Elements* of a *διορισμός* to a problem in the sense of a statement of the conditions or limits of the possibility of a solution. The criterion is of course supplied by the preceding proposition.

2. thus it is necessary. This is usually translated (e.g. by Williamson and Simson) "*But it is necessary,*" which is however inaccurate, since the Greek is not *δεῖ δέ* but *δεῖ δὴ*. The words are the same as those used to introduce the *διορισμός* in the other sense of the "definition" or "particular statement" of a construction to be effected. Hence, as in the latter case we say "thus it is required" (e.g. to bisect the finite straight line AB , I. 10), we should here translate "*thus it is necessary.*"

4. To this enunciation all the mss. and Boethius add, after the *διορισμός*, the words "because in any triangle two sides taken together in any manner are greater than the remaining one." But this explanation has the appearance of a gloss, and it is omitted by Proclus and Campanus. Moreover there is no corresponding addition to the *διορισμός* of VI. 28.

It was early observed that Euclid assumes, without giving any reason, that the circles drawn as described will meet if the condition that any two of the straight lines A , B , C are together greater than the third be fulfilled. Proclus (p. 331, 8 sqq.) argues the matter by means of *reductio ad absurdum*, but does not exhaust the possible hypotheses inconsistent with the contention. He says the circles must do one of three things, (1) cut one another, (2) touch one another, (3) stand apart (*διστάσθαι*) from one another. He then considers the hypotheses (a) of their touching *externally*, (b) of their being separated from one another by a space. He should have considered also the hypothesis (c) of one circle touching the other *internally* or lying entirely within the other without touching. These three hypotheses being successively disproved, it follows that the circles must meet (this is the line taken by Camerer and Todhunter).

Simson says: "Some authors blame Euclid because he does not demonstrate that the two circles made use of in the construction of this problem must cut one another: but this is very plain from the determination he has given, namely, that any two of the straight lines DF , FG , GH must be greater than the third. For who is so dull, though only beginning to learn the *Elements*, as not to perceive that the circle described from the centre F , at the distance FD , must meet FH betwixt F and H , because FD is less than FH ; and that, for the like reason, the circle described from the

centre G at the distance GH must meet DG betwixt D and G ; and that these circles must meet one another, because FD and GH are together greater than FG ."

We have in fact only to satisfy ourselves that one of the circles, e.g. that with centre G , has at least one point of its circumference outside the other circle and also at least one point of its circumference inside the same circle; and this is best shown with reference to the points in which the first circle cuts the straight line DE . For (1) FH , being equal to the sum of B and C , is greater than A , i.e. than the radius of the circle with centre F , and therefore H is outside that circle. (2) If GM be measured along GF equal to GH or C , then, since GM is either (a) less or (b) greater than GF , M will fall either (a) between G and F or (b) beyond F towards D ; in the first case (a) the sum of FM and C is equal to FG and therefore less than the sum of A and C , so that FM is less than A or FD ; in the second case (b) the sum of MF and FG , i.e. the sum of MF and B , is equal to GM or C , and therefore less than the sum of A and B , so that MF is less than A or FD ; hence in either case M falls within the circle with centre F .

It being now proved that the circumference of the circle with centre G has at least one point outside, and at least one point inside, the circle with centre F , we have only to invoke the Principle of Continuity, as we have to do in I. 1 (cf. the note on that proposition, p. 242, where the necessary postulate is stated in the form suggested by Killing).

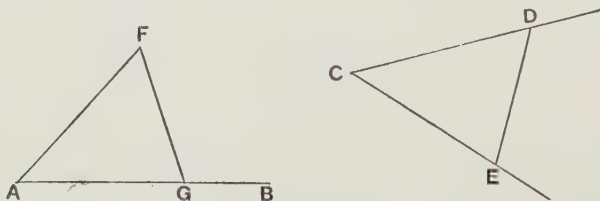
That the construction of the proposition gives only *two* points of intersection between the circles, and therefore only two triangles satisfying the condition, one on each side of FG , is clear from I. 7, which, as before pointed out, takes the place, in Book I., of III. 10 proving that two circles cannot intersect in more points than two.

PROPOSITION 23.

On a given straight line and at a point on it to construct a rectilineal angle equal to a given rectilineal angle.

Let AB be the given straight line, A the point on it, and the angle DCE the given rectilineal angle;

thus it is required to construct on the given straight line AB , and at the point A on it, a rectilineal angle equal to the given rectilineal angle DCE .



On the straight lines CD , CE respectively let the points D , E be taken at random;
let DE be joined,
and out of three straight lines which are equal to the three

straight lines CD , DE , CE let the triangle AFG be constructed in such a way that CD is equal to AF , CE to AG , and further DE to FG . [I. 22]

Then, since the two sides DC , CE are equal to the two sides FA , AG respectively,

and the base DE is equal to the base FG ,

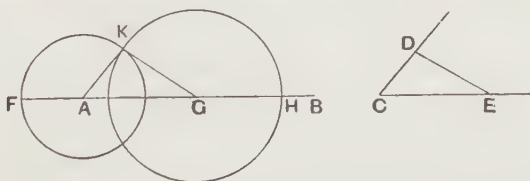
the angle DCE is equal to the angle FAG . [I. 8]

Therefore on the given straight line AB , and at the point A on it, the rectilinear angle FAG has been constructed equal to the given rectilinear angle DCE .

Q. E. F.

This problem was, according to Eudemus (see Proclus, p. 333, 5), "rather the discovery of Oenopides," from which we must apparently infer, not that Oenopides was the first to find any solution of it, but that it was he who discovered the particular solution given by Euclid. (Cf. Bretschneider, p. 65.)

The editors do not seem to have noticed the fact that the construction of the triangle assumed in this proposition is not exactly the construction given in I. 22. We have here to construct a triangle on a certain finite straight line AG as base; in I. 22 we have only to construct a triangle with sides of given length without any restriction as to how it is to be placed. Thus in I. 22 we set out any line whatever and measure successively three lengths along it beginning from the given extremity, and what we must regard as the base is the intermediate length, not the length beginning at the given extremity of the straight line arbitrarily set out. Here the base is a given straight line abutting at a given point. Thus the construction has to be modified somewhat from



that of the preceding proposition. We must measure AG along AB so that AG is equal to CE (or CD), and GH along GB equal to DE ; and then we must produce BA , in the opposite direction, to F , so that AF is equal to CD (or CE , if AG has been made equal to CD).

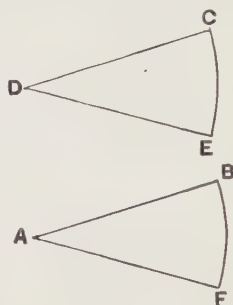
Then, by drawing circles (1) with centre A and radius AF , (2) with centre G and radius GH , we determine K , one of their points of intersection, and we prove that the triangle KAG is equal in all respects to the triangle DCE , and then that the angle at A is equal to the angle DCE .

I think that Proclus must (though he does not say so) have felt the same difficulty with regard to the use in I. 23 of the result of I. 22, and that this is probably the reason why he gives over again the construction which I have given above, with the remark (p. 334, 6) that "you may obtain the construction of the triangle in a more instructive manner (*διδασκαλικώτερον*) as follows."

Proclus objects to the procedure of Apollonius in constructing an angle under the same conditions, and certainly, if he quotes Apollonius correctly, the latter's exposition must have been somewhat slipshod.

"He takes an angle CDE at random," says Proclus (p. 335, 19 sqq.), "and a straight line AB , and with centre D and distance CD describes the circumference CE , and in the same way with centre A and distance AB the circumference FB . Then, cutting off FB equal to CE , he joins AF . And he declares that the angles A, D standing on equal circumferences are equal."

In the first place, as Proclus remarks, it should be premised that AB is equal to CD in order that the circles may be equal; and the use of Book III. for such an elementary construction is objectionable. The omission to state that AB must be taken equal to CD was no doubt a slip, if it occurred. And, as regards the equal angles "standing on equal circumferences," it would seem possible that Apollonius said this in *explanation*, for the sake of brevity, rather than by way of proof. It seems to me probable that his construction was only given from the point of view of *practical*, not *theoretical*, geometry. It really comes to the same thing as Euclid's except that DC is taken equal to DE . For cutting off the arc BF equal to the arc CE can only be meant in the sense of measuring the *chord* CE , say, with a pair of compasses, and then drawing a circle with centre B and radius equal to the chord CE . Apollonius' direction was therefore probably intended as a practical short cut, avoiding the actual drawing of the chords CE, BF , which, as well as a proof of the equality in all respects of the triangles CDE, BAF , would be required to establish *theoretically* the correctness of the construction.



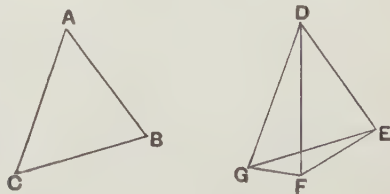
PROPOSITION 24.

If two triangles have the two sides equal to two sides respectively, but have the one of the angles contained by the equal straight lines greater than the other, they will also have the base greater than the base.

- 5 Let ABC, DEF be two triangles having the two sides AB, AC equal to the two sides DE, DF respectively, namely AB to DE , and AC to DF , and let the angle at A be greater than the angle at D ;

I say that the base BC is also greater than the base EF .

- 10 For, since the angle BAC is greater than the angle EDF , let there be constructed, on the straight line DE , and at the point D on it, the angle EDG equal to the angle BAC ; [I. 23] let DG be made equal to either of the two straight lines AC, DF , and let EG, FG be joined.



Then, since AB is equal to DE , and AC to DG ,
 20 the two sides BA , AC are equal to the two sides ED , DG ,
 respectively ;

and the angle BAC is equal to the angle EDG ;

therefore the base BC is equal to the base EG . [I. 4]

Again, since DF is equal to DG ,

25 the angle DGF is also equal to the angle DFG ; [I. 5]

therefore the angle DFG is greater than the angle EGF .

Therefore the angle EFG is much greater than the angle EGF .

And, since EFG is a triangle having the angle EFG
 30 greater than the angle EGF ,

and the greater angle is subtended by the greater side,
 [I. 19]

the side EG is also greater than EF .

But EG is equal to BC .

Therefore BC is also greater than EF .

35 Therefore etc.

Q. E. D.

10. I have naturally left out the well-known words added by Simson in order to avoid the necessity of considering three cases : "Of the two sides DE , DF let DE be the side which is not greater than the other." I doubt whether Euclid could have been induced to insert the words himself, even if it had been represented to him that their omission meant leaving two possible cases out of consideration. His habit and that of the great Greek geometers was, not to set out all possible cases, but to give as a rule one case, generally the most difficult, as here, and to leave the others to the reader to work out for himself. We have already seen one instance in I. 7.

Proclus of course gives the other two cases which arise if we do not first provide that DE is not greater than DF .

(1) In the first case G may fall on EF produced, and it is then obvious that EG is greater than EF .

(2) In the second case EG may fall below EF .

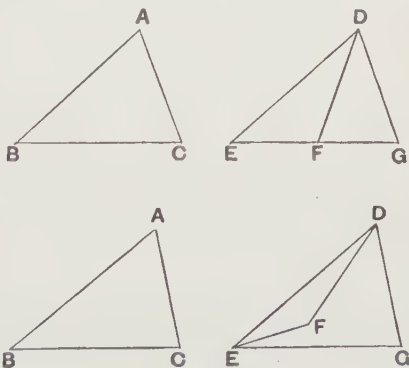
If so, by I. 21, DF , FE are together less than DG , GE .

But DF is equal to DG ; therefore EF is less than EG , i.e. than BC .

These two cases are therefore decidedly simpler than the case taken

by Euclid as typical, and could well be left to the ingenuity of the learner.

If however after all we prefer to insert Simson's words and avoid the latter



two cases, the proof is not complete unless we show that, with his assumption, F must, in the figure of the proposition, fall *below* EG .

De Morgan would make the following proposition precede: *Every straight line drawn from the vertex of a triangle to the base is less than the greater of the two sides, or than either if they are equal*, and he would prove it by means of the proposition relating to perpendicular and obliques given above, p. 291.

But it is easy to prove directly that F falls below EG , if DE is not greater than DG , by the method employed by Pfeiderer, Lardner, and Todhunter.

Let DF , produced if necessary, meet EG in H .

Then the angle DHG is greater than the angle DEG ; [I. 16]

and the angle DEG is not less than the angle DGE ; [I. 18]

therefore the angle DHG is greater than the angle DGH .

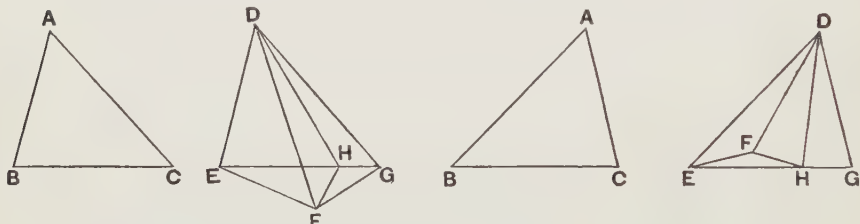
Hence DH is less than DG , [I. 19]

and therefore DH is less than DF .



Alternative proof.

Lastly, the modern alternative proof is worth giving.



Let DH bisect the angle FDG (after the triangle DEG has been made equal in all respects to the triangle ABC , as in the proposition), and let DH meet EG in H . Join HF .

Then, in the triangles FDH , GDH ,

the two sides FD , DH are equal to the two sides GD , DH ,

and the included angles FDH , GDH are equal;

therefore the base HF is equal to the base HG .

Accordingly EG is equal to the sum of EH , HF ;

and EH , HF are together greater than EF ; [I. 20]

therefore EG , or BC , is greater than EF .

Proclus (p. 339, 11 sqq.) answers by anticipation the possible question that might occur to any one on this proposition, viz. why does Euclid not compare the areas of the triangles as he does in I. 4? He observes that inequality of the areas does not follow from the inequality of the angles contained by the equal sides, and that Euclid leaves out all reference to the question both for this reason and because the areas cannot be compared without the help of the theory of parallels. "But if," says Proclus, "we must anticipate what is to come and make our comparison of the areas at once, we assert that (1) if the angles A , D —supposing that our argument proceeds with reference to the figure in the proposition—are (together) equal to two right angles, the triangles

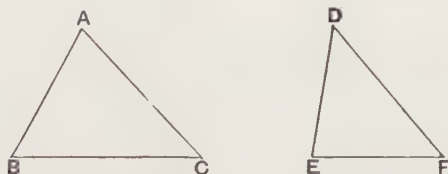
are proved equal, (2) if greater than two right angles, that triangle which has the greater angle is less, and (3) if they are less, greater." Proclus then gives the proof, but without any reference to the source from which he quoted the proposition. Now an-Nairizī adds a similar proposition to I. 38, but definitely attributes it to Heron. I shall accordingly give it in the place where Heron put it

PROPOSITION 25.

If two triangles have the two sides equal to two sides respectively, but have the base greater than the base, they will also have the one of the angles contained by the equal straight lines greater than the other.

Let ABC , DEF be two triangles having the two sides AB , AC equal to the two sides DE , DF respectively, namely AB to DE , and AC to DF ; and let the base BC be greater than the base EF ;

I say that the angle BAC is also greater than the angle EDF .



For, if not, it is either equal to it or less.

Now the angle BAC is not equal to the angle EDF ;
for then the base BC would also have been equal to the base EF , [I. 4]

but it is not ;

therefore the angle BAC is not equal to the angle EDF .

Neither again is the angle BAC less than the angle EDF ;
for then the base BC would also have been less than the base EF , [I. 24]

but it is not ;

therefore the angle BAC is not less than the angle EDF .
But it was proved that it is not equal either ;

therefore the angle BAC is greater than the angle EDF .
Therefore etc.

Q. E. D.

De Morgan points out that this proposition (as also I. 8) is a purely *logical* consequence of I. 4 and I. 24 in the same way as I. 19 and I. 6 are purely *logical* consequences of I. 18 and I. 5. If a, b, c denote the sides, A, B, C the angles opposite to them in a triangle ABC , and a', b', c', A', B', C' the sides and opposite angles respectively in a triangle $A'B'C'$, I. 4 and I. 24 tell us that, b, c being respectively equal to b', c' ,

- (1) if A is equal to A' , then a is equal to a' ,
- (2) if A is less than A' , then a is less than a' ,
- (3) if A is greater than A' , then a is greater than a' ;

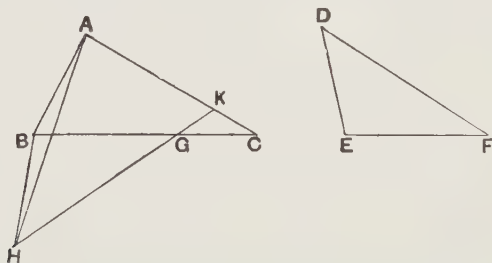
and it follows *logically* that,

- | | |
|--|---------|
| (1) if a is equal to a' , the angle A is equal to the angle A' , | [I. 8] |
| (2) if a is less than a' , A is less than A' , | |
| (3) if a is greater than a' , A is greater than A' . } | [I. 25] |

Two alternative proofs of this theorem are given by Proclus (pp. 345—7), and they are both interesting. Moreover both are *direct*.

I. Proof by Menelaus of Alexandria.

Let ABC, DEF be two triangles having the two sides BA, AC equal to the two sides ED, DF , but the base BC greater than the base EF .



Then shall the angle at A be greater than the angle at D .

From BC cut off BG equal to EF . At B , on the straight line BC , make the angle GBH (on the side of BG remote from A) equal to the angle FED .

Make BH equal to DE ; join HG , and produce it to meet AC in K . Join AH .

Then, since the two sides GB, BH are equal to the two sides FE, ED respectively,

and the angles contained by them are equal,

HG is equal to DF or AC ,

and the angle BHG is equal to the angle EDF .

Now HK is greater than HG or AC ,

and *a fortiori* greater than AK ;

therefore the angle KAH is greater than the angle KHA .

And, since AB is equal to BH ,

the angle BAH is equal to the angle BHA .

Therefore, by addition,

the whole angle BAC is greater than the whole angle BHG ,

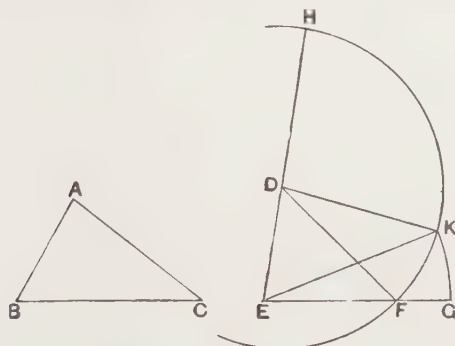
that is, greater than the angle EDF .

II. Heron's proof.

Let the triangles be given as before.

Since BC is greater than EF , produce EF to G so that EG is equal to BC .

Produce ED to H so that DH is equal to DF . The circle with centre D and radius DF will then pass through H . Let it be described, as FKH .



Now, since BA, AC are together greater than BC ,
 and BA, AC are equal to ED, DH respectively,
 while BC is equal to EG ,
 EH is greater than EG .

Therefore the circle with centre E and radius EG will cut EH , and therefore will cut the circle already drawn. Let it cut that circle in K , and join DK, KE .

Then, since D is the centre of the circle FKH ,
 DK is equal to DF or AC .

Similarly, since E is the centre of the circle KG ,
 EK is equal to EG or BC ,

And DE is equal to AB .

Therefore the two sides BA, AC are equal to the two sides ED, DK respectively ;

and the base BC is equal to the base EK ;

therefore the angle BAC is equal to the angle EDK .

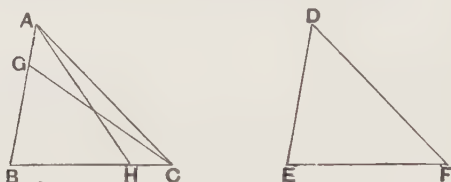
Therefore the angle BAC is greater than the angle EDF .

PROPOSITION 26.

If two triangles have the two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that subtending one of the equal angles, they will also have the remaining sides equal to the remaining sides and the remaining angle to the remaining angle.

Let ABC, DEF be two triangles having the two angles ABC, BCA equal to the two angles DEF, EFD respectively, namely the angle ABC to the angle DEF , and the angle BCA to the angle EFD ; and let them also have one side equal to one side, first that adjoining the equal angles, namely BC to EF ;

I say that they will also have the remaining sides equal to the remaining sides respectively, namely AB to DE and AC to DF , and the remaining angle to the remaining angle, namely the angle BAC to the angle EDF .



For, if AB is unequal to DE , one of them is greater.

Let AB be greater, and let BG be made equal to DE ; and let GC be joined.

Then, since BG is equal to DE , and BC to EF , the two sides GB, BC are equal to the two sides DE, EF respectively; and the angle GBC is equal to the angle DEF ;

therefore the base GC is equal to the base DF ,

and the triangle GBC is equal to the triangle DEF , and the remaining angles will be equal to the remaining angles, namely those which the equal sides subtend; [1. 4]

therefore the angle GCB is equal to the angle DFE .

But the angle DFE is by hypothesis equal to the angle BCA ;

therefore the angle BCG is equal to the angle BCA , the less to the greater: which is impossible.

Therefore AB is not unequal to DE ,

and is therefore equal to it.

But BC is also equal to EF ;

therefore the two sides AB, BC are equal to the two sides DE, EF respectively,

and the angle ABC is equal to the angle DEF ;

therefore the base AC is equal to the base DF ,

and the remaining angle BAC is equal to the remaining angle EDF . [1. 4]

Again, let sides subtending equal angles be equal, as AB to DE ;

I say again that the remaining sides will be equal to the remaining sides, namely AC to DF and BC to EF , and
45 further the remaining angle BAC is equal to the remaining angle EDF .

For, if BC is unequal to EF , one of them is greater.

Let BC be greater, if possible, and let BH be made equal to EF ; let AH be joined.

50 Then, since BH is equal to EF , and AB to DE ,
the two sides AB, BH are equal to the two sides DE, EF respectively, and they contain equal angles;

therefore the base AH is equal to the base DF ,

and the triangle ABH is equal to the triangle DEF ,

55 and the remaining angles will be equal to the remaining angles,
namely those which the equal sides subtend; [I. 4]

therefore the angle BHA is equal to the angle EFD .

But the angle EFD is equal to the angle BCA ;

therefore, in the triangle AHC , the exterior angle BHA is
60 equal to the interior and opposite angle BCA ;

which is impossible.

[I. 16]

Therefore BC is not unequal to EF ,

and is therefore equal to it.

But AB is also equal to DE ;

65 therefore the two sides AB, BC are equal to the two sides
 DE, EF respectively, and they contain equal angles;

therefore the base AC is equal to the base DF ,

the triangle ABC equal to the triangle DEF ,

and the remaining angle BAC equal to the remaining angle
70 EDF . [I. 4]

Therefore etc.

Q. E. D.

2—3. the side adjoining the equal angles, πλευρὰν τὴν πρὸς ταῖς ἴσαις γωνίαις.

29. is by hypothesis equal. ὑπόκειται ἴση, according to the elegant Greek idiom. ὑπόκειται is used for the passive of ὑποτίθημι, as κείμεναι is used for the passive of τίθημι, and so with the other compounds. Cf. προσκείσθαι, "to be added."

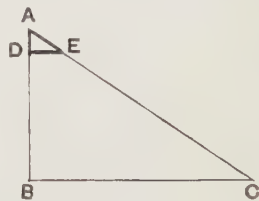
The alternative method of proving this proposition, viz. by applying one triangle to the other, was very early discovered, at least so far as regards the case where the equal sides are adjacent to the equal angles in each. An-Nairizî gives it for this case, observing that the proof is one which he had found, but of which he did not know the author.

Proclus has the following interesting note (p. 352, 13—18): "Eudemos in his geometrical history refers this theorem to Thales. For he says that, in the method by which they say that Thales proved the distance of ships in the sea, it was necessary to make use of this theorem." As, unfortunately, this information is not sufficient of itself to enable us to determine how Thales solved this problem, there is considerable room for conjecture as to his method.

The suggestions of Bretschneider and Cantor agree in the assumption that the necessary observations were probably made from the top of some tower or structure of known height, and that a right-angled triangle was used in which the tower was the perpendicular, and the line connecting the bottom of the tower and the ship was the base, as in the annexed figure, where AB is the tower and C the ship. Bretschneider (*Die Geometrie und die Geometer vor Eukleides*, § 30) says that it was only necessary for the observer to observe the angle CAB , and then the triangle would be completely determined by means of this angle and the known length AB . As Bretschneider says that the result would be obtained "in a moment" by this method, it is not clear in what sense he supposes Thales to have "observed" the angle BAC . Cantor is more definite (*Gesch. d. Math.* 13, p. 145), for he says that the problem was nearly related to that of finding the *Seqt* from given sides. By the *Seqt* in the Papyrus Rhind is meant the ratio to one another of certain lines in pyramids or obelisks. Eisenlohr and Cantor took the one word to be equivalent, sometimes to the *cosine* of the angle made by the *edge* of the pyramid with the co-terminous diagonal of the base, sometimes to the *tangent* of the angle of slope of the *faces* of the pyramid. It is now certain that it meant one thing, viz. the ratio of half the side of the base to the height of the pyramid, i.e. the *cotangent* of the angle of slope. The calculation of the *Seqt* thus implying a sort of theory of similarity, or even of trigonometry, the suggestion of Cantor is apparently that the *Seqt* in this case would be found from a *small* right-angled triangle ADE having a common angle A with ABC as shown in the figure, and that the ascertained value of the *Seqt* with the length AB would determine BC . This amounts to the use of the property of similar triangles; and Bretschneider's suggestion must apparently come to the same thing, since, even if Thales measured the *angle* in our sense (e.g. by its ratio to a right angle), he would, in the absence of something corresponding to a table of trigonometrical ratios, have gained nothing and would have had to work out the proportions all the same.

Max C. P. Schmidt also (*Kulturhistorische Beiträge zur Kenntnis des griechischen und römischen Altertums*, 1906, p. 32) similarly supposes Thales to have had a right angle made of wood or bronze with the legs graduated, to have placed it in the position ADE (A being the position of his eye), and then to have read off the lengths AD , DE respectively, and worked out the length of BC by the rule of three.

How then does the supposed use of similar triangles and their property square with Eudemos' remark about I. 26? As it stands, it asserts the *equality* of *two* triangles which have two angles and one side respectively equal, and the theorem can only be brought into relation with the above explanations by taking it as asserting that, if two angles and one side of *one* triangle are given, the triangle is completely determined. But, if Thales



practically used *proportions*, as supposed, I. 26 is surely not at all the theorem which this procedure would naturally suggest as underlying it and being "necessarily used"; the use of proportions or of similar but not equal triangles would surely have taken attention altogether away from I. 26 and fixed it on VI. 4.

For this reason I think Tannery is on the right road when he tries to find a solution using I. 26 as it stands, and withal as primitive as any recorded solution of such a problem. His suggestion (*La Géométrie grecque*, pp. 90—1) is based on the *fuminis varatio* of the Roman agrimensor Marcus Junius Nipsus and is as follows.

To find the distance from a point *A* to an inaccessible point *B*. From *A* measure along a straight line at right angles to *AB* a length *AC* and bisect it at *D*. From *C* draw *CE* at right angles to *CA* on the side of it remote from *B*, and let *E* be the point on it which is in a straight line with *B* and *D*.

Then, by I. 26, *CE* is obviously equal to *AB*.

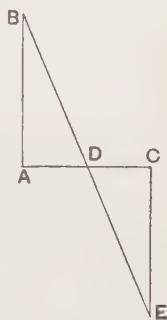
As regards the equality of angles, it is to be observed that those at *D* are equal because they are vertically opposite, and, curiously enough, Thales is expressly credited with the discovery of the equality of such angles.

The only objection which I can see to Tannery's solution is that it would require, in the case of the ship, a certain extent of free and level ground for the construction and measurements.

I suggest therefore that the following may have been Thales' method. Assuming that he was on the top of a tower, he had only to use a rough instrument made of a straight stick and a cross-piece fastened to it so as to be capable of turning about the fastening (say a nail) so that it could form any angle with the stick and would remain where it was put. Then the natural thing would be to fix the stick upright (by means of a plumb-line) and direct the cross-piece towards the ship. Next, leaving the cross-piece at the angle so found, the stick could be turned round, still remaining vertical, until the cross-piece pointed to some visible object on the shore, when the object could be mentally noted and the distance from the bottom of the tower to it could be subsequently measured. This would, by I. 26, give the distance from the bottom of the tower to the ship. This solution has the advantage of corresponding better to the simpler and more probable version of Thales' method of measuring the height of the pyramids; Diogenes Laertius says namely (I. 27, p. 6, ed. Cobet) on the authority of Hieronymus of Rhodes (B.C. 290—230), that he waited for this purpose until the moment when *our shadows are of the same length as ourselves*.

Recapitulation of congruence theorems.

Proclus, like other commentators, gives at this point (p. 347, 20 sqq.) a summary of the cases in which the equality of two triangles in all respects can be established. We may, he says, seek the conditions of such equality by successively considering as hypotheses the equality (1) of sides alone, (2) of angles alone, (3) of sides and angles combined. Taking (1) first, we can only establish the equality of the triangles in all respects if all three sides are respectively equal; we cannot establish the equality of the triangles by any hypothesis of class (2), not even the hypothesis that all the three angles are respectively equal; among the hypotheses of class (3), the equality of one



side and one angle in each triangle is not enough, the equality (*a*) of one side and all three angles is more than enough, as is also the equality (*b*) of two sides and two or three angles, and (*c*) of three sides and one or two angles.

The only hypotheses therefore to be examined from this point of view are the equality of

(*a*) three sides [Eucl. 1. 8].

(*β*) two sides and one angle [1. 4 proves one case of this, where the angle is that contained by the sides which are by hypothesis equal].

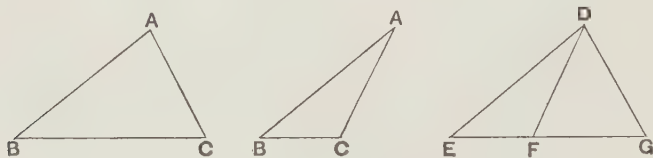
(*γ*) one side and two angles [1. 26 covers all cases].

It is curious that Proclus makes no allusion to what we call the *ambiguous case*, that case namely of (*β*) in which it is an angle opposite to one of the two specified sides in one triangle which is equal to the angle opposite to the equal side in the other triangle. Camerer indeed attributes to Proclus the observation that in this case the equality of the triangles cannot, unless some other condition is added, be asserted generally; but it would appear that Camerer was probably misled by a figure (Proclus, p. 351) which looks like a figure of the ambiguous case but is really only used to show that in 1. 26 the equal sides must be *corresponding* sides, i.e., they must be either adjacent to the equal angles in each triangle, or opposite to corresponding equal angles, and that, e.g., one of the equal sides must not be adjacent to the two angles in one triangle, while the side equal to it in the other triangle is opposite to one of the two corresponding angles in that triangle.

The ambiguous case.

If two triangles have two sides equal to two sides respectively, and if the angles opposite to one pair of equal sides be also equal, then will the angles opposite the other pair of equal sides be either equal or supplementary; and, in the former case, the triangles will be equal in all respects.

Let ABC , DEF be two triangles such that AB is equal to DE , and AC to DF , while the angle ABC is equal to the angle DEF ; it is required to prove that the angles ACB , DFE are either equal or supplementary.



Now (1), if the angle BAC be equal to the angle EDF , it follows, since the two sides AB , AC are equal to the two sides DE , DF respectively, that

the triangles ABC , DEF are equal in all respects, [1. 4]
and the angles ACB , DFE are equal.

(2) If the angles BAC , EDF be not equal, make the angle EDG (on the same side of ED as the angle EDF) equal to the angle BAC .

Let EF , produced if necessary, meet DG in G .

Then, in the triangles ABC , DEG ,
the two angles BAC , ABC are equal to the two angles EDG , DEG respectively,
and the side AB is equal to the side DE ;

therefore the triangles ABC , DEG are equal in all respects, [I. 26]
 so that the side AC is equal to the side DG ,
 and the angle ACB is equal to the angle DGE .

Again, since AC is equal to DF as well as to DG ,

DF is equal to DG ,

and therefore the angles DFG , DGF are equal.

But the angle DFE is supplementary to the angle DFG ; and the angle DGF was proved equal to the angle ACB ;

therefore the angle DFE is supplementary to the angle ACB .

If it is desired to avoid the ambiguity and secure that the triangles may be congruent, we can introduce the necessary conditions into the enunciation, on the analogy of Eucl. vi. 7.

If two triangles have two sides of the one equal to two sides of the other respectively, and the angles opposite to a pair of equal sides equal, then, if the angles opposite to the other pair of equal sides are both acute, or both obtuse, or if one of them is a right angle, the two triangles are equal in all respects.

The proof of the three cases (by *reductio ad absurdum*) was given by Todhunter.

PROPOSITION 27.

If a straight line falling on two straight lines make the alternate angles equal to one another, the straight lines will be parallel to one another.

For let the straight line EF falling on the two straight
 5 lines AB , CD make the alternate angles AEF , EFD equal
 to one another;

I say that AB is parallel to CD .

For, if not, AB , CD when pro-
 duced will meet either in the direction
 10 of B , D or towards A , C .

Let them be produced and meet,
 in the direction of B , D , at G .

Then, in the triangle GEF ,

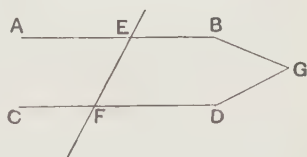
the exterior angle AEF is equal to the interior and opposite
 15 angle EFG :

which is impossible.

[I. 16]

Therefore AB , CD when produced will not meet in the
 direction of B , D .

Similarly it can be proved that neither will they meet
 20 towards A , C .



But straight lines which do not meet in either direction are parallel; [Def. 23]

therefore AB is parallel to CD .

Therefore etc.

Q. E. D.

1. falling on two straight lines, *εἰς δύο εὐθείας ἐμπίπτουσα*, the phrase being the same as that used in Post. 5, meaning a *transversal*.

2. the alternate angles, *αἱ ἐναλλὰξ γωνίαι*. Proclus (p. 357, 9) explains that Euclid uses the word *alternate* (or, more exactly, *alternately*, *ἐναλλάξ*) in two connexions, (1) of a certain transformation of a proportion, as in Book v. and the arithmetical Books, (2) as here, of certain of the angles formed by parallels with a straight line crossing them. *Alternate* angles are, according to Euclid as interpreted by Proclus, those which are not on the same side of the transversal, and are not adjacent, but are separated by the transversal, both being within the parallels but one "above" and the other "below." The meaning is natural enough if we imagine the four internal angles to be taken in cyclic order and *alternate* angles to be any two of them not successive but separated by one angle of the four.

9. in the direction of B , D or towards A , C , literally "towards the parts B , D or towards A , C ," *ἐπὶ τὰ B , D μέρη ἢ ἐπὶ τὰ A , C* .

With this proposition begins the second section of the first Book. Up to this point Euclid has dealt mainly with triangles, their construction and their properties in the sense of the relation of their parts, the sides and angles, to one another, and the comparison of different triangles in respect of their parts, and of their area in the particular cases where they are congruent.

The second section leads up to the third, in which we pass to relations between the areas of triangles, parallelograms and squares, the special feature being a new conception of *equality* of areas, equality not dependent on *congruence*. This whole subject requires the use of parallels. Consequently the second section beginning at I. 27 establishes the theory of parallels, introduces the cognate matter of the equality of the sum of the angles of a triangle to two right angles (I. 32), and ends with two propositions forming the transition to the third section, namely I. 33, 34, which introduce the parallelogram for the first time.

Aristotle on parallels.

We have already seen reason to believe that Euclid's personal contribution to the subject was nothing less than the formulation of the famous Postulate 5 (see the notes on that Postulate and on Def. 23), since Aristotle indicates that the then current theory of parallels contained a *petitio principii*, and presumably therefore it was Euclid who saw the defect and proposed the remedy.

But it is clear that the propositions I. 27, 28 were contained in earlier text-books. They were familiar to Aristotle, as we may judge from two interesting passages.

(1) In *Anal. Post.* I. 5 he is explaining that a scientific demonstration must not only prove a fact of every individual of a class (*κατὰ παντός*) but must prove it primarily and generally true (*πρῶτον καθόλου*) of the *whole* of the class as one; it will not do to prove it first of one part, then of another part, and so on, until the class is exhausted. He illustrates this (74 a 13—16) by a reference to parallels: "If then one were to show that right (angles) do not meet, the proof of this might be thought to depend on the fact that this is true of all (pairs of actual) right angles. But this is not so, inasmuch as the result does not follow because (the two angles are) equal (to two right

angles) *in the particular way* [i.e. because each is a right angle], but by virtue of their being equal (to two right angles) in any way whatever [i.e. because the *sum* only needs to be equal to two right angles, and the angles themselves may vary as much as we please subject to this]."

(2) The second passage has already been quoted in the note on Def. 23: "there is nothing surprising in different hypotheses leading to the same false conclusion; e.g. the conclusion that parallels meet might equally be drawn from either of the assumptions (*a*) that the interior (angle) is greater than the exterior or (*b*) that the sum of the angles of a triangle is greater than two right angles" (*Anal. Prior. II. 17, 66 a 11—15*).

I do not quite concur in the interpretation which Heiberg places upon these passages (*Mathematisches zu Aristoteles*, pp. 18—19). He says, first, that the allusion to the "interior angle" being "greater than the exterior" in the second passage shows that the reference in the first passage must be to Eucl. I. 28 and not to I. 27, and he therefore takes the words $\delta\tau\iota\ \omega\delta\iota\ \acute{\iota}\sigma\alpha\iota$ in the first passage (which I have translated "because the two angles are equal to two right angles in the particular way") as meaning "because the angles, viz. the *exterior* and the *interior*, are equal in the particular way." He also takes $\alpha\iota\ \delta\omicron\rho\theta\alpha\iota\ \omicron\upsilon\ \sigma\upsilon\mu\pi\acute{\iota}\pi\tau\omicron\upsilon\sigma\iota$ (which I have translated "right angles do not meet," an expression quite in Aristotle's manner) to mean "perpendicular *straight lines* do not meet"; this is very awkward, especially as he is obliged to supply *angles* with $\acute{\iota}\sigma\alpha\iota$ in the next sentence.

But I think that the first passage certainly refers to I. 28, although I do not think that the alternative (*a*) in the second passage suggests it. This alternative may, I think, equally with the alternative (*b*) refer to I. 27. That proposition is proved by *reductio ad absurdum* based on the fact that, if the straight lines do meet, they must form a *triangle*, in which case the exterior angle must be greater than the interior (while according to the hypothesis these angles are equal). It is true that Aristotle speaks of the hypothesis that the *interior* angle is greater than the *exterior*; but after all Aristotle had only to state *some* incorrect hypothesis. It is of course only in connexion with straight lines *meeting*, as the hypothesis in I. 27 makes them, that the alternative (*b*) about the sum of the angles of a triangle could come in, and alternative (*a*) implies alternative (*b*).

It seems clear then from Aristotle that I. 27, 28 at least are pre-Euclidean, and that it was only in I. 29 that Euclid made a change by using his Postulate.

De Morgan observes that I. 27 is a *logical* equivalent of I. 16. Thus, if *A* means "straight lines forming a triangle with a transversal," *B* "straight lines making angles with a transversal on the same side which are together less than two right angles," we have

All *A* is *B*,

and it follows *logically* that

All not-*B* is not-*A*.

PROPOSITION 28.

If a straight line falling on two straight lines make the exterior angle equal to the interior and opposite angle on the same side, or the interior angles on the same side equal to two right angles, the straight lines will be parallel to one another.

For let the straight line EF falling on the two straight lines AB , CD make the exterior angle EGB equal to the interior and opposite angle GHD , or the interior angles on the same side, namely BGH , GHD , equal to two right angles;

I say that AB is parallel to CD .

For, since the angle EGB is equal to the angle GHD ,

while the angle EGB is equal to the angle AGH , [I. 15]

the angle AGH is also equal to the angle GHD ;

and they are alternate;

therefore AB is parallel to CD .

[I. 27]

Again, since the angles BGH , GHD are equal to two right angles, and the angles AGH , BGH are also equal to two right angles, [I. 13]

the angles AGH , BGH are equal to the angles BGH , GHD .

Let the angle BGH be subtracted from each;

therefore the remaining angle AGH is equal to the remaining angle GHD ;

and they are alternate;

therefore AB is parallel to CD .

[I. 27]

Therefore etc.

Q. E. D.

One criterion of parallelism, the equality of alternate angles, is given in I. 27; here we have two more, each of which is easily reducible, and is actually reduced, to the other.

Proclus observes (pp. 358—9) that Euclid could have stated six criteria as well as three, by using, in addition, other pairs of angles in the figure (not adjacent) of which it could be predicated that the two angles are equal or that their sum is equal to two right angles. A natural division is to consider, first the pairs which are on the same side of the transversal, and secondly the pairs which are on different sides of it.

Taking (1) the possible pairs on the *same* side, we may have a pair consisting of

(a) two internal angles, viz. the pairs (BGH, GHD) and (AGH, GHC) ;

(b) two external angles, viz. the pairs (EGB, DHF) and (EGA, CHF) ;

(c) one external and one internal angle, viz. the pairs (EGB, GHD) , (FHD, HGB) , (EGA, GHC) and (FHC, HGA) .



And (2) the possible pairs on *different* sides of the transversal may consist respectively of

- (a) two internal angles, viz. the pairs (AGH, GHD) and (CHG, HGB) ;
- (b) two external angles, viz. the pairs (AGE, DHF) and (EGB, CHF) ;
- (c) one external and one internal, viz. the pairs (AGE, GHD) , (EGB, GHC) , (FHC, HGB) and (FHD, HGA) .

The angles are equal in the pairs (1) (c), (2) (a) and (2) (b), and the sum is equal to two right angles in the case of the pairs (1) (a), (1) (b) and (2) (c). For his criteria Euclid selects the cases (2) (a) [I. 27] and (1) (c), (1) (a) [I. 28], leaving out the other three, which are of course equivalent but are not quite so easily expressed.

From Proclus' note on I. 28 (p. 361) we learn that one Aigeias (? Aineias) of Hierapolis wrote an epitome or abridgment of the *Elements*. This seems to be the only mention of this editor and his work; and they are only mentioned as having combined Eucl. I. 27, 28 into one proposition. To do this, or to make the three hypotheses the subject of *three* separate theorems, would, Proclus thinks, have been more natural than to deal with them, as Euclid does, in two propositions. Proclus has no suggestion for explaining Euclid's arrangement unless the ground were that I. 27 deals with angles on different sides, I. 28 with angles on one and the same side, of the transversal. But may not the reason have been one of convenience, namely that the criterion of I. 27 is that actually used to prove parallelism, and is moreover the basis of the construction of parallels in I. 31, while I. 28 only reduces the other two hypotheses to that of I. 27, so that precision of reference, as well as clearness of exposition, is better secured by the arrangement adopted?

PROPOSITION 29.

A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles.

5 For let the straight line EF fall on the parallel straight lines AB, CD ;

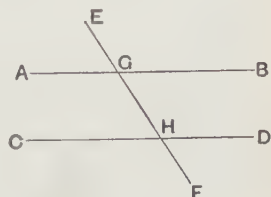
I say that it makes the alternate angles AGH, GHD equal, the exterior angle EGB equal to the interior and opposite angle GHD , and the interior angles on the same
10 side, namely BGH, GHD , equal to two right angles.

For, if the angle AGH is unequal to the angle GHD , one of them is greater.

Let the angle AGH be greater.

15 Let the angle BGH be added to each;

therefore the angles AGH, BGH are greater than the angles BGH, GHD .



20 But the angles AGH , BGH are equal to two right angles;
[I. 13]

therefore the angles BGH , GHD are less than two right angles.

But straight lines produced indefinitely from angles less than two right angles meet; [Post. 5]

25 therefore AB , CD , if produced indefinitely, will meet; but they do not meet, because they are by hypothesis parallel.

Therefore the angle AGH is not unequal to the angle GHD ,

and is therefore equal to it.

30 Again, the angle AGH is equal to the angle EGB ; [I. 15]
therefore the angle EGB is also equal to the angle GHD .
[C. N. 1]

Let the angle BGH be added to each;

therefore the angles EGB , BGH are equal to the
35 angles BGH , GHD . [C. N. 2]

But the angles EGB , BGH are equal to two right angles;
[I. 13]

therefore the angles BGH , GHD are also equal to two right angles.

Therefore etc.

Q. E. D.

23. straight lines produced indefinitely from angles less than two right angles, *αἱ δὲ ἀπ' ἐλασσόνων ἢ δύο ὀρθῶν ἐκβαλλόμεναι εἰς ἄπειρον συμπίπτουσιν*, a variation from the more explicit language of Postulate 5. A good deal is left to be understood, namely that the straight lines begin from points at which they meet a transversal, and make with it internal angles on the same side the sum of which is less than two right angles.

26. because they are by hypothesis parallel, literally "because they are supposed parallel," *διὰ τὸ παραλλήλους αὐτὰς ὑποκείσθαι*.

Proof by "Playfair's" axiom.

If, instead of Postulate 5, it is preferred to use "Playfair's" axiom in the proof of this proposition, we proceed thus.

To prove that the alternate angles AGH , GHD are equal.

If they are not equal, draw another straight line KL through G making the angle KGH equal to the angle GHD .

Then, since the angles KGH , GHD are equal,

KL is parallel to CD .

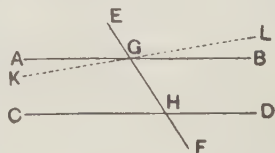
[I. 27]

Therefore two straight lines KL , AB intersecting at G are both parallel to the straight line CD :

which is impossible (by the axiom).

Therefore the angle AGH cannot but be equal to the angle GHD .

The rest of the proposition follows as in Euclid.



Proof of Euclid's Postulate 5 from "Playfair's" axiom.

Let AB , CD make with the transversal EF the angles AEF , EFC together less than two right angles.

To prove that AB , CD meet towards A , C .

Through E draw GH making with EF the angle GEF equal (and alternate) to the angle EFD .

Thus GH is parallel to CD . [I. 27]



Then (1) AB must meet CD in one direction or the other.

For, if it does not, AB must be parallel to CD ; hence we have two straight lines AB , GH intersecting at E and both parallel to CD :

which is impossible.

Therefore AB , CD must meet.

(2) Since AB , CD meet, they must form a triangle with EF .

But in any triangle any two angles are together less than two right angles.

Therefore the angles AEF , EFC (which are less than two right angles), and not the angles BEF , EFD (which are together greater than two right angles, by I. 13), are the angles of the triangle;

that is, EA , FC meet in the direction of A , C , or on the side of EF on which are the angles together less than two right angles.

The usual course in modern text-books which use "Playfair's" axiom in lieu of Euclid's Postulate is apparently to prove I. 29 by means of the axiom, and then Euclid's Postulate by means of I. 29.

De Morgan would introduce the proof of Postulate 5 by means of "Playfair's" axiom *before* I. 29, and would therefore apparently prove I. 29 as Euclid does, without any change.

As between Euclid's Postulate 5 and "Playfair's" axiom, it would appear that the tendency in modern text-books is rather in favour of the latter. Thus, to take a few noteworthy foreign writers, we find that Rausenberger stands almost alone in using Euclid's Postulate, while Hilbert, Henrici and Treutlein, Rouché and De Comberousse, Enriques and Amaldi all use "Playfair's" axiom.

Yet the case for preferring Euclid's Postulate is argued with some force by Dodgson (*Euclid and his modern Rivals*, pp. 44—6). He maintains (1) that "Playfair's" axiom in fact involves Euclid's Postulate, but at the same time involves *more* than the latter, so that, to that extent, it is a needless strain on the faith of the learner. This is shown as follows.

Given AB , CD making with EF the angles AEF , EFC together less than two right angles, draw GH through E so that the angles GEF , EFC are together *equal* to two right angles.

Then, by I. 28, GH , CD are "separational."

We see then that any lines which have the property (α) that they make with a transversal angles less than two right angles have also the property (β) that one of them intersects a straight line which is "separational" from the other.

Now Playfair's axiom asserts that the lines which have property (β) meet if produced: for, if they did not, we should have two intersecting straight lines both "separational" from a third, which is impossible.

We then argue that lines having property (α) meet because lines having property (α) are lines having property (β). But we do not know, until we have proved I. 29, that all pairs of lines having property (β) have also property

(α). For anything we know to the contrary, class (β) *may* be greater than class (α). Hence, if you assert anything of class (β), the logical effect is more extensive than if you assert it of class (α); for you assert it, not only of that portion of class (β) which is known to be included in class (α), but also of the unknown (but possibly existing) portion which is *not* so included.

(2) Euclid's Postulate puts before the beginner clear and *positive* conceptions, a pair of straight lines, a transversal, and two angles together less than two right angles, whereas "Playfair's" axiom requires him to realise a pair of straight lines which never meet though produced to infinity: a *negative* conception which does not convey to the mind any clear notion of the relative position of the lines. And (p. 68) Euclid's Postulate gives a direct criterion for judging that two straight lines meet, a criterion which is constantly required, e.g. in 1. 44. It is true that the Postulate can be *deduced* from "Playfair's" axiom, but editors frequently omit to deduce it, and then tacitly assume it afterwards: which is the least justifiable course of all.

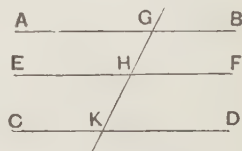
PROPOSITION 30.

Straight lines parallel to the same straight line are also parallel to one another.

Let each of the straight lines AB , CD be parallel to EF ;
I say that AB is also parallel to CD .

5 For let the straight line GK fall upon them,

Then, since the straight line GK has fallen on the parallel straight lines AB , EF ,



10 the angle AGK is equal to the angle GHF .

[1. 29]

Again, since the straight line GK has fallen on the parallel straight lines EF , CD ,

the angle GHF is equal to the angle GKD . [1. 29]

15 But the angle AGK was also proved equal to the angle GHF ;

therefore the angle AGK is also equal to the angle GKD ;

[C. N. 1]

and they are alternate.

20 Therefore AB is parallel to CD .

Q. E. D.

20. The usual *conclusion* in general terms ("Therefore etc.") repeating the enunciation is, curiously enough, wanting at the end of this proposition.

The proposition is, as De Morgan points out, the *logical* equivalent of "Playfair's" axiom. Thus, if X denote "pairs of straight lines intersecting one

another," Y "pairs of straight lines parallel to one and the same straight line," we have

No X is Y ,

and it follows logically that

No Y is X .

De Morgan adds that a proposition is much wanted about parallels (or perpendiculars) to two straight lines respectively making the same angles with one another as the latter do. The proposition may be enunciated thus:

If the sides of one angle be respectively (1) parallel or (2) perpendicular to the sides of another angle, the two angles are either equal or supplementary.

(1) Let DE be parallel to AB and GEF parallel to BC .

To prove that the angles ABC , DEG are equal and the angles ABC , DEF supplementary.

Produce DE to meet BC in H .

Then [I. 29] the angle DEG is equal to the angle DHC ,

and the angle ABC is equal to the angle DHC .

Therefore the angle DEG is equal to the angle ABC ; whence also the angle DEF is supplementary to the angle ABC .

(2) Let ED be perpendicular to AB , and GEF perpendicular to BC .

To prove that the angles ABC , DEG are equal, and the angles ABC , DEF supplementary.

Draw ED' at right angles to ED on the side of it opposite to B , and draw EG' at right angles to EF on the side of it opposite to B .

Then, since the angles BDE , DED' , being right angles, are equal,

ED' is parallel to BA . [I. 27]

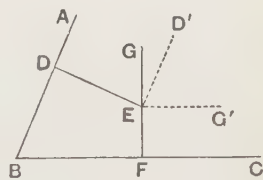
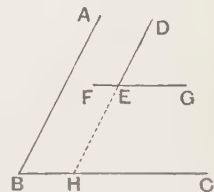
Similarly EG' is parallel to BC .

Therefore [Part (1)] the angle $D'EG'$ is equal to the angle ABC .

But, the right angle DED' being equal to the right angle GEG' , if the common angle GED' be subtracted,

the angle DEG is equal to the angle $D'EG'$.

Therefore the angle DEG is equal to the angle ABC ; and hence the angle DEF is supplementary to the angle ABC .



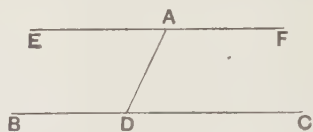
PROPOSITION 31.

Through a given point to draw a straight line parallel to a given straight line.

Let A be the given point, and BC the given straight line;

thus it is required to draw through the point A a straight line parallel to the straight line BC .

Let a point D be taken at random on BC , and let AD be joined; on the straight line DA , and at the point A on it, let the angle DAE be constructed equal to the angle ADC [I. 23]; and let the straight line AF be produced in a straight line with EA .



Then, since the straight line AD falling on the two straight lines BC , EF has made the alternate angles EAD , ADC equal to one another,

therefore EA is parallel to BC . [I. 27]

Therefore through the given point A the straight line EA has been drawn parallel to the given straight line BC .

Q. E. F.

Proclus rightly remarks (p. 376, 14—20) that, as it is implied in I. 12 that only one perpendicular can be drawn to a straight line from an external point, so here it is implied that only one straight line can be drawn through a point parallel to a given straight line. The construction, be it observed, depends only upon I. 27, and might therefore have come directly after that proposition. Why then did Euclid postpone it until after I. 29 and I. 30? Presumably because he considered it necessary, before giving the construction, to place beyond all doubt the fact that only one such parallel can be drawn. Proclus infers this fact from I. 30; for, he says, if two straight lines could be drawn through one and the same point parallel to the same straight line, the two straight lines would be *parallel*, though intersecting at the given point: which is impossible. I think it is a fair inference that Euclid would have considered it necessary to justify the assumption that only one parallel can be drawn by some such argument, and that he deliberately determined that his own assumption was more appropriate to be made the subject of a Postulate than the assumption of the uniqueness of the parallel.

PROPOSITION 32.

In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.

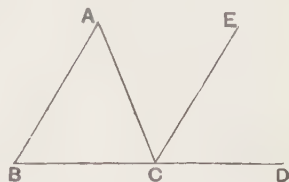
Let ABC be a triangle, and let one side of it BC be produced to D ;

I say that the exterior angle ACD is equal to the two interior and opposite angles CAB , ABC , and the three interior angles of the triangle ABC , BCA , CAB are equal to two right angles.

For let CE be drawn through the point C parallel to the straight line AB . [1. 31]

Then, since AB is parallel to CE ,
and AC has fallen upon them,
the alternate angles BAC , ACE are
equal to one another. [1. 29]

Again, since AB is parallel to CE ,



and the straight line BD has fallen upon them,
the exterior angle ECD is equal to the interior and opposite
angle ABC . [1. 29]

But the angle ACE was also proved equal to the angle BAC ;

therefore the whole angle ACD is equal to the two
interior and opposite angles BAC , ABC .

Let the angle ACB be added to each;

therefore the angles ACD , ACB are equal to the three
angles ABC , BCA , CAB .

But the angles ACD , ACB are equal to two right angles; [1. 13]

therefore the angles ABC , BCA , CAB are also equal
to two right angles.

Therefore etc.

Q. E. D.

This theorem was discovered in the very early stages of Greek geometry. What we know of the history of it is gathered from three allusions found in Eutocius, Proclus and Diogenes Laertius respectively.

1. Eutocius at the beginning of his commentary on the *Conics* of Apollonius (ed. Heiberg, Vol. II, p. 170) quotes Geminus as saying that "the ancients (οἱ ἀρχαῖοι) investigated the theorem of the two right angles in each individual species of triangle, first in the equilateral, again in the isosceles, and afterwards in the scalene triangle, and later geometers demonstrated the general theorem to the effect that in *any* triangle the three interior angles are equal to two right angles."

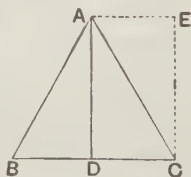
2. Now, according to Proclus (p. 379, 2—5), Eudemus the Peripatetic refers the discovery of this theorem to the Pythagoreans and gives what he affirms to be their demonstration of it. This demonstration will be given below, but it should be remarked that it is general, and therefore that the "later geometers" spoken of by Geminus were presumably the Pythagoreans, whence it appears that the "ancients" contrasted with them must have belonged to the time of Thales, if they were not his Egyptian instructors.

3. That the truth of the theorem was known to Thales might also be inferred from the statement of Pamphile (quoted by Diogenes Laertius, I. 24—5, p. 6, ed. Cobet) that "he, having learnt geometry from the

Egyptians, was the first to inscribe a right-angled triangle in a circle and sacrificed an ox" (on the strength of it); in other words, he discovered that the angle in a semicircle is a right angle. No doubt, when this fact was once discovered (*empirically*, say), the consideration of the two isosceles triangles having the centre for vertex and the sides of the right angle for bases respectively, with the help of the theorem of Eucl. I. 5, also known to Thales, would easily lead to the conclusion that the sum of the angles of a *right-angled* triangle is equal to two right angles, and it could be readily inferred that the angles of *any* triangle were likewise equal to two right angles (by resolving it into two right-angled triangles). But it is not easy to see how the property of the angle in a semicircle could be *proved* except (in the reverse order) by means of the equality of the sum of the angles of a *right-angled* triangle to two right angles; and hence it is most natural to suppose, with Cantor, that Thales proved it (if he did prove it) practically as Euclid does in III. 31, i.e. by means of I. 32 as applied to *right-angled* triangles at all events.

If the theorem of I. 32 was proved before Thales' time, or by Thales himself, by the stages indicated in the note of Geminus, we may be satisfied that the reconstruction of the argument of the older proof by Hankel (pp. 96—7) and Cantor (I₃, pp. 143—4) is not far wrong. First, it must have been observed that six angles equal to an angle of an equilateral triangle would, if placed adjacent to one another round a common vertex, fill up the whole space round that vertex. It is true that Proclus attributes to the Pythagoreans the general theorem that only three kinds of regular polygons, the equilateral triangle, the square and the regular hexagon, can fill up the entire space round a point, but the practical knowledge that equilateral triangles have this property could hardly have escaped the Egyptians, whether they made floors with tiles in the form of equilateral triangles or regular hexagons (Allman, *Greek Geometry from Thales to Euclid*, p. 12) or joined the ends of adjacent radii of a figure like the six-spoked wheel, which was their common form of wheel from the time of Ramses II. of the nineteenth Dynasty, say 1300 B.C. (Cantor, I₃, p. 109). It would then be clear that six angles equal to an angle of an equilateral triangle are equal to four right angles, and therefore that the three angles of an equilateral triangle are equal to two right angles. (It would be as clear or clearer, from observation of a square divided into two triangles by a diagonal, that an isosceles right-angled triangle has each of its equal angles equal to half a right angle, so that an isosceles right-angled triangle must have the sum of its angles equal to two right angles.) Next, with regard

to the equilateral triangle, it could not fail to be observed that, if AD were drawn from the vertex A perpendicular to the base BC , each of the two right-angled triangles so formed would have the sum of its angles equal to two right angles; and this would be confirmed by completing the rectangle $ADCE$, when it would be seen that the rectangle (with its angles equal to four right angles) was divided by its diagonal into two equal triangles, each of which had the sum of its angles equal to two right angles. Next it would be inferred, as the result of drawing the diagonal of *any* rectangle and observing the equality of the triangles forming the two halves, that the sum of the angles of *any* right-angled triangle is equal to two right angles, and hence (the two congruent right-angled triangles being then placed so as to form one isosceles triangle) that the same is true of *any isosceles* triangle. Only the last step remained, namely that of observing that *any* triangle could be regarded as the half of a rectangle (drawn as indicated in the next figure), or



simply that any triangle could be divided into two right-angled triangles, whence it would be inferred that in general the sum of the angles of any triangle is equal to two right angles.

Such would be the probabilities if we could absolutely rely upon the statements attributed to Pamphile and Geminus respectively. But in fact there is considerable ground for doubt in both cases.



1. Pamphile's story of the sacrifice of an ox by Thales for joy at his discovery that the angle in a semicircle is a right angle is too suspiciously like the similar story told with reference to Pythagoras and his discovery of the theorem of Eucl. I. 47 (Proclus, p. 426, 6—9). And, as if this were not enough, Diogenes Laertius immediately adds that "others, among whom is Apollodorus the calculator (ὁ λογιστικός), say it was Pythagoras" (sc. who "inscribed the right-angled triangle in a circle"). Now Pamphile lived in the reign of Nero (A.D. 54—68) and therefore some 700 years after the birth of Thales (about 640 B.C.). I do not know on what Max Schmidt bases his statement (*Kulturhistorische Beiträge zur Kenntnis des griechischen und römischen Altertums*, 1906, p. 31) that "other, much older, sources name Pythagoras as the discoverer of the said proposition," because nothing more seems to be known of Apollodorus than what is stated here by Diogenes Laertius. But it would at least appear that Apollodorus was only one of several authorities who attributed the proposition to Pythagoras, while Pamphile is alone mentioned as referring it to Thales. Again, the connexion of Pythagoras with the investigation of the right-angled triangle makes it *a priori* more likely that it would be he who would discover its relation to a semicircle. On the whole, therefore, the attribution to Thales would seem to be more than doubtful.

2. As regards Geminus' account of the three stages through which the proof of the theorem of I. 32 passed, we note, first, that it is certainly not confirmed by Eudemus, who referred to the Pythagoreans the *discovery* of the theorem that the sum of the angles of *any* triangle is equal to two right angles and says nothing about any gradual stages by which it was proved. Secondly, it must be admitted, I think, that in the evolution of the proof as reconstructed by Hankel the middle stage is rather artificial and unnecessary, since, once it is proved that *any right-angled* triangle has the sum of its angles equal to two right angles, it is just as easy to pass at once to any *scalene* triangle (which is decomposable into two *unequal* right-angled triangles) as to the isosceles triangle made up of two congruent right-angled triangles. Thirdly, as Heiberg has recently pointed out (*Mathematisches zu Aristoteles*, p. 20), it is quite possible that the statement of Geminus from beginning to end is simply due to a misapprehension of a passage of Aristotle (*Anal. Post.* I. 5, 74 a 25). Aristotle is illustrating his contention that a property is not scientifically proved to belong to a class of things unless it is proved to belong *primarily* (πρῶτον) and *generally* (καθόλου) to the *whole* of the class. His first illustration relates to parallels making with a transversal angles on the same side together equal to two right angles, and has been quoted above in the note on I. 27 (pp. 308—9). His second illustration refers to the transformation of a proportion *alternando*, which (he says) "used at one time to be proved separately" for numbers, lines, solids, and times, although it admits of being proved of all at once by one demonstration. The third illustration is: "For the same reason, even if one should prove (οὐδ' ἂν τις δεῖξει) with reference to

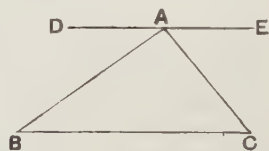
each (sort of) triangle, the equilateral, scalene and isosceles, separately, that each has its angles equal to two right angles, either by one proof or by different proofs, he does not yet know that *the triangle*, i.e. the triangle *in general*, has its angles equal to two right angles, except in a sophistical sense, even though there exists no triangle other than triangles of the kinds mentioned: For he knows it, not *quâ* triangle, nor of *every* triangle, except in a numerical sense (*κατ' ἀριθμὸν*); he does not know it *notionally* (*κατ' εἶδος*) of every triangle, even though there be actually no triangle which he does not know."

The difference between the phrase "used at one time to be proved" in the second illustration and "if any one should prove" in the third appears to indicate that, while the former referred to a historical fact, the latter does not; the reference to a person who should prove the theorem of I. 32 for the three kinds of triangle separately, and then claim that he had proved it generally, states a purely hypothetical case, a mere illustration. Yet, coming after the historical fact stated in the preceding illustration, it might not unnaturally give the impression, at first sight, that it was historical too.

On the whole, therefore, it would seem that we cannot safely go behind the dictum of Eudemus that the discovery and proof of the theorem of I. 32 in all its generality were Pythagorean. This does not however preclude its having been discovered by stages such as those above set out after Hankel and Cantor. Nor need it be doubted that Thales and even his Egyptian instructors had advanced some way on the same road, so far at all events as to see that in an equilateral triangle, and in an isosceles right-angled triangle, the sum of the angles is equal to two right angles.

The Pythagorean proof.

This proof, handed down by Eudemus (Proclus, p. 379, 2—15), is no less elegant than that given by Euclid, and is a natural development from the last figure in the reconstructed argument of Hankel. It would be seen, after the theory of parallels was added to geometry, that the actual drawing of the perpendicular and the complete rectangle on *BC* as base was unnecessary, and that the parallel to *BC* through *A* was all that was required.



Let *ABC* be a triangle, and through *A* draw *DE* parallel to *BC*. [I. 31]

Then, since *BC*, *DE* are parallel,

the alternate angles *DAB*, *ABC* are equal, [I. 29]

and so are the alternate angles *EAC*, *ACB* also.

Therefore the angles *ABC*, *ACB* are together equal to the angles *DAB*, *EAC*.

Add to each the angle *BAC*;

therefore the sum of the angles *ABC*, *ACB*, *BAC* is equal to the sum of the angles *DAB*, *BAC*, *CAE*, that is, to two right angles.

Euclid's proof pre-Euclidean.

The theorem of I. 32 is Aristotle's favourite illustration when he wishes to refer to some truth generally acknowledged, and so often does it occur that it is often indicated by two or three words in themselves hardly intelligible, e.g. τὸ δυνῆν ὁρθαῖς (*Anal. Post.* I. 24, 85 b 5) and ὑπάρχει παντὶ τριγώνῳ τὸ δύο (*ibid.* 85 b 11).

One passage (*Metaph.* 1051 a 24) makes it clear, as Heiberg (*op. cit.*

p. 19) acutely observes, that in the proof as Aristotle knew it Euclid's construction was used. "Why does the triangle make up two right angles? Because the angles about one point are equal to two right angles. If then the parallel to the side had been *drawn up* (*ἀνέκτο*), the fact would at once have been clear from merely looking at the figure." The words "the angles about one point" would equally fit the Pythagorean construction, but "drawn upwards" applied to the parallel to a side can only indicate Euclid's.

Attempts at proof independently of parallels.

The most indefatigable worker on these lines was Legendre, and a sketch of his work has been given in the note on Postulate 5 above.

One other attempted proof needs to be mentioned here because it has found much favour. I allude to

Thibaut's method.

This appeared in Thibaut's *Grundriss der reinen Mathematik*, Göttingen (2 ed. 1809, 3 ed. 1818), and is to the following effect.

Suppose CB produced to D , and let BD (produced to any necessary extent either way) revolve in one direction (say clockwise) first about B into the position BA , then about A into the position of AC produced both ways, and lastly about C into the position CB produced both ways.

The argument then is that the straight line BD has revolved through the sum of the three exterior angles of the triangle. But, since it has at the end of the revolution assumed a position in the same straight line with its original position, it must have revolved *through four right angles*.

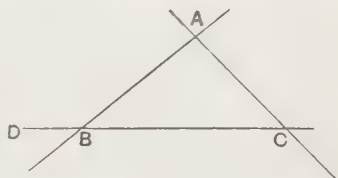
Therefore the sum of the three exterior angles is equal to four right angles;

from which it follows that the sum of the three angles of the triangle is equal to two right angles.

But it is to be observed that the straight line BD revolves about *different points in it*, so that there is *translation* combined with *rotatory* motion, and it is necessary to assume as an axiom that the two motions are independent, and therefore that the *translation* may be neglected.

Schumacher (letter to Gauss of 3 May, 1831) tried to represent the rotatory motion graphically in a second figure as mere motion round a point; but Gauss (letter of 17 May, 1831) pointed out in reply that he really assumed, without proving it, a proposition to the effect that "If two straight lines (1) and (2) which cut one another make angles A, A' with a straight line (3) cutting both of them, and if a straight line (4) in the same plane is likewise cut by (1) at an angle A' , then (4) will be cut by (2) at the angle A'' ". But this proposition not only needs proof, but we may say that it is, in essence, the very proposition to be proved" (see Engel and Stäckel, *Die Theorie der Parallelinien von Euklid bis auf Gauss*, 1895, p. 230).

How easy it is to be deluded in this way is plainly shown by Proclus' attempt on the same lines. He says (p. 384, 13—21) that the truth of the theorem is borne in upon us by the help of "common notions" only. "For, if we conceive a straight line with two perpendiculars drawn to it at its extremities, and if we then suppose the perpendiculars to (revolve about their feet and) approach one another, so as to form a triangle, we see that,



to the extent to which they converge, they diminish the right angles which they made with the straight line, so that the amount taken from the right angles is also the amount added to the vertical angle of the triangle, and the three angles are necessarily made equal to two right angles." But a moment's reflection shows that, so far from being founded on mere "common notions," the supposed proof assumes, to begin with, that, if the perpendiculars approach one another ever so little, they will then form a triangle immediately, i.e., it assumes Postulate 5 itself; and the fact about the vertical angle can only be seen by means of the equality of the alternate angles exhibited by drawing a perpendicular from the vertex of the triangle to the base, i.e. a *parallel* to either of the original perpendiculars.

Extension to polygons.

The two important corollaries added to I. 32 in Simson's edition are given by Proclus; but Proclus' proof of the first is different from, and perhaps somewhat simpler than, Simson's.

1. *The sum of the interior angles of a convex rectilineal figure is equal to twice as many right angles as the figure has sides, less four.*

For let one angular point A be joined to all the other angular points with which it is not connected already.

The figure is then divided into triangles, and mere inspection shows

(1) that the number of triangles is two less than the number of sides in the figure,

(2) that the sum of the angles of all the triangles is equal to the sum of all the interior angles of the figure.

Since then the sum of the angles of each triangle is equal to two right angles the sum of the interior angles of the figure is equal to $2(n-2)$ right angles, i.e. $(2n-4)$ right angles, where n is the number of sides in the figure.

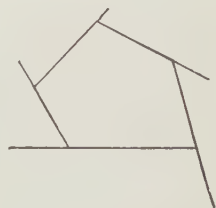
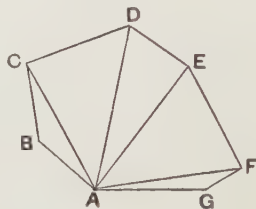
2. *The exterior angles of any convex rectilineal figure are together equal to four right angles.*

For the interior and exterior angles together are equal to $2n$ right angles, where n is the number of sides.

And the interior angles are together equal to $(2n-4)$ right angles.

Therefore the exterior angles are together equal to four right angles.

This last property is already quoted by Aristotle as true of all rectilineal figures in two passages (*Anal. Post.* I. 24, 85 b 38 and II. 17, 99 a 19).



PROPOSITION 33.

The straight lines joining equal and parallel straight lines (at the extremities which are) in the same directions (respectively) are themselves also equal and parallel.

Let AB , CD be equal and parallel, and let the straight
5 lines AC , BD join them (at the extremities which are) in the same directions (respectively);

I say that AC , BD are also equal and parallel.

Let BC be joined.

Then, since AB is parallel to CD ,
10 and BC has fallen upon them,

the alternate angles ABC , BCD
are equal to one another. [I. 29]

And, since AB is equal to CD ,
and BC is common,

15 the two sides AB , BC are equal to the two sides DC , CB ;
and the angle ABC is equal to the angle BCD ;

therefore the base AC is equal to the base BD ,

and the triangle ABC is equal to the triangle DCB ,

and the remaining angles will be equal to the remaining angles
20 respectively, namely those which the equal sides subtend; [I. 4]

therefore the angle ACB is equal to the angle CBD .

And, since the straight line BC falling on the two straight
lines AC , BD has made the alternate angles equal to one
another,

25 AC is parallel to BD . [I. 27]

And it was also proved equal to it.

Therefore etc.

Q. E. D.

1. joining... (at the extremities which are) in the same directions (respectively). I have for clearness' sake inserted the words in brackets though they are not in the original Greek, which has "joining... in the same directions" or "on the same sides," *ἐπὶ τὰ αὐτὰ μέρη ἐπιτενύνουσαι*. The expression "towards the same parts," though usage has sanctioned it, is perhaps not quite satisfactory.

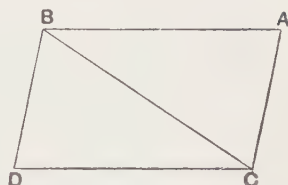
15. DC , CB and 18. DCB . The Greek has " BC , CD " and " BCD " in these places respectively. Euclid is not always careful to write in corresponding order the letters denoting corresponding points in congruent figures. On the contrary, he evidently prefers the alphabetical order, and seems to disdain to alter it for the sake of beginners or others who might be confused by it. In the case of angles alteration is perhaps unnecessary; but in the case of triangles and pairs of corresponding sides I have ventured to alter the order to that which the mathematician of to-day expects.

This proposition is, as Proclus says (p. 385, 5), the connecting link between the exposition of the theory of parallels and the investigation of parallelograms. For, while it only speaks of equal and parallel straight lines connecting those ends of equal and parallel straight lines which are in the same directions, it gives, without expressing the fact, the construction or origin of the parallelogram, so that in the next proposition Euclid is able to speak of "parallelogrammic areas" without any further explanation.

PROPOSITION 34.

In parallelogrammic areas the opposite sides and angles are equal to one another, and the diameter bisects the areas.

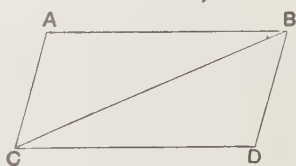
Let $ACDB$ be a parallelogrammic area and BC its diameter;



5 I say that the opposite sides and angles of the parallelogram $ACDB$ are equal to one another, and the diameter BC bisects it.

For, since AB is parallel to CD ,
and the straight line BC has fallen
10 upon them,

the alternate angles ABC , BCD
are equal to one another. [I. 29]



Again, since AC is parallel to BD ,
and BC has fallen upon them,

15 the alternate angles ACB , CBD are equal to one
another. [I. 29]

Therefore ABC , DCB are two triangles having the two
angles ABC , BCA equal to the two angles DCB , CBD
respectively, and one side equal to one side, namely that
20 adjoining the equal angles and common to both of them, BC ;

therefore they will also have the remaining sides equal
to the remaining sides respectively, and the remaining angle
to the remaining angle ; [I. 26]

therefore the side AB is equal to CD ,
25 and AC to BD ,

and further the angle BAC is equal to the angle CDB .

And, since the angle ABC is equal to the angle BCD ,
and the angle CBD to the angle ACB ,

the whole angle ABD is equal to the whole angle ACD .
[C. N. 2]

30 And the angle BAC was also proved equal to the angle CDB .

Therefore in parallelogrammic areas the opposite sides
and angles are equal to one another.

I say, next, that the diameter also bisects the areas.

For, since AB is equal to CD ,

35 and BC is common,

the two sides AB , BC are equal to the two sides DC , CB
respectively ;

and the angle ABC is equal to the angle BCD ;

therefore the base AC is also equal to DB ,

40 and the triangle ABC is equal to the triangle DCB . [I. 4]

Therefore the diameter BC bisects the parallelogram
 $ACDB$.

Q. E. D.

1. It is to be observed that, when parallelograms have to be mentioned for the first time, Euclid calls them "parallelogrammic areas" or, more exactly, "parallelogram" areas (παρλληλόγραμμα χωρία). The meaning is simply areas bounded by parallel straight lines with the further limitation placed upon the term by Euclid that only *four-sided* figures are so called, although of course there are certain regular polygons which have opposite sides parallel, and which therefore might be said to be areas bounded by parallel straight lines. We gather from Proclus (p. 393) that the word "parallelogram" was first introduced by Euclid, that its use was suggested by I. 33, and that the formation of the word παρλληλόγραμμα (parallel-lined) was analogous to that of εὐθύγραμμος (straight-lined or rectilinear).

17, 18, 40. DCB and 36. DC, CB. The Greek has in these places "BCD" and "CD, BC" respectively. Cf. note on I. 33, lines 15, 18.

After specifying the particular kinds of parallelograms (squares and rhombi) in which the diagonals bisect the angles which they join, as well as the areas, and those (rectangles and rhomboids) in which the diagonals do not bisect the angles, Proclus proceeds (pp. 390 sqq.) to analyse this proposition with reference to the distinction in Aristotle's *Anal. Post.* (I. 4, 5, 73 a 21—74 b 4) between attributes which are only predicable of every individual thing (κατὰ παντός) in a class and those which are true of it *primarily* (τούτου πρώτου) and *generally* (καθόλου). We are apt, says Aristotle, to mistake a proof κατὰ παντός for a proof τούτου πρώτου καθόλου because it is either impossible to find a higher generality to comprehend all the particulars of which the predicate is true, or to find a name for it. (Part of this passage of Aristotle has been quoted above in the note on I. 32, pp. 319—320.)

Now, says Proclus, adapting Aristotle's distinction to *theorems*, the present proposition exhibits the distinction between theorems which are *general* and theorems which are *not general*. According to Proclus, the first part of the proposition stating that the opposite sides and angles of a parallelogram are equal is *general* because the property is only true of parallelograms; but the second part which asserts that the diameter bisects the area is *not general* because it does not include all the figures of which this property is true, e.g. circles and ellipses. Indeed, says Proclus, the first attempts upon problems seem usually to have been of this partial character (μερικώτεραι), and generality was only attained by degrees. Thus "the ancients, after investigating the fact that the diameter bisects an ellipse, a circle, and a parallelogram respectively, proceeded to investigate what was common to these cases," though "it is difficult to show what is common to an ellipse, a circle and a parallelogram."

I doubt whether the supposed distinction between the two parts of the proposition, in point of "generality," can be sustained. Proclus himself admits that it is presupposed that the subject of the proposition is a *quadrilateral*, because there are other figures (e.g. regular polygons of an even number of sides) besides parallelograms which have their opposite sides and angles equal; therefore the second part of the theorem is, in this respect, no more *general* than the other, and, if we are entitled to the tacit limitation of the theorem to quadrilaterals in one part, we are equally entitled to it in the other.

It would almost appear as though Proclus had drawn the distinction for the mere purpose of alluding to investigations by Greek geometers on the general subject of *diameters* of all sorts of figures; and it may have been these which brought the subject to the point at which Apollonius could say in the first definitions at the beginning of his *Conics* that "In *any bent line*, such as is in one plane, I give the name *diameter* to any straight line which, being drawn from the bent line, bisects all the straight lines (chords) drawn in the line parallel to any straight line." The term *bent line* (κομπύλη γραμμή) includes, e.g. in Archimedes, not only curves, but any composite line made

up of straight lines and curves joined together in any manner. It is of course clear that either diagonal of a parallelogram bisects all lines drawn within the parallelogram parallel to the other diagonal.

An-Nairizi gives after I. 31 a neat construction for dividing a straight line into any number of equal parts (ed. Curtze, p. 74, ed. Besthorn-Heiberg, pp. 141—3) which requires only one measurement repeated, together with the properties of parallel lines including I. 33, 34. As I. 33, 34 are assumed, I place the problem here. The particular case taken is the problem of dividing a straight line into *three* equal parts.

Let AB be the given straight line. Draw AC , BD at right angles to it on opposite sides.

An-Nairizi takes AC , BD of the same length and then bisects AC at E and BD at F . But of course it is even simpler to measure AE , EC along one perpendicular equal and of any length, and BF , FD along the other also equal and of the same length.

Join ED , CF meeting AB in G , H respectively.

Then shall AG , GH , HB all be equal.

Draw HK parallel to AC , or at right angles to AB .

Since now EC , FD are equal and parallel, ED , CF are equal and parallel. [I. 33]

And HK was drawn parallel to AC .

Therefore $ECHK$ is a parallelogram; whence KH is equal as well as parallel to EC , and therefore to EA .

The triangles EAG , KHG have now two angles respectively equal and the sides AE , HK equal.

Thus the triangles are equal in all respects, and

AG is equal to GH .

Similarly the triangles KHG , FBH are equal in all respects, and

GH is equal to HB .

If now we wish to extend the problem to the case where AB is to be divided into n parts, we have only to measure $(n-1)$ successive equal lengths along AC and $(n-1)$ successive lengths, each equal to the others, along BD . Then join the first point arrived at on AC to the last point on BD , the second on AC to the last but one on BD , and so on; and the joining lines cut AB in points dividing it into n equal parts.

PROPOSITION 35.

Parallelograms which are on the same base and in the same parallels are equal to one another.

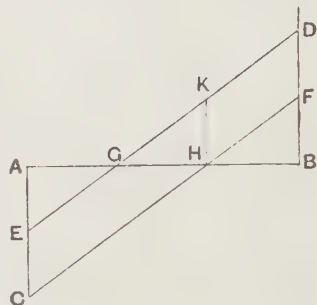
Let $ABCD$, $EBCF$ be parallelograms on the same base BC and in the same parallels AF , BC ;

I say that $ABCD$ is equal to the parallelogram $EBCF$.

For, since $ABCD$ is a parallelogram,

AD is equal to BC .

[I. 34]



For the same reason also

EF is equal to BC ,

so that AD is also equal to EF ; [C. N. 1]

and DE is common;

therefore the whole AE is equal to the whole DF . [C. N. 2]

But AB is also equal to DC ; [I. 34]

therefore the two sides EA , AB are equal to the two sides
15 FD , DC respectively,

and the angle FDC is equal to
the angle EAB , the exterior to the
interior; [I. 29]

therefore the base EB is equal
20 to the base FC ,

and the triangle EAB will be equal to the triangle FDC . [I. 4]

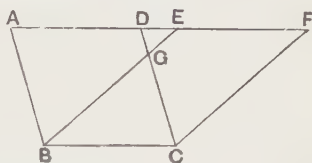
Let DGE be subtracted from each;

therefore the trapezium $ABGD$ which remains is equal to
the trapezium $EGCF$ which remains. [C. N. 3]

25 Let the triangle GBC be added to each;

therefore the whole parallelogram $ABCD$ is equal to the whole
parallelogram $EBCF$. [C. N. 2]

Therefore etc.



Q. E. D.

21. **FDC.** The text has " DFC ."

22. **Let DGE be subtracted.** Euclid speaks of the triangle DGE without any explanation that, in the case which he takes (where AD , EF have no point in common), BE , CD must meet at a point G between the two parallels. He allows this to appear from the figure simply.

Equality in a new sense.

It is important to observe that we are in this proposition introduced for the first time to a new conception of equality between figures. Hitherto we have had equality in the sense of *congruence* only, as applied to straight lines, angles, and even triangles (cf. I. 4). Now, without any explicit reference to any change in the meaning of the term, figures are inferred to be *equal* which are equal in *area* or in *content* but need not be of the same *form*. No *definition* of equality is anywhere given by Euclid; we are left to infer its meaning from the few *axioms* about "equal things." It will be observed that in the above proof the "equality" of two parallelograms on the same base and between the same parallels is inferred by the successive steps (1) of subtracting one and the same area (the triangle DGE) from two areas equal in the sense of *congruence* (the triangles AEB , DFC), and inferring that the remainders (the trapezia $ABGD$, $EGCF$) are "equal"; (2) of adding one and

the same area (the triangle GBC) to each of the latter "equal" trapezia, and inferring the equality of the respective sums (the two given parallelograms).

As is well known, Simson (after Clairaut) slightly altered the proof in order to make it applicable to all the three possible cases. The alteration substituted *one* step of subtracting congruent areas (the triangles AEB , DFC) from one and the same area (the trapezium $ABCF$) for the *two* steps above shown of first subtracting and then adding a certain area.

While, in either case, nothing more is explicitly used than the axioms that, *if equals be added to equals, the wholes are equal* and that, *if equals be subtracted from equals, the remainders are equal*, there is the further *tacit* assumption that it is indifferent to *what part* or from *what part* of the same or equal areas the same or equal areas are added or subtracted. De Morgan observes that the postulate "an area taken from an area leaves the same area from whatever part it may be taken" is particularly important as the key to equality of non-rectilinear areas which could not be cut into coincidence geometrically.

Legendre introduced the word *equivalent* to express this wider sense of equality, restricting the term *equal* to things equal in the sense of congruent; and this distinction has been found convenient.

I do not think it necessary, nor have I the space, to give any account of the recent developments of the theory of equivalence on new lines represented by the researches of W. Bolyai, Duhamel, De Zolt, Stolz, Schur, Veronese, Hilbert and others, and must refer the reader to Ugo Amaldi's article *Sulla teoria dell' equivalenza in Questioni riguardanti le matematiche elementari*, 1. (Bologna, 1912), pp. 145—198, and to Max Simon, *Über die Entwicklung der Elementar-geometrie im XIX. Jahrhundert* (Leipzig, 1906), pp. 115—120, with their full references to the literature of the subject. I may however refer to the suggestive distinction of phraseology used by Hilbert (*Grundlagen der Geometrie*, pp. 39, 40):

(1) "Two polygons are called *divisibly-equal* (*zerlegungsgleich*) if they can be divided into a *finite* number of triangles which are congruent two and two."

(2) "Two polygons are called *equal in content* (*inhaltsgleich*) or *of equal content* if it is possible to add *divisibly-equal* polygons to them in such a way that the two combined polygons are *divisibly-equal*."

(Amaldi suggests as alternatives for the terms in (1) and (2) the expressions *equivalent by sum* and *equivalent by difference* respectively.)

From these definitions it follows that "by combining *divisibly-equal* polygons we again arrive at *divisibly-equal* polygons; and, if we subtract *divisibly-equal* polygons from *divisibly-equal* polygons, the polygons remaining are *equal in content*."

The proposition also follows without difficulty that, "if two polygons are *divisibly-equal* to a third polygon, they are also *divisibly-equal* to one another; and, if two polygons are *equal in content* to a third polygon, they are *equal in content* to one another."

The different cases.

As usual, Proclus (pp. 399—400), observing that Euclid has given only the most difficult of the three possible cases, adds the other two with separate proofs. In the case where E in the figure of the proposition falls between A and D , he *adds* the congruent triangles ABE , DCF respectively to the smaller trapezium $EBCD$, instead of subtracting them (as Simson does) from the larger trapezium $ABCF$.

An ancient "Budget of Paradoxes."

Proclus observes (p. 396, 12 sqq.) that the present theorem and the similar one relating to triangles are among the so-called paradoxical theorems of mathematics, since the uninstructed might well regard it as impossible that the area of the parallelograms should remain the same while the length of the sides other than the base and the side opposite to it may increase indefinitely. He adds that mathematicians had made a collection of such paradoxes, the so-called *treasury of paradoxes* (ὁ παράδοξος τόπος)—cf. the similar expressions τόπος ἀναλυόμενος (treasury of analysis) and τόπος ἀστρονομούμενος—in the same way as the Stoics with their *illustrations* (ὥσπερ οἱ ἀπὸ τῆς Στοᾶς ἐπὶ τῶν δειγμάτων). It may be that this *treasury of paradoxes* was the work of Erycinus quoted by Pappus (III. p. 107, 8) and mentioned above (note on I. 21, p. 290).

Locus-theorems and loci in Greek geometry.

The proposition I. 35 is, says Proclus (pp. 394—6), the first *locus-theorem* (τοπικὸν θεωρήμα) given by Euclid. Accordingly it is in his note on this proposition that Proclus gives us his view of the nature of a locus-theorem and of the meaning of the word *locus* (τόπος); and great importance attaches to his words because he is one of the three writers (Pappus and Eutocius being the two others) upon whom we have to rely for all that is known of the Greek conception of geometrical loci.

Proclus' explanation (pp. 394, 15—395, 2) is as follows. "I call those (theorems) *locus-theorems* (τοπικά) in which the same property is found to exist on the whole of some locus (πρὸς ὅλῳ τινὶ τόπῳ), and (I call) a locus a position of a line or a surface producing one and the same property (γραμμῆς ἢ ἐπιφανείας θέσιν ποιούσαν ἓν καὶ ταὐτὸν σύμπτωμα). For, of locus-theorems, some are constructed on lines and others on surfaces (τῶν γὰρ τοπικῶν τὰ μὲν ἐστὶ πρὸς γραμμαῖς συνιστάμενα, τὰ δὲ πρὸς ἐπιφανείαις). And, since some lines are plane (ἐπιπέδοι) and others solid (στερεαί)—those being plane which are simply conceived of in a plane (ὧν ἐν ἐπιπέδῳ ἀπλῇ ἡ νόησις), and those solid the origin of which is revealed from some section of a solid figure, as the cylindrical helix and the conic lines (ὡς τῆς κυλινδρικῆς ἑλίκος καὶ τῶν κωνικῶν γραμμῶν)—I should say (φαίην ἄν) further that, of locus-theorems on lines, some give a plane locus and others a solid locus."

Leaving out of sight for the moment the class of *loci on surfaces*, we find that the distinction between *plane* and *solid loci*, or *plane* and *solid lines*, was similarly understood by Eutocius, who says (Apollonius, ed. Heiberg, II. p. 184) that "*solid loci* have obtained their name from the fact that the lines used in the solution of problems regarding them have their origin in the section of solids, for example the sections of the cone and several others." Similarly we gather from Pappus that *plane loci* were straight lines and circles, and *solid loci* were conics. Thus he tells us (VII. p. 672, 20) that Aristaeus wrote five books of *Solid Loci* "supplementary to (literally, continuous with) the conics"; and, though Hultsch brackets the passage (VII. p. 662, 10—15) which says plainly that *plane loci* are straight lines and circles, while *solid loci* are sections of cones, i.e. parabolas, ellipses and hyperbolas, we have the exactly corresponding distinction drawn by Pappus (III. p. 54, 7—16) between *plane* and *solid problems*, plane problems being those solved by means of straight lines and circumferences of circles, and solid problems those solved by means of one or more of the sections of the cone. But, whereas Proclus

and Eutocius speak of other *solid loci* besides conics, there is nothing in Pappus to support the wider application of the term. According to Pappus (III. p. 54, 16—21) problems which could not be solved by means of straight lines, circles, or conics were *linear* (γραμμικά) because they used for their construction lines having a more complicated and unnatural origin than those mentioned, namely such curves as *quadratrices*, *conchoids* and *cissoids*. Similarly, in the passage supposed to be interpolated, *linear loci* are distinguished as those which are neither straight lines nor circles nor any of the conic sections (VII. p. 662, 13—15). Thus the classification given by Proclus and Eutocius is less precise than that which we find in Pappus; and the inclusion by Proclus of the cylindrical helix among solid loci, on the ground that it arises from a section of a solid figure, would seem to be, in any case, due to some misapprehension.

Comparing these passages and the hints in Pappus about *loci on surfaces* (τόποι πρὸς ἐπιφανείᾳ) with special reference to Euclid's two books under that title, Heiberg concludes that *loci on lines* and *loci on surfaces* in Proclus' explanation are loci which *are* lines and loci which *are* surfaces respectively. But some qualification is necessary as regards Proclus' conception of *loci on lines*, because he goes on to say (p. 395, 5), with reference to this proposition, that, while the locus is a *locus on lines* and moreover *plane*, it is "the whole space between the parallels" which is the locus of the various parallelograms on the same base proved to be equal in area. Similarly, when he quotes III. 21 about the equality of the angles in the same segment and III. 31 about the right angle in a semicircle as cases where a circumference of a circle takes the place of a straight line in a *plane* locus-theorem, he appears to imply that it is the segment or semicircle as an *area* which is regarded as the locus of an infinite number of *triangles* with the same base and equal vertical angles, rather than that it is the *circumference* which is the locus of the angular *points*. Likewise he gives the equality of parallelograms inscribed in "the asymptotes and the hyperbola" as an example of a *solid* locus-theorem, as if the area included between the curve and its asymptotes was regarded as the *locus* of the equal parallelograms. However this may be, it is clear that the locus in the present proposition can only be either (1) a *line*-locus of a *line*, not a point, or (2) an *area*-locus of an *area*, not a point or a line; and we seem to be thus brought to another and different classification of loci corresponding to that quoted by Pappus (VII. p. 660, 18 sqq.) from the preliminary exposition given by Apollonius in his *Plane Loci*. According to this, loci in general are of three kinds: (1) ἐφεκτικοί, *holding-in*, in which sense the locus of a point is a point, of a line a line, of a surface a surface, and of a solid a solid, (2) διεξοδικοί, *moving along*, a line being in this sense a locus of a point, a surface of a line and a solid of a surface, (3) ἀναστροφικοί, where a surface is a locus of a point and a solid of a line. Thus the locus in this proposition, whether it is the space between the two parallels regarded as the locus of the equal parallelograms, or the line parallel to the base regarded as the locus of the sides opposite to the base, would seem to be of the first class (ἐφεκτικός); and, as Proclus takes the former view of it, a *locus on lines* is apparently not merely a locus which *is* a line but a locus *bounded by lines* also, the locus being *plane* in the particular case because it is bounded by straight lines, or, in the case of III. 21, 31, by straight lines and circles, but not by any higher curves.

Proclus notes lastly (p. 395, 13—21) that, according to Geminus, "Chrysippus likened locus-theorems to the *ideas*. For, as the ideas confine

the genesis of unlimited (particulars) within defined limits, so in such theorems the unlimited (particular figures) are confined within defined *places* or *loci* (τόποι). And it is this boundary which is the cause of the equality; for the height of the parallels, which remains the same, while an infinite number of parallelograms are conceived on the same base, is what makes them all equal to one another."

PROPOSITION 36.

Parallelograms which are on equal bases and in the same parallels are equal to one another.

Let $ABCD$, $EFGH$ be parallelograms which are on equal bases BC , FG and in the same parallels AH , BG ;



I say that the parallelogram $ABCD$ is equal to $EFGH$.

For let BE , CH be joined.

Then, since BC is equal to FG ,

while FG is equal to EH ,

BC is also equal to EH .

[C. N. 1]

But they are also parallel.

And EB , HC join them;

but straight lines joining equal and parallel straight lines (at the extremities which are) in the same directions (respectively) are equal and parallel.

[I. 33]

Therefore $EBCH$ is a parallelogram.

[I. 34]

And it is equal to $ABCD$;

for it has the same base BC with it, and is in the same parallels BC , AH with it.

[I. 35]

For the same reason also $EFGH$ is equal to the same $EBCH$;

[I. 35]

so that the parallelogram $ABCD$ is also equal to $EFGH$.

[C. N. 1]

Therefore etc.

Q. E. D.

PROPOSITION 37.

Triangles which are on the same base and in the same parallels are equal to one another.

Let ABC , DBC be triangles on the same base BC and in the same parallels AD , BC ;

I say that the triangle ABC is equal to the triangle DBC .

Let AD be produced in both directions to E , F ;

through B let BE be drawn parallel to CA ,

[I. 31]

and through C let CF be drawn parallel to BD .

[I. 31]

Then each of the figures $EBCA$, $DBCF$ is a parallelogram; and they are equal,

for they are on the same base BC and in the same parallels BC , EF .

[I. 35]

Moreover the triangle ABC is half of the parallelogram $EBCA$; for the diameter AB bisects it.

[I. 34]

And the triangle DBC is half of the parallelogram $DBCF$;

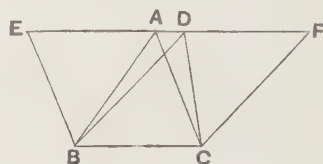
for the diameter DC bisects it.

[I. 34]

[But the halves of equal things are equal to one another.]

Therefore the triangle ABC is equal to the triangle DBC . Therefore etc.

Q. E. D.



21. Here and in the next proposition Heiberg brackets the words "But the halves of equal things are equal to one another" on the ground that, since the *Common Notion* which asserted this fact was interpolated at a very early date (before the time of Theon), it is probable that the words here were interpolated at the same time. Cf. note above (p. 224) on the interpolated *Common Notion*.

There is a lacuna in the text of Proclus' notes to I. 36 and I. 37. Apparently the end of the former and the beginning of the latter are missing, the mss. and the *editio princeps* showing no separate note for I. 37 and no lacuna, but going straight on without regard to sense. Proclus had evidently remarked again in the missing passage that, in the case of both parallelograms and triangles between the same parallels, the two sides which stretch from one parallel to the other may increase in length to any extent, while the area remains the same. Thus the *perimeter* in parallelograms or triangles is of itself no criterion as to their area. Misconception on this subject was rife among non-mathematicians; and Proclus (p. 403, 5 sqq.) tells us (1) of describers of countries (*χωρογράφοι*) who drew conclusions regarding the size of cities from their perimeters, and (2) of certain members of communistic

societies in his own time who cheated their fellow members by giving them land of greater perimeter but less area than they took themselves, so that, on the one hand, they got a reputation for greater honesty while, on the other, they took more than their share of produce. Cantor (*Gesch. d. Math.* 13, p. 172) quotes several remarks of ancient authors which show the prevalence of the same misconception. Thus Thucydides estimates the size of Sicily according to the time required for circumnavigating it. About 130 B.C. Polybius said that there were people who could not understand that camps of the same periphery might have different capacities. Quintilian has a similar remark, and Cantor thinks he may have had in his mind the calculations of Pliny, who compares the size of different parts of the earth by adding their length to their breadth.

The comparison however of the areas of different figures of equal contour had not been neglected by mathematicians. Theon of Alexandria, in his commentary on Book I. of Ptolemy's *Syntaxis*, has preserved a number of propositions on the subject taken from a treatise by Zenodorus *περὶ ἰσομέτρων σχημάτων* (reproduced in Latin on pp. 1190—1211 of Hultsch's edition of Pappus) which was written at some date between, say, 200 B.C. and 90 A.D., and probably not long after the former date. Pappus too has at the beginning of Book v. of his *Collection* (pp. 308 sqq.) the same propositions, in which he appears to have followed Zenodorus pretty closely while making some changes in detail. The propositions proved by Zenodorus and Pappus include the following: (1) that, *of all polygons of the same number of sides and equal perimeter, the equilateral and equiangular polygon is the greatest in area*, (2) that, *of regular polygons of equal perimeter, that is the greatest in area which has the most angles*, (3) that *a circle is greater than any regular polygon of equal contour*, (4) that, *of all circular segments in which the arcs are equal in length, the semicircle is the greatest*. The treatise of Zenodorus was not confined to propositions about plane figures, but gave also the theorem that, *of all solid figures the surfaces of which are equal, the sphere is the greatest in volume*.

PROPOSITION 38.

Triangles which are on equal bases and in the same parallels are equal to one another.

Let ABC , DEF be triangles on equal bases BC , EF and in the same parallels BF , AD ;

I say that the triangle ABC is equal to the triangle DEF .

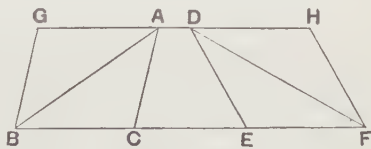
For let AD be produced in both directions to G , H ;

through B let BG be drawn parallel to CA , [I. 31]

and through F let FH be drawn parallel to DE .

Then each of the figures $GBCA$, $DEFH$ is a parallelogram;

and $GBCA$ is equal to $DEFH$;



for they are on equal bases BC , EF and in the same parallels BF , GH . [I. 36]

Moreover the triangle ABC is half of the parallelogram $GBCA$; for the diameter AB bisects it. [I. 34]

And the triangle FED is half of the parallelogram $DEFH$; for the diameter DF bisects it. [I. 34]

[But the halves of equal things are equal to one another.]

Therefore the triangle ABC is equal to the triangle DEF .
Therefore etc.

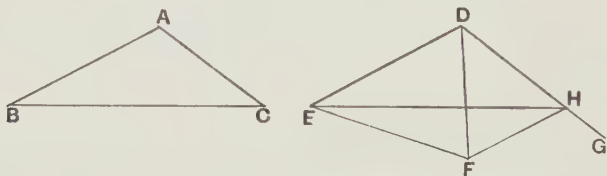
Q. E. D.

On this proposition Proclus remarks (pp. 405—6) that Euclid seems to him to have given in VI. 1 one proof including all the four theorems from I. 35 to I. 38, and that most people had failed to notice this. When Euclid, he says, proves that triangles and parallelograms of the same altitude have to one another the same ratio as their bases, he simply proves all these propositions more generally by the use of proportion; for of course to be of the same altitude is equivalent to being in the same parallels. It is true that VI. 1 generalises these propositions, but it must be observed that it does not prove the propositions themselves, as Proclus seems to imply; they are in fact assumed in order to prove VI. 1.

Comparison of areas of triangles of I. 24.

The theorem already mentioned as given by Proclus on I. 24 (pp. 340—4) is placed here by Heron, who also enunciates it more clearly (an-Nairizī, ed. Besthorn-Heiberg, pp. 155—161, ed. Curtze, pp. 75—8).

If in two triangles two sides of the one be equal to two sides of the other respectively, and the angle of the one be greater than the angle of the other, namely the angles contained by the equal sides, then, (1) if the sum of the two angles contained by the equal sides is equal to two right angles, the two triangles are equal to one another; (2) if less than two right angles, the triangle which has the greater angle is also itself greater than the other; (3) if greater than two right angles, the triangle which has the less angle is greater than the other triangle.



Let two triangles ABC , DEF have the sides AB , AC respectively equal to DE , DF .

(1) First, suppose that the angles at A and D in the triangles ABC , DEF are together equal to two right angles.

Heron's construction is now as follows.

Make the angle EDG equal to the angle BAC .

Draw FH parallel to ED meeting DG in H .

Join EH .

Then, since the angles BAC , EDF are equal to two right angles, the angles EDH , EDF are equal to two right angles.

But so are the angles EDH , DHF .

Therefore the angles EDF , DHF are equal.

And the alternate angles EDF , DFH are equal.

[I. 29]

Therefore the angles DHF , DFH are equal,

and DF is equal to DH .

[I. 6]

Hence the two sides ED , DH are equal to the two sides BA , AC ; and the included angles are equal.

Therefore the triangles ABC , DEH are equal in all respects.

And the triangles DEF , DEH between the same parallels are equal.

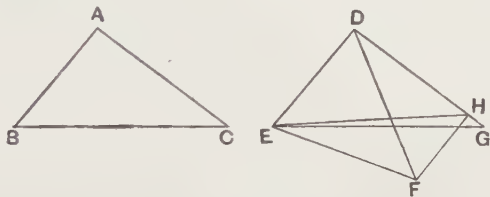
[I. 37]

Therefore the triangles ABC , DEF are equal.

[Proclus takes the construction of Eucl. I. 24, i.e., he makes DH equal to DF and then proves that ED , FH are parallel.]

(2) Suppose the angles BAC , EDF together less than two right angles.

As before, make the angle EDG equal to the angle BAC , draw FH parallel to ED , and join EH .



In this case the angles EDH , EDF are together less than two right angles, while the angles EDH , DHF are equal to two right angles. [I. 29]

Hence the angle EDF , and therefore the angle DFH , is less than the angle DHF .

Therefore DH is less than DF .

[I. 19]

Produce DH to G so that DG is equal to DF or AC , and join EG .

Then the triangle DEG , which is equal to the triangle ABC , is greater than the triangle DEH , and therefore greater than the triangle DEF .

(3) Suppose the angles BAC , EDF together greater than two right angles.



We make the same construction in this case, and we prove in like manner that the angle DHF is less than the angle DFH ,

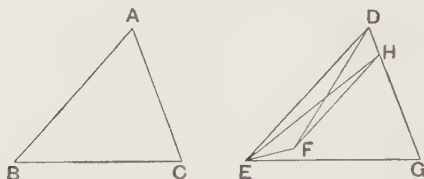
whence DH is greater than DF or AC .

Make DG equal to AC , and join EG .

It then follows that the triangle DEG is greater than the triangle ABC .

[In the second and third cases again Proclus starts from the construction in I. 24, and proves, in the second case, that the parallel, FH , to ED cuts DG and, in the third case, that it cuts DG produced.]

There is no necessity for Heron to take account of the position of F in relation to the side opposite D . For in the first and third cases F *must* fall



in the position in which Euclid draws it in I. 24, whatever be the relative lengths of AB , AC . In the second case the figure *may* be as annexed, but the proof is the same, or rather the case needs no proof at all.

PROPOSITION 39.

Equal triangles which are on the same base and on the same side are also in the same parallels.

Let ABC , DBC be equal triangles which are on the same base BC and on the same side of it ;

5 [I say that they are also in the same parallels.]

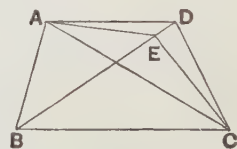
And [For] let AD be joined ;

I say that AD is parallel to BC .

For, if not, let AE be drawn through the point A parallel to the straight line
10 BC ,

[I. 31]

and let EC be joined.



Therefore the triangle ABC is equal to the triangle EBC ;

for it is on the same base BC with it and in the same
15 parallels.

[I. 37]

But ABC is equal to DBC ;

therefore DBC is also equal to EBC ,

[C. N. 1]

the greater to the less : which is impossible.

Therefore AE is not parallel to BC .

20 Similarly we can prove that neither is any other straight line except AD ;

therefore AD is parallel to BC .

Therefore etc.

5. [I say that they are also in the same parallels.] Heiberg has proved (*Hermes*, XXXVIII., 1903, p. 50) from a recently discovered papyrus-fragment (*Fayūm towns and their papyri*, p. 96, No. 1X.) that these words are an interpolation by some one who did not observe that the words "And let AD be joined" are part of the *setting-out* ($\epsilon\kappa\theta\epsilon\sigma\iota\varsigma$), but took them as belonging to the *construction* ($\kappa\alpha\tau\alpha\sigma\kappa\epsilon\nu\eta$) and consequently thought that a $\delta\iota\omicron\rho\iota\sigma\mu\omicron\varsigma$ or "definition" (of the thing to be proved) should precede. The interpolator then altered "And" into "For" in the next sentence.

This theorem is of course the *partial* converse of I. 37. In I. 37 we have triangles which are (1) on the same base, (2) in the same parallels, and the theorem proves (3) that the triangles are equal. Here the hypothesis (1) and the conclusion (3) are combined as hypotheses, and the conclusion is the hypothesis (2) of I. 37, that the triangles are in the same parallels. The additional qualification in this proposition that the triangles must be *on the same side* of the base is necessary because it is not, as in I. 37, involved in the other hypotheses.

Proclus (p. 407, 4—17) remarks that Euclid only converts I. 37 and I. 38 relative to triangles, and omits the converses of I. 35, 36 about parallelograms as unnecessary because it is easy to see that the method would be the same, and therefore the reader may properly be left to prove them for himself.

The proof is, as Proclus points out (p. 408, 5—21), equally easy on the supposition that the assumed parallel AE meets BD or CD produced beyond D .

[PROPOSITION 40.

Equal triangles which are on equal bases and on the same side are also in the same parallels.

Let ABC , CDE be equal triangles on equal bases BC , CE and on the same side.

I say that they are also in the same parallels.

For let AD be joined;

I say that AD is parallel to BE .

For, if not, let AF be drawn through A parallel to BE [I. 31], and let FE be joined.

Therefore the triangle ABC is equal to the triangle FCE ;

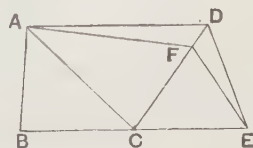
for they are on equal bases BC , CE and in the same parallels BE , AF . [I. 38]

But the triangle ABC is equal to the triangle DCE ;

therefore the triangle DCE is also equal to the triangle FCE , [C. N. 1]

the greater to the less: which is impossible.

Therefore AF is not parallel to BE .



Similarly we can prove that neither is any other straight line except AD ;

therefore AD is parallel to BE .

Therefore etc.

Q. E. D.]

Heiberg has proved by means of the papyrus-fragment mentioned in the last note that this proposition is an interpolation by some one who thought that there should be a proposition following I. 39 and related to it in the same way as I. 38 is related to I. 37, and I. 36 to I. 35.

PROPOSITION 41.

If a parallelogram have the same base with a triangle and be in the same parallels, the parallelogram is double of the triangle.

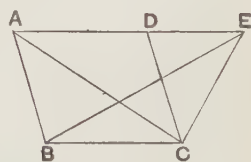
For let the parallelogram $ABCD$ have the same base BC with the triangle EBC , and let it be in the same parallels BC , AE ;

I say that the parallelogram $ABCD$ is double of the triangle BEC .

For let AC be joined.

Then the triangle ABC is equal to the triangle EBC ;

for it is on the same base BC with it and in the same parallels BC , AE .



[I. 37]

But the parallelogram $ABCD$ is double of the triangle ABC ;

for the diameter AC bisects it; [I. 34]

so that the parallelogram $ABCD$ is also double of the triangle EBC .

Therefore etc.

Q. E. D.

On this proposition Proclus (pp. 414, 15—415, 16), “by way of practice” (*γυμνασίας ἕνεκα*), considers the area of a *trapezium* (a quadrilateral with only one pair of opposite sides parallel) in comparison with that of the triangles in the same parallels and having the greater and less of the parallel sides of the trapezium for bases respectively, and proves that the trapezium is less than double of the former triangle and more than double of the latter.

He next (pp. 415, 22—416, 14) proves the proposition that,

If a triangle be formed by joining the middle point of either of the non-parallel sides to the extremities of the opposite side, the area of the trapezium is always double of that of the triangle.

Let $ABCD$ be a trapezium in which AD , BC are the parallel sides, and E the middle point of one of the non-parallel sides, say DC .

Join EA , EB and produce BE to meet AD produced in F .

Then the triangles BEC , FED have two angles equal respectively, and one side CE equal to one side DE ;

therefore the triangles are equal in all respects.

[1. 26]

Add to each the quadrilateral $ABED$;

therefore the trapezium $ABCD$ is equal to the triangle ABF ,

that is, to twice the triangle AEB , since BE is equal to EF . [1. 38]

The three properties proved by Proclus may be combined in one enunciation thus:

If a triangle be formed by joining the middle point of one side of a trapezium to the extremities of the opposite side, the area of the trapezium is (1) greater than, (2) equal to, or (3) less than, double the area of the triangle according as the side the middle point of which is taken is (1) the greater of the parallel sides, (2) either of the non-parallel sides, or (3) the lesser of the parallel sides.

PROPOSITION 42.

To construct, in a given rectilinear angle, a parallelogram equal to a given triangle.

Let ABC be the given triangle, and D the given rectilinear angle;

thus it is required to construct in the rectilinear angle D a parallelogram equal to the triangle ABC .

Let BC be bisected at E , and let AE be joined;

on the straight line EC , and at the point E on it, let the angle CEF be constructed equal to the angle D ;

[1. 23]

through A let AG be drawn parallel to EC , and

[1. 31]

through C let CG be drawn parallel to EF .

Then $FECG$ is a parallelogram.

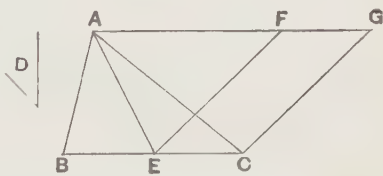
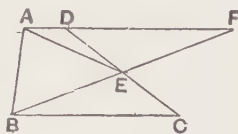
And, since BE is equal to EC ,

the triangle ABE is also equal to the triangle AEC ,

for they are on equal bases BE , EC and in the same parallels BC , AG ;

[1. 38]

therefore the triangle ABC is double of the triangle AEC .



But the parallelogram $FECG$ is also double of the triangle AEC , for it has the same base with it and is in the same parallels with it; [I. 41]

therefore the parallelogram $FECG$ is equal to the triangle ABC .

And it has the angle CEF equal to the given angle D .

Therefore the parallelogram $FECG$ has been constructed equal to the given triangle ABC , in the angle CEF which is equal to D . Q. E. F.

PROPOSITION 43.

In any parallelogram the complements of the parallelograms about the diameter are equal to one another.

Let $ABCD$ be a parallelogram, and AC its diameter; and about AC let EH , FG be parallelograms, and BK , KD the so-called complements;

I say that the complement BK is equal to the complement KD .

For, since $ABCD$ is a parallelogram, and AC its diameter,

the triangle ABC is equal to the triangle ACD . [I. 34]

Again, since EH is a parallelogram, and AK is its diameter,

the triangle AEK is equal to the triangle AHK .

For the same reason

the triangle KFC is also equal to KGC .

Now, since the triangle AEK is equal to the triangle AHK ,

and KFC to KGC ,

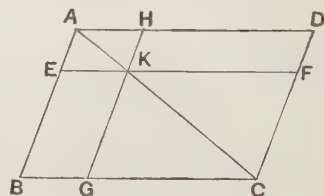
the triangle AEK together with KGC is equal to the triangle AHK together with KFC . [C. N. 2]

And the whole triangle ABC is also equal to the whole ADC ;

therefore the complement BK which remains is equal to the complement KD which remains. [C. N. 3]

Therefore etc.

Q. E. D.



1. complements, *παραπληρώματα*, the figures put in to fill up (interstices).

4. and about AC.... Euclid's phraseology here and in the next proposition implies that the complements as well as the other parallelograms are "about" the diagonal. The words are here *περὶ δὲ τὴν ΑΙ' παραλληλόγραμμα μὲν ἔστω τὰ ΕΘ, ΖΗ, τὰ δὲ λεγόμενα παραπληρώματα τὰ ΒΚ, ΚΔ*. The expression "the so-called complements," indicates that this technical use of *παραπληρώματα* was not new, though it might not be universally known.

In the text of Proclus' commentary as we have it, the end of the note on I. 41, the whole of that on I. 42, and the beginning of that on I. 43 are missing.

Proclus remarks (p. 418, 15—20) that Euclid did not need to give a formal definition of *complement* because the name was simply suggested by the facts; when once we have the two "parallelograms about the diameter," the *complements* are necessarily the areas remaining over on each side of the diameter, which fill up the complete parallelogram. Thus (p. 417, 1 sqq.) the complements need not be parallelograms. They are so if the two "parallelograms about the diameter" are formed by straight lines drawn through *one point* of the diameter parallel to the sides of the original parallelogram, but not otherwise. If, as in the first of the accompanying figures, the parallelograms have no common point, the complements are five-sided figures as shown. When the parallelograms overlap, as in the second figure, Proclus regards the complements as being the small parallelograms *FG, EH*. But, if complements are strictly the areas required to fill up the original parallelogram, Proclus is inaccurate in describing *FG, EH* as the complements. The complements are really (1) the parallelogram *FG* minus the triangle *LMN*, and (2) the parallelogram *EH* minus the triangle *KMN*, respectively; the possibility that the respective differences may be negative merely means the possibility that the sum of the two parallelograms about the diameter may be together greater than the original parallelogram.

In all the cases it is easy to show, as Proclus does, that the complements are still equal.

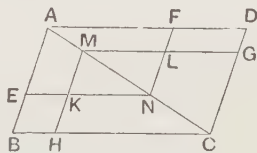
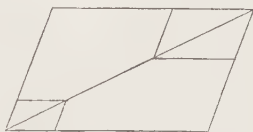
PROPOSITION 44.

To a given straight line to apply, in a given rectilineal angle, a parallelogram equal to a given triangle.

Let *AB* be the given straight line, *C* the given triangle and *D* the given rectilineal angle;

5 thus it is required to apply to the given straight line *AB*, in an angle equal to the angle *D*, a parallelogram equal to the given triangle *C*.

Let the parallelogram *BEFG* be constructed equal to the triangle *C*, in the angle *EBG* which is equal to *D* [I. 42];
10 let it be placed so that *BE* is in a straight line with *AB*; let



obtained, the transformation of a parallelogram of any shape into another with the same angle and of equal area but with one side of any given length, e.g. a *unit* length, and (2) of the simplicity of the means employed, namely the mere application of the property that the complements of the "parallelograms about the diameter" of a parallelogram are equal. The marvellous ingenuity of the solution is indeed worthy of the "godlike men of old," as Proclus calls the discoverers of the method of "application of areas"; and there would seem to be no reason to doubt that the particular solution, like the whole theory, was Pythagorean, and not a new solution due to Euclid himself.

Application of areas.

On this proposition Proclus gives (pp. 419, 15—420, 23) a valuable note on the method of "application of areas" here introduced, which was one of the most powerful methods on which Greek geometry relied. The note runs as follows:

"These things, says Eudemus (οἱ περὶ τὸν Εὐδήμον), are ancient and are discoveries of the Muse of the Pythagoreans, I mean the *application of areas* (παραβολή τῶν χωρίων), their *exceeding* (ὑπερβολή) and their *falling-short* (ἔλλειψις). It was from the Pythagoreans that later geometers [i.e. Apollonius] took the names, which they again transferred to the so-called *conic* lines, designating one of these a *parabola* (application), another a *hyperbola* (exceeding) and another an *ellipse* (falling-short), whereas those godlike men of old saw the things signified by these names in the construction, in a plane, of areas upon a finite straight line. For, when you have a straight line set out and lay the given area exactly alongside the whole of the straight line, then they say that you *apply* (παραβάλλειν) the said area; when however you make the length of the area greater than the straight line itself, it is said to *exceed* (ὑπερβάλλειν), and when you make it less, in which case, after the area has been drawn, there is some part of the straight line extending beyond it, it is said to *fall short* (ἐλλείπειν). Euclid too, in the sixth book, speaks in this way both of *exceeding* and *falling-short*; but in this place he needed the *application* simply, as he sought to apply to a given straight line an area equal to a given triangle in order that we might have in our power, not only the *construction* (σύστασις) of a parallelogram equal to a given triangle, but also the *application* of it to a finite straight line. For example, given a triangle with an area of 12 feet, and a straight line set out the length of which is 4 feet, we apply to the straight line the area equal to the triangle if we take the whole length of 4 feet and find how many feet the breadth must be in order that the parallelogram may be equal to the triangle. In the particular case, if we find a breadth of 3 feet and multiply the length into the breadth, supposing that the angle set out is a right angle, we shall have the area. Such then is the *application* handed down from early times by the Pythagoreans."

Other passages to a similar effect are quoted from Plutarch. (1) "Pythagoras sacrificed an ox on the strength of his proposition (διάγραμμα) as Apollodotus (?-rus) says...whether it was the theorem of the hypotenuse, viz. that the square on it is equal to the squares on the sides containing the right angle, or the problem about the *application of an area*." (*Non posse suaviter vivi secundum Epicurum*, c. 11.) (2) "Among the most geometrical theorems, or rather problems, is the following: given two figures, to *apply* a third equal to the one and similar to the other, on the strength of which discovery they say moreover that Pythagoras sacrificed. This is indeed unquestionably more subtle and more scientific than the theorem which

demonstrated that the square on the hypotenuse is equal to the squares on the sides about the right angle" (*Symp.* viii. 2, 4).

The story of the sacrifice must (as noted by Bretschneider and Hankel) be given up as inconsistent with Pythagorean ritual, which forbade such sacrifices; but there is no reason to doubt that the first distinct formulation and introduction into Greek geometry of the method of *application of areas* was due to the Pythagoreans. The complete exposition of the *application* of areas, their *exceeding* and their *falling-short*, and of the construction of a rectilineal figure equal to one given figure and similar to another, takes us into the sixth Book of Euclid; but it will be convenient to note here the general features of the theory of *application, exceeding and falling-short*.

The simple *application* of a parallelogram of given area to a given straight line as one of its sides is what we have in I. 44 and 45; the general form of the problem with regard to *exceeding* and *falling-short* may be stated thus:

"To apply to a given straight line a rectangle (or, more generally, a parallelogram) equal to a given rectilineal figure and (1) *exceeding* or (2) *falling-short* by a square (or, in the more general case, a parallelogram similar to a given parallelogram)."

What is meant by saying that the applied parallelogram (1) *exceeds* or (2) *falls short* is that, while its base coincides and is coterminous *at one end* with the straight line, the said base (1) overlaps or (2) falls short of the straight line *at the other end*, and the portion by which the applied parallelogram exceeds a parallelogram of the same angle and height on the given straight line (exactly) as base is a parallelogram similar to a given parallelogram (or, in particular cases, a square). In the case where the parallelogram is to *fall short*, a *διορισμός* is necessary to express the condition of possibility of solution.

We shall have occasion to see, when we come to the relative propositions in the second and sixth Books, that the general problem here stated is equivalent to that of solving geometrically a mixed quadratic equation. We shall see that, even by means of II. 5 and 6, we can solve geometrically the equations

$$\begin{aligned} ax \pm x^2 &= b^2, \\ x^2 - ax &= b^2; \end{aligned}$$

but in VI. 28, 29 Euclid gives the equivalent of the solution of the general equations

$$ax \pm \frac{b}{c} x^2 = \frac{C}{m}.$$

We are now in a position to understand the application of the terms *parabola* (application), *hyperbola* (exceeding) and *ellipse* (falling-short) to conic sections. These names were first so applied by Apollonius as expressing in each case the fundamental property of the curves as stated by him. This fundamental property is the geometrical equivalent of the Cartesian equation referred to any diameter of the conic and the tangent at its extremity as (in general, oblique) axes. If the *parameter* of the ordinates from the several points of the conic drawn to the given diameter be denoted by p (p being accordingly, in the case of the hyperbola and ellipse, equal to $\frac{d'^2}{d}$, where d is the length of the given diameter and d' that of its conjugate), Apollonius gives the properties of the three conics in the following form.

(1) For the *parabola*, the square on the ordinate at any point is equal to a rectangle applied to p as base with altitude equal to the corresponding abscissa. That is to say, with the usual notation,

$$y^2 = px.$$

(2) For the *hyperbola* and *ellipse*, the square on the ordinate is equal to the rectangle applied to p having as its width the abscissa and *exceeding* (for the hyperbola) or *falling-short* (for the ellipse) by a figure similar and similarly situated to the rectangle contained by the given diameter and p .

That is, in the *hyperbola* $y^2 = px + \frac{x^2}{d^2} pd$,

or $y^2 = px + \frac{p}{d} x^2$;

and in the *ellipse* $y^2 = px - \frac{p}{d} x^2$.

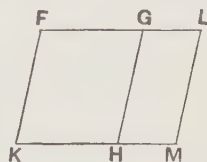
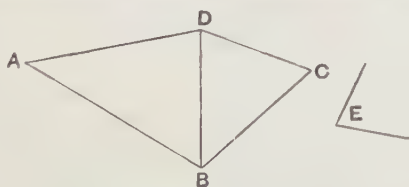
The form of these equations will be seen to be exactly the same as that of the general equations above given, and thus Apollonius' nomenclature followed exactly the traditional theory of *application*, *exceeding*, and *falling-short*.

PROPOSITION 45.

To construct, in a given rectilineal angle, a parallelogram equal to a given rectilineal figure.

Let $ABCD$ be the given rectilineal figure and E the given rectilineal angle;

thus it is required to construct, in the given angle E , a parallelogram equal to the rectilineal figure $ABCD$.



Let DB be joined, and let the parallelogram FH be constructed equal to the triangle ABD , in the angle HKF which is equal to E ;

[I. 42]

let the parallelogram GM equal to the triangle DBC be applied to the straight line GH , in the angle GHM which is equal to E .

[I. 44]

Then, since the angle E is equal to each of the angles HKF , GHM ,

the angle HKF is also equal to the angle GHM . [C. N. 1]

Let the angle KHG be added to each ;
therefore the angles FKH , KHG are equal to the angles
 KHG , GHM .

But the angles FKH , KHG are equal to two right angles ;
[1. 29]

20 therefore the angles KHG , GHM are also equal to two right
angles.

Thus, with a straight line GH , and at the point H on it,
two straight lines KH , HM not lying on the same side make
the adjacent angles equal to two right angles ;

25 therefore KH is in a straight line with HM . [1. 14]

And, since the straight line HG falls upon the parallels
 KM , FG , the alternate angles MHG , HGF are equal to one
another. [1. 29]

Let the angle HGL be added to each ;

30 therefore the angles MHG , HGL are equal to the angles
 HGF , HGL . [C. N. 2]

But the angles MHG , HGL are equal to two right angles ;
[1. 29]

therefore the angles HGF , HGL are also equal to two right
angles. [C. N. 1]

35 Therefore FG is in a straight line with GL . [1. 14]

And, since FK is equal and parallel to HG , [1. 34]
and HG to ML also,

KF is also equal and parallel to ML ; [C. N. 1 ; 1. 30]

and the straight lines KM , FL join them (at their extremities) ;

40 therefore KM , FL are also equal and parallel. [1. 33]

Therefore $KFLM$ is a parallelogram.

And, since the triangle ABD is equal to the parallelogram
 FH ,
and DBC to GM ,

45 the whole rectilinear figure $ABCD$ is equal to the whole
parallelogram $KFLM$.

Therefore the parallelogram $KFLM$ has been constructed
equal to the given rectilinear figure $ABCD$, in the angle FKM
which is equal to the given angle E . Q. E. F.

2, 3, 6, 45, 48. rectilinear figure, in the Greek "rectilinear" simply, without "figure,"
 $\epsilonὐθύγραμμον$ being here used as a substantive, like the similarly formed $\piαραλληλόγραμμον$.

Transformation of areas.

We can now take stock of how far the propositions 1. 43—45 bring us in
the matter of *transformation of areas*, which constitutes so important a part of

what has been fitly called the *geometrical algebra* of the Greeks. We have now learnt how to represent any rectilinear area, which can of course be resolved into triangles, by a single parallelogram having one side equal to any given straight line and one angle equal to any given rectilinear angle. Most important of all such parallelograms is the rectangle, which is one of the simplest forms in which an area can be shown. Since a rectangle corresponds to the product of two magnitudes in algebra, we see that *application* to a given straight line of a rectangle equal to a given area is the geometrical equivalent of algebraical *division* of the product of two quantities by a third. Further than this, it enables us to *add* or *subtract* any rectilinear areas and to represent the sum or difference by *one* rectangle with one side of any given length, the process being the equivalent of obtaining a common factor. But one step still remains, the finding of a *square* equal to a given rectangle, i.e. to a given rectilinear figure; and this step is not taken till II. 14. In general, the transformation of combinations of rectangles and squares into other combinations of rectangles and squares is the subject-matter of Book II., with the exception of the expression of the sum of two squares as a single square which appears earlier in the other Pythagorean theorem I. 47. Thus the transformation of rectilinear areas is made complete *in one direction*, i.e. in the direction of their simplest expression in terms of rectangles and squares, by the end of Book II. The reverse process of transforming the simpler rectangular area into an equal area which shall be similar to any rectilinear figure requires, of course, the use of proportions, and therefore does not appear till VI. 25.

Proclus adds to his note on this proposition the remark (pp. 422, 24—423, 6): "I conceive that it was in consequence of this problem that the ancient geometers were led to investigate the squaring of the circle as well. For, if a parallelogram can be found equal to any rectilinear figure, it is worth inquiring whether it be not also possible to prove rectilinear figures equal to circular. And Archimedes actually proved that any circle is equal to the right-angled triangle which has one of its sides about the right angle [the perpendicular] equal to the radius of the circle and its base equal to the perimeter of the circle. But of this elsewhere."

PROPOSITION 46.

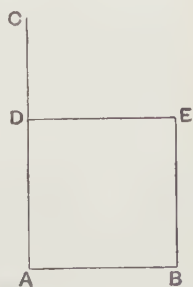
On a given straight line to describe a square.

Let AB be the given straight line; thus it is required to describe a square on the straight line AB .

5 Let AC be drawn at right angles to the straight line AB from the point A on it [I. 11], and let AD be made equal to AB ;

through the point D let DE be drawn
10 parallel to AB ,

and through the point B let BE be drawn parallel to AD .



[I. 31]

Therefore $ADEB$ is a parallelogram ;

therefore AB is equal to DE , and AD to BE . [I. 34]

But AB is equal to AD ;

15 therefore the four straight lines BA , AD , DE , EB
are equal to one another ;

therefore the parallelogram $ADEB$ is equilateral.

I say next that it is also right-angled.

For, since the straight line AD falls upon the parallels
20 AB , DE ,

the angles BAD , ADE are equal to two right angles.

[I. 29]

But the angle BAD is right ;

therefore the angle ADE is also right.

And in parallelogrammic areas the opposite sides and
25 angles are equal to one another ; [I. 34]

therefore each of the opposite angles ABE , BED is also
right.

Therefore $ADEB$ is right-angled.

And it was also proved equilateral.

30 Therefore it is a square ; and it is described on the straight
line AB .

Q. E. F.

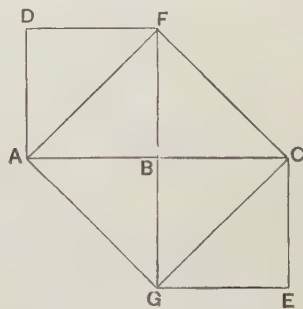
1, 3, 30. Proclus (p. 423, 18 sqq.) notes the difference between the word *construct* ($\sigmaυστησασθαι$) applied by Euclid to the construction of a *triangle* (and, he might have added, of an *angle*) and the words *describe on* ($\alpha\nuαγράφειν ἀπὸ$) used of drawing a square on a given straight line as one side. The *triangle* (or *angle*) is, so to say, pieced together, while the describing of a square on a given straight line is the making of a figure "from" *one* side, and corresponds to the multiplication of the number representing the side by itself.

Proclus (pp. 424—5) proves that, *if squares are described on equal straight lines, the squares are equal*; and, *conversely*, that, *if two squares are equal, the straight lines are equal on which they are described*. The first proposition is immediately obvious if we divide the squares into two triangles by drawing a diagonal in each. The converse is proved as follows.

Place the two equal squares AF , CG so that AB , BC are in a straight line. Then, since the angles are right, FB , BG will also be in a straight line. Join AF , FC , CG , GA .

Now, since the squares are equal, the triangles ABF , CBG are equal.

Add to each the triangle FBC ; therefore the triangles AFC , GFC are equal, and hence they must be in the same parallels.



Therefore AG , CF are parallel.

Also, since each of the alternate angles AFG , FGC is half a right angle,
 AF , CG are parallel.

Hence $AFCG$ is a parallelogram; and AF , CG are equal.

Thus the triangles ABF , CBG have two angles and one side respectively equal;

therefore AB is equal to BC , and BF to BG .

PROPOSITION 47.

In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.

Let ABC be a right-angled triangle having the angle
 5 BAC right;

I say that the square on BC is equal to the squares on
 BA , AC .

For let there be described
 on BC the square $BDEC$,
 10 and on BA , AC the squares
 GB , HC ; [I. 46]
 through A let AL be drawn
 parallel to either BD or CE ,
 and let AD , FC be joined.

15 Then, since each of the
 angles BAC , BAG is right,
 it follows that with a straight
 line BA , and at the point A
 on it, the two straight lines
 20 AC , AG not lying on the
 same side make the adjacent
 angles equal to two right
 angles;

therefore CA is in a straight line with AG . [I. 14]

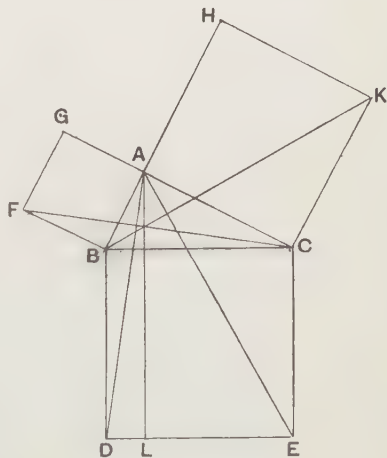
25 For the same reason

BA is also in a straight line with AH .

And, since the angle DBC is equal to the angle FBA : for
 each is right:

let the angle ABC be added to each;

30 therefore the whole angle DBA is equal to the whole
 angle FBC . [C. N. 2]



And, since DB is equal to BC , and FB to BA ,
the two sides AB , BD are equal to the two sides FB , BC
respectively ;

35 and the angle ABD is equal to the angle FBC ;
therefore the base AD is equal to the base FC ,
and the triangle ABD is equal to the triangle FBC . [I. 4]
Now the parallelogram BL is double of the triangle ABD ,
for they have the same base BD and are in the same parallels
40 BD , AL . [I. 41]

And the square GB is double of the triangle FBC ,
for they again have the same base FB and are in the same
parallels FB , GC . [I. 41]

[But the doubles of equals are equal to one another.]

45 Therefore the parallelogram BL is also equal to the
square GB .

Similarly, if AE , BK be joined,
the parallelogram CL can also be proved equal to the square
 HC ;

50 therefore the whole square $BDEC$ is equal to the two
squares GB , HC . [C. N. 2]

And the square $BDEC$ is described on BC ,

and the squares GB , HC on BA , AC .

Therefore the square on the side BC is equal to the
55 squares on the sides BA , AC .

Therefore etc.

Q. E. D.

1. the square on, τὸ ἀπὸ...τετράγωνον, the word ἀναγραφέν or ἀναγεγραμμένον being understood.

subtending the right angle. Here ὑποτεωούσης, "subtending," is used with the simple accusative (τὴν ὀρθὴν γωνίαν) instead of being followed by ὑπὸ and the accusative, which seems to be the original and more orthodox construction. Cf. I. 18, note.

33. the two sides AB , BD Euclid actually writes " DB , BA ," and therefore the equal sides in the two triangles are not mentioned in corresponding order, though he adheres to the words ἐκάτερα ἐκάτερα "respectively." Here DB is equal to BC and BA to FB .

44. [But the doubles of equals are equal to one another.] Heiberg brackets these words as an interpolation, since it quotes a *Common Notion* which is itself interpolated. Cf. notes on I. 37, p. 332, and on interpolated *Common Notions*, pp. 223-4.

"If we listen," says Proclus (p. 426, 6 sqq.), "to those who wish to recount ancient history, we may find some of them referring this theorem to Pythagoras and saying that he sacrificed an ox in honour of his discovery. But for my part, while I admire those who first observed the truth of this theorem, I marvel more at the writer of the *Elements*, not only because he made it fast (κατεδύρατο) by a most lucid demonstration, but because he compelled assent to the still more general theorem by the irrefragable arguments of science in the sixth Book. For in that Book he proves generally that, in right-angled triangles, the figure on the side subtending the right angle is equal to the similar and similarly situated figures described on the sides about the right angle."

In addition, Plutarch (in the passages quoted above in the note on I. 44), Diogenes Laertius (VIII. 12) and Athenaeus (x. 13) agree in attributing this proposition to Pythagoras. It is easy to point out, as does G. Junge ("Wann haben die Griechen das Irrationale entdeckt?" in *Novae Symbolae Joachimicae*, Halle a. S., 1907, pp. 221—264), that these are late witnesses, and that the Greek literature which we possess belonging to the first five centuries after Pythagoras contains no statement specifying this or any other particular great geometrical discovery as due to him. Yet the distich of Apollodorus the "calculator," whose date (though it cannot be fixed) is at least earlier than that of Plutarch and presumably of Cicero, is quite definite as to the existence of *one* "famous proposition" discovered by Pythagoras, whatever it was. Nor does Cicero, in commenting apparently on the verses (*De nat. deor.* III. c. 36, § 88), seem to dispute the fact of the geometrical discovery, but only the story of the sacrifice. Junge naturally emphasises the apparent uncertainty in the statements of Plutarch and Proclus. But, as I read the passages of Plutarch, I see nothing in them inconsistent with the supposition that Plutarch unhesitatingly accepted as discoveries of Pythagoras *both* the theorem of the square of the hypotenuse and the problem of the application of an area, and the only doubt he felt was as to which of the two discoveries was the more appropriate occasion for the supposed sacrifice. There is also other evidence not without bearing on the question. The theorem is closely connected with the whole of the matter of Eucl. Book II., in which one of the most prominent features is the use of the *gnomon*. Now the *gnomon* was a well-understood term with the Pythagoreans (cf. the fragment of Philolaus quoted on p. 141 of Boeckh's *Philolaos des Pythagoreers Lehren*, 1819). Aristotle also (*Physics* III. 4, 203 a 10—15) clearly attributes to the Pythagoreans the placing of odd numbers as *gnomons* round successive squares beginning with 1, thereby forming new squares, while in another place (*Categ.* 14, 15 a 30) the word *gnomon* occurs in the same (obviously familiar) sense: "e.g. a square, when a *gnomon* is placed round it, is increased in size but is not altered in form." The inference must therefore be that practically the whole doctrine of Book II. is Pythagorean. Again Heron (? 3rd cent. A.D.), like Proclus, credits Pythagoras with a general rule for forming right-angled triangles with rational whole numbers for sides. Lastly, the "summary" of Proclus appears to credit Pythagoras with the discovery of the theory, or study, of irrationals (*τὴν τῶν ἀλόγων πραγματείαν*). But it is now more or less agreed that the reading here should be, not *τῶν ἀλόγων*, but *τῶν ἀναλόγων*, or rather *τῶν ἀνὰ λόγον* ("of proportionals"), and that the author intended to attribute to Pythagoras a theory of *proportion*, i.e. the (arithmetical) theory of proportion applicable only to commensurable magnitudes, as distinct from the theory of Eucl. Book V., which was due to Eudoxus. It is not however disputed that the *Pythagoreans* discovered the irrational (cf. the scholium No. 1 to Book x.). Now everything goes to show that this discovery of the irrational was made with reference to $\sqrt{2}$, the ratio of the diagonal of a square to its side. It is clear that this presupposes the knowledge that I. 47 is true of an isosceles right-angled triangle; and the fact that some triangles of which it had been discovered to be true were *rational* right-angled triangles was doubtless what suggested the inquiry whether the ratio between the lengths of the diagonal and the side of a square could also be expressed in whole numbers. On the whole, therefore, I see no sufficient reason to question the tradition that, *so far as Greek geometry is concerned* (the possible priority of the discovery of the same proposition in India will be considered later), Pythagoras

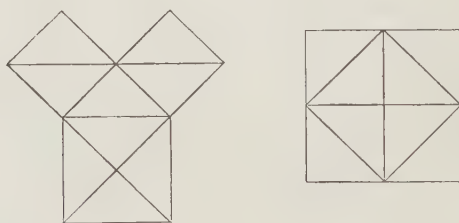
was the first to introduce the theorem of I. 47 and to give a general proof of it.

On this assumption, how was Pythagoras led to this discovery? It has been suggested and commonly assumed that the Egyptians were aware that a triangle with its sides in the ratio 3, 4, 5 was right-angled. Cantor inferred this from the fact that this was precisely the triangle with which Pythagoras began, if we may accept the testimony of Vitruvius (ix. 2) that Pythagoras taught how to make a right angle by means of three lengths measured by the numbers 3, 4, 5. If then he took from the Egyptians the triangle 3, 4, 5, he presumably learnt its property from them also. Now the Egyptians must certainly be credited from a period at least as far back as 2000 B.C. with the knowledge that $4^2 + 3^2 = 5^2$. Cantor finds proof of this in a fragment of papyrus belonging to the time of the 12th Dynasty newly discovered at Kahun. In this papyrus we have extractions of square roots: e.g. that of 16 is 4, that of $1\frac{9}{16}$ is $1\frac{1}{4}$, that of $6\frac{1}{4}$ is $2\frac{1}{2}$, and the following equations can be traced:

$$\begin{aligned} 1^2 + \left(\frac{3}{4}\right)^2 &= \left(1\frac{1}{4}\right)^2 \\ 8^2 + 6^2 &= 10^2 \\ 2^2 + \left(1\frac{1}{2}\right)^2 &= \left(2\frac{1}{2}\right)^2 \\ 16^2 + 12^2 &= 20^2. \end{aligned}$$

It will be seen that $4^2 + 3^2 = 5^2$ can be derived from each of these by multiplying, or dividing out, by one and the same factor. We may therefore admit that the Egyptians knew that $3^2 + 4^2 = 5^2$. But there seems to be no evidence that they knew that the triangle (3, 4, 5) is *right-angled*; indeed, according to the latest authority (T. Eric Peet, *The Rhind Mathematical Papyrus*, 1923), nothing in Egyptian mathematics suggests that the Egyptians were acquainted with this or any special cases of the Pythagorean theorem.

How then did Pythagoras discover the general theorem? Observing that 3, 4, 5 was a right-angled triangle, while $3^2 + 4^2 = 5^2$, he was probably led to



consider whether a similar relation was true of the sides of right-angled triangles other than the particular one. The simplest case (geometrically) to investigate was that of the *isosceles* right-angled triangle; and the truth of the theorem in this particular case would easily appear from the mere construction of a figure. Cantor (I, p. 185) and Allman (*Greek Geometry from Thales to Euclid*, p. 29) illustrate by a figure in which the squares are drawn outwards, as in I. 47, and divided by diagonals into equal triangles; but I think that the truth was more likely to be first observed from a figure of the kind suggested by Bürk (*Das Apastamba-Sulba-Sūtra in Zeitschrift der deutschen morgenländ. Gesellschaft*, LV., 1901, p. 557) to explain how the Indians arrived at the same thing. The two figures are as shown above. When the geometrical

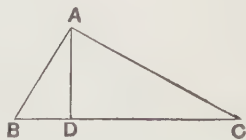
consideration of the figure had shown that the isosceles right-angled triangle had the property in question, the investigation of the same fact from the arithmetical point of view would ultimately lead to the other momentous discovery of the irrationality of the length of the diagonal of a square expressed in terms of its side.

The *irrational* will come up for discussion later; and our next question is: Assuming that Pythagoras had observed the geometrical truth of the theorem in the case of the two particular triangles, and doubtless of other rational right-angled triangles, how did he establish it generally?

There is no positive evidence on this point. Two possible lines are however marked out. (1) Tannery says (*La Géométrie grecque*, p. 105) that the geometry of Pythagoras was sufficiently advanced to make it possible for him to prove the theorem by *similar triangles*. He does not say in what particular manner similar triangles would be used, but their use must apparently have involved the use of *proportions*, and, in order that the proof should be conclusive, of the theory of proportions in its complete form applicable to incommensurable as well as commensurable magnitudes. Now Eudoxus was the first to make the theory of proportion independent of the hypothesis of commensurability; and as, before Eudoxus' time, this had not been done, any proof of the general theorem by means of proportions given by Pythagoras must at least have been inconclusive. But this does not constitute any objection to the supposition that the truth of the general theorem may have been discovered in such a manner; on the contrary, the supposition that Pythagoras proved it by means of an imperfect theory of proportions would better than anything else account for the fact that Euclid had to devise an entirely new proof, as Proclus says he did in I. 47. This proof had to be independent of the theory of proportion even in its rigorous form, because the plan of the *Elements* postponed that theory to Books v. and vi., while the Pythagorean theorem was required as early as Book II. On the other hand, if the Pythagorean proof had been based on the doctrine of Books I. and II. only, it would scarcely have been necessary for Euclid to supply a new proof.

The possible proofs by means of proportion would seem to be practically limited to two.

(a) One method is to prove, from the similarity of the triangles ABC , DBA , that the rectangle CB , BD is equal to the square on BA , and, from the similarity of the triangles ABC , DAC , that the rectangle BC , CD is equal to the square on CA ; whence the result follows by addition.



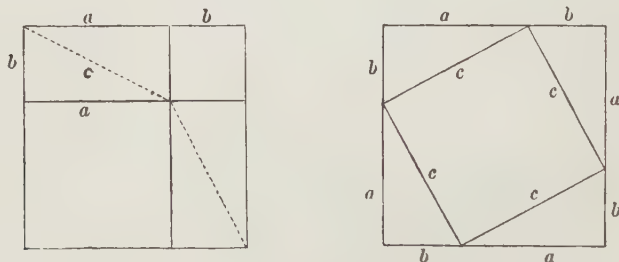
It will be observed that this proof is *in substance* identical with that of Euclid, the only difference being that the equality of the two smaller squares to the respective rectangles is inferred by the method of Book VI. instead of from the relation between the areas of parallelograms and triangles on the same base and between the same parallels established in Book I. It occurred to me whether, if Pythagoras' proof had come, even in substance, so near to Euclid's, Proclus would have emphasised so much as he does the originality of Euclid's, or would have gone so far as to say that he marvelled more at that proof than at the original discovery of the theorem. But on the whole I see no difficulty; for there can be little doubt that the proof by proportion is what suggested to Euclid the method of I. 47, and the transformation of

the method of proportions into one based on Book I. only, effected by a construction and proof so extraordinarily ingenious, is a veritable *tour de force* which compels admiration, notwithstanding the ignorant strictures of Schopenhauer, who wanted something as obvious as the second figure in the case of the isosceles right-angled triangle (p. 352), and accordingly (*Sämmtliche Werke*, III. § 39 and I. § 15) calls Euclid's proof "a mouse-trap proof" and "a proof walking on stilts, nay, a mean, underhand, proof" ("Des Eukleides stelzbeiniger, ja, hinterlistiger Beweis").

(b) The other possible method is this. As it would be seen that the triangles into which the original triangle is divided by the perpendicular from the right angle on the hypotenuse are similar to one another and to the whole triangle, while in these three triangles the two sides about the right angle in the original triangle, and the hypotenuse of the original triangle, are corresponding sides, and that the sum of the two former similar triangles is identically equal to the similar triangle on the hypotenuse, it might be inferred that the same would also be true of *squares* described on the corresponding three sides respectively, because squares as well as similar triangles are to one another in the duplicate ratio of corresponding sides. But the same thing is equally true of any similar rectilineal figures, so that this proof would practically establish the extended theorem of Eucl. VI. 31, which theorem, however, Proclus appears to regard as being entirely Euclid's discovery.

On the whole, the most probable supposition seems to me to be that Pythagoras used the first method (a) of proof by means of the theory of proportion as he knew it, i.e. in the defective form which was in use up to the date of Eudoxus.

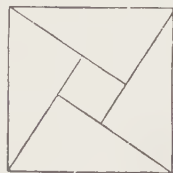
(2) I have pointed out the difficulty in the way of the supposition that Pythagoras' proof depended upon the principles of Eucl. Books I. and II. only.



Were it not for this difficulty, the conjecture of Bretschneider (p. 82), followed by Hankel (p. 98), would be the most tempting hypothesis. According to this suggestion, we are to suppose a figure like that of Eucl. II. 4 in which a , b are the sides of the two inner squares respectively, and $a+b$ is the side of the complete square. Then, if the two complements, which are equal, are divided by their two diagonals into four equal triangles of sides a , b , c , we can place these triangles round another square of the same size as the whole square, in the manner shown in the second figure, so that the sides a , b of successive triangles make up one of the sides of the square and are arranged in cyclic order. It readily follows that the remainder of the square when the four triangles are deducted is, in the one case, a square whose side is c , and in the other the sum of two squares whose sides are a , b respectively. Therefore the square on c is equal

to the sum of the squares on a , b . All that can be said against this conjectural proof is that it has no specifically Greek colouring but rather recalls the Indian method. Thus Bhāskara (born 1114 A.D.; see Cantor, I, p. 656) simply draws four right-angled triangles equal to the original one inwards, one on each side of the square on the hypotenuse, and says "see!", without even adding that inspection shows that

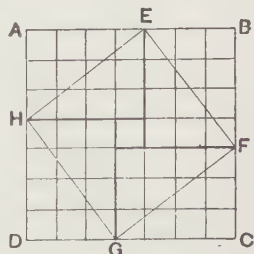
$$c^2 = 4 \frac{ab}{2} + (a-b)^2 = a^2 + b^2.$$



Though, for the reason given, there is difficulty in supposing that Pythagoras used a general proof of this kind, which applies of course to right-angled triangles with sides incommensurable as well as commensurable, there is no objection, I think, to supposing that the truth of the proposition in the case of the first *rational* right-angled triangles discovered, e.g. 3, 4, 5, was proved by a method of this sort. Where the sides are commensurable in this way, the squares can be divided up into small (unit) squares, which would much facilitate the comparison between them. That this subdivision was in fact resorted to in adding and subtracting squares is made probable by Aristotle's allusion to odd numbers as *gnomons* placed round unity to form successive squares in *Physics* III. 4; this must mean that the squares were represented by dots arranged in the form of a square and a gnomon formed of dots put round, or that (if the given square was drawn in the usual way) the gnomon was divided up into unit squares. Zeuthen has shown (*"Théorème de Pythagore," Origine de la Géométrie scientifique in Comptes rendus du II^{me} Congrès international de Philosophie*, Genève, 1904), how easily the proposition could be proved by a method of this kind for the triangle 3, 4, 5. To admit of the two smaller squares being shown side by side, take a square on a line containing 7 units of length ($4 + 3$), and divide it up into 49 small squares. It would be obvious that the whole square could be exhibited as containing four rectangles of sides 4, 3 cyclically arranged round the figure with one unit square in the middle. (This same figure is given by Cantor, I, p. 680. to illustrate the method given in the Chinese "Cháu-peï".) It would be seen that

(i) the whole square (7^2) is made up of two squares 3^2 and 4^2 , and two rectangles 3, 4;

(ii) the same square is made up of the square $EFGH$ and the halves of four of the same rectangles 3, 4, whence the square $EFGH$, being equal to the sum of the squares 3^2 and 4^2 , must contain 25 unit squares and its side, or the diagonal of one of the rectangles, must contain 5 units of length.



Or the result might equally be seen by observing that

(i) the square $EFGH$ on the diagonal of one of the rectangles is made up of the halves of four rectangles and the unit square in the middle, while

(ii) the squares 3^2 and 4^2 placed at adjacent corners of the large square make up two rectangles 3, 4 with the unit square in the middle.

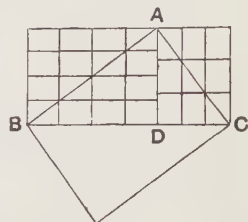
The procedure would be equally easy for any *rational* right-angled triangle, and would be a natural method of trying to *prove* the property when it had

once been *empirically* observed that triangles like 3, 4, 5 did in fact contain a right angle.

Zeuthen has, in the same paper, shown in a most ingenious way how the property of the triangle 3, 4, 5 could be verified by a sort of combination of the second possible method by similar triangles, (b) on p. 354 above, with subdivision of *rectangles* into similar small *rectangles*. I give the method on account of its interest, although it is no doubt too advanced to have been used by those who first proved the property of the particular triangle.

Let ABC be a triangle right-angled at A , and such that the lengths of the sides AB , AC are 4 and 3 units respectively.

Draw the perpendicular AD , divide up AB , AC into unit lengths, complete the rectangle on BC as base and with AD as altitude, and subdivide this rectangle into small rectangles by drawing parallels to BC , AD through the points of division of AB , AC .



Now, since the diagonals of the small rectangles are all equal, each being of unit length, it follows by similar triangles that the small rectangles are all equal. And the rectangle with AB for diagonal contains 16 of the small rectangles, while the rectangle with diagonal AC contains 9 of them.

But the sum of the triangles ABD , ADC is equal to the triangle ABC .

Hence the rectangle with BC as diagonal contains $9 + 16$ or 25 of the small rectangles;

and therefore $BC = 5$.

Rational right-angled triangles from the arithmetical standpoint.

Pythagoras investigated the *arithmetical* problem of finding rational numbers which could be made the sides of right-angled triangles, or of finding square numbers which are the sum of two squares; and herein we find the beginning of the *indeterminate analysis* which reached so high a stage of development in Diophantus. Fortunately Proclus has preserved Pythagoras' method of solution in the following passage (pp. 428, 7—429, 8). "Certain methods for the discovery of triangles of this kind are handed down, one of which they refer to Plato, and another to Pythagoras. [The latter] starts from odd numbers. For it makes the odd number the smaller of the sides about the right angle; then it takes the square of it, subtracts unity, and makes half the difference the greater of the sides about the right angle; lastly it adds unity to this and so forms the remaining side, the hypotenuse. For example, taking 3, squaring it, and subtracting unity from the 9, the method takes half of the 8, namely 4; then, adding unity to it again, it makes 5, and a right-angled triangle has been found with one side 3, another 4 and another 5. But the method of Plato argues from even numbers. For it takes the given even number and makes it one of the sides about the right angle; then, bisecting this number and squaring the half, it adds unity to the square to form the hypotenuse, and subtracts unity from the square to form the other side about the right angle. For example, taking 4, the method squares half of this, or 2, and so makes 4; then, subtracting unity, it produces 3, and adding unity it produces 5. Thus it has formed the same triangle as that which was obtained by the other method."

The formula of Pythagoras amounts, if m be an odd number, to

$$m^2 + \left(\frac{m^2 - 1}{2}\right)^2 = \left(\frac{m^2 + 1}{2}\right)^2,$$

the sides of the right-angled triangle being m , $\frac{m^2 - 1}{2}$, $\frac{m^2 + 1}{2}$. Cantor (I₃, pp. 185—6), taking up an idea of Röth (*Geschichte der abendländischen Philosophie*, II. 527), gives the following as a possible explanation of the way in which Pythagoras arrived at his formula. If $c^2 = a^2 + b^2$, it follows that

$$a^2 = c^2 - b^2 = (c + b)(c - b).$$

Numbers can be found satisfying the first equation if (1) $c + b$ and $c - b$ are either both even or both odd, and if further (2) $c + b$ and $c - b$ are such numbers as, when multiplied together, produce a square number. The first condition is necessary because, in order that c and b may both be whole numbers, the sum and difference of $c + b$ and $c - b$ must both be even. The second condition is satisfied if $c + b$ and $c - b$ are what were called *similar numbers* (ὁμοιοὶ ἀριθμοί); and that such numbers were most probably known in the time before Plato may be inferred from their appearing in Theon of Smyrna (*Expositio rerum mathematicarum ad legendum Platonem utilium*, ed. Hiller, p. 36, 12), who says that similar plane numbers are, first, all square numbers and, secondly, such oblong numbers as have the sides which contain them proportional. Thus 6 is an oblong number with length 3 and breadth 2; 24 is another with length 6 and breadth 4. Since therefore 6 is to 3 as 4 is to 2, the numbers 6 and 24 are similar.

Now the simplest case of two similar numbers is that of 1 and a^2 , and, since 1 is odd, the condition (1) requires that a^2 , and therefore a , is also odd. That is, we may take 1 and $(2n + 1)^2$ and equate them respectively to $c - b$ and $c + b$, whence we have

$$b = \frac{(2n + 1)^2 - 1}{2},$$

$$c = \frac{(2n + 1)^2 + 1}{2},$$

while

$$a = 2n + 1.$$

As Cantor remarks, the form in which c and b appear correspond sufficiently closely to the description in the text of Proclus.

Another obvious possibility would be, instead of equating $c - b$ to unity, to put $c - b = 2$, in which case the similar number $c + b$ must be equated to double of some square, i.e. to a number of the form $2n^2$, or to the half of an even square number, say $\frac{(2n)^2}{2}$. This would give

$$a = 2n,$$

$$b = n^2 - 1,$$

$$c = n^2 + 1,$$

which is Plato's solution, as given by Proclus.

The two solutions supplement each other. It is interesting to observe that the method suggested by Röth and Cantor is very like that of Eucl. x. (Lemma 1 following Prop. 28). We shall come to this later, but it may be mentioned here that the problem is to find two square numbers such that their

sum is also a square. Euclid there uses the property of II. 6 to the effect that, if AB is bisected at C and produced to D ,

$$AD \cdot DB + BC^2 = CD^2.$$

We may write this

$$uv = c^2 - b^2,$$

where

$$u = c + b, \quad v = c - b.$$

In order that uv may be a square, Euclid points out that u and v must be similar numbers, and further that u and v must be either both odd or both even in order that b may be a whole number. We may then put for the similar numbers, say, $a\beta^2$ and $\alpha\gamma^2$, whence (if $a\beta^2$, $\alpha\gamma^2$ are either both odd or both even) we obtain the solution

$$a\beta^2 \cdot \alpha\gamma^2 + \left(\frac{a\beta^2 - \alpha\gamma^2}{2}\right)^2 = \left(\frac{a\beta^2 + \alpha\gamma^2}{2}\right)^2.$$

But I think a serious, and even fatal, objection to the conjecture of Cantor and Rôth is the very fact that the method enables both the Pythagorean and the Platonic series of triangles to be deduced with equal ease. If this had been the case with the method used by Pythagoras, it would not, I think, have been left to Plato to discover the second series of such triangles. It seems to me therefore that Pythagoras must have used some method which would produce his rule *only*; and further it would be some less recondite method, suggested by direct *observation* rather than by argument from general principles.

One solution satisfying these conditions is that of Bretschneider (p. 83), who suggests the following simple method. Pythagoras was certainly aware that the successive odd numbers are *gnomons*, or the differences between successive square numbers. It was then a simple matter to write down in three rows (a) the natural numbers, (b) their squares, (c) the successive odd numbers constituting the differences between the successive squares in (b), thus:

1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	4	9	16	25	36	49	64	81	100	121	144	169	196
1	3	5	7	9	11	13	15	17	19	21	23	25	27

Pythagoras had then only to pick out the numbers in the third row which are squares, and his rule would be obtained by finding the formula connecting the square in the third line with the two adjacent squares in the second line. But even this would require some little argument; and I think a still better suggestion, because making pure observation play a greater part, is that of P. Treutlein (*Zeitschrift für Mathematik und Physik*, xxviii., 1883, Hist.-litt. Abtheilung, pp. 209 sqq.).

We have the best evidence (e.g. in Theon of Smyrna) of the practice of representing square numbers and other figured numbers, e.g. oblong, triangular, hexagonal, by dots or signs arranged in the shape of the particular figure. (Cf. Aristotle, *Metaph.* 1092 b 12). Thus, says Treutlein, it would be easily seen that any square number can be turned into the next higher square by putting a single row of dots round two adjacent sides, in the form of a *gnomon* (see figures on next page).

If a is the side of a particular square, the *gnomon* round it is shown by simple inspection to contain $2a + 1$ dots or units. Now, in order that $2a + 1$ may itself be a square, let us suppose

$$2a + 1 = n^2,$$

whence

$$a = \frac{1}{2}(n^2 - 1),$$

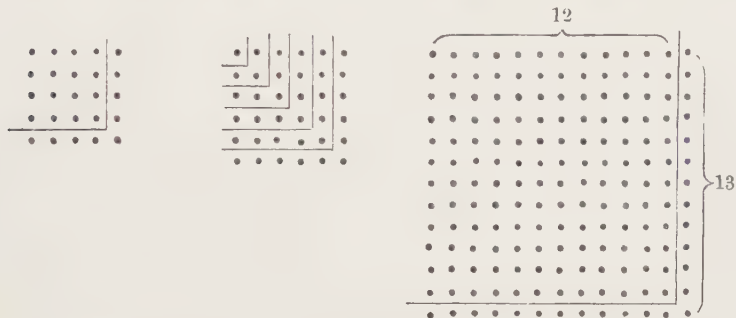
and

$$a + 1 = \frac{1}{2}(n^2 + 1).$$

In order that a and $a + 1$ may be integral, n must be odd, and we have at once the Pythagorean formula

$$n^2 + \left(\frac{n^2 - 1}{2}\right)^2 = \left(\frac{n^2 + 1}{2}\right)^2.$$

I think Treutlein's hypothesis is shown to be the correct one by the passage in Aristotle's *Physics* already quoted, where the reference is undoubtedly to the Pythagoreans, and odd numbers are clearly identified with *gnomons* "placed round 1." But the ancient commentaries on the passage make the matter clearer still. Philoponus says: "As a proof...the Pythagoreans refer to what



happens with the addition of numbers; for when the odd numbers are successively added to a square number they keep it square and equilateral... Odd numbers are accordingly called *gnomons* because, when added to what are already squares, they preserve the square form... Alexander has excellently said in explanation that the phrase 'when gnomons are placed round' means *making a figure* with the odd numbers (*τὴν κατὰ τοὺς περιττοὺς ἀριθμοὺς σχηματογραφίαν*)... for it is the practice with the Pythagoreans to *represent things in figures* (*σχηματογραφεῖν*).

The next question is: assuming this explanation of the Pythagorean formula, what are we to say of the origin of Plato's? It could of course be obtained as a particular case of the general formula of Eucl. x. already referred to; but there are two simple alternative explanations in this case also. (1) Bretschneider observes that, to obtain Plato's formula, we have only to double the sides of the squares in the Pythagorean formula,

$$\text{for } (2n)^2 + (n^2 - 1)^2 = (n^2 + 1)^2,$$

where however n is not necessarily odd.

(2) Treutlein would explain by means of an extension of the gnomon idea. As, he says, the Pythagorean formula was obtained by placing a gnomon consisting of a single row of dots round two adjacent sides of a square, it would be natural to try whether another solution could not be found by placing round the square a gnomon consisting of a *double* row of dots. Such a gnomon would equally turn the square into a larger square; and the question would be whether the double-row gnomon itself could be a square. If the side of the original square was a , it would easily be seen that the number of units in the double-row gnomon would be $4a + 4$, and we have only to put

$$4a + 4 = 4n^2,$$



whence

$$a = n^2 - 1,$$

$$a + 2 = n^2 + 1,$$

and we have the Platonic formula

$$(2n)^2 + (n^2 - 1)^2 = (n^2 + 1)^2.$$

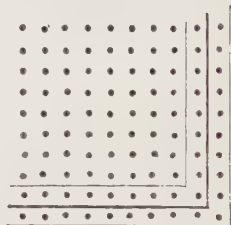
I think this is, in substance, the right explanation, but, in form, not quite correct. The Greeks would not, I think, have treated the *double* row as a gnomon. Their comparison would have been between (1) a certain square *plus* a single-row gnomon and (2) the same square *minus* a single-row gnomon. As the application of Eucl. II. 4 to the case where the segments of the side of the square are $a, 1$ enables the Pythagorean formula to be obtained as Treutlein obtains it, so I think that Eucl. II. 8 confirms the idea that the Platonic formula was obtained by comparing a square *plus* a gnomon with the same square *minus* a gnomon. For II. 8 proves that

$$4ab + (a - b)^2 = (a + b)^2,$$

whence, substituting 1 for b , we have

$$4a + (a - 1)^2 = (a + 1)^2,$$

and we have only to put $a = n^2$ to obtain Plato's formula.



The "theorem of Pythagoras" in India.

This question has been discussed anew in the last few years as the result of the publication of two important papers by Albert Bürk on *Das Āpastamba-Sulba-Sūtra* in the *Zeitschrift der deutschen morgenländischen Gesellschaft* (LV., 1901, pp. 543—591, and LVI., 1902, pp. 327—391). The first of the two papers contains the introduction and the text, the second the translation with notes. A selection of the most important parts of the material was made and issued by G. Thibaut in the *Journal of the Asiatic Society of Bengal*, XLIV., 1875, Part 1. (reprinted also at Calcutta, 1875, as *The Sulvasūtras*, by G. Thibaut). Thibaut in this work gave a most valuable comparison of extracts from the three Sulvasūtras by Bāudhāyana, Āpastamba and Kātyāyana respectively, with a running commentary and an estimate of the date and originality of the geometry of the Indians. Bürk has however done good service by making the Āpastamba-Ś.-S. accessible in its entirety and investigating the whole subject afresh. With the natural enthusiasm of an editor for the work he is editing, he roundly maintains, not only that the Pythagorean theorem was known and proved in all its generality by the Indians long before the date of Pythagoras (about 580—500 B.C.), but that they had also discovered the irrational; and further that, so far from Indian geometry being indebted to the Greek, the much-travelled Pythagoras probably obtained his theory from India (*loc. cit.* LV., p. 575 note). Three important notices and criticisms of Bürk's work have followed, by H. G. Zeuthen ("Théorème de Pythagore," *Origine de la Géométrie scientifique*, 1904, already quoted), by Moritz Cantor (*Über die älteste indische Mathematik* in the *Archiv der Mathematik und Physik*, VIII., 1905, pp. 63—72) and by Heinrich Vogt (*Haben die alten Inder den Pythagoreischen Lehrsatz und das Irrationale gekannt?* in the *Bibliotheca Mathematica*, VII., 1906, pp. 6—23. See also Cantor's *Geschichte der Mathematik*, I., pp. 635—645.

The general effect of the criticisms is, I think, to show the necessity for the greatest caution, to say the least, in accepting Bürk's conclusions.

I proceed to give a short summary of the portions of the contents of the Āpastamba-Ś.-S. which are important in the present connexion. It may be premised that the general object of the book is to show how to construct altars of certain shapes, and to vary the dimensions of altars without altering the form. It is a collection of *rules* for carrying out certain constructions. There are no proofs, the nearest approach to a proof being in the rule for obtaining the area of an isosceles trapezium, which is done by drawing a perpendicular from one extremity of the smaller of the two parallel sides to the greater, and then taking away the triangle so cut off and placing it, the other side up, adjacent to the other equal side of the trapezium, thereby transforming the trapezium into a rectangle. It should also be observed that Āpastamba does not speak of *right-angled triangles*, but of two adjacent sides and the diagonal of a *rectangle*. For brevity, I shall use the expression "rational rectangle" to denote a rectangle the two sides and the diagonal of which can be expressed in terms of rational numbers. The references in brackets are to the chapters and numbers of Āpastamba's work.

(1) Constructions of right angles by means of cords of the following relative lengths respectively :

$$\begin{aligned} & \left\{ \begin{array}{l} 3, 4, 5 \quad (\text{I. 3, v. 3}) \\ 12, 16, 20 \quad (\text{v. 3}) \\ 15, 20, 25 \quad (\text{v. 3}) \end{array} \right. \\ & \left\{ \begin{array}{l} 5, 12, 13 \quad (\text{v. 4}) \\ 15, 36, 39 \quad (\text{I. 2, v. 2, 4}) \\ 8, 15, 17 \quad (\text{v. 5}) \end{array} \right. \\ & 12, 35, 37 \quad (\text{v. 5}) \end{aligned}$$

(2) A general enunciation of the Pythagorean theorem thus: "The diagonal of a rectangle produces [i.e. the square on the diagonal is equal to] the sum of what the longer and shorter sides separately produce [i.e. the squares on the two sides]." (I. 4)

(3) The application of the Pythagorean theorem to a *square* instead of a rectangle [i.e. to an *isosceles* right-angled triangle]: "The diagonal of a square produces an area double [of the original square]." (I. 5)

(4) An approximation to the value of $\sqrt{2}$; the diagonal of a square is $\left(1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34}\right)$ times the side. (I. 6)

(5) Application of this approximate value to the construction of a square with side of any length. (II. 1)

(6) The construction of $a\sqrt{3}$, by means of the Pythagorean theorem, as the diagonal of a rectangle with sides a and $a\sqrt{2}$. (II. 2)

(7) Remarks equivalent to the following :

(a) $a\sqrt{\frac{1}{3}}$ is the side of $\frac{1}{9}(a\sqrt{3})^2$, or $a\sqrt{\frac{1}{3}} = \frac{1}{3}a\sqrt{3}$. (II. 3)

(b) A square on length of 1 unit gives 1 unit square (III. 4)

" " 2 units gives 4 unit squares (III. 6)

" " 3 " 9 " (III. 6)

" " $1\frac{1}{2}$ " $2\frac{1}{4}$ " (III. 8)

A square on length of $2\frac{1}{2}$ units gives $6\frac{1}{4}$ unit squares (III. 8)

" " $\frac{1}{2}$ unit gives $\frac{1}{4}$ unit square (III. 10)

" " $\frac{1}{3}$ " $\frac{1}{9}$ " (III. 10)

(c) Generally, the square on any length contains as many rows (of small, unit, squares) as the length contains units. (III. 7)

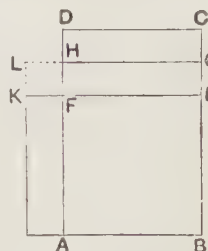
(8) Constructions, by means of the Pythagorean theorem, of

(a) the *sum* of two squares as one square, (II. 4)

(b) the *difference* of two squares as one square. (II. 5)

(9) A transformation of a rectangle into a square. (II. 7)

[This is not directly done as by Euclid in II. 14, but the rectangle is first transformed into a gnomon, i.e. into the difference between two squares, which difference is then transformed into one square by the preceding rule. If $ABCD$ be the given rectangle of which BC is the longer side, cut off the square $ABEF$, bisect the rectangle DE left over by HG parallel to FE , move the upper half DG and place it on AF as base in the position AK . Then the rectangle $ABCD$ is equal to the gnomon which is the difference between the square LB and the square LF . In other words, Apastamba transforms the rectangle ab into the difference between the squares $\left(\frac{a+b}{2}\right)^2$ and $\left(\frac{a-b}{2}\right)^2$.]



(10) An attempt at a transformation of a square (a^2) into a rectangle which shall have one side of given length (b). (III. 1)

[This shows no sign of such a procedure as that of Eucl. I. 44, and indeed does no more than say that we must subtract ab from a^2 and then adapt the remainder $a^2 - ab$ so that it may "fit on" to the rectangle ab . The problem is therefore only reduced to another of the same kind, and presumably it was only solved *arithmetically* in the case where a, b are given numerically. The Indian was therefore far from the general, geometrical, solution.]

(11) Increase of a given square into a larger square. (III. 9)

[This amounts to saying that you must add two rectangles (a, b) and another square (b^2) in order to transform a square a^2 into a square $(a+b)^2$. The formula is therefore that of Eucl. II. 4, $a^2 + 2ab + b^2 = (a+b)^2$.]

The first important question in relation to the above is that of date. Bürk assigns to the *Apastamba-Sulba-Sūtra* a date at least as early as the 5th or 4th century B.C. He observes however (what is likely enough) that the matter of it must have been much older than the book itself. Further, as regards one of the constructions for right angles, that by means of cords of lengths 15, 36, 39, he shows that it was known at the time of the *Tāittirīya-Saṃhitā* and the *Satapatha-Bṛāhmaṇa*, still older works belonging to the 8th century B.C. at latest. It may be that (as Bürk maintains) the discovery that triangles with sides (a, b, c) in rational numbers such that $a^2 + b^2 = c^2$ are right-angled was nowhere made so early as in India. We find however in two ancient Chinese treatises (1) a statement that the diagonal of the rectangle (3, 4) is 5 and (2) a rule for finding the hypotenuse of a "right triangle" from the sides, while tradition connects both works with the name of Chou Kung

who died 1105 B.C. (D. E. Smith, *History of Mathematics*, I. pp. 30—33, II. p. 288).

As regards the various "rational rectangles" used by Apastamba, it is to be observed that two of the seven, viz. 8, 15, 17 and 12, 35, 37, do not belong to the Pythagorean series, the others consist of two which belong to it, viz. 3, 4, 5 and 5, 12, 13, and multiples of these. It is true, as remarked by Zeuthen (*op. cit.* p. 842), that the rules of II. 7 and III. 9, numbered (9) and (11) above respectively, would furnish the means of finding any number of "rational rectangles." But it would not appear that the Indians had been able to formulate any general rule; otherwise their list of such rectangles would hardly have been so meagre. Apastamba mentions seven only, really reducible to four (though one other, 7, 24, 25, appears in the Bāudhāyana-Ś.-S., supposed to be older than Apastamba). These are all that Apastamba knew of, for he adds (v. 6): "So many *recognisable* (erkennbare) constructions are there," implying that he knew of no other "rational rectangles" that could be employed. But the words also imply that the theorem of the square on the diagonal is also true of other rectangles not of the "recognisable" kind, i.e. rectangles in which the sides and the diagonal are not in the ratio of integers; this is indeed implied by the constructions for $\sqrt{2}$, $\sqrt{3}$ etc. up to $\sqrt{6}$ (cf. II. 2, VIII. 5). This is all that can be said. The theorem is, it is true, enunciated as a general proposition, but there is no sign of anything like a general proof; there is nothing to show that the assumption of its universal truth was founded on anything better than an imperfect induction from a certain number of cases, discovered empirically, of triangles with sides in the ratio of whole numbers in which the property (1) that the square on the longest side is equal to the sum of the squares on the other two sides was found to be always accompanied by the property (2) that the latter two sides include a right angle.

It remains to consider Bürk's claim that the Indians had discovered the *irrational*. This is based upon the approximate value of $\sqrt{2}$ given by Apastamba in his rule I. 6 numbered (4) above. There is nothing to show how this was arrived at, but Thibaut's suggestion certainly seems the best and most natural. The Indians may have observed that $17^2 = 289$ is nearly double of $12^2 = 144$. If so, the next question which would naturally occur to them would be, by how much the side 17 must be diminished in order that the square on it may be 288 *exactly*. If, in accordance with the Indian fashion, a gnomon with unit area were to be subtracted from a square with 17 as side, this would approximately be secured by giving the gnomon the breadth $\frac{1}{34}$, for $2 \times 17 \times \frac{1}{34} = 1$. The side of the smaller square thus arrived at would be $17 - \frac{1}{34} = 12 + 4 + 1 - \frac{1}{34}$, whence, dividing out by 12, we have

$$\sqrt{2} \doteq 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34}, \text{ approximately.}$$

But it is a far cry from this calculation of an approximate value to the discovery of the *irrational*. First, we ask, is there any sign that this value was known to be inexact? It comes directly after the statement (I. 6) that the square on the diagonal of a square is double of that square, and the rule is quite boldly stated without any qualification: "lengthen the unit by one-third and the latter by one-quarter of itself less one-thirty-fourth of this part." Further, the approximate value is actually used for the purpose of constructing a square when the side is given (II. 1). So familiar was the formula that it was apparently made the basis of a sub-division of measures of length.

Thibaut observes (*Journal of the Asiatic Society of Bengal*, XLIX., p. 241) that, according to Bāudhāyana, the unit of length was divided into 12 *fingerbreadths*, and that one of two divisions of the *fingerbreadth* was into 34 *sesame-corns*, and he adds that he has no doubt that this division, which he has not elsewhere met, owes its origin to the formula for $\sqrt{2}$. The result of using this sub-division would be that, in a square with side equal to 12 *fingerbreadths*, the diagonal would be 17 *fingerbreadths* less 1 *sesame-corn*. Is it conceivable that a sub-division of a measure of length would be based on an evaluation known to be inexact? No doubt the first discoverer would be aware that the area of a gnomon with breadth $\frac{1}{34}$ and outer side 17 is not exactly equal to 1 but less than it by the square of $\frac{1}{34}$ or by $\frac{1}{1156}$, and therefore that, in taking that gnomon as the proper area to be subtracted from 17^2 , he was leaving out of account the small fraction $\frac{1}{1156}$; as, however, the object of the whole proceeding was purely practical, he would, without hesitation, ignore this as being of no practical importance, and, thereafter, the formula would be handed down and taken as a matter of course without arousing suspicion as to its accuracy. This supposition is confirmed by reference to the sort of rules which the Indians allowed themselves to regard as accurate. Thus Āpastamba himself gives a construction for a circle equal in area to a given square, which is equivalent to taking $\pi = 3.09$, and yet observes that it gives the required circle "*exactly*" (III. 2), while his construction of a square equal to a circle, which he equally calls "*exact*," makes the side of the square equal to $\frac{1}{16}$ ths of the diameter of the circle (III. 3), and is equivalent to taking $\pi = 3.004$. But, even if some who used the approximation for $\sqrt{2}$ were conscious that it was not quite accurate (of which there is no evidence), there is an immeasurable difference between arrival at this consciousness and the discovery of the irrational. As Vogt says, three stages had to be passed through before the irrationality of the diagonal of a square was discovered in any real sense. (1) All values found by direct measurement or calculations based thereon have to be recognised as being inaccurate. Next (2) must supervene the conviction that it is *impossible* to arrive at an accurate arithmetical expression of the value. And lastly (3) the impossibility must be proved. Now there is no real evidence that the Indians, at the date in question, had even reached the first stage, still less the second or third.

The net results then of Bürk's papers and of the criticisms to which they have given rise appear to be these. (1) It must be admitted that Indian geometry had reached the stage at which we find it in Āpastamba quite independently of Greek influence. But (2) the old Indian geometry was purely empirical and practical, far removed from abstractions such as the irrational. The Indians had indeed, by trial in particular cases, persuaded themselves of the truth of the Pythagorean theorem and enunciated it in all its generality; but they had not established it by scientific proof.

Alternative proofs.

I. The well-known proof of 1. 47 obtained by putting two squares side by side, with their bases continuous, and cutting off right-angled triangles which can then be put on again in different positions, is attributed by an-Nairīzī to Thābit b. Qurra (826—901 A.D.).

His actual construction proceeds thus.

Let ABC be the given triangle right-angled at A .

Construct on AB the square AD ;

produce AC to F so that EF may be equal to AC .

Construct on EF the square EG , and produce DH to K so that DK may be equal to AC .

It is then proved that, in the triangles BAC , CFG , KHG , BDK ,

the sides BA , CF , KH , BD are all equal, and

the sides AC , FG , HG , DK are all equal.

The angles included by the equal sides are all right angles; hence the four triangles are equal in all respects. [1. 4]

Hence BC , CG , GK , KB are all equal.

Further the angles DBK , ABC are equal; hence, if we add to each the angle DBC , the angle KBC is equal to the angle ABD and is therefore a right angle.

In the same way the angle CGK is right; therefore $BCGK$ is a square, i.e. the square on BC .

Now the sum of the quadrilateral $GCLH$ and the triangle LDB together with two of the equal triangles make the squares on AB , AC , and together with the other two make the square on BC .

Therefore etc.

II. Another proof is easily arrived at by taking the particular case of Pappus' more general proposition given below in which the given triangle is right-angled and the parallelograms on the sides containing the right angles are squares. If the figure is drawn, it will be seen that, with no more than one additional line inserted, it contains Thābit's figure, so that Thābit's proof may have been practically derived from that of Pappus.

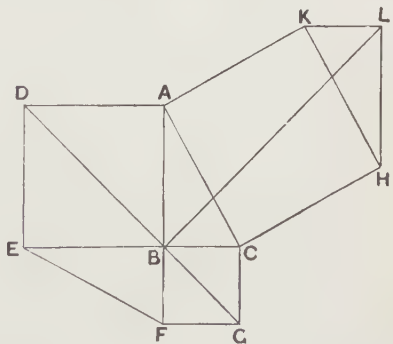
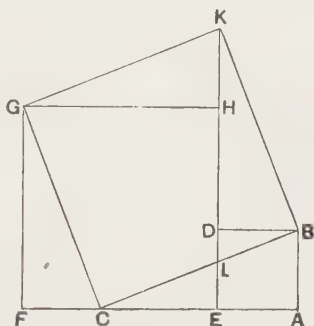
III. The most interesting of the remaining proofs seems to be that shown in the accompanying figure. It is given by J. W. Müller, *Systematische Zusammenstellung der wichtigsten bisher bekannten Beweise des Pythag. Lehrsatzes* (Nürnberg, 1819), and in the second edition (Mainz, 1821) of Ign. Hoffmann, *Der Pythag. Lehrsatz mit 32 theils bekannten theils neuen Beweisen* [3 more in second edition]. It appears to come from one of the scientific papers of Leonardo da Vinci (1452—1519).

The triangle HKL is constructed on the base KH with the side KL equal to BC and the side LH equal to AB .

Then the triangle HLK is equal in all respects to the triangle ABC , and to the triangle EBF .

Now DB , BG , which bisect the angles ABE , CBF respectively, are in a straight line. Join BL .

It is easily proved that the four quadrilaterals $ADGC$, $EDGF$, $ABLK$, $HLBC$ are all equal.



Hence the hexagons $ADEFGC$, $ABCHLK$ are equal.

Subtracting from the former the two triangles ABC , EBF , and from the latter the two equal triangles ABC , HLK , we prove that

the square CK is equal to the sum of the squares AE , CF .

Pappus' extension of I. 47.

In this elegant extension the triangle may be *any* triangle (not necessarily right-angled), and *any* parallelograms take the place of squares on two of the sides.

Pappus (iv. p. 177) enunciates the theorem as follows:

If ABC be a triangle, and any parallelograms whatever ABED, BCFG be described on AB, BC, and if DE, FG be produced to H, and HB be joined, the parallelograms ABED, BCFG are equal to the parallelogram contained by AC, HB in an angle which is equal to the sum of the angles BAC, DHB.

Produce HB to K ; through A , C draw AL , CM parallel to HK , and join LM .

Then, since $ALHB$ is a parallelogram, AL , HB are equal and parallel. Similarly MC , HB are equal and parallel.

Therefore AL , MC are equal and parallel;

whence LM , AC are also equal and parallel,

and $ALMC$ is a parallelogram.

Further, the angle LAC of this parallelogram is equal to the sum of the angles BAC , DHB , since the angle DHB is equal to the angle LAB .

Now, since the parallelogram $DABE$ is equal to the parallelogram $LABH$ (for they are on the same base AB and in the same parallels AB , DH), and likewise $LABH$ is equal to $LAKN$ (for they are on the same base LA and in the same parallels LA , HK),

the parallelogram $DABE$ is equal to the parallelogram $LAKN$.

For the same reason,

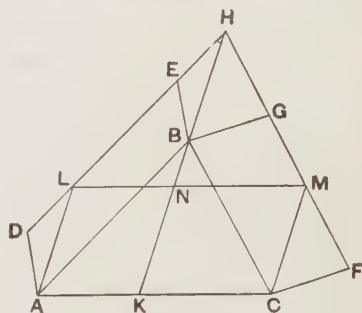
the parallelogram $BGFC$ is equal to the parallelogram $NKCM$.

Therefore the sum of the parallelograms $DABE$, $BGFC$ is equal to the parallelogram $LACM$, that is, to the parallelogram which is contained by AC , HB in an angle LAC which is equal to the sum of the angles BAC , DHB .

"And this is far more general than what is proved in the Elements about squares in the case of right-angled (triangles)."

Heron's proof that AL , BK , CF in Euclid's figure meet in a point.

The final words of Proclus' note on I. 47 (p. 429, 9—15) are historically interesting. He says: "The demonstration by the writer of the Elements being clear, I consider that it is unnecessary to add anything further, and that we may be satisfied with what has been written, since in fact those who have added anything more, like Pappus and Heron, were obliged to draw upon what is proved in the sixth Book, for no really useful object." These words cannot



of course refer to the extension of 1. 47 given by Pappus; but the key to them, so far as Heron is concerned, is to be found in the commentary of an-Nairizī (pp. 175—185, ed. Besthorn-Heiberg; pp. 78—84, ed. Curtze) on 1. 47, wherein he gives Heron's proof that the lines AL , FC , BK in Euclid's figure meet in a point. Heron proved this by means of three lemmas which would most naturally be proved from the principle of similitude as laid down in Book vi., but which Heron, as a *tour de force*, proved on the principles of Book I. only. The *first* lemma is to the following effect.

If, in a triangle ABC , DE be drawn parallel to the base BC , and if AF be drawn from the vertex A to the middle point F of BC , then AF will also bisect DE .

This is proved by drawing HK through A parallel to DE or BC , and HDL , KEM through D , E respectively parallel to AGF , and lastly joining DF , EF .

Then the triangles ABF , AFC are equal (being on equal bases), and the triangles DBF , EFC are also equal (being on equal bases and between the same parallels).

Therefore, by subtraction, the triangles ADF , AEF are equal, and hence the parallelograms AL , AM are equal.

These parallelograms are between the same parallels LM , HK ; therefore LF , FM are equal, whence DG , GE are also equal.

The *second* lemma is an extension of this to the case where DE meets BA , CA produced beyond A .

The *third* lemma proves the converse of Euclid 1. 43, that, If a parallelogram AB is cut into four others $ADGE$, DF , $FGCB$, CE , so that DF , CE are equal, the common vertex G will be on the diagonal AB .

Heron produces AG till it meets CF in H . Then, if we join HB , we have to prove that AHB is one straight line. The proof is as follows. Since the areas DF , EC are equal, the triangles DGF , ECG are equal.

If we add to each the triangle GCF ,

the triangles ECF , DCF are equal;

therefore ED , CF are parallel.

Now it follows from 1. 34, 29 and 26 that the triangles AKE , GKD are equal in all respects;

therefore EK is equal to KD .

Hence, by the second lemma,

CH is equal to HF .

Therefore, in the triangles FHB , CHG ,

the two sides BF , FH are equal to the two sides GC , CH ,

and the angle BFH is equal to the angle GCH ;

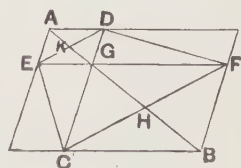
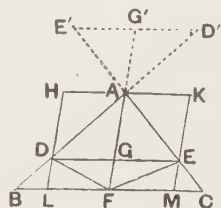
hence the triangles are equal in all respects,

and the angle BHF is equal to the angle GHC .

Adding to each the angle GHF , we find that the angles BHF , FHG are equal to the angles CHG , GHF ,

and therefore to two right angles.

Therefore AHB is a straight line.



therefore the squares on DA , AC are equal to the squares on BA , AC .

But the square on DC is equal to the squares on DA , AC , for the angle DAC is right ; [I. 47]

and the square on BC is equal to the squares on BA , AC , for this is the hypothesis ;

therefore the square on DC is equal to the square on BC ,
so that the side DC is also equal to BC .

And, since DA is equal to AB ,
and AC is common,

the two sides DA , AC are equal to the two sides BA , AC ;

and the base DC is equal to the base BC ;

therefore the angle DAC is equal to the angle BAC . [I. 8]

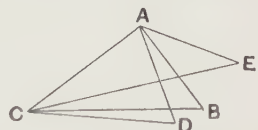
But the angle DAC is right ;

therefore the angle BAC is also right.

Therefore etc.

Q. E. D.

Proclus' note (p. 430) on this proposition, though it does not mention Heron's name, gives an alternative proof, which is the same as that definitely attributed by an-Nairizī to Heron, the only difference being that Proclus demonstrates two cases in full, while Heron dismisses the second with a "similarly." The alternative proof is another instance of the use of I. 7 as a means of answering objections. If, says Proclus, it be not admitted that the perpendicular AD may be drawn on the opposite side of AC from B , we may draw it on the same side as AB , in which case it is impossible that it should not coincide with AB . Proclus takes two cases, first supposing that the perpendicular falls, as AD , within the angle CAB , and secondly that it falls, as AE , outside that angle. In either case the absurdity results that, on the same straight line AC and on the same side of it, AD , DC must be respectively equal to AB , BC , which contradicts I. 7.



Much to the same effect is the note of De Morgan that there is here "an appearance of avoiding indirect demonstration by drawing the triangles on different sides of the base and appealing to I. 8, because drawing them on the same side would make the appeal to I. 7 (on which, however, I. 8 is founded)."

BOOK II.

DEFINITIONS.

1. Any rectangular parallelogram is said to be **contained** by the two straight lines containing the right angle.

2. And in any parallelogrammic area let any one whatever of the parallelograms about its diameter with the two complements be called a **gnomon**.

DEFINITION 1.

Πᾶν παραλληλόγραμμον ὀρθογώνιον περιέχεσθαι λέγεται ὑπὸ δύο τῶν τὴν ὀρθὴν γωνίαν περιεχουσῶν εὐθευῶν.

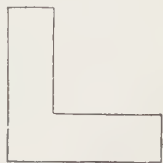
As the full expression in Greek for "the angle BAC " is "the angle contained by the (straight lines) BA, AC ," ἡ ὑπὸ τῶν BA, AC περιεχομένη γωνία, so the full expression for "the rectangle contained by BA, AC " is τὸ ὑπὸ τῶν BA, AC περιεχόμενον ὀρθογώνιον. In this case too BA, AC is commonly abbreviated by the Greek geometers into BAC . Thus in Archimedes and Apollonius τὸ ὑπὸ BAC or τὸ ὑπὸ τῶν BAC means *the rectangle BA, AC* , just as ἡ ὑπὸ BAC means *the angle BAC* ; the gender of the article shows which is meant in each case. In the early Books Euclid uses the full expression τὸ ὑπὸ τῶν BA, AC ; but the shorter form τὸ ὑπὸ τῶν BAC is found from Book x. onwards. Cf. XII. 11, where τὰ (τμήματα) ἐπὶ τῶν $\Theta O E, E \Pi Z, Z P H, H \Sigma \Theta$ means the segments on the eight straight lines $\Theta O, O E, E \Pi, \Pi Z, Z P, P H, H \Sigma, \Sigma \Theta$.

DEFINITION 2.

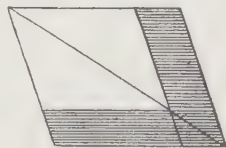
Παντὸς δὲ παραλληλογράμμου χωρίου τῶν περὶ τὴν διάμετρον αὐτοῦ παραλληλογράμμων ἓν ὅποιονοῦν σὺν τοῖς δυσὶ παραπληρώμασι γνῶμων καλεῖσθω.

Meaning literally a thing enabling something to be *known, observed or verified*, a *teller or marker*, as we might say, the word *gnomon* (γνώμων) was first used in the sense (1) in which it appears in a passage of Herodotus (II. 109) stating that "the Greeks learnt the πόλος, the *gnomon* and the twelve parts of the day from the Babylonians." According to Suidas, it was Anaximander (611—545 B.C.) who introduced the *gnomon* into Greece. Whatever may be the details of the construction of the two instruments called the πόλος and the *gnomon*, so much is certain, that the *gnomon* had to do with the

measurement of time by shadows thrown by the sun, and that the word signified the placing of a staff perpendicular to the horizon. This is borne out by the statement of Proclus that Oenopides of Chios, who first investigated the problem (Eucl. I. 12) of drawing a perpendicular from an external point to a given straight line, called the perpendicular a straight line drawn "*gnomon-wise*" (κατὰ γνώμονα). Then (2) we find the term used of a mechanical instrument for drawing right angles, as shown in the figure annexed. This seems to be the meaning in Theognis 805, where it is said that the envoy sent to consult the oracle at Delphi should be "straighter (ἰθύτερος) than the τόρνος, the στάθμη and the *gnomon*," and all three words evidently denote appliances, the τόρνος being an instrument for drawing a circle (probably a string stretched between a fixed and a moving point), and the στάθμη a plumb-line. Next (3) it was natural that the *gnomon*, owing to its shape, should become the figure which remained of a square when a smaller square was cut out of one corner (or the figure, as Aristotle says, which when added to a square increases its size but does not alter its form). We have seen (note on I. 47, p. 351) that the Pythagoreans used the term in this sense, and further applied it, by analogy, to the series of odd numbers as having the same property in relation to square numbers. The earliest evidence for this is the fragment of Philolaus (c. 460 B.C.) already mentioned (see Boeckh, *Philolaos des Pythagoreers Lehren*, p. 141) where he says that "number makes all things knowable and mutually agreeing (ποτάγορα ἀλλάλοις) in the way characteristic of the *gnomon*" (κατὰ γνώμονος φύσιν). As Boeckh says (p. 144), it would appear from the fragment that the connexion between the *gnomon* and the square to which it is added was regarded as symbolical of union and agreement, and that Philolaus used the idea to explain the knowledge of things, making the *knowing* embrace and grasp the *known* as the *gnomon* does the square. Cf. Scholium II. No. 11 (Euclid, ed. Heiberg, Vol. v. p. 225), which says "It is to be noted that the *gnomon* was discovered by geometers with a view to brevity, while the name came from its incidental property, namely that from it the whole is known, whether of the whole area or of the remainder, when it is either placed round or taken away. In sundials too its sole function is to make the actual time of day known."



The geometrical meaning of the word is extended in the definition of *gnomon* given by Euclid, where (4) the *gnomon* has the same relation to *any parallelogram* as it before had to a *square*. From the fact that Euclid says "let" the figure described "*be called a gnomon*" we may infer that he was using the word in the wider sense for the first time. Later still (5) we find Heron of Alexandria defining a *gnomon in general* as any figure which, when added to any figure whatever, makes the whole figure similar to that to which it is added. In this definition of Heron (Def. 58) Hultsch brackets the words which make it apply to any *number* as well; but Theon of Smyrna, who explains that plane, triangular, square, solid and other kinds of numbers are so called after the likeness of the areas which they measure, does make the term in its most general sense apply to numbers. "All the successive numbers which [by being successively added] produce triangles or squares or polygons are called *gnomons*" (p. 37, 11—13, ed. Hiller). Thus the successive odd numbers added



together make square numbers; the gnomons in the case of triangular numbers are the successive numbers 1, 2, 3, 4...; those for pentagonal numbers are the series 1, 4, 7, 10... (the common difference being 3), and so on. In general, the successive *gnomonic* numbers for any polygonal number, say of n sides, have $n - 2$ for their common difference (Theon of Smyrna, p. 34, 13—15).

GEOMETRICAL ALGEBRA.

We have already seen (cf. part of the note on 1. 47 and the above note on the *gnomon*) how the Pythagoreans and later Greek mathematicians exhibited different kinds of numbers as forming different geometrical figures. Thus, says Theon of Smyrna (p. 36, 6—11), "plane numbers, triangular, square and solid numbers, and the rest, are not so called independently (*κνρίως*) but in virtue of their similarity to the areas which they measure; for 4, since it measures a square area, is called square by adaptation from it, and 6 is called oblong for the same reason." A "plane number" is similarly described as a number obtained by multiplying two numbers together, which two numbers are sometimes spoken of as "sides," sometimes as the "length" and "breadth" respectively, of the number which is their product.

The *product* of two numbers was thus represented geometrically by the *rectangle* contained by the straight lines representing the two numbers respectively. It only needed the discovery of incommensurable or irrational straight lines in order to represent geometrically by a rectangle the product of any two quantities whatever, rational or irrational; and it was possible to advance from a geometrical arithmetic to a geometrical *algebra*, which indeed by Euclid's time (and probably long before) had reached such a stage of development that it could solve the same problems as our algebra so far as they do not involve the manipulation of expressions of a degree higher than the second. In order to make the geometrical algebra so generally effective, the theory of proportions was essential. Thus, suppose that x, y, z etc. are quantities which can be represented by straight lines, while α, β, γ etc. are coefficients which can be expressed by ratios between straight lines. We can then by means of Book VI. find a single straight line d such that

$$\alpha x + \beta y + \gamma z + \dots = d.$$

To solve the simple equation in its general form

$$\alpha x + a = b,$$

where α represents any ratio between straight lines, also requires recourse to the sixth Book, though, e.g., if α is $\frac{1}{2}$ or $\frac{1}{3}$ or any submultiple of unity, or if α is 2, 4 or any power of 2, we should not require anything beyond Book I. for solving the equation. Similarly the general form of a quadratic equation requires Book VI. for its geometrical solution, though particular quadratic equations may be so solved by means of Book II. alone.

Besides enabling us to solve geometrically these particular quadratic equations, Book II. gives the geometrical proofs of a number of algebraical formulae. Thus the first ten propositions give the equivalent of the several identities

$$1. \quad a(b + c + d + \dots) = ab + ac + ad + \dots,$$

$$2. \quad (a + b)a + (a + b)b = (a + b)^2,$$

$$3. \quad (a + b)a = ab + a^2,$$

$$4. \quad (a + b)^2 = a^2 + b^2 + 2ab,$$

5. $ab + \left(\frac{a+b}{2} - b\right)^2 = \left(\frac{a+b}{2}\right)^2$,
or $(a + \beta)(a - \beta) + \beta^2 = a^2$,
6. $(2a + b)b + a^2 = (a + b)^2$,
or $(a + \beta)(\beta - a) + a^2 = \beta^2$,
7. $(a + b)^2 + a^2 = 2(a + b)a + b^2$,
or $a^2 + \beta^2 = 2a\beta + (a - \beta)^2$,
8. $4(a + b)a + b^2 = \{(a + b) + a\}^2$,
or $4a\beta + (a - \beta)^2 = (a + \beta)^2$,
9. $a^2 + b^2 = 2\left\{\left(\frac{a+b}{2}\right)^2 + \left(\frac{a+b}{2} - b\right)^2\right\}$,
or $(a + \beta)^2 + (a - \beta)^2 = 2(a^2 + \beta^2)$,
10. $(2a + b)^2 + b^2 = 2\{a^2 + (a + b)^2\}$,
or $(a + \beta)^2 + (\beta - a)^2 = 2(a^2 + \beta^2)$.

The form of these identities may of course be varied according to the different symbols which we may use to denote particular portions of the lines given in Euclid's figures. They are, for the most part, simple identities, but there is no reason to suppose that these were the only applications of the geometrical algebra that Euclid and his predecessors had been able to make. We may infer the very contrary from the fact that Apollonius in his *Conics* frequently states without proof much more complicated propositions of the kind.

It is important however to bear in mind that the whole procedure of Book II. is *geometrical*; rectangles and squares are shown in the figures, and the equality of certain combinations to other combinations is proved by those figures. We gather that this was the classical or standard method of proving such propositions, and that the *algebraical* method of proving them, with no figure except a line with points marked thereon, was a later introduction. Accordingly Eutocius' method of proving certain lemmas assumed by Apollonius (*Conics*, II. 23 and III. 29) probably represents more nearly than Pappus' proof of the same the point of view from which Apollonius regarded them.

It would appear that Heron was the first to adopt the *algebraical* method of demonstrating the propositions of Book II., beginning from the second, without figures, as consequences of the first proposition corresponding to

$$a(b + c + d) = ab + ac + ad.$$

According to an-Nairizi (ed. Curtze, p. 89), Heron explains that it is not possible to prove II. 1 without drawing a number of lines (i.e. without actually drawing the rectangles), but that the following propositions up to II. 10 inclusive can be proved by merely drawing one line. He distinguishes two varieties of the method, one by *dissolutio*, the other by *compositio*, by which he seems to mean *splitting-up* of rectangles and squares, and *combination* of them into others. But in his proofs he sometimes combines the two varieties.

When he comes to II. 11, he says that it is not possible to do without a figure because the proposition is a problem, which accordingly requires an *operation* and therefore the drawing of a figure.

The algebraical method has been preferred to Euclid's by some English editors; but it should not find favour with those who wish to preserve the

essential features of Greek geometry as presented by its greatest exponents, or to appreciate their point of view.

It may not be out of place to add a word with reference to the geometrical equivalent of the algebraical operations. The addition and subtraction of quantities represented in the geometrical algebra by lines is of course effected by producing the line to the required extent or cutting off a portion of it. The equivalent of multiplication is the construction of the rectangle of which the given lines are adjacent sides. The equivalent of the division of one quantity represented by a line by another quantity represented by a line is simply the statement of a *ratio* between lines on the principles of Books v. and vi. The division of a product of two quantities by a third is represented in the geometrical algebra by the finding of a rectangle with one side of a given length and equal to a given rectangle or square. This is the problem of *application of areas* solved in I. 44, 45. The addition and subtraction of products is, in the geometrical algebra, the addition and subtraction of rectangles or squares; the sum or difference can be transformed into a single rectangle by means of the *application of areas* to any line of given length, corresponding to the algebraical process of finding a common measure. Lastly, the extraction of the square root is, in the geometrical algebra, the finding of a square equal to a given rectangle, which is done in II. 14 with the help of I. 47.

BOOK II. PROPOSITIONS.

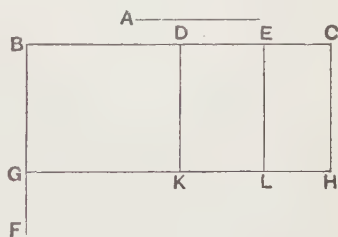
PROPOSITION 1.

If there be two straight lines, and one of them be cut into any number of segments whatever, the rectangle contained by the two straight lines is equal to the rectangles contained by the uncut straight line and each of the segments.

5 Let A, BC be two straight lines, and let BC be cut at random at the points D, E ;

I say that the rectangle contained by A, BC is equal to the rectangle contained by A, BD ,
that contained by A, DE and
10 that contained by A, EC .

For let BF be drawn from B
at right angles to BC ; [I. 11]
let BG be made equal to A , [I. 3]
through G let GH be drawn
15 parallel to BC , [I. 31]
and through D, E, C let $DK,$
 EL, CH be drawn parallel to
 BG .



Then BH is equal to BK, DL, EH .

20 Now BH is the rectangle A, BC , for it is contained by GB, BC , and BG is equal to A ;

BK is the rectangle A, BD , for it is contained by GB, BD , and BG is equal to A ;

and DL is the rectangle A, DE , for DK , that is BG [I. 34],
25 is equal to A .

Similarly also EH is the rectangle A, EC .

Therefore the rectangle A, BC is equal to the rectangle A, BD , the rectangle A, DE and the rectangle A, EC .

Therefore etc.

Q. E. D.

20. the rectangle *A, BC*. From this point onward I shall translate thus in cases where Euclid leaves out the word *contained* (περιεχόμενον). Though the word "rectangle" is also omitted in the Greek (the neuter article being sufficient to show that the rectangle is meant), it cannot be dispensed with in English. De Morgan advises the use of the expression "the rectangle *under* two lines." This does not seem to me a very good expression, and, if used in a translation from the Greek, it might suggest that *ὑπὸ* in *τὸ ὑπὸ* meant *under*, which it does not.

This proposition, the geometrical equivalent of the algebraical formula

$$a(b + c + d + \dots) = ab + ac + ad + \dots,$$

can, of course, easily be extended so as to correspond to the more general algebraical proposition that the product of an expression consisting of any number of terms added together and another expression also consisting of any number of terms added together is equal to the sum of all the products obtained by multiplying each term of one expression by all the terms of the other expression, one after another. The geometrical proof of the more general proposition would be effected by means of a figure showing all the rectangles corresponding to the partial products, in the same way as they are shown in the simpler case of II. 1; the difference would be that a series of parallels to *BC* would have to be drawn as well as the series of parallels to *BF*.

PROPOSITION 2.

If a straight line be cut at random, the rectangle contained by the whole and both of the segments is equal to the square on the whole.

For let the straight line *AB* be cut at random at the point *C*;

I say that the rectangle contained by *AB, BC* together with the rectangle contained by *BA, AC* is equal to the square on *AB*.

For let the square *ADEB* be described on *AB* [I. 46], and let *CF* be drawn through *C* parallel to either *AD* or *BE*. [I. 31]

Then *AE* is equal to *AF, CE*.

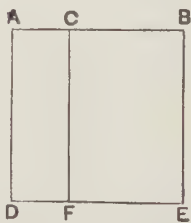
Now *AE* is the square on *AB*;

AF is the rectangle contained by *BA, AC*, for it is contained by *DA, AC*, and *AD* is equal to *AB*;

and *CE* is the rectangle *AB, BC*, for *BE* is equal to *AB*.

Therefore the rectangle *BA, AC* together with the rectangle *AB, BC* is equal to the square on *AB*.

Therefore etc.



The *fact* asserted in the enunciation of this proposition has already been used in the proof of I. 47; but there was no occasion in that proof to observe that the two rectangles BL , CL making up the square on BC are the rectangles contained by BC and the two parts, respectively, into which it is divided by the perpendicular from A on BC . It is this fact which it is necessary to state in this proposition, in accordance with the plan of Book II.

The second and third propositions are of course particular cases of the first. They were no doubt separately enunciated by Euclid in order that they might be immediately available for use hereafter, instead of having to be deduced for the particular occasion from II. 1. For, if they had not been thus separately stated, it would scarcely have been practicable to quote them later without explaining at the same time that they are included in II. 1 as particular cases. And, though the propositions are not used by Euclid in the later propositions of Book II., they are used afterwards in XIII. 10 and IX. 15 respectively; and they are of extreme importance for geometry generally, being constantly used by Pappus, for example, who frequently quotes the third proposition by the Book and number.

Attention has been called to the fact that II. 1 is never used by Euclid; and this may seem no less remarkable than the fact that II. 2, 3 are not again used in Book II. But it is important, I think, to observe that the proofs of all the first ten propositions of Book II. are practically independent of each other, though the results are really so interwoven that they can often be deduced from each other in a variety of ways. What then was Euclid's intention, first in inserting some propositions not immediately required, and secondly in making the proofs of the first ten practically independent of each other? Surely the object was to show the power of the *method* of geometrical algebra as much as to arrive at results. From the point of view of illustrating the *method*, there can be no doubt that Euclid's procedure is far more instructive than the semi-algebraical substitutes which seem to find a good deal of favour; practically it means that, instead of relying on our memory of a few standard formulae, we can use the machinery given us by Euclid's method to prove immediately *ab initio* any of the propositions taken at random.

Let us contrast with Euclid's plan the semi-algebraical alternative. One editor, for example, thinks that, as II. 1 is not used by Euclid afterwards, it seems more logical to deduce from it those of the subsequent propositions which can be readily so deduced. Putting this idea into practice, he proves II. 2 and 3 by quoting II. 1, then proves II. 4 by means of II. 1 and 3, II. 5 and 6 by means of II. 1, 3 and 4, and so on. The result is ultimately to deduce the whole of the first ten propositions from II. 1, which Euclid does not use at all; and this is to give an importance to II. 1 which is altogether disproportionate and, by starting with such a narrow foundation, to make the whole structure of Book II. top-heavy.

Editors have of course been much influenced by a desire to make the proofs of the propositions of Book II. easier, as they think, for schoolboys. But, even from this point of view, is it an improvement to deduce II. 2 and 3 from II. 1 as corollaries? I doubt it. For, in the first place, Euclid's figures *visualise* the results and so make it easier to grasp their meaning; the truth of the propositions is made clear even to the eye. Then, in the matter of brevity, to which such an exaggerated importance is attached, Euclid's proof positively has the advantage. Counting a capital letter or a collocation of such as one word, I find, e.g., that Mr H. M. Taylor's proof of II. 2 contains

120 words, of which 8 represent the construction. Euclid's as above translated has 126 words, of which 22 are descriptive of the construction; therefore the actual *proof* by Euclid has 8 words fewer than Mr Taylor's, and the extra words due to the construction in Euclid are much more than atoned for by the advantage of picturing the result in the figure.

The advantages then which Euclid's method may claim are, I think, these: in the case of II. 2, 3 it produces the result more easily and clearly than does the alternative proof by means of II. 1, and, in its general application, it is more powerful in that it makes us independent of any recollection of results.

PROPOSITION 3.

If a straight line be cut at random, the rectangle contained by the whole and one of the segments is equal to the rectangle contained by the segments and the square on the aforesaid segment.

For let the straight line AB be cut at random at C ;
I say that the rectangle contained by AB , BC is equal to the rectangle contained by AC , CB together with the square on BC .

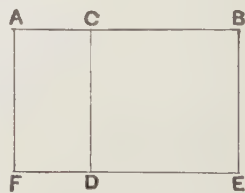
For let the square $CDEB$ be described on CB ;

[I. 46]

let ED be drawn through to F ,

and through A let AF be drawn parallel to either CD or BE .

[I. 31]



Then AE is equal to AD , CE .

Now AE is the rectangle contained by AB , BC , for it is contained by AB , BE , and BE is equal to BC ;

AD is the rectangle AC , CB , for DC is equal to CB ;

and DB is the square on CB .

Therefore the rectangle contained by AB , BC is equal to the rectangle contained by AC , CB together with the square on BC .

Therefore etc.

Q. E. D.

If we leave out of account the contents of Book II. itself and merely look to the applicability of propositions to general use, this proposition and the preceding are, as already indicated, of great importance, and particularly so to the semi-algebraical method just described, which seems to have found its first exponents in Heron and Pappus. Thus the proposition that *the difference of the squares on two straight lines is equal to the rectangle contained by the sum*

and the difference of the straight lines, which is generally given as equivalent to II. 5, 6, can be proved by means of II. 1, 2, 3, as shown by Lardner. For suppose the given straight lines are AB, BC , the latter being measured along BA .

Then, by II. 2, the square on AB is equal to the sum of the rectangles AB, BC and AB, AC .

By II. 3, the rectangle AB, BC is equal to the sum of the square on BC and the rectangle AC, CB .

Therefore the square on AB is equal to the square BC together with the sum of the rectangles AC, AB and AC, CB .

But, by II. 1, the sum of the latter rectangles is equal to the rectangle contained by AC and the sum of AB, BC , i.e. the rectangle contained by the sum and difference of AB, BC .

Hence the square on AB is equal to the square on BC and the rectangle contained by the sum and difference of AB, BC ;

that is, the difference of the squares on AB, BC is equal to the rectangle contained by the sum and difference of AB, BC .

PROPOSITION 4.

If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments.

For let the straight line AB be cut at random at C ;

I say that the square on AB is equal to the squares on AC, CB and twice the rectangle contained by AC, CB .

For let the square $ADEB$ be described on AB , [I. 46]

let BD be joined;

through C let CF be drawn parallel to either AD or EB ,

and through G let HK be drawn parallel to either AB or DE . [I. 31]

Then, since CF is parallel to AD , and BD has fallen on them,

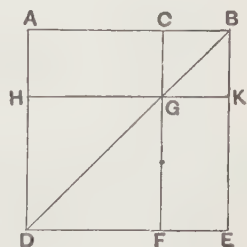
the exterior angle CGB is equal to the interior and opposite angle ADB . [I. 29]

But the angle ADB is equal to the angle ABD ,

since the side BA is also equal to AD ; [I. 5]

therefore the angle CGB is also equal to the angle GBC ,

so that the side BC is also equal to the side CG . [I. 6]



But CB is equal to GK , and CG to KB ; [I. 34]

therefore GK is also equal to KB ;

25 therefore $CGKB$ is equilateral.

I say next that it is also right-angled.

For, since CG is parallel to BK ,

the angles KBC , GCB are equal to two right angles.

[I. 29]

But the angle KBC is right;

30 therefore the angle BCG is also right,

so that the opposite angles CGK , GKB are also right.

[I. 34]

Therefore $CGKB$ is right-angled;

and it was also proved equilateral;

therefore it is a square;

35 and it is described on CB .

For the same reason

HF is also a square;

and it is described on HG , that is AC .

[I. 34]

Therefore the squares HF , KC are the squares on AC , CB .

40 Now, since AG is equal to GE ,

and AG is the rectangle AC , CB , for GC is equal to CB ,

therefore GE is also equal to the rectangle AC , CB .

Therefore AG , GE are equal to twice the rectangle AC , CB .

45 But the squares HF , CK are also the squares on AC , CB ;

therefore the four areas HF , CK , AG , GE are equal to the squares on AC , CB and twice the rectangle contained by AC , CB .

But HF , CK , AG , GE are the whole $ADEB$,

50 which is the square on AB .

Therefore the square on AB is equal to the squares on AC , CB and twice the rectangle contained by AC , CB .

Therefore etc.

Q. E. D.

2. **twice the rectangle contained by the segments.** By a curious idiom this is in Greek "the rectangle *twice contained* by the segments." Similarly "twice the rectangle contained by AC , CB " is expressed as "the rectangle *twice contained* by AC , CB " ($\tauὸ δις ὑπὸ τῶν ΑΓ, ΒΒ περιεχόμενον ὀρθογώνιον$).

35, 38. **described.** 39, 45. **the squares** (before "on"). These words are not in the Greek, which simply says that the squares "are on" ($\epsilonἰσὶν ἀπὸ$) their respective sides.

46. **areas.** It is necessary to supply some substantive (the Greek leaves it to be understood); and I prefer "areas" to "figures."

The editions of the Greek text which preceded that of E. F. August (Berlin, 1826—9) give a second proof of this proposition introduced by the usual word ἄλλως or “otherwise thus.” Heiberg follows August in omitting this proof, which is attributed to Theon, and which is indeed not worth reproducing, since it only differs from the genuine proof in that portion of it which proves that $CGKB$ is a square. The proof that $CGKB$ is equilateral is rather longer than Euclid’s, and the only interesting point to notice is that, whereas Euclid still, as in I. 46, seems to regard it as necessary to prove that *all* the angles of $CGKB$ are right angles before he concludes that it is *right-angled*, Theon says simply “And it also has the angle CBK right; therefore CK is a square.” The shorter form indicates a legitimate abbreviation of the genuine proof; because there can be no need to repeat exactly that part of the proof of I. 46 which shows that *all* the angles of the figure there constructed are right when one is.

There is also in the Greek text a Porism which is undoubtedly interpolated: “From this it is manifest that in square areas the parallelograms about the diameter are squares.” Heiberg doubted its genuineness when preparing his edition, and conjectured that it too may have been added by Theon; but the matter is placed beyond doubt by a papyrus-fragment referred to already (see Heiberg, *Paralipomena zu Euklid*, in *Hermes* xxxviii., 1903, p. 48) in which the Porism was evidently wanting. It is the only Porism in Book II., but does not correspond to Proclus’ remark (p. 304, 2) that “the Porism found in the second book belongs to a *problem*.” Heiberg regards these words as referring to the Porism to IV. 15, the correct reading having probably been not δευτέρῳ but δ’, i.e. τετάρτῳ.

The semi-algebraical proof of this proposition is very easy, and is of course old enough, being found in Clavius and in most later editions. It proceeds thus.

By II. 2, the square on AB is equal to the sum of the rectangles AB, AC and AB, CB .

But, by II. 3, the rectangle AB, AC is equal to the sum of the square on AC and the rectangle AC, CB ;

while, by II. 3, the rectangle AB, CB is equal to the sum of the square on BC and the rectangle AC, CB .

Therefore the square on AB is equal to the sum of the squares on AC, CB and twice the rectangle AC, CB .

The figure of the proposition also helps to visualise, in the orthodox manner, the proof of the theorem deduced above from II. 1—3, viz. that *the difference of the squares on two given straight lines is equal to the rectangle contained by the sum and the difference of the lines*.

For, if the lines be AB, BC respectively, the shorter of the lines being measured along BA , the figure shows that

the square AE is equal to the sum of the square CK and the rectangles AF, FK ;

that is, the square on AB is equal to the sum of the square on BC and the rectangles AB, AC and AC, BC .

But the rectangles AB, AC and BC, AC are, by II. 1, together equal to the rectangle contained by AC and the sum of AB, BC ,

i.e. to the rectangle contained by the sum and difference of AB, BC .

Whence the result follows as before.

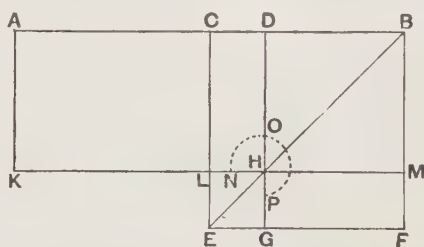
The proposition II. 4 can also be extended to the case where a straight line is divided into any number of segments; for the figure will show in like manner that the square on the whole line is equal to the sum of the squares on all the parts together with twice the rectangles contained by every pair of the parts.

PROPOSITION 5.

If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half.

For let a straight line AB be cut into equal segments at C and into unequal segments at D ;

I say that the rectangle contained by AD , DB together with the square on CD is equal to the square on CB .



For let the square $CEFB$ be described on CB , [I. 46]
and let BE be joined;

through D let DG be drawn parallel to either CE or BF ,
through H again let KM be drawn parallel to either AB or EF ,

and again through A let AK be drawn parallel to either CL or BM . [I. 31]

Then, since the complement CH is equal to the complement HF , [I. 43]

let DM be added to each;

therefore the whole CM is equal to the whole DF .

But CM is equal to AL ,

since AC is also equal to CB ; [I. 36]

therefore AL is also equal to DF .

Let CH be added to each;

therefore the whole AH is equal to the gnomon NOP .

But AH is the rectangle AD, DB , for DH is equal to DB ,

therefore the gnomon NOP is also equal to the rectangle AD, DB .

Let LG , which is equal to the square on CD , be added to each;

therefore the gnomon NOP and LG are equal to the rectangle contained by AD, DB and the square on CD .

But the gnomon NOP and LG are the whole square $CEFB$, which is described on CB ;

therefore the rectangle contained by AD, DB together with the square on CD is equal to the square on CB .

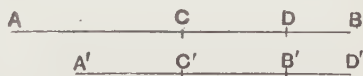
Therefore etc.

Q. E. D.

3. between the points of section, literally "between the sections," the word being the same (*τομή*) as that used of a conic section.

It will be observed that the gnomon is indicated in the figure by three separate letters and a dotted curve. This is no doubt a clearer way of showing what exactly the gnomon is than the method usual in our text-books. In this particular case the figure of the mss. has two M 's in it, the gnomon being MNE . I have corrected the lettering to avoid confusion.

It is easily seen that this proposition and the next give exactly the theorem already alluded to under the last propositions, namely that *the difference of the squares on two straight lines is equal to the rectangle contained by their sum and difference*. The two given lines are, in II. 5, the lines CB and CD , and their sum and difference are respectively equal to AD and DB . To show that II. 6 gives the same theorem we have only to make CD the greater line and CB the less, i.e. to draw $C'D'$ equal to CB , measure $C'B'$ along it equal to CD , and then produce $B'C'$ to A' , making $A'C'$ equal to $B'C'$, whence it is immediately clear that $A'D'$ on the second line is equal to AD on the first, while $D'B'$ is also equal to DB , so that the rectangles AD, DB and $A'D', D'B'$ are equal, while the difference of the squares on CB, CD is equal to the difference of the squares on $C'D', C'B'$.



Perhaps the most important fact about II. 5, 6 is however their bearing on the

Geometrical solution of a quadratic equation.

Suppose, in the figure of II. 5, that $AB = a$, $DB = x$;

then

$$\begin{aligned} ax - x^2 &= \text{the rectangle } AH \\ &= \text{the gnomon } NOP. \end{aligned}$$

Thus, if the area of the gnomon is given ($=b^2$, say), and if a is given ($=AB$), the problem of solving the equation

$$ax - x^2 = b^2$$

is, in the language of geometry, *To a given straight line (a) to apply a rectangle which shall be equal to a given square (b^2) and shall fall short by a square figure, i.e. to construct the rectangle AH or the gnomon NOP .*

Now we are told by Proclus (on I. 44) that "these propositions are ancient

and the discoveries of the Muse of the Pythagoreans, the application of areas, their exceeding and their falling-short." We can therefore hardly avoid crediting the Pythagoreans with the geometrical solution, based upon II. 5, 6, of the problems corresponding to the quadratic equations which are directly obtainable from them. It is certain that the Pythagoreans solved the problem in II. 11, which corresponds to the quadratic equation

$$a(a-x) = x^2,$$

and Simson has suggested the following easy solution of the equation now in question,

$$ax - x^2 = b^2,$$

on exactly similar lines.

Draw CO perpendicular to AB and equal to b ; produce OC to N so that $ON = CB$ (or $\frac{1}{2}a$); and with O as centre and radius ON describe a circle cutting CB in D .

Then DB (or x) is found, and therefore the required rectangle AH .

For the rectangle AD , DB together with the square on CD is equal to the square on CB , [II. 5]

i.e. to the square on OD ,

i.e. to the squares on OC , CD ; [I. 47]

whence the rectangle AD , DB is equal to the square on OC ,

or

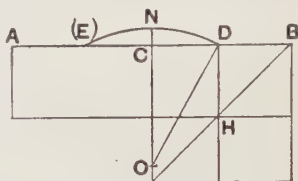
$$ax - x^2 = b^2.$$

It is of course a necessary condition of the possibility of a real solution that b^2 must not be greater than $(\frac{1}{2}a)^2$. This condition itself can easily be obtained from Euclid's proposition; for, since the sum of the rectangle AD , DB and the square on CD is equal to the square on CB , which is constant, it follows that, as CD diminishes, i.e. as D moves nearer to C , the rectangle AD , DB increases and, when D actually coincides with C , so that CD vanishes, the rectangle AD , DB becomes the rectangle AC , CB , i.e. the square on CB , and is a maximum. It will be seen also that the geometrical solution of the quadratic equation derived from Euclid does not differ from our practice of solving a quadratic by completing the square on the side containing the terms in x^2 and x .

But, while in this case there are two geometrically real solutions (because the circle described with ON as radius will not only cut CB in D but will also cut AC in another point E), Euclid's figure corresponds to one only of the two solutions. Not that there is any doubt that Euclid was aware that the method of solving the quadratic gives two solutions; he could not fail to see that $x = BE$ satisfies the equation as well as $x = BD$. If however he had actually given us the solution of the equation, he would probably have omitted to specify the solution $x = BE$ because the rectangle found by means of it, which would be a rectangle on the base AE (equal to BD) and with altitude EB (equal to AD), is really an equal rectangle to that corresponding to the other solution $x = BD$; there is therefore no real object in distinguishing two solutions. This is easily understood when we regard the equation as a statement of the problem of finding two magnitudes when their sum (a) and product (b^2) are given, i.e. as equivalent to the simultaneous equations

$$x + y = a,$$

$$xy = b^2.$$



These symmetrical equations have really only one solution, as the two apparent solutions are simply the result of interchanging the values of x and y . This form of the problem was known to Euclid, as appears from the *Data*, Prop. 85, which states that, *If two straight lines contain a parallelogram given in magnitude in a given angle, and if the sum of them be given, then shall each of them be given.*

This proposition then enables us to solve the problem of finding a rectangle the area and perimeter of which are both given; and it also enables us to infer that, of all rectangles of given perimeter, the square has the greatest area, while, the more unequal the sides are, the less is the area.

If in the figure of 11. 5 we suppose that $AD=a$, $BD=b$, we find that $CB=(a+b)/2$ and $CD=(a-b)/2$, and we may state the result of the proposition in the following algebraical form

$$\left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 = ab.$$

This way of stating it (which could hardly have escaped the Pythagoreans) gives a ready means of obtaining the two rules, respectively attributed to the Pythagoreans and Plato, for finding integral square numbers which are the sum of two other integral square numbers. We have only to make ab a perfect square in the above formula. The simplest way in which this can be done is to put $a=n^2$, $b=1$, whence we have

$$\left(\frac{n^2+1}{2}\right)^2 - \left(\frac{n^2-1}{2}\right)^2 = n^2,$$

and in order that the first two squares may be integral n^2 , and therefore n , must be odd. Hence the Pythagorean rule.

Suppose next that $a=2n^2$, $b=2$, and we have

$$(n^2+1)^2 - (n^2-1)^2 = 4n^2,$$

whence Plato's rule starting from an *even* number $2n$.

PROPOSITION 6.

If a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line.

For let a straight line AB be bisected at the point C , and let a straight line BD be added to it in a straight line;

I say that the rectangle contained by AD , DB together with the square on CB is equal to the square on CD .

For let the square $CEFD$ be described on CD , [I. 46] and let DE be joined;

through the point B let BG be drawn parallel to either EC or DF ,

through the point H let KM be drawn parallel to either AB or EF ,

and further through A let AK be drawn parallel to either CL or DM . [I. 31]

Then, since AC is equal to CB ,

AL is also equal to CH . [I. 36]

But CH is equal to HF . [I. 43]

Therefore AL is also equal to HF .

Let CM be added to each ;

therefore the whole AM is equal to the gnomon NOP .

But AM is the rectangle AD, DB ,

for DM is equal to DB ;

therefore the gnomon NOP is also equal to the rectangle AD, DB .

Let LG , which is equal to the square on BC , be added to each ;

therefore the rectangle contained by AD, DB together with the square on CB is equal to the gnomon NOP and LG .

But the gnomon NOP and LG are the whole square $CEFD$, which is described on CD ;

therefore the rectangle contained by AD, DB together with the square on CB is equal to the square on CD .

Therefore etc.

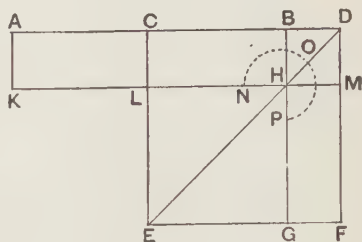
Q. E. D.

In this case the rectangle AD, DB is "a rectangle applied to a given straight line (AB) but *exceeding* by a square (the side of which is equal to BD)"; and the problem suggested by II. 6 is to find a rectangle of this description equal to a given area, which we will, for convenience, suppose to be a square; i.e., in the language of geometry, *to apply to a given straight line a rectangle which shall be equal to a given square and shall exceed by a square figure*.

We suppose that in Euclid's figure $AB = a$, $BD = x$; then, if the given square be b^2 , the problem is to solve géométrically the equation

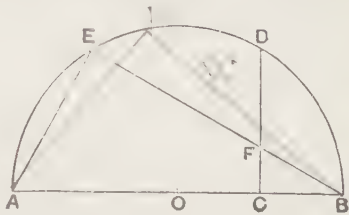
$$ax + x^2 = b^2.$$

The solution of a problem theoretically equivalent to the solution of a quadratic equation of this kind is presupposed in the fragment of Hippocrates' *Quadrature of lunes* preserved in a quotation by Simplicius (*Comment. in*



Aristot. Phys. pp. 61—68, ed. Diels) from Eudemos' *History of Geometry*. In this fragment Hippocrates (5th cent. B.C.) assumes the following construction.

AB being the diameter and O the centre of a semicircle, and C being the middle point of OB and CD at right angles to AB , a straight line of length such that its square is $1\frac{1}{2}$ times the square on the radius (i.e. of length $a\sqrt{\frac{3}{2}}$, where a is the radius) is to be so placed, as EF , between CD and the circumference AD that it "verges towards B ," that is, EF when produced passes through B .



Now the right-angled triangles BFC , BAE are similar, so that

$$BF : BC = BA : BE,$$

and therefore the rectangle BE , BF = rect. BA , BC
= sq. on BO .

In other words, EF ($= a\sqrt{\frac{3}{2}}$) being given in length, BF ($= x$, say) has to be found such that

$$(\sqrt{\frac{3}{2}}a + x)x = a^2;$$

or the quadratic equation

$$\sqrt{\frac{3}{2}}ax + x^2 = a^2$$

has to be solved.

A straight line of length $a\sqrt{\frac{3}{2}}$ would easily be constructed, for, in the figure, $CD^2 = AC \cdot CB = \frac{3}{4}a^2$, or $CD = \frac{1}{2}a\sqrt{3}$, and $a\sqrt{\frac{3}{2}}$ is the diagonal of a square of which CD is the side.

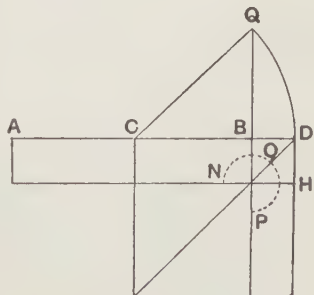
There is no doubt that Hippocrates could have solved the equation by the geometrical construction given below, but he may have contemplated, on this occasion, the merely *mechanical* process of *placing* the straight line of the length required between CD and the circumference AD and *moving* it until E , F , B were in a straight line. Zeuthen (*Die Lehre von den Kegelschnitten im Altertum*, pp. 270, 271) thinks this probable because, curiously enough, the fragment speaks immediately afterwards of "joining B to F ."

To solve the equation

$$ax + x^2 = b^2$$

we have to find the rectangle AH , or the gnomon NOP , which is equal in area to b^2 and has one of the sides containing the inner right angle equal to CB or $\frac{1}{2}a$. Thus we know $(\frac{1}{2}a)^2$ and b^2 , and we have to find, by I. 47, a square equal to the sum of two given squares.

To do this Simson draws BQ at right angles to AB and equal to b , joins CQ and, with centre C and radius CQ , describes a circle cutting AB produced in D . Thus BD , or x , is found.



Now the rectangle AD , DB together with the square on CB
is equal to the square on CD ,
i.e. to the square on CQ ,
i.e. to the squares on CB , BQ .

Therefore the rectangle AD, DB is equal to the square on BQ , that is,

$$xx + x^2 = b^2.$$

From Euclid's point of view there would only be one solution in this case.

This proposition enables us also to solve the equation

$$x^2 - ax = b^2$$

in a similar manner.

We have only to suppose that $AB = a$, and AD (instead of BD) $= x$; then

$$x^2 - ax = \text{the gnomon.}$$

To find the gnomon we have its area (b^2) and the area, CB^2 or $(\frac{1}{2}a)^2$, by which the gnomon differs from CD^2 . Thus we can find D (and therefore AD or x) by the same construction as that just given.

Converse propositions to II. 5, 6 are given by Pappus (vii. pp. 948—950) among his lemmas to the *Conics* of Apollonius to the effect that,

- (1) if D be a point dividing AB unequally, and C another point on AB such that the rectangle AD, DB together with the square on CD is equal to the square on AC , then

AC is equal to CB ;

- (2) if D be a point on AB produced, and C a point on AB such that the rectangle AD, DB together with the square on CB is equal to the square on CD , then

AC is equal to CB .

PROPOSITION 7.

If a straight line be cut at random, the square on the whole and that on one of the segments both together are equal to twice the rectangle contained by the whole and the said segment and the square on the remaining segment.

For let a straight line AB be cut at random at the point C ;

I say that the squares on AB, BC are equal to twice the rectangle contained by AB, BC and the square on CA .

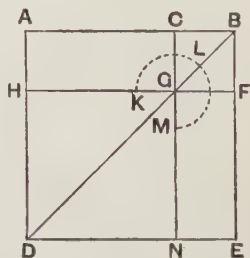
For let the square $ADEB$ be described on AB , [I. 46]
and let the figure be drawn.

Then, since AG is equal to GE , [I. 43]
let CF be added to each;

therefore the whole AF is equal to the whole CE .

Therefore AF, CE are double of AF .

But AF, CE are the gnomon KLM and the square CF ; therefore the gnomon KLM and the square CF are double of AF .



But twice the rectangle AB, BC is also double of AF ; for BF is equal to BC ; therefore the gnomon KLM and the square CF are equal to twice the rectangle AB, BC .

Let DG , which is the square on AC , be added to each; therefore the gnomon KLM and the squares BG, GD are equal to twice the rectangle contained by AB, BC and the square on AC .

But the gnomon KLM and the squares BG, GD are the whole $ADEB$ and CF ,

which are squares described on AB, BC ;

therefore the squares on AB, BC are equal to twice the rectangle contained by AB, BC together with the square on AC .

Therefore etc.

Q. E. D.

An interesting variation of the form of this proposition may be obtained by regarding AB, BC as two given straight lines of which AB is the greater, and AC as the *difference* between the two straight lines. Thus the proposition shows that the squares on two straight lines are together equal to twice the rectangle contained by them and the square on their difference. That is, *the square on the difference of two straight lines is equal to the sum of the squares on the straight lines diminished by twice the rectangle contained by them*. In other words, just as II. 4 is the geometrical equivalent of the identity

$$(a + b)^2 = a^2 + b^2 + 2ab,$$

so II. 7 proves that

$$(a - b)^2 = a^2 + b^2 - 2ab.$$

The addition and subtraction of these formulae give the algebraical equivalent of the propositions II. 9, 10 and II. 8 respectively; and we have accordingly a suggestion of alternative methods of proving those propositions.

PROPOSITION 8.

If a straight line be cut at random, four times the rectangle contained by the whole and one of the segments together with the square on the remaining segment is equal to the square described on the whole and the aforesaid segment as on one straight line.

For let a straight line AB be cut at random at the point C ;

I say that four times the rectangle contained by AB, BC together with the square on AC is equal to the square described on AB, BC as on one straight line.

For let [the straight line] BD be produced in a straight line [with AB], and let BD be made equal to CB ;

let the square $AEFD$ be described on AD , and let the figure be drawn double.

Then, since CB is equal to BD , while CB is equal to GK , and BD to KN ,

therefore GK is also equal to KN .

For the same reason

QR is also equal to RP .

And, since BC is equal to BD , and GK to KN , therefore CK is also equal to KD , and GR to RN . [I. 36]

But CK is equal to RN , for they are complements of the parallelogram CP ; [I. 43]

therefore KD is also equal to GR ;

therefore the four areas DK , CK , GR , RN are equal to one another.

Therefore the four are quadruple of CK .

Again, since CB is equal to BD , while BD is equal to BK , that is CG , and CB is equal to GK , that is GQ ,

therefore CG is also equal to GQ .

And, since CG is equal to GQ , and QR to RP ,

AG is also equal to MQ , and QL to RF . [I. 36]

But MQ is equal to QL , for they are complements of the parallelogram ML ; [I. 43]

therefore AG is also equal to RF ;

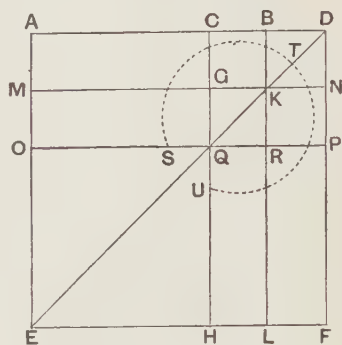
therefore the four areas AG , MQ , QL , RF are equal to one another.

Therefore the four are quadruple of AG .

But the four areas CK , KD , GR , RN were proved to be quadruple of CK ;

therefore the eight areas, which contain the gnomon STU , are quadruple of AK .

Now, since AK is the rectangle AB , BD , for BK is equal to BD ,



therefore four times the rectangle AB, BD is quadruple of AK .

But the gnomon STU was also proved to be quadruple of AK ;

therefore four times the rectangle AB, BD is equal to the gnomon STU .

Let OH , which is equal to the square on AC , be added to each;

therefore four times the rectangle AB, BD together with the square on AC is equal to the gnomon STU and OH .

But the gnomon STU and OH are the whole square $AEFD$,

which is described on AD ;

therefore four times the rectangle AB, BD together with the square on AC is equal to the square on AD .

But BD is equal to BC ;

therefore four times the rectangle contained by AB, BC together with the square on AC is equal to the square on AD , that is to the square described on AB and BC as on one straight line.

Therefore etc.

Q. E. D.

This proposition is quoted by Pappus (p. 428, ed. Hultsch) and is used also by Euclid himself in the *Data*, Prop. 86. Further, it is of decided use in proving the fundamental property of a parabola.

Two alternative proofs are worth giving.

The first is that suggested by the consideration mentioned in the last note, though the proof is old enough, being given by Clavius and others. It is of the semi-algebraical type.

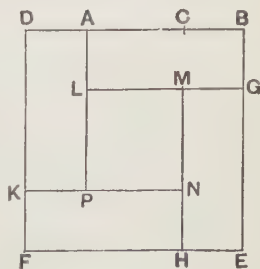
Produce AB to D (in the figure of the proposition), so that BD is equal to BC .

By 11. 4, the square on AD is equal to the squares on AB, BD and twice the rectangle AB, BD , i.e. to the squares on AB, BC and twice the rectangle AB, BC .

By 11. 7, the squares on AB, BC are equal to twice the rectangle AB, BC together with the square on AC .

Therefore the square on AD is equal to four times the rectangle AB, BC together with the square on AC .

The second proof is after the manner of Euclid but with a difference. Produce BA to D so that AD is equal to BC . On BD construct the square $BEFD$.



Take BG, EH, FK each equal to BC or AD , and draw ALP, HNM parallel to BE and GML, KPN parallel to BD .

Then it can be shown that each of the rectangles BL, AK, FN, EM is equal to the rectangle AB, BC , and that PM is equal to the square on AC .

Therefore the square on BD is equal to four times the rectangle AB, BC together with the square on AC .

PROPOSITION 9.

If a straight line be cut into equal and unequal segments, the squares on the unequal segments of the whole are double of the square on the half and of the square on the straight line between the points of section.

For let a straight line AB be cut into equal segments at C , and into unequal segments at D ;

I say that the squares on AD, DB are double of the squares on AC, CD .

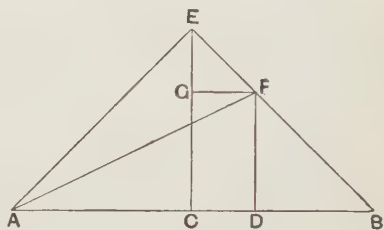
For let CE be drawn from C at right angles to AB , and let it be made equal to either AC or CB ;

let EA, EB be joined,

let DF be drawn through D parallel to EC ,

and FG through F parallel to AB ,

and let AF be joined.



Then, since AC is equal to CE ,

the angle EAC is also equal to the angle AEC .

And, since the angle at C is right,

the remaining angles EAC, AEC are equal to one right angle. [I. 32]

And they are equal;

therefore each of the angles CEA, CAE is half a right angle.

For the same reason

each of the angles CEB, EBC is also half a right angle;

therefore the whole angle AEB is right.

And, since the angle GEF is half a right angle,

and the angle EGF is right, for it is equal to the interior and opposite angle ECB , [I. 29]

the remaining angle EFG is half a right angle; [I. 32]

therefore the angle GEF is equal to the angle EFG ,

so that the side EG is also equal to GF . [I. 6]

Again, since the angle at B is half a right angle,

and the angle FDB is right, for it is again equal to the interior and opposite angle ECB , [I. 29]

the remaining angle BFD is half a right angle; [I. 32]

therefore the angle at B is equal to the angle DFB ,

so that the side FD is also equal to the side DB . [I. 6]

Now, since AC is equal to CE ,

the square on AC is also equal to the square on CE ;

therefore the squares on AC , CE are double of the square on AC .

But the square on EA is equal to the squares on AC , CE , for the angle ACE is right; [I. 47]

therefore the square on EA is double of the square on AC .

Again, since EG is equal to GF ,

the square on EG is also equal to the square on GF ;

therefore the squares on EG , GF are double of the square on GF .

But the square on EF is equal to the squares on EG , GF ;

therefore the square on EF is double of the square on GF .

But GF is equal to CD ; [I. 34]

therefore the square on EF is double of the square on CD .

But the square on EA is also double of the square on AC ;

therefore the squares on AE , EF are double of the squares on AC , CD .

And the square on AF is equal to the squares on AE , EF , for the angle AEF is right; [I. 47]

therefore the square on AF is double of the squares on AC , CD .

But the squares on AD , DF are equal to the square on AF , for the angle at D is right; [I. 47]

therefore the squares on AD , DF are double of the squares on AC , CD .

And DF is equal to DB ;
therefore the squares on AD , DB are double of the squares
on AC , CD .

Therefore etc.

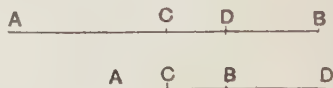
Q. E. D.

It is noteworthy that, while the first eight propositions of Book II. are proved independently of the Pythagorean theorem I. 47, all the remaining propositions beginning with the 9th are proved by means of it. Also the 9th and 10th propositions mark a new departure in another respect; the method of demonstration by showing in the figures the various rectangles and squares to which the theorems relate is here abandoned.

The 9th and 10th propositions are related to one another in the same way as the 5th and 6th; they really prove the same result which can, as in the earlier case, be comprised in a single enunciation thus: *The sum of the squares on the sum and difference of two given straight lines is equal to twice the sum of the squares on the lines.*

The semi-algebraical proof of Prop. 9 is that suggested by the remark on the algebraical formulae given at the end of the note on II. 7. It applies with a very slight modification to both II. 9 and II. 10. We will put in brackets the variations belonging to II. 10. The first of the annexed lines is the figure for II. 9 and the second for II. 10.

By II. 4, the square on AD is equal to the squares on AC , CD and twice the rectangle AC , CD .



By II. 7, the squares on CB , CD (CD , CB) are equal to
twice the rectangle CB , CD together with the square on BD .

By addition of these equals crosswise,
the squares on AD , DB together with twice the rectangle CB , CD are
equal to the squares on AC , CD , CB , CD together with twice
the rectangle AC , CD .

But AC , CB are equal, and therefore the rectangles AC , CD and CB , CD are equal.

Taking away the equals, we see that

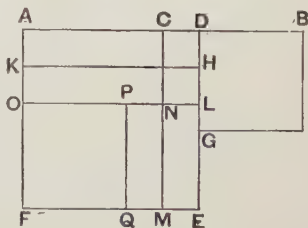
the squares on AD , DB are equal to the squares on AC , CD , CB , CD ,
i.e. to twice the squares on AC , CD .

To show also that the method of geometrical algebra illustrated by II. 1—8 is still effective for the purpose of proving II. 9, 10, we will now prove II. 9 in that manner.

Draw squares on AD , DB respectively as shown in the figure. Measure DH along DE equal to CD , and HL along HE also equal to CD .

Draw HK , LNO parallel to EF , and CNM parallel to DE .

Measure NP along NO equal to CD , and draw PQ parallel to DE .



Now, since AD , CD are respectively equal to DE , DH ,

HE is equal to AC or CB ;

and, since HL is equal to CD , LE is equal to DB .

Similarly, since each of the segments EM , MQ is equal to CD ,

FQ is equal to EL or BD .

Therefore OQ is equal to the square on DB .

We have to prove that the squares on AD , DB are equal to twice the squares on AC , CD .

Now the square on AD includes KM (the square on AC) and CH , HN (that is, twice the square on CD).

Therefore we have to prove that what is left over of the square on AD together with the square on DB is equal to the square on AC .

The parts left over are the rectangles CK and NE , which are equal to KN , PM respectively.

But the latter with the square on DB are equal to the rectangles KN , PM and the square OQ ,

i.e. to the square KM , or the square on AC .

Hence the required result follows.

PROPOSITION 10.

If a straight line be bisected, and a straight line be added to it in a straight line, the square on the whole with the added straight line and the square on the added straight line both together are double of the square on the half and of the square described on the straight line made up of the half and the added straight line as on one straight line.

For let a straight line AB be bisected at C , and let a straight line BD be added to it in a straight line;

I say that the squares on AD , DB are double of the squares on AC , CD .

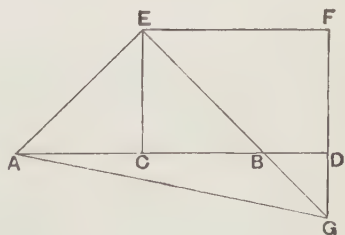
For let CE be drawn from the point C at right angles to AB [I. 11], and let it be made equal to either AC or CB [I. 3];

let EA , EB be joined;

through E let EF be drawn parallel to AD ,

and through D let FD be drawn parallel to CE . [I. 31]

Then, since a straight line EF falls on the parallel straight lines EC , FD ,



the angles CEF , EFD are equal to two right angles; [I. 29]
therefore the angles FEB , EFD are less than two right angles.

But straight lines produced from angles less than two right angles meet; [I. Post. 5]

therefore EB , FD , if produced in the direction B , D , will meet.

Let them be produced and meet at G ,
and let AG be joined.

Then, since AC is equal to CE ,
the angle EAC is also equal to the angle AEC ; [I. 5]
and the angle at C is right;

therefore each of the angles EAC , AEC is half a right angle. [I. 32]

For the same reason

each of the angles CEB , EBC is also half a right angle;
therefore the angle AEB is right.

And, since the angle EBC is half a right angle,
the angle DBG is also half a right angle. [I. 15]

But the angle BDG is also right,
for it is equal to the angle DCE , they being alternate; [I. 29]

therefore the remaining angle DGB is half a right angle; [I. 32]

therefore the angle DGB is equal to the angle DBG ,
so that the side BD is also equal to the side GD . [I. 6]

Again, since the angle EGF is half a right angle,
and the angle at F is right, for it is equal to the opposite angle, the angle at C , [I. 34]

the remaining angle FEG is half a right angle; [I. 32]
therefore the angle EGF is equal to the angle FEG ,

so that the side GF is also equal to the side EF . [I. 6]

Now, since the square on EC is equal to the square on CA ,

the squares on EC , CA are double of the square on CA .

But the square on EA is equal to the squares on EC , CA ; [I. 47]

therefore the square on EA is double of the square on AC . [C. N. 1]

Again, since FG is equal to EF ,
the square on FG is also equal to the square on FE ;
therefore the squares on GF, FE are double of the square on EF .

But the square on EG is equal to the squares on GF, FE ;
[I. 47]
therefore the square on EG is double of the square on EF .

And EF is equal to CD ; [I. 34]

therefore the square on EG is double of the square on CD .

But the square on EA was also proved double of the square on AC ;

therefore the squares on AE, EG are double of the squares on AC, CD .

And the square on AG is equal to the squares on AE, EG ; [I. 47]

therefore the square on AG is double of the squares on AC, CD .

But the squares on AD, DG are equal to the square on AG ; [I. 47]

therefore the squares on AD, DG are double of the squares on AC, CD .

And DG is equal to DB ;

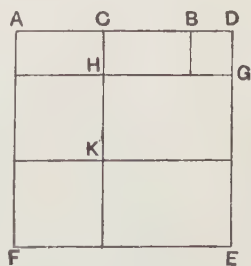
therefore the squares on AD, DB are double of the squares on AC, CD .

Therefore etc.

Q. E. D.

The alternative proof of this proposition by means of the principles exhibited in II. 1—8 follows the lines of that which I have given for the preceding proposition.

It is at once obvious from the figure that the square on AD includes within it twice the square on AC together with once the square on CD . What is left over is the sum of the rectangles AH, KE . These, which are equivalent to BH, GK , make up the square on CD less the square on BD . Adding therefore the square BG to each side, we have the required result.



Another alternative proof of the theorem which includes both II. 9 and 10 is worth giving. The theorem states that the sum of the squares on the sum and difference of two given straight lines is equal to twice the sum of the squares on the lines.

Let AD , DB be the two given straight lines (of which AD is the greater), placed so as to be in one straight line. Make AC equal to DB and complete the figure as shown, each of the segments CG and DH being equal to AC or DB .

Now, AD , DB being the given straight lines, AB is their sum and CD is equal to their difference.

Also AD is equal to BC .

And AE is the square on AB , GK is equal to the square on CD , AK or FH is the square on AD , and BL the square on CB , while each of the small squares AG , BH , EK , FL is equal to the square on AC or DB .

We have to prove that twice the squares on AD , DB are equal to the squares on AB , CD .

Now twice the square on AD is the sum of the squares on AD , CB , which is equal to the sum of the squares BL , FH ; and the figure shows these to be equal to twice the inner square GK and once the remainder of the large square AE excluding the two squares AG , KE , which latter squares are equal to twice the square on AC or DB .

Therefore twice the squares on AD , DB are equal to twice the inner square GK together with once the remainder of the large square AE , that is, to the sum of the squares AE , GK , which are the squares on AB , CD .

“Side” and “diagonal” numbers giving successive approximations to $\sqrt{2}$.

Zeuthen pointed out (*Die Lehre von den Kegelschnitten im Altertum*, 1886, pp. 27, 28) that II. 9, 10 have great interest in connexion with a problem of indeterminate analysis which received much attention from the ancient Greeks. If we take the straight line AB divided at C and D as in II. 9, and if we put $CD = x$, $DB = y$, the result obtained by Euclid, namely:

$$AD^2 + DB^2 = 2AC^2 + 2CD^2,$$

$$\text{or } AD^2 - 2AC^2 = 2CD^2 - DB^2,$$

becomes the formula

$$(2x + y)^2 - 2(x + y)^2 = 2x^2 - y^2.$$

If therefore x , y be numbers which satisfy one of the two equations

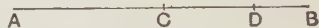
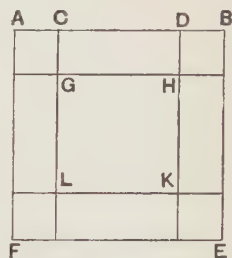
$$2x^2 - y^2 = \pm 1,$$

the formula gives us two higher numbers, $x + y$ and $2x + y$, which satisfy the other of the two equations.

Euclid's propositions thus give a general proof of the very formula used for the formation of the succession of what were called “side” and “diagonal” numbers.”

As is well known, Theon of Smyrna (pp. 43, 44, ed. Hiller) describes this system of numbers. The unit, being the beginning of all things, must be potentially both a *side* and a *diameter*. Consequently we begin with two units, the one being the first *side* and the other the first *diameter*, and (a) from the sum of them, (b) from the sum of twice the first unit and once the second, we form two new numbers

$$1 \cdot 1 + 1 = 2, \quad 2 \cdot 1 + 1 = 3.$$



Of these new numbers the first is a *side*- and the second a *diagonal*-number, or (as we may say)

$$a_2 = 2, \quad d_2 = 3.$$

In the same way as these numbers were formed from $a_1 = 1$, $d_1 = 1$, successive pairs of numbers are formed from a_2 , d_2 , and so on, according to the formula

$$a_{n+1} = a_n + d_n, \quad d_{n+1} = 2a_n + d_n.$$

Thus

$$a_3 = 2 + 3 = 5, \quad d_3 = 2 \cdot 2 + 3 = 7,$$

$$a_4 = 5 + 7 = 12, \quad d_4 = 2 \cdot 5 + 7 = 17,$$

and so on.

Theon states, with reference to these numbers, the general proposition that

$$d_n^2 = 2a_n^2 \pm 1,$$

and he observes (1) that the signs alternate as successive d 's and a 's are taken, $d_1^2 - 2a_1^2$ being equal to -1 , $d_2^2 - 2a_2^2$ equal to $+1$, $d_3^2 - 2a_3^2$ equal to -1 , and so on, (2) that the sum of the squares of *all* the d 's will be double of the sum of the squares of *all* the a 's. [If the number of successive terms in each series is *finite*, it is of course necessary that the number should be even.] The proof, no doubt omitted because it was well known, may be put algebraically thus

$$\begin{aligned} d_n^2 - 2a_n^2 &= (2a_{n-1} + d_{n-1})^2 - 2(a_{n-1} + d_{n-1})^2 \\ &= 2a_{n-1}^2 - d_{n-1}^2 \\ &= -(d_{n-1}^2 - 2a_{n-1}^2) \\ &= + (d_{n-2}^2 - 2a_{n-2}^2), \text{ in like manner,} \end{aligned}$$

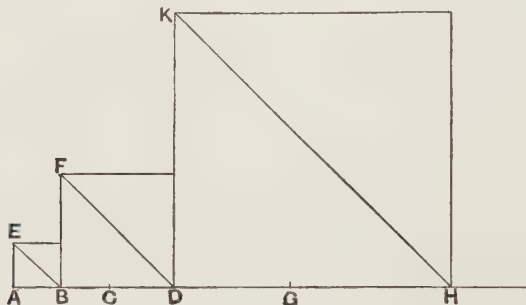
and so on, while $d_1^2 - 2a_1^2 = -1$. Thus the theorem is established.

Euclid's propositions enable us to establish the theorem geometrically; and this fact might well be thought to confirm the conjecture that the investigation of the indeterminate equation $2x^2 - y^2 = \pm 1$ in the manner explained by Theon was no new thing but began at a period long before Euclid's time. No one familiar with the truth of the proposition stated by Theon could have failed to observe that, as the corresponding *side*- and *diagonal*-numbers were successively formed, the value of d_n^2/a_n^2 would approach more and more nearly to 2, and consequently that the successive fractions d_n/a_n would give nearer and nearer approximations to the value of $\sqrt{2}$, viz. $\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \dots$.

It is fairly clear that in the famous passage of Plato's *Republic* (546 c) about the "geometrical number" some such system of approximations is hinted at. Plato there contrasts the "*rational diameter* of five" ($\rho\eta\tau\eta\ \delta\acute{\iota}\alpha\mu\epsilon\tau\rho\omicron\varsigma\ \tau\eta\varsigma\ \pi\epsilon\mu\pi\acute{\alpha}\delta\omicron\varsigma$) with the "*irrational*" (diameter). This was certainly taken from the Pythagorean theory of numbers (cf. the expression immediately preceding, 546 B, $\varsigma\ \pi\acute{\alpha}\nu\tau\alpha\ \pi\rho\omicron\sigma\eta\gamma\omicron\rho\alpha\ \kappa\alpha\iota\ \rho\eta\tau\acute{\alpha}\ \pi\rho\acute{\omicron}\varsigma\ \acute{\alpha}\lambda\lambda\eta\lambda\alpha\ \acute{\alpha}\pi\acute{\epsilon}\phi\eta\gamma\eta\nu$, with the phrase $\pi\acute{\alpha}\nu\tau\alpha\ \gamma\eta\nu\sigma\tau\acute{\alpha}\ \kappa\alpha\iota\ \pi\omicron\tau\acute{\alpha}\gamma\omicron\rho\alpha\ \acute{\alpha}\lambda\lambda\acute{\alpha}\lambda\omicron\iota\varsigma\ \acute{\alpha}\pi\epsilon\rho\gamma\acute{\alpha}\zeta\epsilon\tau\alpha\iota$ in the fragment of Philolaus). The reference of Plato is to the following consideration. If the square of side 5 be taken, the diagonal is $\sqrt{2 \cdot 25}$ or $\sqrt{50}$. This is the Pythagorean "*irrational diameter*" of 5; and the "*rational diameter*" was the approximation $\sqrt{50} - 1$, or 7.

But the conjecture of Zeuthen, and the attribution of the whole theory of *side*- and *diagonal*-numbers to the Pythagoreans, have now been fully confirmed by the publication of Kroll's edition of *Procli Diadochi in Platonis rempublicam commentarii* (Teubner), Vol. II., 1901. The passages (cc. 23 and 27, pp. 24, 25 and 27—29) which there saw the light for the first time describe the same

system of forming *side-* and *diagonal-numbers* and definitely attribute it, as well as the distinction between the "rational" and "irrational diameter," to the Pythagoreans. Proclus further says (p. 27, 16—22) that the property of the *side-* and *diagonal-numbers* "is proved graphically (*γραμματικῶς*) in the second book of the Elements by 'him' (*ἄπ' ἐκείνου*). For, *if a straight line be bisected and a straight line be added to it, the square on the whole line including the added straight line and the square on the latter alone are double of the square on the half of the original straight line and of the square on the straight line made up of the half and of the added straight line.*" And this is simply Eucl. II. 10. Proclus then goes on to show specifically how this proposition was used to prove that, with the notation above used, the *diameter* corresponding to the *side* $a + d$ is $2a + d$. Let AB be a *side* and BC equal to it, while CD is the *diameter* corresponding to AB , i.e. a straight line such that the square on it is double of the square on AB . (I use the figure supplied by Hultsch on p. 397 of Kroll's Vol. II.)



Then, by the theorem of Eucl. II. 10, the squares on AD , DC are double of the squares on AB , BD .

But the square on DC (i.e. BE) is double of the square on AB ; therefore, by subtraction, the square on AD is double of the square on BD .

And the square on DF , the *diagonal* corresponding to the *side* BD , is double the square of BD .

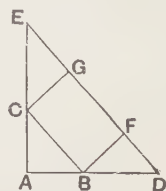
Therefore the square on DF is equal to the square on AD , so that DF is equal to AD .

That is, while the *side* BD is, with our notation, $a + d$, the corresponding *diagonal*, being equal to AD , is $2a + d$.

In the above reference by Proclus to II. 10 *ἄπ' ἐκείνου* "by him" must apparently mean *ὑπ' Εὐκλείδου*, "by Euclid," although Euclid's name has not been mentioned in the chapter; the phrase would be equivalent to saying "in the second Book of the famous Elements." But, when Proclus says "this *is proved* in the second Book of the Elements," he does not imply that it had not been proved before; on the contrary, it is clear that the theorem had been proved by the Pythagoreans, and we have therefore here a confirmation of the inference from the part played by the *gnomon* and by 1. 47 in Book II. that the whole of the substance of that Book was Pythagorean. For further detailed explanation of the passages of Proclus reference should be made to Hultsch's note in Kroll's Vol. II. pp. 393—400, and to the separate article, also by Hultsch, in the *Bibliotheca Mathematica* 13, 1900, pp. 8—12.

P. Bergh has an ingenious suggestion (see *Zeitschrift für Math. u. Physik*

xxx1. Hist.-litt. Abt. p. 135, and Cantor, *Geschichte der Mathematik*, I₃, p. 437) as to the way in which the formation of the successive *side-* and *diagonal-*numbers may have been discovered, namely by observation from a very simple geometrical figure. Let ABC be an isosceles triangle, right-angled at A , with sides a_{n-1} , a_{n-1} , d_{n-1} respectively. If now the two sides AB , AC about the right angle be lengthened by adding d_{n-1} to each, and the extremities D , E be joined, it is easily seen by means of the figure (in which BF , CG are perpendicular to DE) that the new *diagonal* d_n is equal to $2a_{n-1} + d_{n-1}$, while the equal *sides* a_n are, by construction, equal to $a_{n-1} + d_{n-1}$.

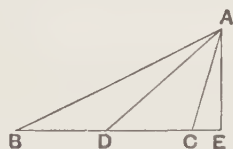
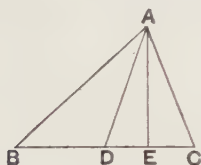


Important deductions from II. 9, 10.

I. Pappus (vii. pp. 856—8) uses II. 9, 10 for proving the well-known theorem that

The sum of the squares on two sides of a triangle is equal to twice the square on half the base together with twice the square on the straight line joining the middle point of the base to the opposite vertex.

Let ABC be the given triangle and D the middle point of the base BC . Join AD , and draw AE perpendicular to BC (produced if necessary).



Now, by II. 9, 10,

the squares on BE , EC are equal to twice the squares on BD , DE .

Add to each twice the square on AE .

Then, remembering that

the squares on BE , EA are equal to the square on BA ,

the squares on AE , EC are equal to the square on AC ,

and the squares on AE , ED are equal to the square on AD ,

we find that

the squares on BA , AC are equal to twice the squares on AD , BD .

The proposition is generally proved by means of II. 12, 13, but not, I think, so conveniently as by the method of Pappus.

II. The inference was early made by Gregory of St. Vincent (1584–1667) and Viviani (1622–1703) that *In any parallelogram the squares on the diagonals are together equal to the squares on the sides, or to twice the squares on adjacent sides.*

III. It appears that Leonhard Euler (1707–83) was the first to discover the corresponding theorem with reference to any quadrilateral, namely that *In any quadrilateral the sum of the squares on the sides is equal to the sum of the squares on the diagonals and four times the square on the line joining the middle*

points of the diagonals. Euler seems however to have proved the property from the corresponding theorem for parallelograms just quoted (cf. Camerer's Euclid, Vol. I. pp. 468, 469) and not from the property of the triangle, though the latter brings out the result more easily.

PROPOSITION 11.

To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.

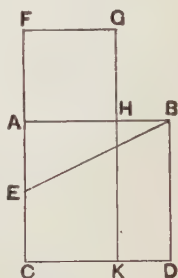
Let AB be the given straight line ;
thus it is required to cut AB so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.

For let the square $ABDC$ be described on AB ; [I. 46]

let AC be bisected at the point E , and let BE be joined ;

let CA be drawn through to F , and let EF be made equal to BE ;

let the square FH be described on AF , and let GH be drawn through to K .



I say that AB has been cut at H so as to make the rectangle contained by AB , BH equal to the square on AH .

For, since the straight line AC has been bisected at E , and FA is added to it,

the rectangle contained by CF , FA together with the square on AE is equal to the square on EF . [II. 6]

But EF is equal to EB ;

therefore the rectangle CF , FA together with the square on AE is equal to the square on EB .

But the squares on BA , AE are equal to the square on EB , for the angle at A is right ; [I. 47]

therefore the rectangle CF , FA together with the square on AE is equal to the squares on BA , AE .

Let the square on AE be subtracted from each ;

therefore the rectangle CF , FA which remains is equal to the square on AB .

Now the rectangle CF, FA is FK , for AF is equal to FG ;

and the square on AB is AD ;

therefore FK is equal to AD .

Let AK be subtracted from each;

therefore FH which remains is equal to HD .

And HD is the rectangle AB, BH , for AB is equal to BD ;

and FH is the square on AH ;

therefore the rectangle contained by AB, BH is equal to the square on HA .

therefore the given straight line AB has been cut at H so as to make the rectangle contained by AB, BH equal to the square on HA .

Q. E. F.

As the solution of this problem is necessary to that of inscribing a regular pentagon in a circle (Eucl. iv. 10, 11), we must necessarily conclude that it was solved by the Pythagoreans, or, in other words, that they discovered the geometrical solution of the quadratic equation

$$\begin{array}{l} a(a-x) = x^2, \\ \text{or} \qquad \qquad \qquad x^2 + ax = a^2. \end{array}$$

The solution in II. 11, too, exactly corresponds to the solution of the more general equation

$$x^2 + ax = b^2,$$

which, as shown above (pp. 387—8), Simson based upon II. 6. Only Simson's solution, if applied here, gives us the point F on CA produced and does not directly find the point H . It takes E the middle point of CA , draws AB at right angles to CA and of length equal to CA , and then describes a circle with EB as radius cutting EA produced in F . The only difference between the solution in this case and in the more general case is that AB is here equal to CA instead of being equal to another given straight line b .

As in the more general case, there is, from Euclid's point of view, only one solution.

The construction shows that CF is also divided at A in the manner described in the enunciation, since the rectangle CF, FA is equal to the square on CA .

The problem in II. 11 reappears in vi. 30 in the form of *cutting a given straight line in extreme and mean ratio*.

PROPOSITION 12.

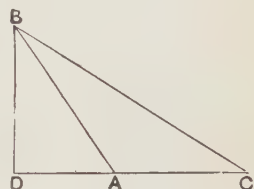
In obtuse-angled triangles the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle by twice the rectangle contained by one of the sides about the obtuse angle, namely that on which the

perpendicular falls, and the straight line cut off outside by the perpendicular towards the obtuse angle.

Let ABC be an obtuse-angled triangle having the angle BAC obtuse, and let BD be drawn from the point B perpendicular to CA produced;

I say that the square on BC is greater than the squares on BA , AC by twice the rectangle contained by CA , AD .

For, since the straight line CD has been cut at random at the point A , the square on DC is equal to the squares on CA , AD and twice the rectangle contained by CA , AD . [II. 4]



Let the square on DB be added to each;

therefore the squares on CD , DB are equal to the squares on CA , AD , DB and twice the rectangle CA , AD .

But the square on CB is equal to the squares on CD , DB , for the angle at D is right; [I. 47]

and the square on AB is equal to the squares on AD , DB ; [I. 47]

therefore the square on CB is equal to the squares on CA , AB and twice the rectangle contained by CA , AD ;

so that the square on CB is greater than the squares on CA , AB by twice the rectangle contained by CA , AD .

Therefore etc.

Q. E. D.

Since in this proposition and the next we have to do with the squares on the sides of triangles, the particular form of graphic representation of areas which we have had in Book II. up to this point does not help us to visualise the results of the propositions in the same way, and only two lines of proof are possible, (1) by means of the *results* of certain earlier propositions in Book II. combined with the *result* of I. 47 and (2) by means of the procedure in Euclid's proof of I. 47 itself. The alternative proofs of II. 12, 13 after the manner of Euclid's proof of I. 47 are therefore alone worth giving.

These proofs appear in certain modern text-books (e.g. Mehler, Henrici and Treutlein, H. M. Taylor, Smith and Bryant). Smith and Bryant are not correct in saying (p. 142) that they cannot be traced further back than Lardner's Euclid (1828); they are to be found in Gregory of St. Vincent's work (published in 1647) *Opus geometricum quadraturae circuli et sectionum conicarum*, Book I. Pt. 2, Props. 44, 45 (pp. 31, 32).

To prove II. 12, take an obtuse-angled triangle ABC in which the angle at A is the obtuse angle

Describe squares on BC , CA , AB , as $BCED$, $CAGF$, $ABKH$.

Draw AL , BM , CN , perpendicular to BC , CA , AB (produced if necessary), and produce them to meet the further sides of the squares on them in P , Q , R respectively.

Join AD , CK .

Then, as in I. 47, the triangles KBC , ABD are equal in all respects;

therefore their doubles, the parallelograms in the same parallels respectively, are equal;

that is, the rectangle BP is equal to the rectangle BR .

Similarly the rectangle CP is equal to the rectangle CQ .

Also, if BG , CH be joined, we see that

the triangles BAG , HAC are equal in all respects;

therefore their doubles, the rectangles AQ , AR , are equal.

Now the square on BC is equal to the sum of the rectangles BP , CP ,

i.e. to the sum of the rectangles BR , CQ ,

i.e. to the sum of the squares BH , CG and

the rectangles AR , AQ .

But the rectangles AR , AQ are equal, and they are respectively the rectangle contained by BA , AN and the rectangle contained by CA , AM .

Therefore the square on BC is equal to the squares on BA , AC together with twice the rectangle BA , AN or CA , AM .

Incidentally this proof shows that the rectangle BA , AN is equal to the rectangle CA , AM : a result which will be seen later on to be a particular case of the theorem in III. 35.

Heron (in an-Nairizī, ed. Curtze, p. 109) gives a "converse" of II. 12 related to it as I. 48 is related to I. 47.

In any triangle, if the square on one of the sides is greater than the squares on the other two sides, the angle contained by the latter is obtuse.

Let ABC be a triangle such that the square on BC is greater than the squares on BA , AC .

Draw AD at right angles to AC and of length equal to AB .

Join DC .

Then, since DAC is a right angle, the square on DC is equal to the squares on DA , AC ,

[I. 47]

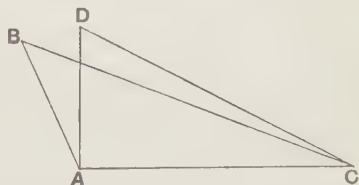
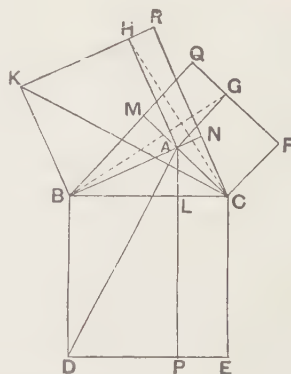
i.e. to the squares on BA , AC .

But the square on BC is greater than the squares on BA , AC ; therefore the square on BC is greater than the square on DC .

Therefore BC is greater than DC .

Thus, in the triangles BAC , DAC ,

the two sides BA , AC are equal to the two sides DA , AC respectively, but the base BC is greater than the base DC .



Therefore the angle BAC is greater than the angle DAC ;
that is, the angle BAC is *obtuse*.

[I. 25]

PROPOSITION 13.

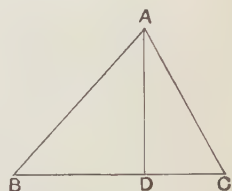
In acute-angled triangles the square on the side subtending the acute angle is less than the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides about the acute angle, namely that on which the perpendicular falls, and the straight line cut off within by the perpendicular towards the acute angle.

Let ABC be an acute-angled triangle having the angle at B acute, and let AD be drawn from the point A perpendicular to BC ;

I say that the square on AC is less than the squares on CB , BA by twice the rectangle contained by CB , BD .

For, since the straight line CB has been cut at random at D ,

the squares on CB , BD are equal to twice the rectangle contained by CB , BD and the square on DC . [II. 7]



Let the square on DA be added to each;

therefore the squares on CB , BD , DA are equal to twice the rectangle contained by CB , BD and the squares on AD , DC .

But the square on AB is equal to the squares on BD , DA , for the angle at D is right; [I. 47]

and the square on AC is equal to the squares on AD , DC ;
therefore the squares on CB , BA are equal to the square on AC and twice the rectangle CB , BD ,

so that the square on AC alone is less than the squares on CB , BA by twice the rectangle contained by CB , BD .

Therefore etc.

Q. E. D.

As the text stands, this proposition is unequivocally enunciated of *acute-angled* triangles; and, as if to obviate any doubt as to whether the restriction was fully intended, the enunciation speaks of the rectangle contained by one of the sides containing the acute angle and the straight line intercepted *within* by the perpendicular towards the acute angle. On the other hand, it

is curious that it speaks of the square on the side subtending *the* acute angle; and again the *setting-out* begins "let ABC be an acute-angled triangle *having the angle at B acute*," though the last words have no point if all the angles of the triangle are necessarily acute.

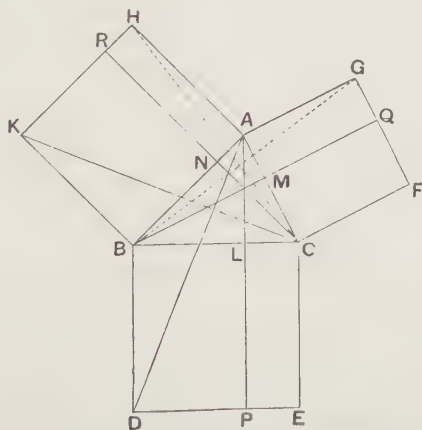
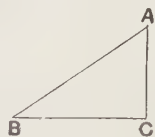
It was however very early noticed, not only by Isaacus Monachus, Campanus, Peletarius, Clavius, Commandinus and the rest, but by the Greek scholiast (Heiberg, Vol. v. p. 253), that the relation between the sides of a triangle established by this theorem is true of the side opposite to, and the sides about, an acute angle respectively in any sort of triangle whether acute-angled, right-angled or obtuse-angled. The scholiast tries to explain away the word "acute-angled" in the enunciation: "Since in the definitions he calls acute-angled the triangle which has three acute angles, you must know that he does not mean that here, but calls all triangles acute-angled because all have an acute angle, one at least, if not all. The enunciation therefore is: 'In any triangle the square on the side subtending the acute angle is less than the squares on the sides containing the acute angle by twice the rectangle, etc.'"

We may judge too by Heron's enunciation of his "converse" of the proposition that he would have left the word "acute-angled" out of the enunciation. His converse is: *In any triangle in which the square on one of the sides is less than the squares on the other two sides, the angle contained by the latter sides is acute.*

If the triangle that we take is a right-angled triangle, and the perpendicular is drawn, not from the right angle, but from the acute angle *not* referred to in the enunciation, the proposition reduces to I. 47, and this case need not detain us.

The other cases can be proved, like II. 12, after the manner of I. 47.

Let us take first the case where all the angles of the triangle are acute.



As before, if we draw ALP , BMQ , CNR perpendicular to BC , CA , AB and meeting the further sides of the squares on BC , CA , AB in P , Q , R , and if we join KC , AD , we have

the triangles KBC , ABD equal in all respects,

and consequently the rectangles BP , BR equal to one another.

Similarly the rectangles CP , CQ are equal to one another.

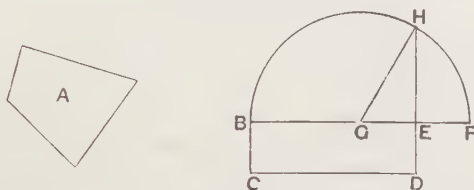
Therefore the angle DBC (a right angle) is greater than the angle ABC [I. 25], which latter is therefore *acute*.

It may be noted, lastly, that II. 12, 13 are supplementary to I. 47 and complete the theory of the relations between the squares on the sides of *any* triangle, whether right-angled or not.

PROPOSITION 14.

To construct a square equal to a given rectilineal figure.

Let A be the given rectilineal figure ;
thus it is required to construct a square equal to the rectilineal figure A .



5 For let there be constructed the rectangular parallelogram BD equal to the rectilineal figure A . [I. 45]

Then, if BE is equal to ED , that which was enjoined will have been done ; for a square BD has been constructed equal to the rectilineal figure A .

10 But, if not, one of the straight lines BE , ED is greater.
Let BE be greater, and let it be produced to F ;

let EF be made equal to ED , and let BF be bisected at G .

With centre G and distance one of the straight lines GB , GF let the semicircle BHF be described ; let DE be produced
15 to H , and let GH be joined.

Then, since the straight line BF has been cut into equal segments at G , and into unequal segments at E ,

the rectangle contained by BE , EF together with the square on EG is equal to the square on GF . [II. 5]

20 But GF is equal to GH ;

therefore the rectangle BE , EF together with the square on GE is equal to the square on GH .

But the squares on HE , EG are equal to the square on GH ; [I. 47]

25 therefore the rectangle BE , EF together with the square on GE is equal to the squares on HE , EG .

Let the square on GE be subtracted from each ;

therefore the rectangle contained by BE , EF which remains is equal to the square on EH .

30 But the rectangle BE , EF is BD , for EF is equal to ED ; therefore the parallelogram BD is equal to the square on HE .

And BD is equal to the rectilineal figure A .

Therefore the rectilineal figure A is also equal to the square
35 which can be described on EH .

Therefore a square, namely that which can be described on EH , has been constructed equal to the given rectilineal figure A Q. E. F.

7. that which was enjoined will have been done, literally "would have been done," *γεγονὸς ἂν εἴη τὸ ἐπιταχθέν*.

35, 36. which can be described, expressed by the future passive participle, *ἀναγραφη-σόμενος*, *ἀναγραφησόμενον*.

Heiberg (*Mathematisches zu Aristoteles*, p. 20) quotes as bearing on this proposition Aristotle's remark (*De anima* II. 2, 413 a 19: cf. *Metaph.* 996 b 21) that "squaring" (*τετραγωνισμός*) is better defined as the "finding of the mean (proportional)" than as "the making of an equilateral rectangle equal to a given oblong," because the former definition states the *cause*, the latter the *conclusion* only. This, Heiberg thinks, implies that in the text-books which were in Aristotle's hands the problem of II. 14 was solved by means of proportions. As a matter of fact, the actual *construction* is the same in II. 14 as in VI. 13; and the change made by Euclid must have been confined to substituting in the *proof* of the correctness of the construction an argument based on the principles of Books I. and II. instead of Book VI.

As II. 12, 13 are supplementary to I. 47, so II. 14 completes the theory of *transformation of areas* so far as it can be carried without the use of proportions. As we have seen, the propositions I. 42, 44, 45 enable us to construct a parallelogram having a given side and angle, and equal to any given rectilineal figure. The parallelogram can also be transformed into an equal triangle with the same given side and angle by making the other side about the angle twice the length. Thus we can, as a particular case, construct a rectangle on a given base (or a right-angled triangle with one of the sides about the right angle of given length) equal to a given square. Further, I. 47 enables us to make a square equal to the sum of any number of squares or to the difference between any two squares. The problem still remaining unsolved is to transform any rectangle (as representing an area equal to that of any rectilineal figure) into a square of equal area. The solution of this problem, given in II. 14, is of course the equivalent of the extraction of the square root, or of the solution of the pure quadratic equation

$$x^2 = ab.$$

Simson pointed out that, in the construction given by Euclid in this case, it was not necessary to put in the words "*Let BE be greater*," since the construction is not affected by the question whether BE or ED is the greater. This is true, but after all the words do little harm, and perhaps Euclid may have regarded it as conducive to clearness to have the points B , G , E , F in the same relative positions as the corresponding points A , C , D , B in the figure of II. 5 which he quotes in the proof.

EXCURSUS I.

PYTHAGORAS AND THE PYTHAGOREANS.

The problem of determining how much of the Pythagorean discoveries in mathematics can be attributed to Pythagoras himself is not only difficult; it may be said to be insoluble. Tradition on the subject is very meagre and uncertain, and further doubt is thrown upon it by the well-known tendency of the later Pythagoreans to ascribe everything to the Master himself (*αὐτὸς ἔφα, Ipse dixit*). Pythagoras himself left no written exposition of his doctrines, nor did any of his immediate successors, not even Hippasus, about whom the different stories ran (1) that he was expelled from the school because he published doctrines of Pythagoras, and (2) that he was drowned at sea for revealing the construction of the dodecahedron in the sphere and claiming it as his own, or (as others have it) for making known the discovery of the irrational or incommensurable. Nor is the absence of any written record of Pythagorean doctrines down to the time of Philolaus to be put down to a pledge of secrecy binding the school; at all events this did not apply to their mathematics or their physics; and it may be that the supposed secrecy was invented to account for the absence of documents. The fact seems to be that oral communication was the tradition of the school, while their doctrines would in the main be too abstruse to be understood by the generality of people outside. Even Aristotle felt the difficulty; he evidently knew nothing for certain about any ethical or physical doctrines going back to Pythagoras himself; when he speaks of the Pythagorean system, he always refers it to "the Pythagoreans." sometimes even to "the so-called Pythagoreans."

Since my note on Eucl. I. 47 was originally written the part of Pythagoras in the Pythagorean mathematical discoveries has been further discussed and every scrap of evidence closely, and even meticulously, examined in two long articles by Heinrich Vogt, "Die Geometrie des Pythagoras" (*Bibliotheca Mathematica* 1x₃, 1908/9, pp. 15—54) and "Die Entstehungsgeschichte des Irrationalen nach Plato und anderen Quellen des 4. Jahrhunderts" (*Bibliotheca Mathematica* x₃, 1910, pp. 97—155). These papers would not indeed have enabled me to modify greatly what I have written regarding the supposed discoveries of Pythagoras and the early Pythagoreans, because I have throughout been careful to give the traditions on the subject for what they are worth and no more, and not to build too much upon them. It is right however to give, in a separate note, a few details of Vogt's arguments.

G. Junge had, in his paper "Wann haben die Griechen das Irrationale entdeckt?" mentioned above (p. 351), tried to prove that Pythagoras himself could not have discovered the irrational; and the object of Vogt's papers is to go further on the same lines and to show (1) that it was only the later Pythagoreans who (before 410 B.C.) recognised the incommensurability of the diagonal with the side of a square, (2) that the *theory* of the irrational was first discovered by Theodorus, to whom Plato refers (*Theaetetus* 147 D), and (3) that Pythagoras himself could not have been the discoverer of any one of

the things specifically attributed to him, namely (*a*) the theorem of Eucl. I. 47, (*b*) the construction of the five regular solids in the sense in which they are respectively constructed in Eucl. XIII, (*c*) the application of an area in its widest sense, equivalent to the solution of a quadratic equation in its most general form.

Vogt's main argument as regards (*a*) the theorem of I. 47 is based on a new translation which he gives of the well-known passage of Proclus' note on the proposition (p. 426, 6—9), *Τῶν μὲν ἱστορεῖν τὰ ἀρχαῖα βουλομένων ἀκούοντας τὸ θεώρημα τοῦτο εἰς Πυθαγόραν ἀναπεμπόντων ἐστὶν εὐρεῖν καὶ βουθῆτην λεγόντων αὐτὸν ἐπὶ τῇ εὐρέσει*. Vogt translates this as follows: "Unter denen, welche das Altertum erforschen wollen, kann man einige finden, welche denen Gehör geben, die dieses Theorem auf Pythagoras zurückführen und ihn als Stieropferer bei dieser Gelegenheit bezeichnen," "Among those who have a taste for research into antiquity, we can find some who give ear to those who refer this theorem to Pythagoras and describe him as sacrificing an ox on the strength of the discovery." According to this version the words *τῶν... βουλομένων* and the words *ἀναπεμπόντων...καὶ...λεγόντων* refer respectively to two different sets of persons, in fact two different generations; the latter are older authorities who are supposed to be cited by the former; the former are a later generation, perhaps contemporaries of Proclus, some of whom accepted the view of the older authorities while others did not. But this would have required the article *τῶν* before *ἀναπεμπόντων*, or some such expression as *ἄλλων τινῶν οἱ ἀναπέμπονσι* instead of *ἀναπεμπόντων*. Vogt's interpretation is therefore quite inadmissible. The persons denoted by *ἀναπεμπόντων* are *some* of the persons denoted by *τῶν βουλομένων*; hence Tannery's translation, to which mine (p. 350 above) is equivalent, is the only possible one, namely "Si l'on écoute ceux qui veulent raconter l'histoire des anciens temps, on peut en trouver qui attribuent ce théorème à Pythagore et lui font sacrifier un bœuf après sa découverte" (*La Géométrie grecque*, p. 103). *ἀκούοντας* agrees with the assumed *subject* of *εὐρεῖν*; *ἀναπεμπόντων* and *λεγόντων* should, strictly speaking, have been *ἀναπέμποντας* and *λέγοντας* agreeing with *τινῶς* (the direct *object* of *εὐρεῖν*) understood, but are simply attracted into the case of *βουλομένων*; the construction is quite intelligible. I agree with Vogt that Eudemos' history contained nothing attributing the theorem to Pythagoras. The words of Proclus imply this; but I do not think that they imply (as Vogt maintains) any pronouncement by Proclus himself *against* such attribution. In my opinion, Proclus is simply determined not to commit himself to any view; his way of evading a decision is the sentence following, *ἐγὼ δὲ θαυμάζω μὲν καὶ τοὺς πρῶτους ἐπιστάνας τῇ τοῦδε τοῦ θεωρήματος ἀληθείᾳ, μειζόνως δὲ ἄγμαι τὸν στοιχειωτήν...*; the plural *τοὺς πρῶτους ἐπιστάνας* is, I hold, used for the very purpose of making the statement as vague as possible; he will not even allow it to be inferred that he attributed the discovery to any single person. Returning to *ἡ τῶν ἀλόγων πραγματεία* (Proclus, p. 65, 19), we may now concede (following Diels) that we should read *τῶν ἀνὰ λόγον* ("proportionals") instead of *τῶν ἀλόγων* ("irrationals") and that the author intended to attribute to Pythagoras a theory of *proportion* (the arithmetical theory applicable to commensurable magnitudes only) rather than the theory of irrationals. But I do not agree in Vogt's contention that the theory of the irrational was first discovered by Theodorus. It seems to me that we have evidence to the contrary in the very passage of Plato referred to. Plato (*Theaetetus* 147 D) mentions $\sqrt{3}$, $\sqrt{5}$, ... up to $\sqrt{17}$ as dealt with by Theodorus, but *omits* $\sqrt{2}$. This fact, along with Plato's allusions elsewhere to the irrationality of $\sqrt{2}$, and

to approximations to it, in the expressions ἄρρητος and ῥητὴ διάμετρος τῆς πεμπάδος, as if those expressions had a well-known signification, implies that the discovery of the irrationality of $\sqrt{2}$ had been made before the time of Theodorus. The words ἡ τῶν ἀλόγων πραγματεία might well be used even if the reference is only to $\sqrt{2}$, because the first step would be the most difficult, and πραγματεία need not mean the establishment of a complete theory or anything more than "investigation" of a subject.

Junge and Vogt hold that the theory of the irrational was not discovered by the early Pythagoreans any more than Pythagoras because, if it had been so discovered, an impossibly long period would intervene between the investigation of the particular case of $\sqrt{2}$ and the extension of the theory by Theodorus to the cases of $\sqrt{3}$, $\sqrt{5}$ etc. But might not this well be due to the fact that in the meantime the minds of geometers were engrossed by other problems of importance, namely the quadrature of the circle (Hippocrates of Chios and his quadratures of lunes), the trisection of any angle (Hippias of Elis and his curve, afterwards known as the *quadratrix*), and the doubling of the cube (reduced by Hippocrates to the problem of finding two mean proportionals in continued proportion between two given straight lines), the last of which problems, which meant finding geometrically the equivalent of $\sqrt[3]{2}$, would naturally follow the investigation of $\sqrt{2}$? Now Hippias was probably born about 460 B.C., while Hippocrates seems to have been in Athens during a considerable portion of the second half of the fifth century, perhaps from about 450 to 430 B.C. Moreover Vogt has to get over the fact that Democritus (born 470/469 B.C.) wrote a book περὶ ἀλόγων γραμμῶν καὶ ναστῶν, *On irrational lines and solids* (or *atoms*). This difficulty he seeks to overcome by maintaining that ἀλόγων does not here mean "irrational" at all, but "without ratio" ("verhältnisslos"), in the sense that any two straight lines are "without ratio" because they both contain an infinite number of the indivisible (or atomic) lines, and therefore their ratio, being of the form ∞/∞ , is indeterminate. But, if these were so, *all* lines (including commensurable lines) would be "without a ratio" to one another, whereas the title of Democritus' work clearly implies that ἄλογοι γραμμαὶ are a class or classes of lines distinguished from other lines. The fact is that Democritus was too good a mathematician to have anything to do with "indivisible lines." This is confirmed by a scholium to Aristotle's *De caelo* (p. 469 b 14, Brandis) which implicitly denies to Democritus any theory of indivisible lines: "of those who have maintained the existence of indivisibles, some, as for example Leucippus and Democritus, believe in indivisible bodies, others, like Xenocrates, in indivisible lines." Moreover Simplicius tells us that, according to Democritus himself, even the atoms were, in a mathematical sense, divisible further and in fact *ad infinitum*.

Coming now to (δ) the construction of the cosmic figures, ἡ τῶν κοσμικῶν σχημάτων σύστασις (Proclus, p. 65, 20), I agree with Vogt to the following extent. It is unlikely that Pythagoras or even the early Pythagoreans "constructed" the five regular solids in the sense of a complete theoretical construction such as we find, say, in Eucl. XIII.; and it is possible that Theaetetus was the first to give these constructions, whether ἔγραψε in Suidas' notice, πρῶτος δὲ τὰ πέντε καλούμενα στερεὰ ἔγραψε, means "constructed" or "wrote upon." But σύστασις in the above phrase of Proclus may well mean something less than the theoretical constructions and proofs of Eucl. XIII.; it may mean, as Vogt says, simply the "putting together" of the figures in the same way as Plato puts them together in the *Timaeus*, i.e. by bringing a certain number of

angles of equilateral triangles and of regular pentagons together at one point. There is no reason why the early Pythagoreans should not have "constructed" the five regular solids in this sense; in fact the supposition that they did so agrees well with what we know of their having put angles of certain regular figures together round a point (in connexion with the theorem of Eucl. I. 32) and shown that only three kinds of such angles would fill up the space *in one plane* round the point. But I do not agree in the apparent refusal of Vogt to credit the Pythagoreans with the knowledge of the theoretical construction of the regular pentagon as we find it in Eucl. IV. 10, 11. I do not know of any reason for rejecting the evidence of the Scholia IV. Nos. 2 and 4 which say categorically that "this Book" (Book IV.) and "the whole of the theorems" in it (including therefore Props. 10, 11) are discoveries of the Pythagoreans. And the division of a straight line in extreme and mean ratio, on which the construction of the regular pentagon depends, comes in Eucl. Book II. (Prop. 11), while we have sufficient grounds for regarding the whole of the substance of Book II. as Pythagorean.

I will permit myself one more criticism on Vogt's first paper. I think he bases too much on the fact that it was left for Oenopides (in the period from, say, 470 to 450 B.C.) to discover two elementary constructions (with ruler and compasses only), namely that of a perpendicular to a straight line from an external point (Eucl. I. 12), and that of an angle equal to a given rectilineal angle (Eucl. I. 23). Vogt infers that geometry must have been in a very rudimentary condition at the time. I do not think this follows; the explanation would seem to be rather that, the restriction of the instruments used in constructions to the ruler and compasses not having been definitely established before the time when Oenopides wrote, it had not previously occurred to anyone to substitute new constructions based on that principle for others previously in vogue. In the case of the perpendicular, for example, the construction would no doubt, in earlier days, have been made by means of a set square.

EXCURSUS II.

POPULAR NAMES FOR EUCLIDEAN PROPOSITIONS.

Although some of these time-honoured names are familiar to most educated people, it seems to be impossible to trace them to their original sources, or to say who applied them for the first time respectively. It may be that they were handed down by oral tradition for long periods in each case before they found their way into written documents.

We begin with

I. 5.

1. This proposition is in this country universally known as the *Pons Asinorum*, "Asses' Bridge." Even in this case opinion is not unanimous as to the exact implication of the term. Perhaps the more general view is that taken in the *Stanford Dictionary of Anglicised Words and Phrases* (by C. A. M. Fennell) where the description is: "Name of the fifth proposition of the first Book of Euclid, suggested by the figure and the difficulties which poor geometricians find in mastering it." This is certainly the equivalent of what I gathered, in my early days at school, from a former Fellow of St John's, the Reverend Anthony Bower, who was a high Wrangler in 1846 and a friend of Todhunter's. The "ass" on this interpretation is a synonym for "fool." But there is another view (as I have learnt lately) which is more complimentary to the ass. It is that, the figure of the proposition being like that of a trestle-bridge, with a ramp at each end which is the more practicable the flatter the figure is drawn, the bridge is such that, while a horse could not surmount the ramp, an ass could; in other words, the term is meant to refer to the surefootedness of the ass rather than to any want of intelligence on his part. (I may perhaps mention that Sir George Greenhill is a strong supporter of this view.)

An epigram of 1780 is the earliest reference to the term in Murray's English Dictionary:

"If this be rightly called the bridge of asses,
He's not the fool that sticks but he that passes."

The writer's own view is not too clear. He seems to imply that, while the inventor of the name meant that only the fool finds the bridge difficult to pass, the more proper view would be that, since the ass can get over, and "ass" is synonymous with "fool," therefore it must be the fool who *can* get over; in other words, he seems to object to the phrase as being a contradiction in terms.

But we have also to take account of the fact that the French apply the term to I. 47. Now in Euclid's figure for I. 47 there is no suggestion of a bridge, and the reference can only be to the nature of the theorem, its difficulty or otherwise. It is curious that the French dictionaries give two different explanations of *Pont aux ânes*. Littré makes it "ce que personne ne doit ni ne peut ignorer; ce qui est si facile que tout le monde doit y réussir." Now no intelligent person could have applied the name to Eucl. I. 47 for this reason, namely that it was so easy that even a fool could not help knowing it. Larousse is better informed; there we find "*Pont aux ânes*, certaine difficulté, certaine question qui n'arrête que les ignorants, et qui sert de critérium

pour juger l'intelligence de quelqu'un, et particulièrement d'un écolier. C'est ainsi que, dans les classes de mathématiques, on ne manque jamais de dire que le carré de l'hypoténuse est le *pont aux ânes* de la géométrie. La plupart des dictionnaires entendent par ce mot une chose si simple, si facile, que personne ne doit l'ignorer : c'est une erreur évidente." Larousse is clearly right. But it will be observed that, so far as it goes, Larousse's interpretation rather supports the first of the two alternative explanations of the meaning of "Asses' Bridge" as applied to 1. 5, namely that it is difficult for the fool (= "ass") to master.

In the *Stanford Dictionary* it is added that "in logic the term was in the 16 c. applied to the conversion of propositions by the aid of a difficult diagram for finding middle terms"; and if the mathematicians borrowed the term from logic, this again would be rather in favour of the first explanation of its use for 1. 5.

If it is permitted *desipere in loco*, I would add for the benefit of future generations (in the hope that they will still be able to appreciate the joke or, in the alternative, will be tempted to discuss learnedly what could possibly have been meant) a very topical allusion in a recent *Punch* (14 Oct. 1925): "When they film Euclid, as is suggested, we shall no doubt see a very thrilling rescue over the burning Pons Asinorum."—And yet it is safe to prophesy that the "Asses' Bridge" will outlive the "film"!

2. *Elefuga*.

This name for Eucl. 1. 5 is mentioned by Roger Bacon (about 1250), who also gives an explanation of it (*Opus Tertium*, c. vi). He observes that in his day people in general, finding no utility in any science such as geometry, for example, recoiled from the idea of studying it unless they were boys forced to it by the rod, so that they would hardly learn so much as three or four propositions. Hence it is, he says, that the fifth proposition is called "Elefuga, id est, fuga miserorum; elegia enim Graece dicitur, Latine miseria; et elegi sunt miseri." That is, according to Roger Bacon, Elefuga is "flight of the miserable." This explanation no doubt accounts for the verses about *Dulcarnon* in Chaucer's *Troilus and Criseyde*, III, ll. 933-5:

"Dulcarnon called is 'fleminge of wrecches';
It seemeth hard, for wrecches wol not lere
For verray slouth or othere wilful teches";

since "fleminge of wrecches," "banishment of the miserable," is a translation of "fuga miserorum." Only Dulcarnon is there wrongly taken to be the same proposition as Elefuga, i.e. 1. 5, whereas, as we shall see, Dulcarnon was really the name for the Pythagorean theorem 1. 47.

Etymologically, Roger Bacon's explanation leaves something to be desired. The word would really seem to be an attempt to compound the two Greek words *ἔλεος*, pity (or the object of pity), and *φύγη*, flight (cf. note *ad loc.* in Skeat's edition of Chaucer). Notwithstanding the confusion of tongues, the object seems to be a play upon the two words *Elementa* and *ἔλεος*, which both begin with the same three letters, and the implication is that "escape from the Elements" (which normally came when Prop. 5 was reached) was equivalent to "escape from misery" or "trouble." A better form for the word would perhaps be Eleufuga; and this form actually occurs in Alanus' *Anticlaudianus*, III, c. 6 (cited by Du Cange, *Glossarium*, s.v.). The word also occurs, according to Skeat's note, in Richard of Bury's *Philobiblon*, c. xiii, where it was somewhat oddly translated by J. B. Inglis in 1832 "How many scholars has the Helleflight of Euclid repelled!"

I. 47.

The Pythagorean proposition about the square on the hypotenuse has taken even a deeper hold of the minds of men, and has been known by a number of names.

1. *The Theorem of the Bride* (θεώρημα τῆς νύμφης).

This name is found in a ms. of Georgius Pachymeres (1242-1310) in the Bibliothèque Nationale at Paris; there is a note to this effect by Tannery (*La Géométrie grecque*, p. 105), but, as he says nothing more, it is probable that the passage gives the mere name without any explanation of it. We have, however, much earlier evidence of the supposed connexion of the proposition with marriage. Plutarch (born about 46 A.D.) says (*De Iside et Osiride* 56, p. 373 F) "We may imagine the Egyptians (thinking of) the most beautiful of triangles (and) likening the nature of the All to this triangle most particularly, for it is this same triangle which Plato is thought to have employed in the *Republic*, when he put together the Nuptial Figure (γαμήλιον διάγραμμα)"—διάγραμμα, though literally meaning "diagram" or "figure," was commonly used in the sense of "proposition"—"and in that triangle the perpendicular side is 3, the base 4, and the hypotenuse, the square on which is equal to the sum of the squares on the sides containing (the right angle), 5. We must, then, liken the perpendicular to the male, the base to the female and the hypotenuse to the offspring of both.... For 3 is the first odd number and is perfect, 4 is the square on an even side, 2, while the 5 partly resembles the father and partly the mother, being the sum of 3 and 2."

Plato used the three numbers 3, 4, 5 of the Pythagorean triangle in the formation of his famous Geometrical Number; but Plato himself does not call the triangle the Nuptial Triangle nor the number the Nuptial Number. It is later writers, Plutarch, Nicomachus and Iamblichus, who connect the passage about the Geometrical Number with marriage; Nicomachus (*Introd. Ar.*, II, 24, 11) merely alludes to "the passage in the *Republic* connected with the so-called Marriage," while Iamblichus (*In Nicom.*, p. 82, 20 Pistelli) only speaks of "the Nuptial Number in the *Republic*."

It would appear, then, that the name "Nuptial Figure" or "Theorem of the Bride" was originally used of one particular right-angled triangle, namely (3, 4, 5). A late Arabian writer Behā-ad-din (1547-1622) seems to have applied the term "Figure of the Bride" to the same triangle; the Arabs therefore seemingly followed the Greeks. The idea underlying the use of the term, first for the triangle (3, 4, 5), and then for the general theorem of I. 47, seems to be roughly that of the two parties to a marriage becoming one, just as the two squares on the sides containing the right angle become the one square on the hypotenuse in the said theorem.

2. *The "Bride's Chair."*

The origin of this name is more obscure. It must presumably have been suggested by a supposed resemblance between the figure of the proposition and such a chair. D. E. Smith (*History of Mathematics*, II, pp. 289-90) remarks that the "Bride's Chair" may be so-called "because the Euclid figure is not unlike the chair which a slave carries on his back and in which the Eastern bride is sometimes transported to the ceremony," and he cites a note from Edouard Lucas' *Récréations Mathématiques*, II, p. 130: "La démonstration que nous venons de donner du théorème de Pythagore sur le carré de l'hypoténuse ne diffère pas essentiellement de la démonstration hindoue, connue sous le nom de la *Chaise de la petite mariée*, que l'on rencontre dans

l'ouvrage de Bhascara (Bija-Ganita, § 146).” The figure of Bhāskara is not that of Euclid but that shown at the top of p. 355 above; I have however not been able to find the name “Bride’s Chair” in Colebrooke’s translation of the work of Bhāskara.

Notwithstanding the apparent frivolity of the setting, I venture to suggest that light may be thrown on the question by a very modern version of the “Bride’s Chair” which appeared during or since the War in *La Vie Parisienne*. The illustration represents Euclid’s figure for 1. 47 and, drawn over it, as on a frame, a *poilu* in full fighting kit carrying on his back his bride and his household belongings. Roughly speaking, the soldier is standing (or rather walking) in the middle of the large square, his head and shoulders are bending to the right within the contour of one of the small squares, while the lady, with mirror and powder-puff in action, is sitting with her back to him in the right angle between the two smaller squares (*HAG* in the figure on p. 349 above)¹. I am informed by Sir George Greenhill that there was also an earlier version “showing the chair as it is in use to-day in Cairo and Egypt, the earliest version of a taxi-chair, a pattern as early as Euclid and suggesting the nickname of the proposition.” This recalls to my mind the remark of a friend to whom I mentioned the subject and showed the figure of the proposition; he observed at once on seeing it “But I should have said it was more like a sedan chair,” the large square suggesting to him the actual chair and the two smaller squares the two bearers.

3. *Dulcarnon*.

This name for 1. 47 appears, as above mentioned, in Chaucer’s *Troilus and Criseyde*, III, ll. 930–3, where Criseyde says:

‘I am, til God me better minde sende,
At dulcarnon, right at my wittes ende.’
Quod Pandarus, ‘ye, nece, wol ye here?
Dulcarnon called is “fleminge of wrecches.”’

Billingsley, too, in his edition of Euclid (1570) observes of 1. 47 that “it hath bene commonly called of barbarous writers of the latter time Dulcarnon.”

Dulcarnon (see Skeat’s note *ad loc.*) seems to represent the Persian and Arabic *du’l-karnayn*, lit. *two-horned*, from Pers. *du*, two, and *karn*, horn. The name was applied to 1. 47 because the two smaller squares stick up like two horns and, as the proposition is difficult, the word here takes the sense of “puzzle”; hence Criseyde was “at dulcarnon” because she was perplexed and at her wit’s end.

4. *Francisci tunica* = “Franciskaner Kutte,” “Franciscan’s cowl.”

This name is quoted by Weissenborn (*Die Uebersetzungen des Euklid durch Campano und Zamberti*, p. 42) as given in a *Geometrie* by one Kunze. The name is quite appropriate, one of the squares representing the hood thrown back.

III. 7, 8.

I have already mentioned the names “Goose’s Foot” (*Pes anseris*) and “Peacock’s Tail” (*Cauda pavonis*) applied, suitably enough, to these propositions respectively. They come from Luca Paciolo’s edition of Euclid published in 1509 (*vide* Weissenborn, *ibid.*).

¹ Old Cambridge men will recall a picture in some respects not unlike, though less artistic than, the cartoon in *La Vie Parisienne*, I mean the painting of “The Man Loaded with Mischief” which used to be over the door of the former inn of that name on the St Neots Road, a short distance from Cambridge.

BOOK III.

DEFINITIONS.

1. **Equal circles** are those the diameters of which are equal, or the radii of which are equal.

2. A straight line is said to **touch a circle** which, meeting the circle and being produced, does not cut the circle.

3. **Circles** are said to **touch one another** which, meeting one another, do not cut one another.

4. In a circle straight lines are said to **be equally distant from the centre** when the perpendiculars drawn to them from the centre are equal.

5. And that straight line is said to be **at a greater distance** on which the greater perpendicular falls.

6. A **segment of a circle** is the figure contained by a straight line and a circumference of a circle.

7. An **angle of a segment** is that contained by a straight line and a circumference of a circle.

8. An **angle in a segment** is the angle which, when a point is taken on the circumference of the segment and straight lines are joined from it to the extremities of the straight line which is the **base of the segment**, is contained by the straight lines so joined.

9. And, when the straight lines containing the angle cut off a circumference, the angle is said to **stand upon** that circumference.

10. A **sector of a circle** is the figure which, when an angle is constructed at the centre of the circle, is contained by the straight lines containing the angle and the circumference cut off by them.

11. **Similar segments of circles** are those which admit equal angles, or in which the angles are equal to one another.

DEFINITION 1.

ἴσοι κύκλοι εἰσίν, ὧν αἱ διάμετροι ἴσαι εἰσίν, ἢ ὧν αἱ ἐκ τῶν κέντρων ἴσαι εἰσίν.

Many editors have held that this should not have been included among definitions. Some, e.g. Tartaglia, would call it a *postulate*; others, e.g. Borelli and Playfair, would call it an *axiom*; others again, as Billingsley and Clavius, while admitting it as a *definition*, add explanations based on the mode of constructing a circle; Simson and Pflaiderer hold that it is a *theorem*. I think however that Euclid would have maintained that it is a definition in the proper sense of the term; and certainly it satisfies Aristotle's requirement that a "definitional statement" (ὁριστικὸς λόγος) should not only state the *fact* (τὸ ὅτι) but should indicate the *cause* as well (*De anima* II. 2, 413 a 13). The equality of circles with equal radii can of course be proved by superposition, but, as we have seen, Euclid avoided this method wherever he could, and there is nothing technically wrong in saying "By *equal circles* I mean circles with equal radii." No flaw is thereby introduced into the system of the *Elements*; for the definition could only be objected to if it could be proved that the equality predicated of the two circles in the definition was not the same thing as the equality predicated of other equal figures in the *Elements* on the basis of the Congruence-Axiom, and, needless to say, this cannot be proved because it is not true. The *existence* of equal circles (in the sense of the definition) follows from the existence of equal straight lines and I. Post. 3.

The Greeks had no distinct word for *radius*, which is with them, as here, *the (straight line drawn) from the centre* ἡ ἐκ τοῦ κέντρου (εὐθεΐα); and so definitely was the expression appropriated to the radius that ἐκ τοῦ κέντρου was used without the article as a predicate, just as if it were one word. Thus, e.g., in III. 1 ἐκ κέντρου γάρ means "for they are radii": cf. Archimedes, *On the Sphere and Cylinder* II. 2, ἡ BE ἐκ τοῦ κέντρου ἐστὶ τοῦ...κύκλου, *BE is a radius of the circle*.

DEFINITION 2.

Εὐθεΐα κύκλον ἐφάπτεσθαι λέγεται, ἥτις ἀποτομένη τοῦ κύκλου καὶ ἐκβαλλομένη οὐ τέμνει τὸν κύκλον.

Euclid's phraseology here shows the regular distinction between ἀπτεσθαι and its compound ἐφάπτεσθαι, the former meaning "to *meet*" and the latter "to *touch*." The distinction was generally observed, by Greek geometers from Euclid onwards. There are however exceptions so far as ἀπτεσθαι is concerned; thus it means "to *touch*" in Eucl. IV. Def. 5 and sometimes in Archimedes. On the other hand, ἐφάπτεσθαι is used by Aristotle in certain

cases where the orthodox geometrical term would be ἀπτεσθαι. Thus in *Meteorologica* III. 5 (376 b 9) he says a certain circle *will pass through all the angles* (ἀπασῶν ἐφάπεται τῶν γωνιῶν), and (376 a 6) M *will lie on a given* (circular) *circumference* (δεδομένης περιφερείας ἐφάπεται τὸ M). We shall find ἀπτεσθαι used in these senses in Book IV. Deff. 2, 6 and Deff. 1, 3 respectively. The latter of the two expressions quoted from Aristotle means that *the locus of M is a given circle*, just as in Pappus ἄπεται τὸ σημεῖον θέσει δεδομένης εὐθείας means that *the locus of the point is a straight line given in position*.

DEFINITION 3.

Κύκλοι ἐφάπτεσθαι ἀλλήλων λέγονται οἷτινες ἀπτόμενοι ἀλλήλων οὐ τέμνουσιν ἀλλήλους.

Todhunter remarks that different opinions have been held as to what is, or should be, included in this definition, one opinion being that it only means that the circles do not cut in the neighbourhood of the point of contact, and that it must be shown that they do not cut elsewhere, while another opinion is that the definition means that the circles do not cut at all. Todhunter thinks the latter opinion correct. I do not think this is proved; and I prefer to read the definition as meaning simply that the circles meet at a point but do not cut *at that point*. I think this interpretation preferable for the reason that, although Euclid does practically assume in III. 11—13, without stating, the theorem that circles touching at one point do not intersect anywhere else, he has given us, before reaching that point in the Book, means for proving for ourselves the truth of that statement. In particular, he has given us the propositions III. 7, 8 which, taken as a whole, give us more information as to the general nature of a circle than any other propositions that have preceded, and which can be used, as will be seen in the sequel, to solve any doubts arising out of Euclid's unproved assumptions. Now, as a matter of fact, the propositions are not used in any of the genuine proofs of the theorems in Book III.; III. 8 is required for the second proof of III. 9 which Simson selected in preference to the first proof, but the first proof only is regarded by Heiberg as genuine. Hence it would not be easy to account for the appearance of III. 7, 8 at all unless as affording means of answering possible *objections* (cf. Proclus' explanation of Euclid's reason for inserting the second part of I. 5).

External and *internal* contact are not distinguished in Euclid until III. 11, 12, though the *figure* of III. 6 (not the *enunciation* in the original text) represents the case of internal contact only. But the definition of touching circles here given must be taken to imply so much about *internal* and *external* contact respectively as that (a) a circle touching another internally must, immediately before "meeting" it, have passed through points *within* the circle that it touches, and (b) a circle touching another externally must, immediately before meeting it, have passed through points *outside* the circle which it touches. These facts must indeed be admitted if *internal* and *external* are to have any meaning at all in this connexion, and they constitute a minimum admission necessary to the proof of III. 6.

DEFINITION 4.

Ἐν κύκλῳ ἴσον ἀπέχειν ἀπὸ τοῦ κέντρου εὐθεῖαι λέγονται, ὅταν αἱ ἀπὸ τοῦ κέντρου ἐπ' αὐτὰς καθέτοι ἀγόμεναι ἴσαι ὦσιν.

DEFINITION 5.

Μείζον δὲ ἀπέχειν λέγεται, ἐφ' ἣν ἡ μείζων κάθετος πίπτει.

DEFINITION 6.

Τμήμα κύκλου ἐστὶ τὸ περιεχόμενον σχῆμα ὑπὸ τε εὐθείας καὶ κύκλου περιφερείας.

DEFINITION 7.

Τμήματος δὲ γωνία ἐστὶν ἡ περιεχομένη ὑπὸ τε εὐθείας καὶ κύκλου περιφερείας.

This definition is only interesting historically. The *angle of a segment*, being the “angle” formed by a straight line and a “circumference,” is of the kind described by Proclus as “mixed.” A particular “angle” of this sort is the “*angle of a semicircle*,” which we meet with again in III. 16, along with the so-called “horn-like angle” (κερατοειδής), the supposed “angle” between a tangent to a circle and the circle itself. The “angle of a semicircle” occurs once in Pappus (VII. p. 670, 19), but it there means scarcely more than the *corner* of a semicircle regarded as a point to which a straight line is directed. Heron does not give the definition of the *angle of a segment*, and we may conclude that the mention of it and of the *angle of a semicircle* in Euclid is a survival from earlier text-books rather than an indication that Euclid considered either to be of importance in elementary geometry (cf. the note on III. 16 below).

We have however, in the note on I. 5 above (Vol. I. pp. 252—3), seen evidence that the *angle of a segment* had played some part in geometrical proofs up to Euclid's time. It would appear from the passage of Aristotle there quoted (*Anal. prior.* I. 24, 41 b 13 sqq.) that the theorem of I. 5 was, in the text-books immediately preceding Euclid, proved by means of the equality of the two “*angles of*” any one segment. This latter property must therefore have been regarded as more elementary (for whatever reason) than the theorem of I. 5; indeed the definition as given by Euclid practically implies the same thing, since it speaks of only *one* “angle of a segment,” namely “*the* angle contained by a straight line and a circumference of a circle.” Euclid abandoned the actual use of the “angle” in question, but no doubt thought it unnecessary to break with tradition so far as to strike the definition out also.

DEFINITION 8.

Ἐν τμήματι δὲ γωνία ἐστίν, ὅταν ἐπὶ τῆς περιφερείας τοῦ τμήματος ληφθῇ τι σημείον καὶ ἀπ' αὐτοῦ ἐπὶ τὰ πέρατα τῆς εὐθείας, ἣ ἐστι βάσις τοῦ τμήματος, ἐπιζευχθῶσιν εὐθείαι, ἡ περιεχομένη γωνία ὑπὸ τῶν ἐπιζευχθεισῶν εὐθειῶν.

DEFINITION 9.

Ὅταν δὲ αἱ περιέχουσιν τὴν γωνίαν εὐθείαι ἀπολαμβάνωσιν τινα περιφέρειαν, ἐπ' ἐκείνης λέγεται βεβηκέναι ἡ γωνία.

DEFINITION 10.

Τομεὺς δὲ κύκλου ἐστίν, ὅταν πρὸς τῷ κέντρῳ τοῦ κύκλου συσταθῇ γωνία, τὸ περιεχόμενον σχῆμα ὑπὸ τε τῶν τήν γωνίαν περιεχουσῶν εὐθειῶν καὶ τῆς ἀπολαμβανομένης ὑπ' αὐτῶν περιφερείας.

A scholiast says that it was the *shoemaker's knife*, σκυτοτομικὸς τομεὺς, which suggested the name τομεὺς for a sector of a circle. The derivation of the name from a resemblance of shape is parallel to the use of ἀρβηλος (also a *shoemaker's knife*) to denote the well known figure of the Book of Lemmas partly attributed to Archimedes.

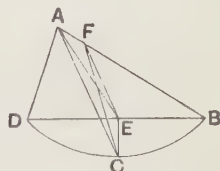
A wider definition of a sector than that given by Euclid is found in a Greek scholiast (Heiberg's Euclid, Vol. v. p. 260) and in an-Nairizi (ed. Curtze, p. 112). "There are two varieties of sectors; the one kind have the angular vertices at the centres, the other at the circumferences. Those others which have their vertices neither at the circumferences nor at the centres, but at some other points, are for that reason not called sectors but sector-like figures (τομοειδῆ σχήματα)." The exact agreement between the scholiast and an-Nairizi suggests that Heron was the authority for this explanation.

The *sector-like figure* bounded by an arc of a circle and two lines drawn from its extremities to meet at any point actually appears in Euclid's book *On divisions* (περὶ διαιρέσεων) discovered in an Arabic ms. and edited by Woepcke (cf. Vol. i. pp. 8—10 above). This treatise, alluded to by Proclus, had for its object the division of figures such as triangles, trapezia, quadrilaterals and circles, by means of straight lines, into parts equal or in given ratios. One proposition e.g. is, *To divide a triangle into two equal parts by a straight line passing through a given point on one side*. The proposition (28) in which the *quasi-sector* occurs is, *To divide such a figure by a straight line into two equal parts*. The solution in this case is given by Cantor (*Gesch. d. Math.* 1, pp. 287—8).

If $ABCD$ be the given figure, E the middle point of BD and EC at right angles to BD , the broken line AEC clearly divides the figure into two equal parts.

Join AC , and draw EF parallel to it meeting AB in F .

Join CF , when it is seen that CF divides the figure into two equal parts.



DEFINITION 11.

Ὅμοια τμήματα κύκλων ἐστὶ τὰ δεχόμενα γωνίας ἴσας, ἥ ἐν οἷς αἱ γωνίαι ἴσαι ἀλλήλαις εἰσίν.

De Morgan remarks that the use of the word *similar* in "similar segments" is an anticipation, and that similarity of *form* is meant. He adds that the definition is a theorem, or would be if "similar" had taken its final meaning.

BOOK III. PROPOSITIONS.

PROPOSITION I.

To find the centre of a given circle.

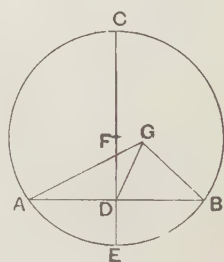
Let ABC be the given circle ;

thus it is required to find the centre of the circle ABC .

Let a straight line AB be drawn
5 through it at random, and let it be bisected
at the point D ;

from D let DC be drawn at right angles
to AB and let it be drawn through to E ;
let CE be bisected at F ;

10 I say that F is the centre of the circle
 ABC .



For suppose it is not, but, if possible,
let G be the centre,
and let GA, GD, GB be joined.

15 Then, since AD is equal to DB ,
and DG is common,

the two sides AD, DG are equal to the two sides
 BD, DG respectively ;

and the base GA is equal to the base GB , for they are
20 radii ;

therefore the angle ADG is equal to the angle GDB . [I. 8]

But, when a straight line set up on a straight line makes
the adjacent angles equal to one another, each of the equal
angles is right ;

[I. Def. 10]

25 therefore the angle GDB is right.

But the angle FDB is also right ;
therefore the angle FDB is equal to the angle GDB , the
greater to the less : which is impossible.

Therefore G is not the centre of the circle ABC .

30 Similarly we can prove that neither is any other point
except F .

Therefore the point F is the centre of the circle ABC .

PORISM. From this it is manifest that, if in a circle a
straight line cut a straight line into two equal parts and at
35 right angles, the centre of the circle is on the cutting straight
line.

Q. E. F.

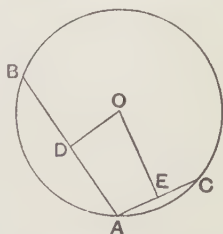
12. For suppose it is not. This is expressed in the Greek by the two words Μή γάρ , but such an elliptical phrase is impossible in English.

17. the two sides AD , DG are equal to the two sides BD , DG respectively. As before observed, Euclid is not always careful to put the equals in corresponding order. The text here has " GD , DB ."

Todhunter observes that, when, in the construction, DC is said to be *produced* to E , it is assumed that D is within the circle, a fact which Euclid first demonstrates in III. 2. This is no doubt true, although the word δυσήχθω , "let it be *drawn through*," is used instead of ἐκβεβλήσθω , "let it be *produced*." And, although it is not necessary to assume that D is within the circle, it is necessary for the success of the construction that the straight line drawn through D at right angles to AB shall meet the circle in two points (and no more): an assumption which we are not entitled to make on the basis of what has gone before only.

Hence there is much to be said for the alternative procedure recommended by De Morgan as preferable to that of Euclid. De Morgan would first prove the fundamental theorem that "the line which bisects a chord perpendicularly must contain the centre," and then make III. 1, III. 25 and IV. 5 immediate corollaries of it. The fundamental theorem is a direct consequence of the theorem that, if P is *any* point equidistant from A and B , then P lies on the straight line bisecting AB perpendicularly. We then take any two chords AB , AC of the given circle and draw DO , EO bisecting them perpendicularly. Unless BA , AC are in one straight line, the straight lines DO , EO must meet in some point O (see note on IV. 5 for possible methods of proving this). And, since both DO , EO must contain the centre, O must be the centre.

This method, which seems now to be generally preferred to Euclid's, has the advantage of showing that, in order to find the centre of a circle, it is sufficient to know three points on the circumference. If therefore two circles have three points in common, they must have the same centre and radius, so that two circles cannot have three points in common without coinciding entirely. Also, as indicated by De Morgan, the same construction enables us (1) to draw the complete circle of which a segment or arc only is given (III. 25), and (2) to circumscribe a circle to any triangle (IV. 5).



But, if the Greeks had used this construction for finding the centre of a circle, they would have considered it necessary to add a proof that no other point than that obtained by the construction can be the centre, as is clear both from the similar *reductio ad absurdum* in III. 1 and also from the fact that Euclid thinks it necessary to prove as a separate theorem (III. 9) that, if a point within a circle be such that three straight lines (at least) drawn from it to the circumference are equal, that point must be the centre. In fact, however, the proof amounts to no more than the remark that the two perpendicular bisectors can have no more than one point common.

And even in De Morgan's method there is a yet unproved assumption. In order that DO , EO may meet, it is necessary that AB , AC should not be in one straight line or, in other words, that BC should not pass through A . This results from III. 2, which therefore, strictly speaking, should precede.

To return to Euclid's own proposition III. 1, it will be observed that the demonstration only shows that the centre of the circle cannot lie on either side of CD , so that it must lie on CD or CD produced. It is however taken for granted rather than proved that the centre must be the middle point of CE . The proof of this by *reductio ad absurdum* is however so obvious as to be scarcely worth giving. The same consideration which would prove it may be used to show that *a circle cannot have more than one centre*, a proposition which, if thought necessary, may be added to III. 1 as a corollary.

Simson observed that the proof of III. 1 could not but be by *reductio ad absurdum*. At the beginning of Book III. we have nothing more to base the proof upon than the definition of a circle, and this cannot be made use of unless we assume some point to be the centre. We cannot however assume that the point found by the construction is the centre, because that is the thing to be proved. Nothing is therefore left to us but to assume that some other point is the centre and then to prove that, whatever other point is taken, an absurdity results; whence we can infer that the point found is the centre.

The Porism to III. 1 is inserted, as usual, parenthetically before the words ὅπερ ἔδει ποιῆσαι , which of course refer to the problem itself.

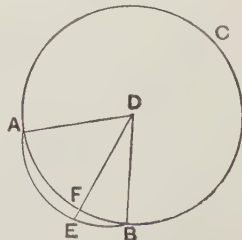
PROPOSITION 2.

If on the circumference of a circle two points be taken at random, the straight line joining the points will fall within the circle.

Let ABC be a circle, and let two points A , B be taken at random on its circumference;

I say that the straight line joined from A to B will fall within the circle.

For suppose it does not, but, if possible, let it fall outside, as AEB ; let the centre of the circle ABC be taken [III. 1], and let it be D ; let DA , DB be joined, and let DFE be drawn through.



Then, since DA is equal to DB ,

the angle DAE is also equal to the angle DBE . [I. 5]

And, since one side AEB of the triangle DAE is produced,
the angle DEB is greater than the angle DAE . [I. 16]

But the angle DAE is equal to the angle DBE ;

therefore the angle DEB is greater than the angle DBE .

And the greater angle is subtended by the greater side; [I. 19]
therefore DB is greater than DE .

But DB is equal to DF ;

therefore DF is greater than DE ,

the less than the greater: which is impossible.

Therefore the straight line joined from A to B will not fall outside the circle.

Similarly we can prove that neither will it fall on the circumference itself;

therefore it will fall within.

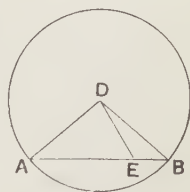
Therefore etc.

Q. E. D.

The *reductio ad absurdum* form of proof is not really necessary in this case, and it has the additional disadvantage that it requires the destruction of two hypotheses, namely that the chord is (1) outside, (2) on the circle. To prove the proposition directly, we have only to show that, if E be any point on the straight line AB between A and B , DE is less than the radius of the circle. This may be done by the method shown above, under I. 24, for proving what is assumed in that proposition, namely that, in the figure of the proposition, F falls below EG if DE is not greater than DF . The assumption amounts to the following proposition, which De Morgan would make to precede I. 24: "Every straight line drawn from the vertex of a triangle to the base is less than the greater of the two sides, or than either if they be equal." The case here is that in which the two sides are equal; and, since the angle DAB is equal to the angle DBA , while the exterior angle DEA is greater than the interior and opposite angle DBA , it follows that the angle DEA is greater than the angle DAE , whence DE must be less than DA or DB .

Camerer points out that we may add to this proposition the further statement that all points on AB produced in either direction are outside the circle. This follows from the proposition (also proved by means of the theorems that the exterior angle of a triangle is greater than either of the interior and opposite angles and that the greater angle is subtended by the greater side) which De Morgan proposes to introduce after I. 21, namely,

"The perpendicular is the shortest straight line that can be drawn from a



given point to a given straight line, and of others that which is nearer to the perpendicular is less than the more remote, and the converse; also not more than two equal straight lines can be drawn from the point to the line, one on each side of the perpendicular."

The fact that not more than two equal straight lines can be drawn from a given point to a given straight line not passing through it is proved by Proclus on I. 16 (see the note to that proposition) and can alternatively be proved by means of I. 7, as shown above in the note on I. 12. It follows that

A straight line cannot cut a circle in more than two points

a proposition which De Morgan would introduce here after III. 2. The proof given does not apply to a straight line *passing through the centre*; but that such a line only cuts the circle in two points is self-evident.

PROPOSITION 3.

If in a circle a straight line through the centre bisect a straight line not through the centre, it also cuts it at right angles; and if it cut it at right angles, it also bisects it.

Let ABC be a circle, and in it let a straight line CD through the centre bisect a straight line AB not through the centre at the point F ;

I say that it also cuts it at right angles.

For let the centre of the circle ABC be taken, and let it be E ; let EA , EB be joined.

Then, since AF is equal to FB ,
and FE is common,

two sides are equal to two sides;

and the base EA is equal to the base EB ;

therefore the angle AFE is equal to the angle BFE . [I. 8]

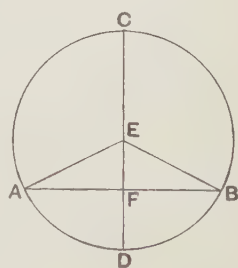
But, when a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right; [I. Def. 10]

therefore each of the angles AFE , BFE is right.

Therefore CD , which is through the centre, and bisects AB which is not through the centre, also cuts it at right angles.

Again, let CD cut AB at right angles;

I say that it also bisects it, that is, that AF is equal to FB .



For, with the same construction,

since EA is equal to EB ,

the angle EAF is also equal to the angle EBF . [1. 5]

But the right angle AFE is equal to the right angle BFE ,
 30 therefore EAF , EBF are two triangles having two angles
 equal to two angles and one side equal to one side, namely
 EF , which is common to them, and subtends one of the equal
 angles;

therefore they will also have the remaining sides equal to
 35 the remaining sides; [1. 26]

therefore AF is equal to FB .

Therefore etc.

Q. E. D.

26. with the same construction, τῶν αὐτῶν κατασκευασθέντων.

This proposition asserts the two *partial* converses (cf. note on I. 6) of the Porism to III. 1. De Morgan would place it next to III. 1.

PROPOSITION 4.

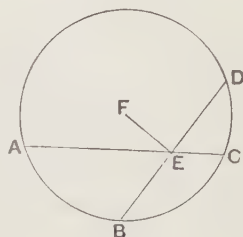
If in a circle two straight lines cut one another which are not through the centre, they do not bisect one another.

Let $ABCD$ be a circle, and in it let the two straight lines AC , BD , which are not through the centre, cut one another at E ;

I say that they do not bisect one another.

For, if possible, let them bisect one another, so that AE is equal to EC , and BE to ED ;

let the centre of the circle $ABCD$ be taken [III. 1], and let it be F ; let FE be joined.



Then, since a straight line FE through the centre bisects a straight line AC not through the centre,

it also cuts it at right angles; [III. 3]

therefore the angle FEA is right.

Again, since a straight line FE bisects a straight line BD ,

it also cuts it at right angles; [III. 3]

therefore the angle FEB is right.

But the angle FEA was also proved right ;

therefore the angle FEA is equal to the angle FEB , the less to the greater : which is impossible.

Therefore AC , BD do not bisect one another.

Therefore etc.

Q. E. D.

PROPOSITION 5.

If two circles cut one another, they will not have the same centre.

For let the circles ABC , CDG cut one another at the points B , C ;

I say that they will not have the same centre.

For, if possible, let it be E ; let EC be joined, and let EFG be drawn through at random.

Then, since the point E is the centre of the circle ABC ,

EC is equal to EF . [1. Def. 15]

Again, since the point E is the centre of the circle CDG ,
 EC is equal to EG .

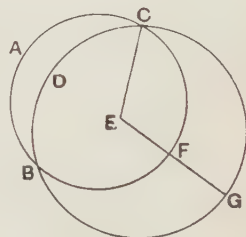
But EC was proved equal to EF also ;

therefore EF is also equal to EG , the less to the greater : which is impossible.

Therefore the point E is not the centre of the circles ABC , CDG .

Therefore etc.

Q. E. D.



The propositions III. 5, 6 could be combined in one. It makes no difference whether the circles cut, or meet without cutting, so long as they do not coincide altogether; in either case they cannot have the same centre. The two cases are covered by the enunciation : *If the circumferences of two circles meet at a point they cannot have the same centre.* On the other hand, *If two circles have the same centre and one point in their circumferences common, they must coincide altogether.*

PROPOSITION 6.

If two circles touch one another, they will not have the same centre.

For let the two circles ABC , CDE touch one another at the point C ;

I say that they will not have the same centre.

For, if possible, let it be F ; let FC be joined, and let FEB be drawn through at random.

Then, since the point F is the centre of the circle ABC ,

FC is equal to FB .

Again, since the point F is the centre of the circle CDE ,

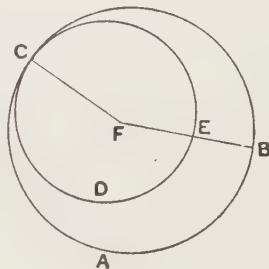
FC is equal to FE .

But FC was proved equal to FB ;

therefore FE is also equal to FB , the less to the greater: which is impossible.

Therefore F is not the centre of the circles ABC , CDE .

Therefore etc.



Q. E. D.

The English editions enunciate this proposition of circles touching *internally*, but the word (*ἐντός*) is a mere interpolation, which was no doubt made because Euclid's figure showed only the case of *internal* contact. The fact is that, in his usual manner, he chose for demonstration the more difficult case, and left the other case (that of *external* contact) to the intelligence of the reader. It is indeed sufficiently self-evident that circles touching externally cannot have the same centre; but Euclid's proof can really be used for *this* case too.

Cramer remarks that the proof of III. 6 seems to assume tacitly that the points E and B cannot coincide, or that circles which touch internally at C cannot meet in any other point, whereas this fact is not proved by Euclid till III. 13. But no such general assumption is necessary here; it is only necessary that *one* line drawn from the assumed common centre should meet the circles in different points; and the very notion of internal contact requires that, before one circle *meets* the other on its inner side, it must have passed through points *within* the latter circle.

PROPOSITION 7.

If on the diameter of a circle a point be taken which is not the centre of the circle, and from the point straight lines fall upon the circle, that will be greatest on which the centre is, the remainder of the same diameter will be least, and of the rest the nearer to the straight line through the centre is always greater than the more remote, and only two equal straight lines will fall from the point on the circle, one on each side of the least straight line.

Let $ABCD$ be a circle, and let AD be a diameter of it ;
 10 on AD let a point F be taken which is not the centre of the circle, let E be the centre of the circle,
 and from F let straight lines FB , FC , FG fall upon the circle $ABCD$;

I say that FA is greatest, FD is least, and of the rest FB is
 15 greater than FC , and FC than FG .

For let BE , CE , GE be joined.

Then, since in any triangle two sides are greater than the remaining one,

20 EB , EF are greater than BF . [I. 20]

But AE is equal to BE ;

therefore AF is greater than BF .

Again, since BE is equal to CE ,
 and FE is common,

25 the two sides BE , EF are equal to the two sides CE , EF .

But the angle BEF is also greater than the angle CEF ;
 therefore the base BF is greater than the base CF . [I. 24]

For the same reason

CF is also greater than FG .

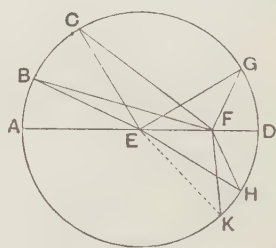
30 Again, since GF , FE are greater than EG ,
 and EG is equal to ED ,

GF , FE are greater than ED .

Let EF be subtracted from each ;

therefore the remainder GF is greater than the remainder
 35 FD .

Therefore FA is greatest, FD is least, and FB is greater than FC , and FC than FG .



I say also that from the point F only two equal straight lines will fall on the circle $ABCD$, one on each side of the
 40 least FD .

For on the straight line EF , and at the point E on it, let the angle FEH be constructed equal to the angle GEF [1. 23], and let FH be joined.

Then, since GE is equal to EH ,
 45 and EF is common,

the two sides GE , EF are equal to the two sides HE , EF ;
 and the angle GEF is equal to the angle HEF ;

therefore the base FG is equal to the base FH . [1. 4]

I say again that another straight line equal to FG will not
 50 fall on the circle from the point F .

For, if possible, let FK so fall.

Then, since FK is equal to FG , and FH to FG ,

FK is also equal to FH ,

the nearer to the straight line through the centre being
 55 thus equal to the more remote : which is impossible.

Therefore another straight line equal to GF will not fall from the point F upon the circle ;

therefore only one straight line will so fall.

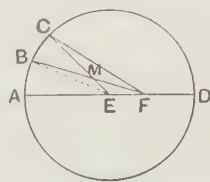
Therefore etc.

Q. E. D.

4. of the same diameter. I have inserted these words for clearness' sake. The text has simply *ελαχίστη δὲ ἡ λοιπή*, "and the remaining (straight line) least."

7. 39. one on each side. The word "one" is not in the Greek, but is necessary to give the force of *ἐφ' ἑκάτερα τῆς ἐλαχίστης*, literally "on both sides," or "on each of the two sides, of the least."

De Morgan points out that there is an unproved assumption in this demonstration. We draw straight lines from F , as FB , FC , such that the angle DFB is greater than the angle DFC and then assume, with respect to the straight lines drawn from the centre E to B , C , that the angle DEB is greater than the angle DEC . This is most easily proved, I think, by means of the converse of part of the theorem about the lengths of different straight lines drawn to a given straight line from an external point which was mentioned above in the note on III. 2. This converse would be to the effect that, *If two unequal straight lines be drawn from a point to a given straight line which are not perpendicular to the straight line, the greater of the two is the further from the perpendicular from the point to the given straight line.* This can either be proved from its converse by *reductio ad absurdum*, or established directly by means of I. 47. Thus, in the accompanying figure, FB must cut EC in some point M , since the angle BFE is less than the angle CFE .



Therefore EM is less than EC , and therefore than EB .

Hence the point B in which FB meets the circle is further from the foot of the perpendicular from E on FB than M is ;

therefore the angle BEF is greater than the angle CEF .

Another way of enunciating the first part of the proposition is that of Mr H. M. Taylor, viz. "Of all straight lines drawn to a circle from an internal point not the centre, the one which passes through the centre is the greatest, and the one which when produced passes through the centre is the least; and of any two others the one which *subtends the greater angle at the centre* is the greater." The substitution of the *angle subtended at the centre* as the criterion no doubt has the effect of avoiding the necessity of dealing with the unproved assumption in Euclid's proof referred to above, and the similar substitution in the enunciation of the first part of III. 8 has the effect of avoiding the necessity for dealing with like unproved assumptions in Euclid's proof, as well as the complication caused by the distinction in Euclid's enunciation between lines falling from an external point on the *convex circumference* and on the *concave circumference* of a circle respectively, terms which are not defined but taken as understood.

Mr Nixon (*Euclid Revised*) similarly substitutes as the criterion the angle subtended at the centre, but gives as his reason that the words "nearer" and "more remote" in Euclid's enunciation are scarcely clear enough without some definition of the sense in which they are used. Smith and Bryant make the substitution in III. 8, but follow Euclid in III. 7.

On the whole, I think that Euclid's plan of taking straight lines drawn from the point which is not the centre direct to the circumference and making greater or less angles *at that point* with the straight line containing it and the centre is the more instructive and useful of the two, since it is such lines drawn in any manner to the circle from the point which are immediately useful in the proofs of later propositions or in resolving difficulties connected with those proofs.

Heron again (an-Nairizi, ed. Curtze, pp. 114—5) has a note on this proposition which is curious. He first of all says that Euclid proves that lines *nearer the centre* are greater than those more remote *from it*. This is a different view of the question from that taken in Euclid's proposition as we have it, in which the lines are not nearer to and more remote from the *centre* but from *the line through the centre*. Euclid takes lines inclined to the latter line at a greater or less angle ; Heron introduces distance *from the centre* in the sense of Deff. 4, 5, i.e. in the sense of *the length of the perpendicular* drawn to the line from the centre, which Euclid does not use till III. 14, 15. Heron then observes that in Euclid's proposition the lines compared are all drawn on one side of the line through the centre, and sets himself to prove the same truth of lines on *opposite* sides which are more or less distant *from the centre*. The new point of view necessitates a quite different line of proof, anticipating the methods of later propositions.

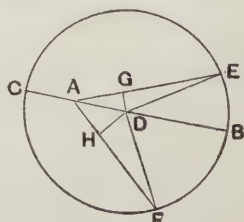
The first case taken by Heron is that of two straight lines such that the perpendiculars from the centre on them fall on the lines themselves and not in either case on the line produced.

Let A be the given point, D the centre, and let AE be nearer the centre than AF , so that the perpendicular DG on AE is less than the perpendicular DH on AF .

Then sqs. on DG , GE = sqs. on DH , HF ,

and sqs. on DG , GA = sqs. on DH , HA .

But sq. on DG < sq. on DH .



Therefore sq. on $GE >$ sq. on HF ,
and sq. on $GA >$ sq. on HA ,
whence $GE > HF$,
 $GA > HA$.

Therefore, by addition, $AE > AF$.

The other case taken by Heron is that where one perpendicular falls on the line produced, as in the annexed figure. In this case we prove in like manner that

$GE > HF$,
and $GA > AH$.

Thus AE is greater than the sum of HF , AH , whence, *a fortiori*, AE is greater than the difference of HF , AH , i.e. than AF .

Heron does not give the third possible case, that, namely, where *both* perpendiculars fall on the lines produced. The fact is that, in this case, the foregoing method breaks down. Though AE be nearer to the centre than AF in the sense that DG is less than DH ,

AE is not greater but *less* than AF .

Moreover this cannot be proved by the same method as before.

For, while we can prove that

$GE > HF$,
 $GA > AH$,

we cannot make any inference as to the comparative length of AE , AF .

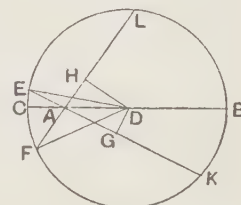
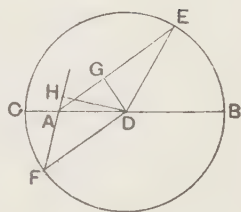
To judge by Heron's corresponding note to III. 8, he would, to prove this case, practically prove III. 35 first, i.e. prove that, if EA be produced to K and FA to L ,

rect. FA , $AL =$ rect. EA , AK ,

from which he would infer that, since $AK > AL$ by the first case,

$AE < AF$.

An excellent moral can, I think, be drawn from the note of Heron. Having the appearance of supplementing, or giving an alternative for, Euclid's proposition, it cannot be said to do more than confuse the subject. Nor was it necessary to find a new proof for the case where the two lines which are compared are on *opposite* sides of the diameter, since Euclid shows that for each line from the point to the circumference on one side of the diameter there is another of the same length equally inclined to it on the other side.



PROPOSITION 8.

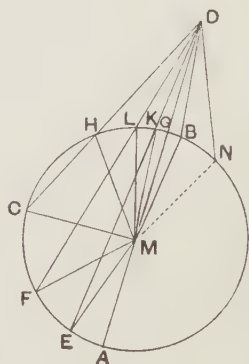
If a point be taken outside a circle and from the point straight lines be drawn through to the circle, one of which is through the centre and the others are drawn at random, then, of the straight lines which fall on the concave circumference, that through the centre is greatest, while of the rest

the nearer to that through the centre is always greater than the more remote, but, of the straight lines falling on the convex circumference, that between the point and the diameter is least, while of the rest the nearer to the least is always less than the more remote, and only two equal straight lines will fall on the circle from the point, one on each side of the least.

Let ABC be a circle, and let a point D be taken outside ABC ; let there be drawn through from it straight lines DA, DE, DF, DC , and let DA be through the centre; I say that, of the straight lines falling on the concave circumference $AEFC$, the straight line DA through the centre is greatest,

while DE is greater than DF and DF than DC ;

but, of the straight lines falling on the convex circumference $HLKG$, the straight line DG between the point and the diameter AG is least; and the nearer to the least DG is always less than the more remote, namely DK than DL , and DL than DH .



For let the centre of the circle ABC be taken [III. 1], and let it be M ; let ME, MF, MC, MK, ML, MH be joined.

Then, since AM is equal to EM , let MD be added to each;

therefore AD is equal to EM, MD .

But EM, MD are greater than ED ;

[I. 20]

therefore AD is also greater than ED .

Again, since ME is equal to MF ,

and MD is common,

therefore EM, MD are equal to FM, MD ;

and the angle EMD is greater than the angle FMD ;

therefore the base ED is greater than the base FD .

[I. 24]

Similarly we can prove that FD is greater than CD ; therefore DA is greatest, while DE is greater than DF , and DF than DC .

Next, since MK , KD are greater than MD , [I. 20]
 and MG is equal to MK ,
 therefore the remainder KD is greater than the remainder
 GD ,

so that GD is less than KD .

And, since on MD , one of the sides of the triangle MLD ,
 two straight lines MK , KD were constructed meeting within
 the triangle,

therefore MK , KD are less than ML , LD ; [I. 21]
 and MK is equal to ML ;

therefore the remainder DK is less than the remainder
 DL .

Similarly we can prove that DL is also less than DH ;

therefore DG is least, while DK is less than DL , and
 DL than DH .

I say also that only two equal straight lines will fall from
 the point D on the circle, one on each side of the least DG .

On the straight line MD , and at the point M on it,
 let the angle DMB be constructed equal to the angle KMD ,
 and let DB be joined.

Then, since MK is equal to MB ,
 and MD is common,

the two sides KM , MD are equal to the two sides BM ,
 MD respectively;

and the angle KMD is equal to the angle BMD ;

therefore the base DK is equal to the base DB . [I. 4]

I say that no other straight line equal to the straight line
 DK will fall on the circle from the point D .

For, if possible, let a straight line so fall, and let it be DN .

Then, since DK is equal to DN ,

while DK is equal to DB ,

DB is also equal to DN ,

that is, the nearer to the least DG equal to the more remote:
 which was proved impossible.

Therefore no more than two equal straight lines will fall
 on the circle ABC from the point D , one on each side of
 DG the least.

Therefore etc.

Q. E. D.

The converses of III. 7, 8 and of the proposition just given are also true and can easily be proved by *reductio ad absurdum*. They could be employed to throw light on such questions as that of internal contact, and the relative position of the centres of circles so touching. This is clear when part of the converses is stated: thus (1) if from any point in the plane of a circle a number of straight lines be drawn to the circumference of the circle, and one of these is greater than any other, the centre of the circle must lie on that one, (2) if one of them is less than any other, then, (a) if the point is within the circle, the centre is on the minimum straight line produced *beyond the point*, (b) if the point is outside the circle, the centre is on the minimum straight line produced *beyond the point in which it meets the circle*.

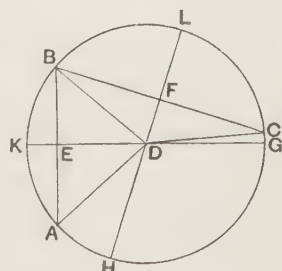
PROPOSITION 9.

If a point be taken within a circle, and more than two equal straight lines fall from the point on the circle, the point taken is the centre of the circle.

Let ABC be a circle and D a point within it, and from D let more than two equal straight lines, namely DA , DB , DC , fall on the circle ABC ;

I say that the point D is the centre of the circle ABC .

For let AB , BC be joined and bisected at the points E , F , and let ED , FD be joined and drawn through to the points G , K , H , L .



Then, since AE is equal to EB , and ED is common,

the two sides AE , ED are equal to the two sides BE , ED ;
and the base DA is equal to the base DB ;

therefore the angle AED is equal to the angle BED .

[I. 8]

Therefore each of the angles AED , BED is right;

[I. Def. 10]

therefore GK cuts AB into two equal parts and at right angles.

And since, if in a circle a straight line cut a straight line into two equal parts and at right angles, the centre of the circle is on the cutting straight line,

[III. 1, Por.]

the centre of the circle is on GK .

For the same reason

the centre of the circle ABC is also on HL .

And the straight lines GK , HL have no other point common but the point D ;

therefore the point D is the centre of the circle ABC .

Therefore etc.

Q. E. D.

The result of this proposition is quoted by Aristotle, *Meteorologica* III. 3, 373 a 13—16 (cf. note on I. 8).

III. 9 is, as De Morgan remarks, a *logical* equivalent of part of III. 7, where it is proved that every *non-central* point is *not* a point from which three equal straight lines can be drawn to the circle. Thus III. 7 says that every *not-A* is *not-B*, and III. 9 states the equivalent fact that every *B* is *A*. Mr H. M. Taylor does in effect make a *logical* inference of the theorem that, *If from a point three equal straight lines can be drawn to a circle, that point is the centre*, by making it a corollary to his proposition which includes the part of III. 7 referred to. Euclid does not allow himself these logical inferences, as we shall have occasion to observe elsewhere also.

Of the two proofs of this proposition given in earlier texts of Euclid, August and Heiberg regard that translated above as genuine, relegating the other, which Simson gave alone, to a place in an Appendix. Camerer remarks that the genuine proof should also have contemplated the case in which one or other of the straight lines AB , BC passes through D . This would however have been a departure from Euclid's manner of taking the most obscure case for proof and leaving others to the reader.

The other proof, that selected by Simson, is as follows:

"For let a point D be taken within the circle ABC , and from D let more than two equal straight lines, namely AD , DB , DC , fall on the circle ABC ;

I say that the point D so taken is the centre of the circle ABC .

For suppose it is not; but, if possible, let it be E , and let DE be joined and carried through to the points F , G .

Therefore FG is a diameter of the circle ABC .

Since, then, on the diameter FG of the circle ABC a point has been taken which is not the centre of the circle, namely D ,

DG is greatest, and DC is greater than DB , and DB than DA .

But the latter are also equal: which is impossible

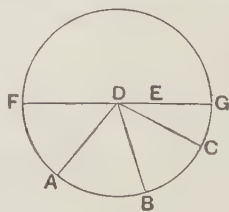
Therefore E is not the centre of the circle.

Similarly we can prove that neither is any other point except D ;

therefore the point D is the centre of the circle ABC .

Q. E. D."

On this Todhunter correctly points out that the point E might be supposed to fall *within* the angle ADC . It cannot then be shown that DC is greater than DB and DB than DA , but only that either DC or DA is less than DB ; this however is sufficient for establishing the proposition.

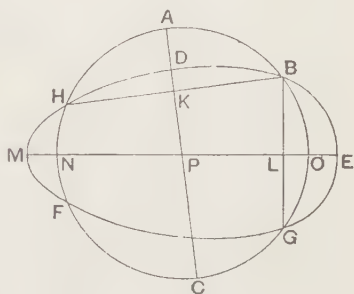


PROPOSITION 10.

A circle does not cut a circle at more points than two.

For, if possible, let the circle ABC cut the circle DEF at more points than two, namely B, C, F, H ;

let BH, BG be joined and bisected at the points K, L , and from K, L let KC, LM be drawn at right angles to BH, BG and carried through to the points A, E .



Then, since in the circle ABC a straight line AC cuts a straight line BH into two equal parts and at right angles,

the centre of the circle ABC is on AC . [III. 1, Por.]

Again, since in the same circle ABC a straight line NO cuts a straight line BG into two equal parts and at right angles,

the centre of the circle ABC is on NO .

But it was also proved to be on AC , and the straight lines AC, NO meet at no point except at P ;

therefore the point P is the centre of the circle ABC .

Similarly we can prove that P is also the centre of the circle DEF ;

therefore the two circles ABC, DEF which cut one another have the same centre P : which is impossible. [III. 5]

Therefore etc.

Q. E. D.

1. The word circle ($\kappaύκλος$) is here employed in the unusual sense of the *circumference* ($\piεριφέρεια$) of a circle. Cf. note on I. Def. 15.

There is nothing in the demonstration of this proposition which assumes that the circles *cut* one another; it proves that two circles cannot *meet* at more than two points, whether they cut or meet without cutting, i.e. *touch* one another.

Here again, of two demonstrations given in the earlier texts, Simson chose the second, which August and Heiberg relegate to an Appendix and which is as follows:

“For again let the circle ABC cut the circle DEF at more points than two, namely B, G, H, F ;

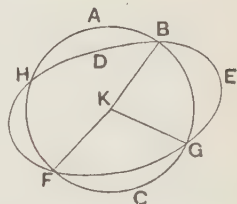
let the centre K of the circle ABC be taken, and let KB, KG, KF be joined.

Since then a point K has been taken within the circle DEF , and from K more than two straight lines, namely KB , KF , KG , have fallen on the circle DEF , the point K is the centre of the circle DEF . [III. 9]

But K is also the centre of the circle ABC .

Therefore two circles cutting one another have the same centre K : which is impossible. [III. 5]

Therefore a circle does not cut a circle at more points than two.



Q. E. D."

This demonstration is claimed by Heron (see an-Nairizī, ed. Curtze, pp. 120—1). It is incomplete because it assumes that the point K which is taken as the centre of the circle ABC is *within* the circle DEF . It can however be completed by means of III. 8 and the corresponding proposition with reference to a point *on* the circumference of a circle which was enunciated in the note on III. 8. For (1) if the point K is *on* the circumference of the circle DEF , we obtain a contradiction of the latter proposition which asserts that only *two* equal straight lines can be drawn from K to the circumference of the circle DEF ; (2) if the point K is *outside* the circle DEF , we obtain a contradiction of the corresponding part of III. 8.

Euclid's proof contains an unproved assumption, namely that the lines bisecting BG , BH at right angles *will* meet in a point P . For a discussion of this assumption see note on IV. 5.

PROPOSITION 11.

If two circles touch one another internally, and their centres be taken, the straight line joining their centres, if it be also produced, will fall on the point of contact of the circles.

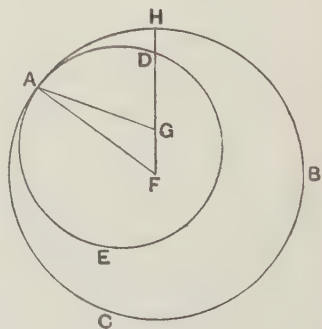
For let the two circles ABC , ADE touch one another internally at the point A , and let the centre F of the circle ABC , and the centre G of ADE , be taken;

I say that the straight line joined from G to F and produced will fall on A .

For suppose it does not, but, if possible, let it fall as FGH , and let AF , AG be joined.

Then, since AG , GF are greater than FA , that is, than FH ,

let FG be subtracted from each;
therefore the remainder AG is greater than the remainder GH .



But AG is equal to GD ;

therefore GD is also greater than GH ,

the less than the greater: which is impossible.

Therefore the straight line joined from F to G will not fall outside;

therefore it will fall at A on the point of contact.

Therefore etc.

Q. E. D.

2. the straight line joining their centres, literally "the straight line joined to their centres" (ἡ ἐπὶ τὰ κέντρα αὐτῶν ἐπεζευγμένη εὐθεῖα).

3. point of contact is here *συναφή*, and in the enunciation of the next proposition *ἐπαφή*.

Again August and Heiberg give in an Appendix the additional or alternative proof, which however shows little or no variation from the genuine proof and can therefore well be dispensed with.

The genuine proof is beset with difficulties in consequence of what it tacitly assumes in the figure, on the ground, probably, of its being obvious to the eye. Camerer has set out these difficulties in a most careful note, the heads of which may be given as follows:

He observes, first, that the straight line joining the centres, when produced, must necessarily (though this is not stated by Euclid) be produced in the direction of the centre of the circle which touches the other internally. (For brevity, I shall call this circle the "inner circle," though I shall imply nothing by that term except that it is the circle which touches the other on the inner side of the latter, and therefore that, in accordance with the definition of *touching*, points on it in the immediate neighbourhood of the point of contact are necessarily *within* the circle which it touches.) Camerer then proceeds by the following steps.

1. The two circles, touching at the given point, cannot *intersect* at any point. For, since points on the "inner" in the immediate neighbourhood of the point of contact are within the "outer" circle, the inner circle, if it intersects the other anywhere, must pass outside it and then return. This is only possible (a) if it passes out at one point and returns at another point, or (b) if it passes out and returns through one and the same point. (a) is impossible because it would require two circles to have *three* common points; (b) would require that the inner circle should have a *node* at the point where it passes outside the other, and this is proved to be impossible by drawing any radius cutting both loops.

2. Since the circles cannot intersect, one must be *entirely* within the other.

3. Therefore the outer circle must be greater than the inner, and the radius of the outer greater than that of the inner.

4. Now, if F be the centre of the greater and G of the inner circle, and if FG produced beyond G does *not* pass through A , the given point of contact, then there are three possible hypotheses.

(a) A may lie on GF produced beyond F .

(b) A may lie outside the line FG altogether, in which case FG produced beyond G must, in consequence of result 2 above, either

- (i) meet the circles in a point common to both, or
- (ii) meet the circles in two points, of which that which is on the inner circle is nearer to G than the other is.

(a) is then proved to be impossible by means of the fact that the radius of the inner circle is less than the radius of the outer.

(b) (ii) is Euclid's case; and his proof holds equally of (b) (i), the hypothesis, namely, that D and H in the figure coincide.

Thus all alternative hypotheses are successively shown to be impossible, and the proposition is completely established.

I think, however, that this procedure may be somewhat shortened in the following manner.

In order to make Euclid's proof absolutely conclusive we have only (1) to take care to produce FG beyond G , the centre of the "inner" circle, and then (2) to prove that the point in which FG so produced meets the "inner" circle is *not further* from G than is the point in which it meets the other circle. Euclid's proof is equally valid whether the first point is nearer to G than the second or the first point and the second coincide.

If FG produced beyond G does not pass through A , there are two



conceivable hypotheses: (a) A may lie on GF produced beyond F , or (b) A may be outside FG produced either way. In either case, if FG produced meets the "inner" circle in D and the other in H , and if GD is greater than GH , then the "inner" circle must cut the "outer" circle at some point between A and D , say X .

But, if two circles have a common point X lying on one side of the line of centres, they must have another corresponding point on the other side of the line of centres. This is clear from III. 7, 8; for the point is determined by drawing from F and G , on the opposite side to that where X is, straight lines FY , GY making with FD angles equal to the angles DFX , DGX respectively.

Hence the two circles will have at least three points common: which is impossible.

Therefore GD cannot be greater than GH ; accordingly GD must be either equal to, or less than, GH , and Euclid's proof is valid.

The particular hypothesis in which FG is supposed to be in the same straight line with A but G is on the side of F away from A is easily disposed of, and would in any case have been left to the reader by Euclid.

For GD is either equal to or less than GH .

Therefore GD is less than FH , and therefore less than FA .

But GD is equal to GA , and therefore greater than FA : which is impossible.

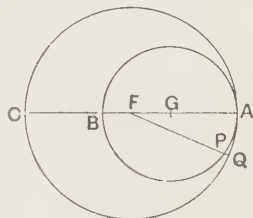
Subject to the same preliminary investigation as that required by Euclid's proof, the proposition can also be proved directly from III. 7.

For, by III. 7, GH is the shortest straight line that can be drawn from G to the circle with centre F ;

therefore GH is less than GA ,

and therefore less than GD : which is absurd.

This proposition is the crucial one as regards circles which touch internally; and, when it is once established, the relative position of the circles can be completely elucidated by means of it and the propositions which have preceded it. Thus, in the annexed figure, if F be the centre of the outer circle and G the centre of the inner, and if any radius FQ of the outer circle meet the two circles in Q, P respectively, it follows, from III. 7, III. 8, or the corresponding theorem with reference to a point *on* the circumference, that FA is the maximum straight line from F to the circumference of the inner circle, FP is less than FA , and FP diminishes in length as FQ moves round from FA until FP reaches its minimum length FB . Hence the circles do not meet at any other point than A , and the distance PQ cut off between them on any radius FQ of the outer circle becomes greater and greater as FQ moves round from FA to FC and is a maximum when FQ coincides with FC , after which it diminishes again on the other side of FC .



The same consideration gives the partial converse of III. 11 which forms the 6th lemma of Pappus to the first book of the *Tactiones* of Apollonius (Pappus, vii. p. 826). This is to the effect that, *if AB, AC are in one straight line, and on one side of A, the circles described on AB, AC as diameters touch (internally at the point A).* Pappus concludes this from the fact that the circles have a common tangent at A ; but the truth of it is clear from the fact that FP diminishes as FQ moves away from FA on either side; whence the circles meet at A but do not cut one another.

Pappus' 5th lemma (vii. p. 824) is another partial converse, namely that, *given two circles touching internally at A, and a line ABC drawn from A cutting both, then, if the centre of the outer circle lies on ABC, so does the centre of the inner.* Pappus himself proves this, by means of the common tangent to the circles at A , in two ways. (1) The tangent is at right angles to AC and therefore to AB : therefore the centre of the inner circle lies on AB . (2) By III. 32, the angles in the alternate segments of both circles are right angles, so that ABC is a diameter of both.

[PROPOSITION 12.

If two circles touch one another externally, the straight line joining their centres will pass through the point of contact.

For let the two circles ABC, ADE touch one another externally at the point A , and let the centre F of ABC , and the centre G of ADE , be taken;

I say that the straight line joined from F to G will pass through the point of contact at A .

For suppose it does not,
 10 but, if possible, let it pass as $FCDG$, and let AF , AG be joined.

Then, since the point F is the centre of the circle ABC ,

15 FA is equal to FC .

Again, since the point G is the centre of the circle ADE ,

GA is equal to GD .

But FA was also proved equal to FC ;

20 therefore FA , AG are equal to FC , GD ,

so that the whole FG is greater than FA , AG ;

but it is also less [I. 20]: which is impossible.

Therefore the straight line joined from F to G will not fail to pass through the point of contact at A ;

25 therefore it will pass through it.

Therefore etc.

Q. E. D.]

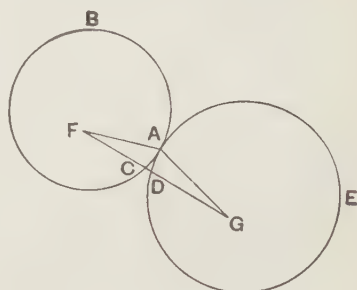
23. will not fail to pass. The Greek has the double negative, οὐκ ἄρα ἡ...εὐθεία... οὐκ ἐλεύσεται, literally "the straight line...will not not-pass...."

Heron says on III. 11: "Euclid in proposition 11 has supposed the two circles to touch internally, made his proposition deal with this case and proved what was sought in it. But I will show how it is to be proved if the contact is external." He then gives substantially the proof and figure of III. 12. It seems clear that neither Heron nor an-Nairizi had III. 12 in this place.

Campanus and the Arabic edition of Naṣīraddin aṭ-Ṭūsī have nothing more of III. 12 than the following addition to III. 11. "In the case of external contact the two lines ac and cb will be greater than ab , whence ad and cb will be greater than the whole ab , which is false." (The points a, b, c, d, e correspond respectively to G, F, C, D, A in the above figure.) It is most probable that Theon or some other editor added Heron's proof in his edition and made Prop. 12 out of it (an-Nairizi, ed. Curtze, pp. 121—2). An-Nairizi and Campanus, conformably with what has been said, number Prop. 13 of Heiberg's text Prop. 12, and so on through the Book.

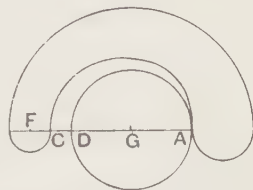
What was said in the note on the last proposition applies, *mutatis mutandis*, to this. Camerer proceeds in the same manner as before; and we may use the same alternative argument in this case also.

Euclid's proof is valid provided only that, if FG , joining the assumed centres, meets the circle with centre F in C and the other circle in D , C is not within the circle ADE and D is not within the circle ABC . (The proof is equally valid whether C, D coincide or the successive points are, as drawn in the figure, in the order F, C, D, G .) Now, if C is within the circle ADE



and D within the circle ABC , the circles must have cut between A and C and between A and D . Hence, as before, they must also have another corresponding point common on the other side of CD . That is, the circles must have *three* common points: which is impossible.

Hence Euclid's proof is valid if F, A, G form a triangle, and the only hypothesis which has still to be disproved is the hypothesis which he would in any case have left to the reader, namely that A does not lie on FG but on FG produced in either direction. In this case, as before, either C, D must coincide or C is nearer F than D is. Then the radius FC must be equal to FA : which is impossible, since FC cannot be greater than FD , and must therefore be *less* than FA .



Given the same preliminaries, III. 12 can be proved by means of III. 8.

Again, when the proposition III. 12 is once proved, III. 8 helps us to prove at once that the circles lie entirely outside each other and have no other common point than the point of contact.

Among Pappus' lemmas to Apollonius' *Tactiones* are the two partial converses of this proposition corresponding to those given in the last note. Lemma 4 (VII. p. 824) is to the effect that, *if AB, AC be in one straight line, B and C being on opposite sides of A, the circles drawn on AB, AC as diameters touch externally at A*. Lemma 3 (VII. p. 822) states that, *if two circles touch externally at A and BAC is drawn through A cutting both circles and containing the centre of one, BAC will also contain the centre of the other*. The proofs, as before, use the common tangent at A .

Mr H. M. Taylor gets over the difficulties involved by III. 11, 12 in a manner which is most ingenious but not Euclidean. He first proves that, *if two circles meet at a point not in the same straight line with their centres, the circles intersect at that point*; this is very easily established by means of III. 7, 8 and the third similar theorem. Then he gives as a corollary the statement that, *if two circles touch, the point of contact is in the same straight line with their centres*. It is not explained how this is inferred from the substantive proposition; it seems, however, to be a *logical* inference simply. By the proposition, every A (circles meeting at a point not in the same straight line with the centre) is B (circles which intersect); therefore every not- B is not- A , i.e. circles which do not intersect do not meet at a point not in the same straight line with the centres. Now non-intersecting circles may either meet (i.e. touch) or not meet. In the former case they must meet *on* the line of centres: for, if they met at a point not in that line, they would intersect. But such a purely *logical* inference is foreign to Euclid's manner. As De Morgan says, "Euclid may have been ignorant of the identity of 'Every X is Y ' and 'Every not- Y is not- X ,' for anything that appears in his writings; he makes the one follow from the other by a new proof each time" (quoted in Keynes' *Formal Logic*, p. 81).

There is no difficulty in proving, by means of I. 20, Mr Taylor's next proposition that, *if two circles meet at a point which lies in the same straight line as their centres and is between the centres, the circles touch at that point, and each circle lies without the other*. But the similar proof, by means of I. 20, of the corresponding theorem for internal contact seems to be open to the same objection as Euclid's proof of III. 11 in that it assumes without proof that the circle which has its centre nearest to the point of meeting is the "inner" circle. Lastly, in order to prove that, *if two circles have a point of contact, they*

do not meet at any other point, Mr Taylor uses the questionable corollary. Therefore in any case his alternative procedure does not seem preferable to Euclid's.

The alternative to Eucl. III. 11—13 which finds most favour in modern continental text-books (e.g. Legendre, Baltzer, Henrici and Treutlein, Veronese, Ingrami, Enriques and Amaldi) connects the number, position and nature of the coincidences between points on two circles with the relation in which the distance between their centres stands to the length of their radii. Enriques and Amaldi, whose treatment of the different cases is typical, give the following propositions (Veronese gives them in the converse form).

1. *If the distance between the centres of two circles is greater than the sum of the radii, the two circles have no point common and are external to one another.*

Let O, O' be the centres of the circles (which we will call "the circles O, O' "), r, r' their radii respectively.

Since then $OO' > r + r'$, *a fortiori* $OO' > r$, and O' is therefore exterior to the circle O .

Next, the circumference of the circle O intersects OO' in a point A , and since $OO' > r + r'$, $AO' > r'$, and A is external to the circle O' .

But $O'A$ is less than any straight line, as $O'B$, drawn to the circumference of the circle O [III. 8]; hence all points, as B , on the circumference of the circle O are external to the circle O' .

Lastly, if C be any point internal to the circle O , the sum of $OC, O'C$ is greater than $O'O$, and *a fortiori* greater than $r + r'$.

But OC is less than r : therefore $O'C$ is greater than r' , or C is external to O .

Similarly we prove that any point on or within the circumference of the circle O' is external to the circle O .

2. *If the distance between the centres of two unequal circles is less than the difference of the radii, the two circumferences have no common point and the lesser circle is entirely within the greater.*

Let O, O' be the centres of the two circles, r, r' their radii respectively ($r < r'$).

Since $OO' < r' - r$, *a fortiori* $OO' < r'$, so that O is internal to the circle O' .

If A, A' be the points in which the straight line OO' intersects respectively the circumferences of the circles O, O' ,

$O'O$ is less than $O'A' - OA$,

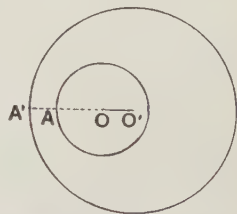
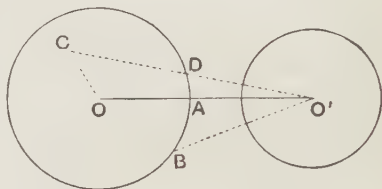
so that $O'O + OA$, or $O'A$, is less than $O'A'$,

and therefore A is internal to the circle O' .

But, of all the straight lines from O' to the circumference of the circle O , $O'A$ passing through the centre O is the greatest [III. 7];

whence all the points of the circumference of O are internal to the circle O' .

A similar argument to the preceding will show that all points within the circle O are internal to the circle O' .

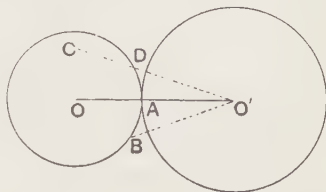


3. *If the distance between the centres of two circles is equal to the sum of the radii, the two circumferences have one point common and one only, and that point is on the line of centres. Each circle is external to the other.*

Let O , O' be the centres, r , r' the radii of the circles, so that OO' is equal to $r + r'$.

Thus OO' is greater than r , so that O' is external to the circle O , and the circumference of the circle O cuts OO' in a point A .

And, since OO' is equal to $r + r'$, and OA to r , it follows that $O'A$ is equal to r' , so that A belongs also to the circumference of the circle O' .



The proof that all other points on, and all points within, the circumference of the circle O are external to the circle O' follows the similar proof of prop. 1 above. And similarly all points (except A) on, and all points within, the circumference of the circle O' are external to the circle O .

The two circles, having one common point only, *touch* at that point, which lies, as shown, on the line of centres. And, since the circles are external to one another, they touch *externally*.

4. *If the distance between the centres of two unequal circles is equal to the difference between the radii, the two circumferences have one point and one only in common, and that point lies on the line of centres. The lesser circle is within the other.*

The proof is that of prop. 2 above, *mutatis mutandis*.

The circles here touch *internally* at the point on the line of centres.

5. *If the distance between the centres of two circles is less than the sum, and greater than the difference, of the radii, the two circumferences have two common points symmetrically situated with respect to the line of centres but not lying on that line.*

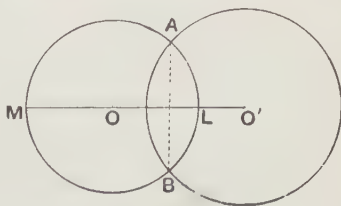
Let O , O' be the centres of the two circles, r , r' their radii, r' being the greater, so that

$$r' - r < OO' < r + r'.$$

It follows that in any case $OO' + r > r'$, so that, if OM be taken on $O'O$ produced equal to r (so that M is on the circumference of the circle O), M is external to the circle O' .

We have to use the same Postulate as in Eucl. I. 1 that

An arc of a circle which has one extremity within and the other without a given circle has one point common with the latter and only one; from which it follows, if we consider two such arcs making a complete circumference, that, if a circumference of a circle passes through one point internal to, and one point external to a given circle, it cuts the latter circle in two points.



We have then to prove that the circle O , besides having one point M of its circumference external to the circle O' , has one other point of its circumference (L) internal to the latter circle.

Three cases have to be distinguished according as OO' is greater than, equal to, or less than, the radius r of the lesser circle.

(1) $OO' > r$. (See the preceding figure.)

Measure OL along OO' equal to r , so that L lies on the circumference of the circle O .

Then, since $OO' < r + r'$, $O'L$ will be less than r' , so that L is within the circle O' .

(2) $OO' = r$.

In this case the circumference of the circle O passes through O' , or L coincides with O' .

(3) $OO' < r$.

If we measure OL along OO' equal to r , the point L will lie on the circumference of the circle O .

Then $O'L = r - OO'$,

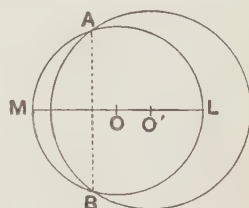
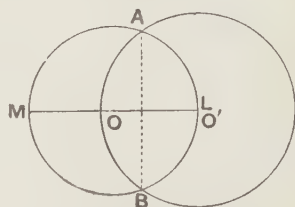
so that $O'L < r$, and *a fortiori* $O'L < r'$, so that L lies within the circle O' .

Thus, in all three cases, since the circumference of O passes through one point (M) external to, and one point (L) internal to, the circle O' , the two circumferences intersect in two points A, B [Post.]

And A, B cannot lie on the line of centres OO' , since this straight line intersects the circle O in L, M only, and of these points one is inside, the other outside, the circle O' .

Since AB is a common chord of both circles, the straight line bisecting it at right angles passes through both centres, i.e. is identical with OO' .

And again by means of III. 7, 8 we prove that all points except A, B on the arc ALB lie within the circle O' , and all points except A, B on the arc AMB outside that circle; and so on.



PROPOSITION 13.

A circle does not touch a circle at more points than one, whether it touch it internally or externally.

For, if possible, let the circle $ABDC$ touch the circle $EBFD$, first internally, at more
5 points than one, namely D, B .

Let the centre G of the circle $ABDC$, and the centre H of $EBFD$, be taken.

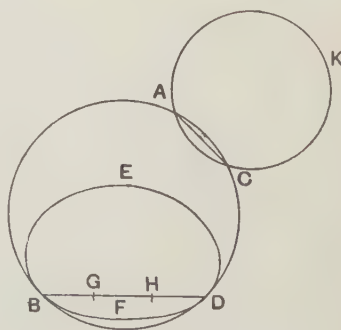
Therefore the straight line
10 joined from G to H will fall on B, D .

[III. 11]

Let it so fall, as $BGHD$.

Then, since the point G is the centre of the circle $ABCD$,

15 BG is equal to GD ;



therefore BG is greater than HD ;

therefore BH is much greater than HD .

Again, since the point H is the centre of the circle $EBFD$,

20 BH is equal to HD ;

but it was also proved much greater than it: which is impossible.

Therefore a circle does not touch a circle internally at more points than one.

25 I say further that neither does it so touch it externally.

For, if possible, let the circle ACK touch the circle $ABDC$ at more points than one, namely A, C , and let AC be joined.

Then, since on the circumference of each of the circles 30 $ABDC, ACK$ two points A, C have been taken at random, the straight line joining the points will fall within each circle; [III. 2]

but it fell within the circle $ABCD$ and outside ACK [III. Def. 3]: which is absurd.

35 Therefore a circle does not touch a circle externally at more points than one.

And it was proved that neither does it so touch it internally.

Therefore etc.

Q. E. D.

3, 7, 14, 27, 30, 33. **ABDC.** Euclid writes $ABCD$ (here and in the next proposition), notwithstanding the order in which the points are placed in the figure.

25, 37. **does it so touch it.** It is necessary to supply these words which the Greek ($\delta\tau\iota\ \o\upsilon\delta\epsilon\ \epsilon\kappa\tau\acute{o}\varsigma$ and $\delta\tau\iota\ \o\upsilon\delta\epsilon\ \epsilon\nu\tau\acute{o}\varsigma$) leaves to be understood.

The difficulties which have been felt in regard to the proofs of this proposition need not trouble us now, because they have already been disposed of in the discussion of the more crucial propositions III. 11, 12.

Euclid's proof of the first part of the proposition differs from Simson's; and we will deal with Euclid's first. On this Camerer remarks that it is assumed that the supposed second point of contact lies on the line of centres *produced beyond the centre of the "outer" circle*, whereas all that is proved in III. 11 is that the line of centres *produced beyond the centre of the "inner" circle* passes through a point of contact. But, by the same argument as that given on III. 11, we show that the circles cannot have a point of contact, or even any common point, outside the line of centres, because, if there were such a point, there would be a corresponding common point on the other side of the line, and the circles would have *three* common points. Hence the only hypothesis left is that the second point of contact may be *on* the line of centres but in the direction of the centre of the "outer" circle; and Euclid's proof disposes of this hypothesis.

Heron (in *an-Nairizī*, ed. Curtze, pp. 122—4), curiously enough, does not question Euclid's assumption that the line of centres passes through both points of contact (if double contact is possible); but he devotes some space to proving that the centre of the "outer" circle must lie within the "inner" circle, a fact which he represents Euclid as asserting ("sicut dixit Euclides"), though there is no such assertion in our text. The proof of the fact is of course easy. If the line of centres passes through *both* points of contact, and the centre of the "outer" circle lies either on or outside the "inner" circle, the line of centres must cut the "inner" circle in *three* points in all: which is impossible, as Heron shows by the lemma, which he places here (and proves by I. 16), that *a straight line cannot cut the circumference of a circle in more points than two*.

Simson's proof is as follows (there is no real need for giving two figures as he does).

"If it be possible, let the circle *EBF* touch the circle *ABC* in more points than one, and first on the inside, in the points *B, D*; join *BD*, and draw *GH* bisecting *BD* at right angles.

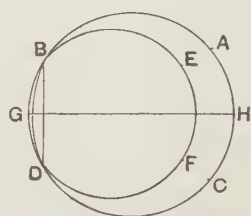
Therefore, because the points *B, D* are in the circumference of each of the circles, the straight line *BD* falls within each of them: And their centres are in the straight line *GH* which bisects *BD* at right angles:

Therefore *GH* passes through the point of contact [III. 11]; but it does not pass through it, because the points *B, D* are without the straight line *GH*: which is absurd.

Therefore one circle cannot touch another on the inside in more points than one."

On this Camerer remarks that, unless III. 11 be more completely elucidated than it is by Euclid's demonstration, which Simson has, it is not sufficiently clear that, besides the point of contact in which *GH* meets the circles, they cannot have another point of contact either (1) on *GH* or (2) outside it. Here again the latter supposition (2) is rendered impossible because in that case there would be a third common point on the opposite side of *GH*; and the former supposition (1) is that which Euclid's proof destroys.

Simson retains Euclid's proof of the second part of the proposition, though his own proof of the first part would apply to the second part also if a reference to III. 12 were substituted for the reference to III. 11. Euclid might also have proved the second part by the same method as that which he employs for the first part.



PROPOSITION 14.

In a circle equal straight lines are equally distant from the centre, and those which are equally distant from the centre are equal to one another.

Let *ABDC* be a circle, and let *AB, CD* be equal straight lines in it;

I say that *AB, CD* are equally distant from the centre.

For let the centre of the circle *ABDC* be taken [III. 1],

and let it be E ; from E let EF , EG be drawn perpendicular to AB , CD , and let AE , EC be joined.

Then, since a straight line EF through the centre cuts a straight line AB not through the centre at right angles, it also bisects it.

[III. 3]

Therefore AF is equal to FB ;

therefore AB is double of AF .

For the same reason

CD is also double of CG ;

and AB is equal to CD ;

therefore AF is also equal to CG .

And, since AE is equal to EC ,

the square on AE is also equal to the square on EC .

But the squares on AF , EF are equal to the square on AE , for the angle at F is right;

and the squares on EG , GC are equal to the square on EC , for the angle at G is right;

[I. 47]

therefore the squares on AF , FE are equal to the squares on CG , GE ,

of which the square on AF is equal to the square on CG , for AF is equal to CG ;

therefore the square on FE which remains is equal to the square on EG ,

therefore EF is equal to EG .

But in a circle straight lines are said to be equally distant from the centre when the perpendiculars drawn to them from the centre are equal;

[III. Def. 4]

therefore AB , CD are equally distant from the centre.

Next, let the straight lines AB , CD be equally distant from the centre; that is, let EF be equal to EG .

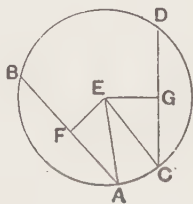
I say that AB is also equal to CD .

For, with the same construction, we can prove, similarly, that AB is double of AF , and CD of CG .

And, since AE is equal to CE ,

the square on AE is equal to the square on CE .

But the squares on EF , FA are equal to the square on AE , and the squares on EG , GC equal to the square on CE . [I. 47]



Therefore the squares on EF , FA are equal to the squares on EG , GC ,

of which the square on EF is equal to the square on EG , for EF is equal to EG ;

therefore the square on AF which remains is equal to the square on CG ;

therefore AF is equal to CG .

And AB is double of AF , and CD double of CG ;

therefore AB is equal to CD .

Therefore etc.

Q. E. D.

Heron (an-Nairizī, pp. 125—7) has an elaborate addition to this proposition in which he proves, first by *reductio ad absurdum*, and then directly, that the centre of the circle falls between the two chords.

PROPOSITION 15.

Of straight lines in a circle the diameter is greatest, and of the rest the nearer to the centre is always greater than the more remote.

Let $ABCD$ be a circle, let AD be its diameter and E the centre; and let BC be nearer to the diameter AD , and FG more remote;

I say that AD is greatest and BC greater than FG .

For from the centre E let EH , EK be drawn perpendicular to BC , FG .

Then, since BC is nearer to the centre and FG more remote, EK is greater than EH . [III. Def. 5]

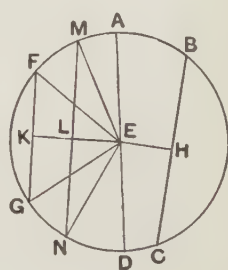
Let EL be made equal to EH , through L let LM be drawn at right angles to EK and carried through to N , and let ME , EN , FE , EG be joined.

Then, since EH is equal to EL ,

BC is also equal to MN . [III. 14]

Again, since AE is equal to EM , and ED to EN ,

AD is equal to ME , EN .



But ME , EN are greater than MN , [I. 20]
and MN is equal to BC ;

therefore AD is greater than BC .

And, since the two sides ME , EN are equal to the two sides FE , EG ,

and the angle MEN greater than the angle FEG ,
therefore the base MN is greater than the base FG . [I. 24]

But MN was proved equal to BC .

Therefore the diameter AD is greatest and BC greater than FG .

Therefore etc.

Q. E. D.

1. Of straight lines. The Greek leaves these words to be understood.

5. Nearer to the diameter AD . As BC , FG are not in general parallel to AD , Euclid should have said "nearer to the centre."

It will be observed that Euclid's proof differs from that given in our text-books (which is Simson's) in that Euclid introduces another line MN , which is drawn so as to be equal to BC but at right angles to EK and therefore parallel to FG . Simson dispenses with MN and bases his proof on a similar proof by Theodosius (*Sphaerica* i. 6). He proves that the sum of the squares on EH , HB is equal to the sum of the squares on EK , KF ; whence he infers that, since the square on EH is less than the square on EK , the square on BH is greater than the square on FK . It may be that Euclid would have regarded this as too complicated an inference to make without explanation or without an increase in the number of his axioms. But, on the other hand, Euclid himself assumes that the angle subtended at the centre by MN is greater than the angle subtended by FG , or, in other words, that M , N both fall outside the triangle FEG . This is a similar assumption to that made in III. 7, 8, as already noticed; and its truth is obvious because EM , EN , being radii of the circle, are greater than the distances from E to the points in which MN cuts EF , EG , and therefore the latter points are nearer than M , N are to L , the foot of the perpendicular from E to MN .

Simson adds the converse of the proposition, proving it in the same way as he proves the proposition itself.

PROPOSITION 16.

The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilineal angle.

Let ABC be a circle about D as centre and AB as diameter;

I say that the straight line drawn from A at right angles to AB from its extremity will fall outside the circle.

For suppose it does not, but, if possible, let it fall within as CA , and let DC be joined.

Since DA is equal to DC ,
the angle DAC is also equal to the angle ACD . [I. 5]

But the angle DAC is right;

therefore the angle ACD is also right:

thus, in the triangle ACD , the two angles DAC , ACD are equal to two right angles: which is impossible. [I. 17]

Therefore the straight line drawn from the point A at right angles to BA will not fall within the circle.

Similarly we can prove that neither will it fall on the circumference;

therefore it will fall outside.

Let it fall as AE ;

I say next that into the space between the straight line AE and the circumference CHA another straight line cannot be interposed.

For, if possible, let another straight line be so interposed, as FA , and let DG be drawn from the point D perpendicular to FA .

Then, since the angle AGD is right,

and the angle DAG is less than a right angle,

AD is greater than DG .

[I. 19]

But DA is equal to DH ;

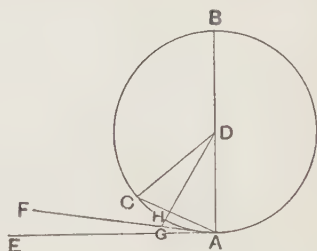
therefore DH is greater than DG , the less than the greater: which is impossible.

Therefore another straight line cannot be interposed into the space between the straight line and the circumference.

I say further that the angle of the semicircle contained by the straight line BA and the circumference CHA is greater than any acute rectilineal angle,

and the remaining angle contained by the circumference CHA and the straight line AE is less than any acute rectilineal angle.

For, if there is any rectilineal angle greater than the angle contained by the straight line BA and the circumference



CHA , and any rectilinear angle less than the angle contained by the circumference CHA and the straight line AE , then into the space between the circumference and the straight line AE a straight line will be interposed such as will make an angle contained by straight lines which is greater than the angle contained by the straight line BA and the circumference CHA , and another angle contained by straight lines which is less than the angle contained by the circumference CHA and the straight line AE .

But such a straight line cannot be interposed ;

therefore there will not be any acute angle contained by straight lines which is greater than the angle contained by the straight line BA and the circumference CHA , nor yet any acute angle contained by straight lines which is less than the angle contained by the circumference CHA and the straight line AE .—

PORISM. . From this it is manifest that the straight line drawn at right angles to the diameter of a circle from its extremity touches the circle.

Q. E. D.

4. cannot be interposed, literally "will not fall in between" (οὐ παρεμπεσείται).

This proposition is historically interesting because of the controversies to which the last part of it gave rise from the 13th to the 17th centuries. History was here repeating itself, for it is certain that, in ancient Greece, both before and after Euclid's time, there had been a great deal of the same sort of contention about the nature of the "angle of a semicircle" and the "remaining angle" between the circumference of the semicircle and the tangent at its extremity. As we have seen (note on 1. Def. 8), the latter angle had a recognised name, *κερατοειδὴς γωνία*, *horn-like* or *cornicular* angle; though this term does not appear in Euclid, it is often used by Proclus, evidently as a term well understood. While it is from Proclus that we get the best idea of the ancient controversies on this subject, we may, I think, infer their prevalence in Euclid's time from this solitary appearance of the two "angles" in the *Elements*. Along with the definition of the angle of a segment, it seems to show that, although these angles are only mentioned to be dropped again immediately, and are of no use in elementary geometry, or even at all, Euclid thought that an allusion to them would be expected of him; it is as if he merely meant to guard himself against appearing to ignore a subject which the geometers of his time regarded with interest. If this conjecture is right, the mention of these angles would correspond to the insertion of definitions of which he makes no use, e.g. those of a rhombus and a rhomboid.

Proclus has no hesitation in speaking of the "angle of a semicircle" and the "horn-like angle" as true *angles*. Thus he says that "angles are contained by a straight line and a circumference in two ways; for they are either contained by a straight line and a convex circumference, like that of the semi-

circle, or by a straight line and a concave circumference, like the *κερατοειδής*" (p. 127, 11—14). "There are *mixed* lines, as spirals, and angles, as the angle of a semicircle and the *κερατοειδής*" (p. 104, 16—18). The difficulty which the ancients felt arose from the very fact which Euclid embodies in this proposition. Since an angle can be divided by a line, it would seem to be a magnitude; "but if it is a magnitude, and all homogeneous magnitudes which are finite have a ratio to one another, then all homogeneous angles, or rather all those on surfaces, will have a ratio to one another, so that the *cornicular* will also have a ratio to the rectilinear. But things which have a ratio to one another can, if multiplied, exceed one another. Therefore the *cornicular* angle will also sometime exceed the rectilinear; which is impossible, for it is proved that the former is less than any rectilinear angle" (Proclus, p. 121, 24—122, 6). The nature of contact between straight lines and circles was also involved in the question, and that this was the subject of controversy before Euclid's time is clear from the title of a work attributed to Democritus (fl. 420—400 B.C.) *περὶ διαφορῆς γνώμονος ἢ περὶ ψαύσιος κύκλου καὶ σφαίρης*, *On a difference in a gnomon or on contact of a circle and a sphere*. There is, however, another reading of the first words of this title as given by Diogenes Laertius (ix. 47), namely *περὶ διαφορῆς γνώμης*, *On a difference of opinion*, etc. May it not be that neither reading is correct, but that the words should be *περὶ διαφορῆς γωνιῆς ἢ περὶ ψαύσιος κύκλου καὶ σφαίρης*, *On a difference in an angle or on contact with a circle and a sphere*? There would, of course, hardly be any "angle" in connexion with the sphere; but I do not think that this constitutes any difficulty, because the sphere might easily be tacked on as a kindred subject to the circle. A curiously similar collocation of words appears in a passage of Proclus, though this may be an accident. He says (p. 50, 4) *πὼς δὲ γωνιῶν διαφορὰς λέγομεν καὶ αὐξήσεις αὐτῶν* ... and then, in the next line but one, *πὼς δὲ τὰς ἀφὰς τῶν κύκλων ἢ τῶν εὐθειῶν*, "In what sense do we speak of *differences of angles and of increases of them* ... and in what sense of the *contacts* (or meetings) of circles or of straight lines?" I cannot help thinking that this subject of *cornicular* angles would have had a fascination for Democritus as being akin to the question of infinitesimals, and very much of the same character as the other question which Plutarch (*On Common Notions*, xxxix. 3) says that he raised, namely that of the relation between the base of a cone and a section of it by a plane parallel to the base and apparently, to judge by the context, infinitely near to it: "if a cone were cut by a plane parallel to its base, what must we think of the surfaces of the sections, that they are equal or unequal? For, if they are unequal, they will make the cone irregular, as having many indentations like steps, and unevennesses; but, if they are equal, the sections will be equal, and the cone will appear to have the property of the cylinder, as being made up of equal and not unequal circles, which is the height of absurdity."

The contributions by Democritus to such investigations are further attested by a passage in the *Method* of Archimedes discovered by Heiberg in 1906 (*Archimedes*, ed. Heiberg, Vol. II. 1913, p. 430; T. L. Heath, *The Method of Archimedes*, 1912, p. 13), which says that, though Eudoxus was the first to discover the scientific proof of the propositions (attributed to him) that the cone and the pyramid are one-third of the cylinder and prism respectively which have the same base and equal height, they were first *stated*, without proof, by Democritus.

A full history of the later controversies about the *cornicular* "angle" cannot be given here; more on the subject will be found in Camerer's *Euclid* (Excursus IV. on III. 16) or in Cantor's *Geschichte der Mathematik*.

Vol. II. (see *Contingenzwinkel* in the index). But the following short note about the attitude of certain well-known mathematicians to the question will perhaps not be out of place. Johannes Campanus, who edited Euclid in the 13th century, inferred from III. 16 that there was a flaw in the principle that *the transition from the less to the greater, or vice versa, takes place through all intermediate quantities and therefore through the equal*. If a diameter of a circle, he says, be moved about its extremity until it takes the position of the tangent to that circle, then, as long as it cuts the circle, it makes an acute angle *less* than the "angle of a semicircle"; but the moment it ceases to cut, it makes a right angle *greater* than the same "angle of a semicircle." The rectilineal angle is never, during the transition, *equal* to the "angle of a semicircle." There is therefore an apparent inconsistency with X. 1, and Campanus could only observe (as he does on that proposition), in explanation of the paradox, that "these are not angles in the same sense (univoce), for the curved and the straight are not things of the same kind without qualification (simpliciter)." The argument assumes, of course, that the right angle is greater than the "angle of a semicircle."

Very similar is the statement of the paradox by Cardano (1501—1576), who observed that *a quantity may continually increase without limit, and another diminish without limit; and yet the first, however increased, may be less than the second, however diminished*. The first quantity is of course the *angle of contact*, as he calls it, which may be "increased" indefinitely by drawing smaller and smaller circles touching the same straight line at the same point, but will always be less than any acute rectilineal angle however small.

We next come to the French geometer, Peletier (Peletarius), who edited the *Elements* in 1557, and whose views on this subject seem to mark a great advance. Peletier's opinions and arguments are most easily accessible in the account of them given by Clavius (Christoph Klau [?], 1537—1612) in the 1607 edition of his Euclid. The violence of the controversy between the two will be understood from the fact that the arguments and counter-arguments (which sometimes run into other matters than the particular question at issue) cover, in that book, 26 pages of small print. Peletier held that the "angle of contact" was not an angle at all, that the "contact of two circles," i.e. the "angle" between the circumferences of two circles touching one another internally or externally, is not a *quantity*, and that the "contact of a straight line with a circle" is not a *quantity* either; that angles contained by a diameter and a circumference whether inside or outside the circle are *right angles* and equal to rectilineal right angles, and that angles contained by a diameter and the circumference in *all* circles are *equal*. The proof which Peletier gave of the latter proposition in a letter to Cardano is sufficiently ingenious. If a greater and a less semicircle be placed with their diameters terminating at a common point and lying in a straight line, then (1) the *angle* of the larger obviously cannot be *less* than the *angle* of the smaller. Neither (2) can the former be *greater* than the latter; for, if it were, we could obtain another *angle* of a semicircle greater still by drawing a still larger semicircle, and so on, until we should ultimately have an *angle* of a semicircle greater than a right angle: which is impossible. Hence the *angles* of semicircles must all be *equal*, and the differences between them *nothing*. Having satisfied himself that all *angles of contact* are *not*-angles, *not*-quantities, and therefore *nothings*, Peletier holds the difficulty about X. 1 to be at an end. He adds the interesting remark that the essence of an angle is in *cutting*, not contact, and that a tangent is not *inclined* to the circle at the point of contact but is, as it were, *immersed* in it at that point, just as much as if the circle did not diverge from it on either side.

The reply of Clavius need not detain us. He argues, evidently appealing to the eye, that the angle of contact can be *divided* by the arc of a circle greater than the given one, that the angles of two semicircles of different sizes cannot be equal, since they do not coincide if they are applied to one another, that there is nothing to prevent *angles of contact* from being *quantities*, it being only necessary, in view of x. 1, to admit that they are not of the same kind as rectilinear angles; lastly that, if the angle of contact had been a *nothing*, Euclid would not have given himself so much trouble to prove that it is less than any acute angle. (The word is *desudasset*, which is certainly an exaggeration as applied to what is little more than an *obiter dictum* in III. 16.)

Vieta (1540—1603) ranged himself on the side of Peletier, maintaining that the *angle of contact* is no angle; only he uses a new method of proof. The circle, he says, may be regarded as a plane figure with an infinite number of sides and angles; but *a straight line touching a straight line, however short it may be, will coincide with that straight line and will not make an angle*. Never before, says Cantor (II, p. 540), had it been so plainly declared what exactly was to be understood by *contact*.

Galileo Galilei (1564—1642) seems to have held the same view as Vieta and to have supported it by a very similar argument derived from the comparison of the circle and an inscribed polygon with an infinite number of sides.

The last writer on the question who must be mentioned is John Wallis (1616—1703). He published in 1656 a paper entitled *De angulo contactus et semicirculi tractatus* in which he also maintained that the so-called angle was not a true angle, and was not a *quantity*. Vincent Leotaud (1595—1672) took up the cudgels for Clavius in his *Cyclomathia* which appeared in 1663. This brought a reply from Wallis in a letter to Leotaud dated 17 February, 1667, but not apparently published till it appeared in *A defense of the treatise of the angle of contact* which, with a separate title-page, and date 1684, was included in the English edition of his *Algebra* dated 1685. The essence of Wallis' position may be put as follows. According to Euclid's definition, a plane angle is an *inclination* of two lines; therefore two lines forming an angle must *incline* to one another, and, if two lines meet without being *inclined* to one another at the point of meeting (which is the case when a circumference is touched by a straight line), the lines do not form an *angle*. The "angle of contact" is therefore no angle, because *at the point of contact* the straight line is not inclined to the circle but lies on it ἀκλινώς, or is coincident with it. Again, as a point is not a line but a *beginning* of a line, and a line is not a surface but a *beginning* of a surface, so an angle is not the distance between two lines, but their initial tendency towards separation: *Angulus (seu gradus divaricationis) Distantia non est sed Inceptivus distantiae*. How far lines, which at their point of meeting do not form an angle, separate from one another as they pass on depends on the *degree of curvature* (gradus curvitatatis), and it is the latter which has to be compared in the case of two lines so meeting. The arc of a smaller circle is more curved as having as much curvature in a lesser length, and is therefore curved in a greater degree. Thus what Clavius called *angulus contactus* becomes with Wallis *gradus curvitatatis*, the use of which expression shows that curvature and curvature can be compared according to one and the same standard. A straight line has the least possible curvature; but of the "angle" made by it with a curve which it touches we cannot say that it is greater or less than the "angle" which a second curve touching the same straight line at the same point makes with the first curve; for in both cases there is no true angle at all (cf. Cantor III, p. 24).

The words usually given as a part of the corollary "and that a straight line touches a circle at one point only, since in fact the straight line meeting it in two points was proved to fall within it" are omitted by Heiberg as being an undoubted addition of Theon's. It was Simson who added the further remark that "it is evident that there can be but one straight line which touches the circle at the same point."

PROPOSITION 17.

From a given point to draw a straight line touching a given circle.

Let A be the given point, and BCD the given circle ; thus it is required to draw from the point A a straight line touching the circle BCD .

For let the centre E of the circle be taken ;

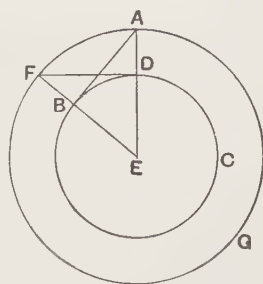
[III. 1]

let AE be joined, and with centre E and distance EA let the circle AFG be described ;

from D let DF be drawn at right angles to EA ,

and let EF , AB be joined ;

I say that AB has been drawn from the point A touching the circle BCD .



For, since E is the centre of the circles BCD , AFG ,

EA is equal to EF , and ED to EB ;

therefore the two sides AE , EB are equal to the two sides FE , ED ;

and they contain a common angle, the angle at E ;

therefore the base DF is equal to the base AB ,

and the triangle DEF is equal to the triangle BEA ,

and the remaining angles to the remaining angles ; [I. 4]

therefore the angle EDF is equal to the angle EBA .

But the angle EDF is right ;

therefore the angle EBA is also right.

Now EB is a radius ;

and the straight line drawn at right angles to the diameter of a circle, from its extremity, touches the circle ; [III. 16, Por.]

therefore AB touches the circle BCD .

Therefore from the given point A the straight line AB has been drawn touching the circle BCD .

Q. E. F.

The construction shows, of course, that two straight lines can be drawn from a given external point to touch a given circle; and it is equally obvious that these two straight lines are equal in length and equally inclined to the straight line joining the external point to the centre of the given circle. These facts are given by Heron (*an-Nairīzī*, p. 130).

It is true that Euclid leaves out the case where the given point lies *on* the circumference of the circle, doubtless because the construction is so directly indicated by III. 16, Por. as to be scarcely worth a separate statement.

An easier solution is of course possible as soon as we know (III. 31) that the angle in a semicircle is a right angle; for we have only to describe a circle on AE as diameter, and this circle cuts the given circle in the two points of contact.

PROPOSITION 18.

If a straight line touch a circle, and a straight line be joined from the centre to the point of contact, the straight line so joined will be perpendicular to the tangent.

For let a straight line DE touch the circle ABC at the point C , let the centre F of the circle ABC be taken, and let FC be joined from F to C ;

I say that FC is perpendicular to DE .

For, if not, let FG be drawn from F perpendicular to DE .

Then, since the angle FGC is right,

the angle FCG is acute; [I. 17]
and the greater angle is subtended by the greater side;

therefore FC is greater than FG .

But FC is equal to FB ;

therefore FB is also greater than FG ,

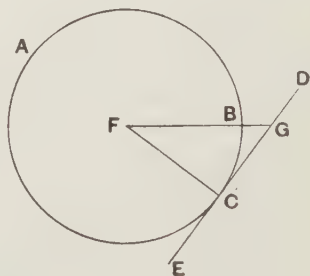
the less than the greater: which is impossible.

Therefore FG is not perpendicular to DE .

Similarly we can prove that neither is any other straight line except FC ;

therefore FC is perpendicular to DE .

Therefore etc.



[I. 19]

3. the tangent, ἡ ἐφαπτομένη.

Just as III. 3 contains two *partial* converses of the Porism to III. 1, so the present proposition and the next give two partial converses of the corollary to III. 16. We may show their relation thus: suppose three things, (1) a tangent at a point of a circle, (2) a straight line drawn from the centre to the point of contact, (3) right angles made at the point of contact [with (1) or (2) as the case may be]. Then the corollary to III. 16 asserts that (2) and (3) together give (1), III. 18 that (1) and (2) give (3), and III. 19 that (1) and (3) give (2), i.e. that the straight line drawn from the point of contact at right angles to the tangent passes through the centre.

PROPOSITION 19.

If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the tangent, the centre of the circle will be on the straight line so drawn.

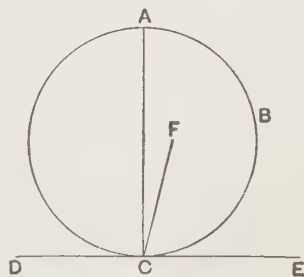
For let a straight line DE touch the circle ABC at the point C , and from C let CA be drawn at right angles to DE ;

I say that the centre of the circle is on AC .

For suppose it is not, but, if possible, let F be the centre, and let CF be joined.

Since a straight line DE touches the circle ABC ,

and FC has been joined from the centre to the point of contact,



FC is perpendicular to DE ; [III. 18]

therefore the angle FCE is right.

But the angle ACE is also right;

therefore the angle FCE is equal to the angle ACE ,

the less to the greater: which is impossible.

Therefore F is not the centre of the circle ABC .

Similarly we can prove that neither is any other point except a point on AC .

Therefore etc.

Q. E. D.

We may also regard III. 19 as a partial converse of III. 18. Thus suppose (1) a straight line through the centre, (2) a straight line through the point of contact, and suppose (3) to mean perpendicular to the tangent; then III. 18 asserts that (1) and (2) combined produce (3), and III. 19 that (2) and (3)

produce (1); while again we may enunciate a second partial converse of III. 18, corresponding to the statement that (1) and (3) produce (2), to the effect that a straight line drawn through the centre perpendicular to the tangent passes through the point of contact.

We may add at this point, or even after the Porism to III. 16, the theorem that *two circles which touch one another internally or externally have a common tangent at their point of contact*. For the line joining their centres, produced if necessary, passes through their point of contact, and a straight line drawn through that point at right angles to the line of centres is a tangent to both circles.

PROPOSITION 20.

In a circle the angle at the centre is double of the angle at the circumference, when the angles have the same circumference as base.

Let ABC be a circle, let the angle BEC be an angle
5 at its centre, and the angle BAC an
angle at the circumference, and let
them have the same circumference BC
as base;

I say that the angle BEC is double of
10 the angle BAC .

For let AE be joined and drawn
through to F .

Then, since EA is equal to EB ,
the angle EAB is also equal to the
15 angle EBA ; [I. 5]
therefore the angles EAB , EBA are double of the angle
 EAB .

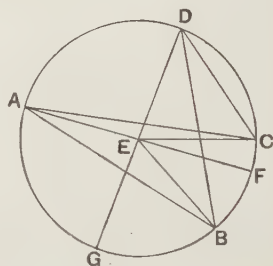
But the angle BEF is equal to the angles EAB , EBA ;
[I. 32]
therefore the angle BEF is also double of the angle
20 EAB .

For the same reason

the angle FEC is also double of the angle EAC .

Therefore the whole angle BEC is double of the whole
angle BAC .

25 Again let another straight line be inflected, and let there
be another angle BDC ; let DE be joined and produced
to G .



Similarly then we can prove that the angle GEC is double of the angle EDC ,

30 of which the angle GEB is double of the angle EDB ;

therefore the angle BEC which remains is double of the angle BDC .

Therefore etc.

Q. E. D.

25. let another straight line be inflected, $\kappa\epsilon\kappa\lambda\acute{\alpha}\sigma\theta\omega$ δὴ πάλιν (without $\epsilon\upsilon\theta\epsilon\iota\alpha$). The verb $\kappa\lambda\acute{\alpha}\omega$ (to *break off*) was the regular technical term for drawing from a point a (broken) straight line which first meets another straight line or curve and is then *bent back* from it to another point, or (in other words) for drawing straight lines from two points meeting at a point on a curve or another straight line. $\kappa\epsilon\kappa\lambda\acute{\alpha}\sigma\theta\alpha\iota$ is one of the geometrical terms the definition of which must according to Aristotle be assumed (*Anal. Post.* I. 10, 76 b 9).

The early editors, Tartaglia, Commandinus, Peletarius, Clavius and others, gave the extension of this proposition to the case where the segment is less than a semicircle, and where accordingly the "angle" corresponding to Euclid's "angle at the centre" is greater than two right angles. The convenience of the extension is obvious, and the proof of it is the same as the first part of Euclid's proof. By means of the extension III. 21 is demonstrated without making two cases; III. 22 will follow immediately from the fact that the sum of the "angles at the centre" for two segments making up a whole circle is equal to four right angles; also III. 31 follows immediately from the extended proposition.

But all the editors referred to were forestalled in this matter by Heron, as we now learn from the commentary of an-Nairizi (ed. Curtze, p. 131 sqq.). Heron gives the extension of Euclid's proposition which, he says, it had been left for him to make, but which is necessary in order that the caviller may not be able to say that the next proposition (about the equality of the angles in any segment) is not established generally, i.e. in the case of a segment less than a semicircle as well as in the case of a segment greater than a semicircle, inasmuch as III. 20, as given by Euclid, only enables us to prove it in the latter case. Heron's enunciation is important as showing how he describes what we should now call an "angle" greater than two right angles. (The language of Gherard's translation is, in other respects, a little obscure; but the meaning is made clear by what follows.)

"The angle," Heron says, "which is at the centre of any circle is double of the angle which is at the circumference of it *when one arc is the base of both angles*; and *the remaining angles which are at the centre, and fill up the four right angles*, are double of the angle at the circumference of the arc which is subtended by the [original] angle which is at the centre."

Thus the "angle greater than two right angles" is for Heron the sum of certain "angles" in the Euclidean sense of angles less than two right angles. The particular method of splitting up which Heron adopts will be seen from his proof, which is in substance as follows.

Let CDB be an angle at the centre, CAB that at the circumference.

Produce BD , CD to F , G ;

take any point E on BC , and join BE , EC , ED .

Then any angle in the segment BAC is half of the angle BDC ; and the sum of the angles BDG , GDF , FDC is double of any angle in the segment BEC .

Proof. Since CD is equal to ED ,
the angles DCE , DEC are equal.

Therefore the exterior angle GDE is equal to twice the angle DEC .

Similarly the exterior angle FDE is equal to twice the angle DEB .

By addition, the angles GDE , FDE are double of the angle BEC .

But

the angle BDC is equal to the angle FDG ,
therefore the sum of the angles BDG , GDF , FDC is double of the angle BEC .

And Euclid has proved the first part of the proposition, namely that the angle BDC is double of the angle BAC .

Now, says Heron, BAC is any angle in the segment BAC , and therefore any angle in the segment BAC is half of the angle BDC .

Therefore all the angles in the segment BAC are equal.

Again, BEC is any angle in the segment BEC and is equal to half the sum of the angles BDG , GDF , FDC .

Therefore all the angles in the segment BEC are equal.

Hence III. 21 is proved generally.

Lastly, says Heron,

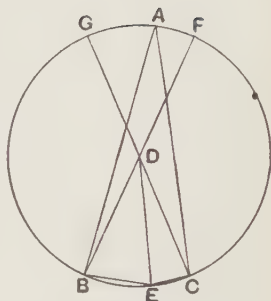
since the sum of the angles BDG , GDF , FDC is double of the angle BEC ,

and the angle BDC is double of the angle BAC ,

therefore, by addition, the sum of four right angles is double of the sum of the angles BAC , BEC .

Hence the angles BAC , BEC are together equal to two right angles, and III. 22 is proved.

The above notes of Heron show conclusively, if proof were wanted, that Euclid had no idea of III. 20 applying *in terms* (either as a matter of enunciation or proof) to the case where the angle at the circumference, or the angle in the segment, is *obtuse*. He would not have recognised the "angle" greater than two right angles or the so-called "straight angle" as being an angle at all. This is indeed clear from his definition of an angle as the *inclination κ.τ.ε.*, and from the language used by other later Greek mathematicians where there would be an opportunity for introducing the extension. Thus Proclus' notion of a "four-sided triangle" (cf. the note above on the definition of a triangle) shows that he did not count a re-entrant angle as an angle, and Zenodorus' application to the same figure of the word "hollow-angled" shows that in that case it was the exterior angle only which he would have called an angle. Further it would have been inconvenient to have introduced at the beginning of the *Elements* an "angle" equal to or greater than two right angles, because other definitions, e.g. that of a *right angle*, would have needed a qualification. If an "angle" might be equal to two right angles, one straight line in a straight line with another would have satisfied Euclid's definition of a right angle. This is noticed by Dodgson (p. 160), but it is practically brought out by Proclus on I. 13. "For he did not merely say that 'any straight line standing on a straight line either makes two right angles or angles equal to two right angles' but 'if it make angles.'



If it stand on the straight line at its extremity and make one angle, is it possible for this to be equal to two right angles? It is of course impossible; *for every rectilineal angle is less than two right angles*, as every solid angle is less than four right angles (p. 292, 13–20).” [It is true that it has been generally held that the meaning of “angle” is tacitly extended in vi. 33, but there is no real ground for this view. See the note on the proposition.]

It will be observed that, following his usual habit, Euclid omits the demonstration of the case which some editors, e.g. Clavius, have thought it necessary to give separately, the case namely where one of the lines forming the angle in the segment passes through the centre. Euclid’s proof gives so obviously the means of proving this that it is properly left out.

Todhunter observes, what Clavius had also remarked, that there are two assumptions in the proof of III. 20, namely that, if A is double of B and C double of D , then the sum, or difference, of A and C is equal to double the sum, or difference, of B and D respectively, the assumptions being particular cases of v. 1 and v. 5. But of course it is easy to satisfy ourselves of the correctness of the assumption without any recourse to Book v.

PROPOSITION 21.

In a circle the angles in the same segment arc equal to one another.

Let $ABCD$ be a circle, and let the angles BAD , BED be angles in the same segment $BAED$; I say that the angles BAD , BED are equal to one another.

For let the centre of the circle $ABCD$ be taken, and let it be F ; let BF , FD be joined.

Now, since the angle BFD is at the centre,

and the angle BAD at the circumference,

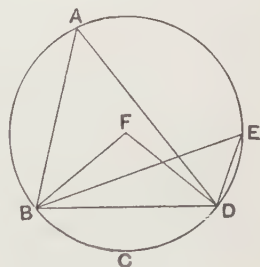
and they have the same circumference BCD as base, therefore the angle BFD is double of the angle BAD . [III. 20]

For the same reason

the angle BFD is also double of the angle BED ;

therefore the angle BAD is equal to the angle BED .

Therefore etc.



Q. E. D.

Under the restriction that the “angle at the centre” used in III. 20 must be less than two right angles, Euclid’s proof of this proposition only applies to the case of a segment greater than a semicircle, and the case of a segment equal to or less than a semicircle has to be considered separately. The simplest proof, of many, seems to be that of Simson.

"But, if the segment $BAED$ be not greater than a semicircle, let BAD , BED be angles in it: these also are equal to one another.

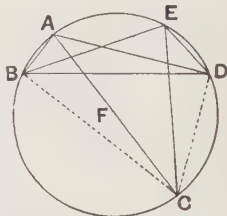
Draw AF to the centre, and produce it to C , and join CE .

Therefore the segment $BADC$ is greater than a semicircle, and the angles in it BAC , BEC are equal, by the first case.

For the same reason, because $CBED$ is greater than a semicircle,

the angles CAD , CED are equal.

Therefore the whole angle BAD is equal to the whole angle BED ."



We can prove, by means of *reductio ad absurdum*, the important converse of this proposition, namely that, *if there be any two triangles on the same base and on the same side of it, and with equal vertical angles, the circle passing through the extremities of the base and the vertex of one triangle will pass through the vertex of the other triangle also*. That a circle can be thus described about a triangle is clear from Euclid's construction in III. 9, which shows how to draw a circle passing through any three points, though it is in IV. 5 only that we have the problem stated. Now, suppose a circle BAC drawn through the angular points of a triangle BAC , and let BDC be another triangle with the same base BC and on the same side of it, and having its vertical angle D equal to the angle A . Then shall the circle pass through D .

For, if it does not, it must pass through some point E on BD or on BD produced. If then EC be joined, the angle BEC is equal to the angle BAC , by III. 21, and therefore equal to the angle BDC . Therefore an exterior angle of a triangle is equal to the interior and opposite angle: which is impossible, by I. 16.

Therefore D lies on the circle BAC .

Similarly for any other triangle on the base BC and with vertical angle equal to A . Thus, *if any number of triangles be constructed on the same base and on the same side of it, with equal vertical angles, the vertices will all lie on the circumference of a segment of a circle*.

A useful theorem derivable from III. 21 is given by Serenus (*De sectione coni*, Props. 52, 53).

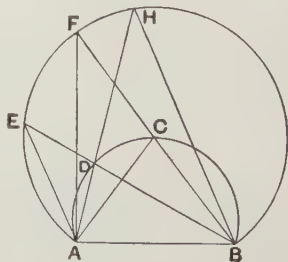
If ADB be any segment of a circle, and C be such a point on the circumference that AC is equal to CB , and if there be described with C as centre and radius CA or CB the circle AHB , then, ADB being any other angle in the segment ACB , and BD being produced to meet the outer segment in E , the sum of AD , DB is equal to BE .

If BC be produced to meet the outer segment in F , and FA be joined,

CA , CB , CF are by hypothesis equal.

Therefore the angle FAC is equal to the angle AFC .

Also, by III. 21, the angles ACB , ADB are equal;



therefore their supplements, the angles ACF , ADE , are equal.

Further, by III. 21, the angles AEB , AFB are equal.

Hence in the triangles ACF , ADE two angles are respectively equal ;
therefore the third angles EAD , FAC are equal.

But the angle FAC is equal to the angle AFC , and therefore equal to the angle AED .

Therefore the angles AED , EAD are equal, or the triangle DEA is isosceles,

and AD is equal to DE .

Adding BD to both, we see that

BE is equal to the sum of AD and DB .

Now, BF being a diameter of the circle of which the outer segment is a part,

BF is greater than BE ;

therefore AC , CB are together greater than AD , DB .

And, generally, of all triangles on the same base and on the same side of it which have equal vertical angles, the isosceles triangle is that which has the greatest perimeter, and of the others that has the lesser perimeter which is further from being isosceles.

The theorem of Serenus gives us the means of solving the following problem given in Todhunter's Euclid, p. 324.

To find a point in the circumference of a given segment of a circle such that the straight lines which join the point to the extremities of the straight line on which the segment stands may be together equal to a given straight line (the length of which is of course subject to limits).

Let ACB in the above figure be the given segment. Find, by bisecting AB at right angles, a point C on it such that AC is equal to CB .

Then with centre C and radius CA or CB describe the segment of a circle AHB on the same side of AB .

Lastly, with A or B as centre and radius equal to the given straight line describe a circle. This circle will, if the given straight line be greater than AB and less than twice AC , meet the outer segment in two points, and if we join those points to the centre of the circle last drawn (whether A or B), the joining straight lines will cut the inner segment in points satisfying the given condition. If the given straight line be equal to twice AC , C is of course the required point. If the given straight line be greater than twice AC , there is no possible solution.

PROPOSITION 22.

The opposite angles of quadrilaterals in circles are equal to two right angles.

Let $ABCD$ be a circle, and let $ABCD$ be a quadrilateral in it ;

I say that the opposite angles are equal to two right angles.

Let AC , BD be joined.

Then, since in any triangle the three angles are equal to two right angles,

[I. 32]

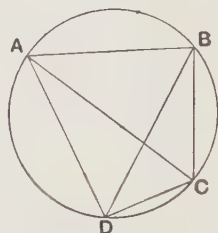
the three angles CAB , ABC , BCA of the triangle ABC are equal to two right angles.

But the angle CAB is equal to the angle BDC , for they are in the same segment $BADC$;

[III. 21]

and the angle ACB is equal to the angle ADB , for they are in the same segment $ADCB$;

therefore the whole angle ADC is equal to the angles BAC , ACB .



Let the angle ABC be added to each ;

therefore the angles ABC , BAC , ACB are equal to the angles ABC , ADC .

But the angles ABC , BAC , ACB are equal to two right angles ;

therefore the angles ABC , ADC are also equal to two right angles.

Similarly we can prove that the angles BAD , DCB are also equal to two right angles.

Therefore etc.

Q. E. D.

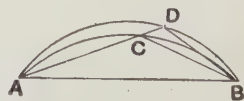
As Todhunter remarks, the converse of this proposition is true and very important : *if two opposite angles of a quadrilateral be together equal to two right angles, a circle may be circumscribed about the quadrilateral.* We can, by the method of III. 9, or by IV. 5, circumscribe a circle about the triangle ABC ; and we can then prove, by *reductio ad absurdum*, that the circle passes through the fourth angular point D .

PROPOSITION 23.

On the same straight line there cannot be constructed two similar and unequal segments of circles on the same side.

For, if possible, on the same straight line AB let two similar and unequal segments of circles ACB , ADB be constructed on the same side ;

let ACD be drawn through, and let CB , DB be joined.



Then, since the segment ACB is similar to the segment ADB ,

and similar segments of circles are those which admit equal angles,

[III. Def. 11]

the angle ACB is equal to the angle ADB , the exterior to the interior : which is impossible.

[I. 16]

Therefore etc.

Q. E. D.

1. cannot be constructed, οὐ συνταθήσεται, the same phrase as in I. 7.

Clavius and the other early editors point out that, while the words "on the same side" in the enunciation are necessary for Euclid's proof, it is equally true that neither can there be two similar and unequal segments on *opposite* sides of the same straight line ; this is at once made clear by causing one of the segments to revolve round the base till it is on the same side with the other.

Simson observes with reason that, while Euclid in the following proposition, III. 24, thinks it necessary to dispose of the hypothesis that, if two similar segments on equal bases are applied to one another with the bases coincident, the segments cannot cut in any other point than the extremities of the base (since otherwise two circles would cut one another in more points than two), this remark is an equally necessary preliminary to III. 23, in order that we may be justified in drawing the segments as being one inside the other. Simson accordingly begins his proof of III. 23 thus :

"Then, because the circle ACB cuts the circle ADB in the two points A, B , they cannot cut one another in any other point :

One of the segments must therefore fall within the other.

Let ACB fall within ADB and draw the straight line ACD , etc."

Simson has also substituted "not coinciding with one another" for "unequal" in Euclid's enunciation.

Then in III. 24 Simson leaves out the words referring to the hypothesis that the segment AEB when applied to the other CFD may be "otherwise placed as CGD "; in fact, after stating that AB must coincide with CD , he merely adds words quoting the result of III. 23 : "Therefore, the straight line AB coinciding with CD , the segment AEB must coincide with the segment CFD , and is therefore equal to it."

PROPOSITION 24.

Similar segments of circles on equal straight lines are equal to one another.

For let AEB, CFD be similar segments of circles on equal straight lines AB, CD ;

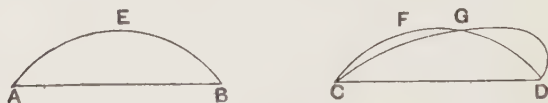
5 I say that the segment AEB is equal to the segment CFD .

For, if the segment AEB be applied to CFD , and if the point A be placed on C and the straight line AB on CD ,

the point B will also coincide with the point D , because AB is equal to CD ;

10 and, AB coinciding with CD ,

the segment AEB will also coincide with CFD .



For, if the straight line AB coincide with CD but the segment AEB do not coincide with CFD ,

it will either fall within it, or outside it;

15 or it will fall awry, as CGD , and a circle cuts a circle at more points than two: which is impossible. [III. 10]

Therefore, if the straight line AB be applied to CD , the segment AEB will not fail to coincide with CFD also;

therefore it will coincide with it and will be equal to it.

20 Therefore etc.

Q. E. D.

15. **fall awry**, *παράλλαξει*, the same word as used in the like case in I. 8. The word implies that the applied figure will partly fall short of, and partly overlap, the figure to which it is applied.

Compare the note on the last proposition. I have put a semicolon instead of the comma which the Greek text has after "outside it," in order the better to indicate that the inference "and a circle cuts a circle in more points than two" only refers to the third hypothesis that the applied segment is "otherwise placed (*παράλλαξει*) as CGD ." The first two hypotheses are disposed of by a *tacit* reference to the preceding proposition III. 23.

PROPOSITION 25.

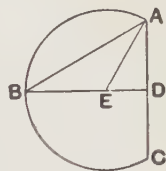
Given a segment of a circle, to describe the complete circle of which it is a segment.

Let ABC be the given segment of a circle;

thus it is required to describe the complete circle belonging to the segment ABC , that is, of which it is a segment.

For let AC be bisected at D , let DB be drawn from the point D at right angles to AC , and let AB be joined;

But, if the angle ABD be less than the angle BAD , and if we construct, on the straight line BA and at the point A on it, an angle equal to the angle ABD , the centre will fall on DB within the segment ABC , and the segment ABC will clearly be greater than a semi-circle.



Therefore, given a segment of a circle, the complete circle has been described.

Q. E. F.

1. to describe the complete circle, *προσαναγράψαι τὸν κύκλον*, literally "to describe the circle *on to it*."

It will be remembered that Simson takes first the case in which the angles ABD , BAD are equal to one another, and then takes the other two cases together, telling us to "produce BD , if necessary." This is a little shorter than Euclid's procedure, though Euclid does not repeat the proof of the first case in giving the third, but only refers to it as equally applicable.

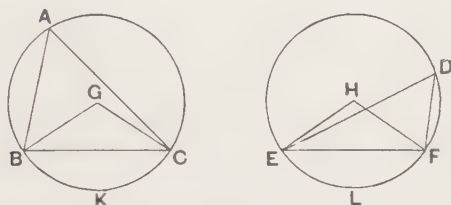
Campanus, Peletarius and others give the solution of this problem in which we take two chords not parallel and bisect each at right angles by straight lines, which must meet in the centre, since each contains the centre and they only intersect in one point. Clavius, Billingsley, Barrow and others give the rather simpler solution in which the two chords have one extremity common (cf. Euclid's proofs of III. 9, 10). This method De Morgan favours, and (as noted on III. 1 above) would make III. 1, this proposition, and IV. 5 all *corollaries* of the theorem that "the line which bisects a chord perpendicularly must contain the centre." Mr H. M. Taylor practically adopts this order and method, though he finds the centre of a circle by means of any two non-parallel chords; but he finds *the centre of the circle of which a given arc is a part* (his proposition corresponding to III. 25) by bisecting at right angles first the base and then the chord joining one extremity of the base to the point in which the line bisecting the base at right angles meets the circumference of the segment. Under De Morgan's alternative the relation between Euclid III. 1 and the Porism to it would be reversed, and Euclid's notion of a Porism or *corollary* would have to be considerably extended.

If the problem is solved after the manner of IV. 5, it is still desirable to state, as Euclid does, after proving AE , EB , EC to be all equal, that "the circle drawn with centre E and distance one of the straight lines AE , EB , EC will also pass through the remaining points of the segment" [III. 9], in order to show that part of the circle described actually coincides with the given segment. This is not so clear if the centre is determined as the intersection of the straight lines bisecting at right angles chords which join pairs of *four* different points.

PROPOSITION 26.

In equal circles equal angles stand on equal circumferences, whether they stand at the centres or at the circumferences.

Let ABC , DEF be equal circles, and in them let there be equal angles, namely at the centres the angles BGC , EHF , and at the circumferences the angles BAC , EDF ; I say that the circumference BKC is equal to the circumference ELF .



For let BC , EF be joined.

Now, since the circles ABC , DEF are equal,
the radii are equal.

Thus the two straight lines BG , GC are equal to the two straight lines EH , HF ;

and the angle at G is equal to the angle at H ;

therefore the base BC is equal to the base EF . [I. 4]

And, since the angle at A is equal to the angle at D ,

the segment BAC is similar to the segment EDF ;
[III. Def. 11]

and they are upon equal straight lines.

But similar segments of circles on equal straight lines are equal to one another; [III. 24]

therefore the segment BAC is equal to EDF .

But the whole circle ABC is also equal to the whole circle DEF ;

therefore the circumference BKC which remains is equal to the circumference ELF .

Therefore etc.

Q. E. D.

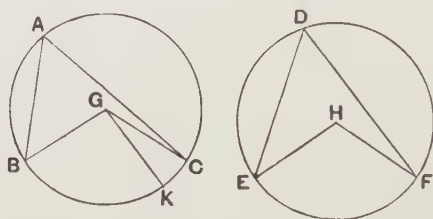
As in III. 21, if Euclid's proof is to cover all cases, it requires us to take cognisance of "angles at the centre" which are equal to or greater than two right angles. Otherwise we must deal separately with the cases where the angle at the circumference is equal to or greater than a right angle. The case of an *obtuse* angle at the circumference can of course be reduced by means of III. 22 to the case of an acute angle at the circumference; and, in case the angle at the circumference is right, it is readily proved, by drawing the radii to the vertex of the angle and to the other extremities of the lines containing it, that the latter two radii are in a straight line, whence they make equal bases in the two circles as in Euclid's proof.

Lardner has another way of dealing with the right angle or obtuse angle at the circumference. In either case, he says, "bisect them, and the halves of them are equal, and it can be proved, as above, that the arcs upon which these halves stand are equal, whence it follows that the arcs on which the given angles stand are equal."

PROPOSITION 27.

In equal circles angles standing on equal circumferences are equal to one another, whether they stand at the centres or at the circumferences.

For in equal circles ABC , DEF , on equal circumferences BC , EF , let the angles BGC , EHF stand at the centres G , H , and the angles BAC , EDF at the circumferences ;
I say that the angle BGC is equal to the angle EHF ,
and the angle BAC is equal to the angle EDF .



For, if the angle BGC is unequal to the angle EHF ,
one of them is greater.

Let the angle BGC be greater : and on the straight line BG ,
and at the point G on it, let the angle BGK be constructed
equal to the angle EHF . [I. 23]

Now equal angles stand on equal circumferences, when
they are at the centres ; [III. 26]

therefore the circumference BK is equal to the circumference EF .

But EF is equal to BC ;

therefore BK is also equal to BC , the less to the greater : which is impossible.

Therefore the angle BGC is not unequal to the angle EHF ;

therefore it is equal to it.

And the angle at A is half of the angle BGC ,
 and the angle at D half of the angle EHF ; [III. 20]
 therefore the angle at A is also equal to the angle at D .
 Therefore etc.

Q. E. D.

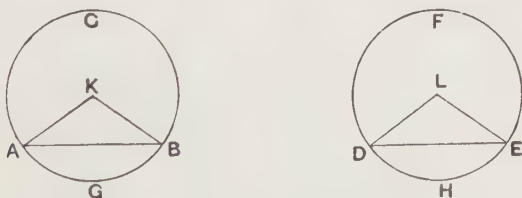
This proposition is the converse of the preceding one, and the remarks about the method of treating the different cases apply here also.

PROPOSITION 28.

In equal circles equal straight lines cut off equal circumferences, the greater equal to the greater and the less to the less.

Let ABC , DEF be equal circles, and in the circles let AB , DE be equal straight lines cutting off ACB , DFE as greater circumferences and AGB , DHE as lesser;

I say that the greater circumference ACB is equal to the greater circumference DFE , and the less circumference AGB to DHE .



For let the centres K , L of the circles be taken, and let AK , KB , DL , LE be joined.

Now, since the circles are equal,

the radii are also equal;

therefore the two sides AK , KB are equal to the two sides DL , LE ;

and the base AB is equal to the base DE ;

therefore the angle AKB is equal to the angle DLE .

[I. 8]

But equal angles stand on equal circumferences, when they are at the centres;

[III. 26]

therefore the circumference AGB is equal to DHE .

And the whole circle ABC is also equal to the whole circle DEF ;

therefore the circumference ACB which remains is also equal to the circumference DFE which remains.

Therefore etc.

Q. E. D.

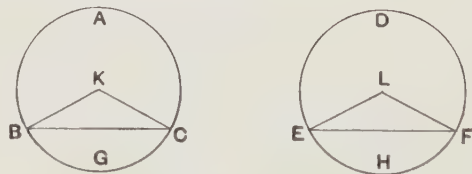
Euclid's proof does not in terms cover the particular case in which the chord in one circle passes through its centre; but indeed this was scarcely worth giving, as the proof can easily be supplied. Since the chord in one circle passes through its centre, the chord in the second circle must also be a diameter of that circle, for equal circles are those which have equal diameters, and all other chords in any circle are less than its diameter [III. 15]; hence the segments cut off in each circle are semicircles, and these must be equal because the circles are equal.

PROPOSITION 29.

In equal circles equal circumferences are subtended by equal straight lines.

Let ABC , DEF be equal circles, and in them let equal circumferences BGC , EHF be cut off; and let the straight lines BC , EF be joined;

I say that BC is equal to EF .



For let the centres of the circles be taken, and let them be K , L ; let BK , KC , EL , LF be joined.

Now, since the circumference BGC is equal to the circumference EHF ,

the angle BKC is also equal to the angle ELF . [III. 27]

And, since the circles ABC , DEF are equal,

the radii are also equal;

therefore the two sides BK , KC are equal to the two sides EL , LF ; and they contain equal angles;

therefore the base BC is equal to the base EF . [I. 4]

Therefore etc.

Q. E. D.

The particular case of this converse of III. 28 in which the given arcs are arcs of semicircles is even easier than the corresponding case of III. 28 itself.

The propositions III. 26—29 are of course equally true if the *same* circle is taken instead of *two equal* circles.

PROPOSITION 30.

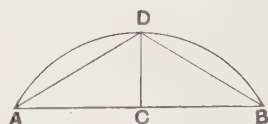
To bisect a given circumference.

Let ADB be the given circumference ;

thus it is required to bisect the circumference ADB .

Let AB be joined and bisected at C ; from the point C let CD be drawn at right angles to the straight line AB , and let AD , DB be joined.

Then, since AC is equal to CB , and CD is common,



the two sides AC , CD are equal to the two sides BC , CD ; and the angle ACD is equal to the angle BCD , for each is right ;

therefore the base AD is equal to the base DB . [I. 4]

But equal straight lines cut off equal circumferences, the greater equal to the greater, and the less to the less ; [III. 28]

and each of the circumferences AD , DB is less than a semicircle ;

therefore the circumference AD is equal to the circumference DB .

Therefore the given circumference has been bisected at the point D .

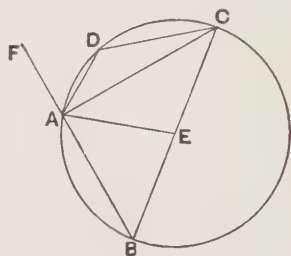
Q. E. F.

PROPOSITION 31.

In a circle the angle in the semicircle is right, that in a greater segment less than a right angle, and that in a less segment greater than a right angle ; and further the angle of the greater segment is greater than a right angle, and the angle of the less segment less than a right angle.

Let $ABCD$ be a circle, let BC be its diameter, and E its centre, and let BA , AC , AD , DC be joined;

I say that the angle BAC in the semicircle BAC is right,
the angle ABC in the segment ABC greater than the semicircle is less than a right angle,
and the angle ADC in the segment ADC less than the semicircle is greater than a right angle.



Let AE be joined, and let BA be carried through to F .

Then, since BE is equal to EA ,
the angle ABE is also equal to the angle BAE . [I. 5]

Again, since CE is equal to EA ,
the angle ACE is also equal to the angle CAE . [I. 5]

Therefore the whole angle BAC is equal to the two angles ABC , ACB .

But the angle FAC exterior to the triangle ABC is also equal to the two angles ABC , ACB ; [I. 32]

therefore the angle BAC is also equal to the angle FAC ;
therefore each is right; [I. Def. 10]

therefore the angle BAC in the semicircle BAC is right.

Next, since in the triangle ABC the two angles ABC , BAC are less than two right angles, [I. 17]
and the angle BAC is a right angle,

the angle ABC is less than a right angle;

and it is the angle in the segment ABC greater than the semicircle.

Next, since $ABCD$ is a quadrilateral in a circle,
and the opposite angles of quadrilaterals in circles are equal to two right angles, [III. 22]

while the angle ABC is less than a right angle,

therefore the angle ADC which remains is greater than a right angle;

and it is the angle in the segment ADC less than the semicircle.

I say further that the angle of the greater segment, namely that contained by the circumference ABC and the straight line AC , is greater than a right angle ;
and the angle of the less segment, namely that contained by the circumference ADC and the straight line AC , is less than a right angle.

This is at once manifest.

For, since the angle contained by the straight lines BA , AC is right,

the angle contained by the circumference ABC and the straight line AC is greater than a right angle.

Again, since the angle contained by the straight lines AC , AF is right,

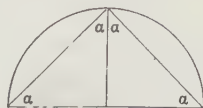
the angle contained by the straight line CA and the circumference ADC is less than a right angle.

Therefore etc.

Q. E. D.

As already stated, this proposition is immediately deducible from III. 20 if that theorem is extended so as to include the case where the segment is equal to or less than a semicircle, and where consequently the "angle at the centre" is equal to two right angles or greater than two right angles respectively.

There are indications in Aristotle that the proof of the first part of the theorem in use before Euclid's time proceeded on different lines. Two passages of Aristotle refer to the proposition that the angle in a semicircle is a right angle. The first passage is *Anal. Post.* II. 11, 94 a 28: "Why is the angle in a semicircle a right angle? Or what makes it a right angle? (τίος οὗτος ὀρθή;) Suppose A to be a right angle, B half of two right angles, C the angle in a semicircle. Then B is the cause of A , the right angle, being an attribute of C , the angle in the semicircle. For B is equal to A , and C to B ; for C is half of two right angles. Therefore it is in virtue of B being half of two right angles that A is an attribute of C ; and the latter means the fact that the angle in a semicircle is right." Now this passage by itself would be consistent with a proof like Euclid's or the alternative interpolated proof next to be mentioned. But the second passage throws a different light on the subject. This is *Metaph.* 1051 a 26: "Why is the angle in a semicircle a right angle invariably (καθόλου)? Because, if there be three straight lines, two forming the base, and the third set up at right angles at its middle point, the fact is obvious by simple inspection to any one who knows the property referred to" (ἐκείνο is the property that the angles of a triangle are together equal to two right angles, mentioned two lines before). That is to say, the angle at the middle point of the circumference of the semicircle was taken and proved, by means of the two isosceles right-angled triangles, to be the sum of two angles each equal to one-fourth of the sum of the angles of the large triangle in the figure, or of two right angles; and the proof must have been completed by means of the theorem of III. 21 (that angles



in the same segment are equal), which Euclid's more general proof does not need.

In the Greek texts before that of August there is an alternative proof that the angle BAC (in a semicircle) is right. August and Heiberg relegate it to an Appendix.

"Since the angle AEC is double of the angle BAE (for it is equal to the two interior and opposite angles), while the angle AEB is also double of the angle EAC ,

the angles AEB , AEC are double of the angle BAC .

But the angles AEB , AEC are equal to two right angles;

therefore the angle BAC is right."

Lardner gives a slightly different proof of the second part of the theorem.

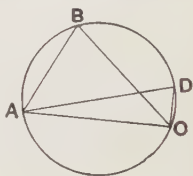
If ABC be a segment greater than a semicircle, draw the diameter AD , and join CD , CA .

Then, in the triangle ACD , the angle ACD is right (being the angle in a semicircle);

therefore the angle ADC is acute.

But the angle ADC is equal to the angle ABC in the same segment;

therefore the angle ABC is acute.



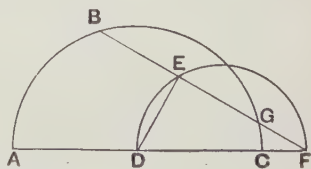
Euclid's references in this proposition to the angle of a segment greater or less than a semicircle respectively seem, like the part of III. 16 relating to the angle of a semicircle, to be a survival of ancient controversies and not to be put in deliberately as being an essential part of elementary geometry. Cf. the notes on III. Def. 7 and III. 16.

The corollary ordinarily attached to this proposition is omitted by Heiberg as an interpolation of date later than Theon. It is to this effect: "From this it is manifest that, if one angle of a triangle be equal to the other two, the first angle is right because the exterior angle to it is also equal to the same angles, and if the adjacent angles be equal, they are right." No doubt the corollary is rightly suspected, because there is no necessity for it here, and the words $\delta\eta\rho\epsilon\iota$ $\epsilon\delta\epsilon\iota$ $\delta\epsilon\iota\chi\alpha\iota$ come before it, not after it, as is usual with Euclid. But, on the other hand, as the fact stated does appear in the proof of III. 31, the Porism would be a Porism after the usual type, and I do not quite follow Heiberg's argument that, "if Euclid had wished to add it, he ought to have placed it after I. 32."

It has already been mentioned above (p. 44) that this proposition supplies us with an alternative construction for the problem in III. 17 of drawing the two tangents to a circle from an external point.

Two theorems of some historical interest which follow directly from III. 31 may be mentioned.

The first is a lemma of Pappus on "the 24th problem" of the second Book of Apollonius' lost treatise on $\nu\epsilon\upsilon\sigma\epsilon\iota\varsigma$ (Pappus VII. p. 812) and is to this effect. If a circle, as DEF , pass through D , the centre of a circle ABC , and if through F , the other point in which the line of centres meets the circle DEF , any straight line be drawn (and produced if necessary) meeting the circle DEF in E and the circle ABC in B , G ,

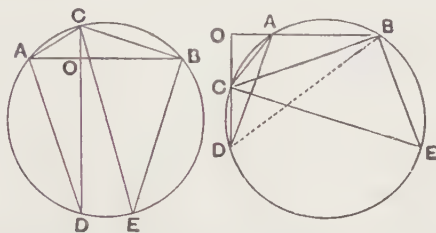


then E is the middle point of BG . For, if DE be joined, the angle DEF (in a semicircle) is a right angle [III. 31]; and DE , being at right angles to the chord BG of the circle ABC , also bisects it [III. 3].

The second is a proposition in the *Liber Assumptorum*, attributed (no doubt erroneously as regards much of it) to Archimedes, which has reached us through the Arabic (Archimedes, ed. Heiberg, II. pp. 520—521).

If two chords AB , CD in a circle intersect at right angles in a point O , then the sum of the squares on AO , BO , CO , DO is equal to the square on the diameter.

For draw the diameter CE , and join AC , CB , AD , BE .



Then the angle CAO is equal to the angle CEB . (This follows, in the first figure, from III. 21 and, in the second, from I. 13 and III. 22.) Also the angle COA , being right, is equal to the angle CBE which, being the angle in a semicircle, is also right [III. 31].

Therefore the triangles AOC , EBC have two angles equal respectively; whence the third angles ACO , ECB are equal. (In the second figure the angle ACO is, by I. 13 and III. 22, equal to the angle ABD , and therefore the angles ABD , ECB are equal.)

Therefore, in both figures, the arcs AD , BE , and consequently the chords AD , BE subtended by them, are equal. [III. 26, 29]

Now the squares on AO , DO are equal to the square on AD [I. 47], that is, to the square on BE .

And the squares on CO , BO are equal to the square on BC .

Therefore, by addition, the squares on AO , BO , CO , DO are equal to the squares on EB , BC , i.e. to the square on CE . [I. 47]

PROPOSITION 32.

If a straight line touch a circle, and from the point of contact there be drawn across, in the circle, a straight line cutting the circle, the angles which it makes with the tangent will be equal to the angles in the alternate segments of the circle.

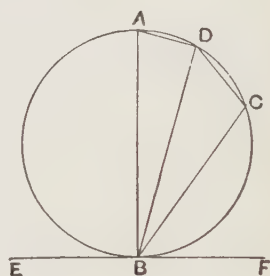
For let a straight line EF touch the circle $ABCD$ at the point B , and from the point B let there be drawn across, in the circle $ABCD$, a straight line BD cutting it;

I say that the angles which BD makes with the tangent EF will be equal to the angles in the alternate segments of the

circle, that is, that the angle FBD is equal to the angle constructed in the segment BAD , and the angle EBD is equal to the angle constructed in the segment DCB .

For let BA be drawn from B at right angles to EF ,
let a point C be taken at random on the circumference BD ,
and let AD , DC , CB be joined.

Then, since a straight line EF touches the circle $ABCD$ at B ,
and BA has been drawn from the point of contact at right angles to the tangent,
the centre of the circle $ABCD$ is on BA .



[III. 19]

Therefore BA is a diameter of the circle $ABCD$;

therefore the angle ADB , being an angle in a semicircle,
is right.

[III. 31]

Therefore the remaining angles BAD , ABD are equal to one right angle.

[I. 32]

But the angle ABF is also right;

therefore the angle ABF is equal to the angles BAD , ABD .

Let the angle ABD be subtracted from each;

therefore the angle DBF which remains is equal to the angle BAD in the alternate segment of the circle.

Next, since $ABCD$ is a quadrilateral in a circle,
its opposite angles are equal to two right angles.

[III. 22]

But the angles DBF , DBE are also equal to two right angles;

therefore the angles DBF , DBE are equal to the angles BAD , BCD ,

of which the angle BAD was proved equal to the angle DBF ;

therefore the angle DBE which remains is equal to the angle DCB in the alternate segment DCB of the circle.

Therefore etc.

Q. E. D.

The converse of this theorem is true, namely that, *If a straight line drawn through one extremity of a chord of a circle make with that chord angles equal respectively to the angles in the alternate segments of the circle, the straight line so drawn touches the circle.*

This can, as Camerer and Todhunter remark, be proved indirectly ; or we may prove it, with Clavius, directly. Let BD be the given chord, and let EF be drawn through B so that it makes with BD angles equal to the angles in the alternate segments of the circle respectively.

Let BA be the diameter through B , and let C be any point on the circumference of the segment DCB which does not contain A . Join AD , DC , CB .

Then, since, by hypothesis, the angle FBD is equal to the angle BAD , let the angle ABD be added to both ;

therefore the angle ABF is equal to the angles ABD , BAD .

But the angle BDA , being the angle in a semicircle, is a right angle ;

therefore the remaining angles ABD , BAD in the triangle ABD are equal to a right angle.

Therefore the angle ABF is right ;

hence, since BA is the diameter through B ,

EF touches the circle at B . [III. 16, Por.]

Pappus assumes in one place (iv. p. 196) the consequence of this proposition that, *If two circles touch, any straight line drawn through the point of contact and terminated by both circles cuts off segments in each which are respectively similar.* Pappus also shows how to prove this (vii. p. 826) by drawing the common tangent at the point of contact and using this proposition, III. 32.

PROPOSITION 33.

On a given straight line to describe a segment of a circle admitting an angle equal to a given rectilineal angle.

Let AB be the given straight line, and the angle at C the given rectilineal angle ;

thus it is required to describe on the given straight line AB a segment of a circle admitting an angle equal to the angle at C .

The angle at C is then acute, or right, or obtuse.

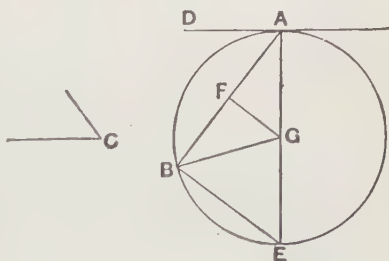
First let it be acute,

and, as in the first figure, on

the straight line AB , and at the point A , let the angle BAD be constructed equal to the angle at C ;

therefore the angle BAD is also acute.

Let AE be drawn at right angles to DA , let AB be



bisected at F , let FG be drawn from the point F at right angles to AB , and let GB be joined.

Then, since AF is equal to FB ,
and FG is common,

the two sides AF, FG are equal to the two sides BF, FG ;
and the angle AFG is equal to the angle BFG ;

therefore the base AG is equal to the base BG . [I. 4]

Therefore the circle described with centre G and distance GA will pass through B also.

Let it be drawn, and let it be ABE ;
let EB be joined.

Now, since AD is drawn from A , the extremity of the diameter AE , at right angles to AE ,

therefore AD touches the circle ABE . [III. 16, Por.]

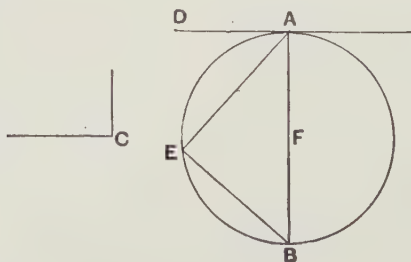
Since then a straight line AD touches the circle ABE ,
and from the point of contact at A a straight line AB is drawn across in the circle ABE ,

the angle DAB is equal to the angle AEB in the alternate segment of the circle. [III. 32]

But the angle DAB is equal to the angle at C ;
therefore the angle at C is also equal to the angle AEB .

Therefore on the given straight line AB the segment AEB of a circle has been described admitting the angle AEB equal to the given angle, the angle at C .

Next let the angle at C be right ;



and let it be again required to describe on AB a segment of a circle admitting an angle equal to the right angle at C .

Let the angle BAD be constructed equal to the right angle at C , as is the case in the second figure ;

let AB be bisected at F , and with centre F and distance either FA or FB let the circle AEB be described.

Therefore the straight line AD touches the circle AEB , because the angle at A is right. [III. 16, Por.]

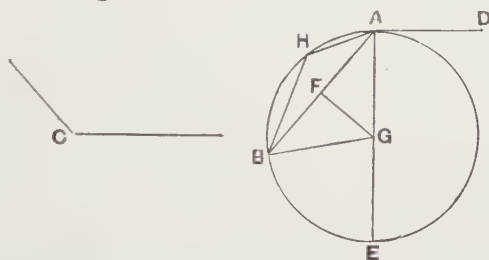
And the angle BAD is equal to the angle in the segment AEB , for the latter too is itself a right angle, being an angle in a semicircle. [III. 31]

But the angle BAD is also equal to the angle at C .

Therefore the angle AEB is also equal to the angle at C .

Therefore again the segment AEB of a circle has been described on AB admitting an angle equal to the angle at C .

Next, let the angle at C be obtuse;



and on the straight line AB , and at the point A , let the angle BAD be constructed equal to it, as is the case in the third figure;

let AE be drawn at right angles to AD , let AB be again bisected at F , let FG be drawn at right angles to AB , and let GB be joined.

Then, since AF is again equal to FB , and FG is common,

the two sides AF, FG are equal to the two sides BF, FG ; and the angle AFG is equal to the angle BFG ;

therefore the base AG is equal to the base BG . [I. 4]

Therefore the circle described with centre G and distance GA will pass through B also; let it so pass, as AEB .

Now, since AD is drawn at right angles to the diameter AE from its extremity,

AD touches the circle AEB . [III. 16, Por.]

And AB has been drawn across from the point of contact at A ;

therefore the angle BAD is equal to the angle constructed in the alternate segment AHB of the circle. [III. 32]

But the angle BAD is equal to the angle at C .

Therefore the angle in the segment AHB is also equal to the angle at C :

Therefore on the given straight line AB the segment AHB of a circle has been described admitting an angle equal to the angle at C .

Q. E. F.

Simson remarks truly that the first and third cases, those namely in which the given angle is acute and obtuse respectively, have exactly the same construction and demonstration, so that there is no advantage in repeating them. Accordingly he deals with the cases as one, merely drawing two different figures. It is also true, as Simson says, that the demonstration of the second case in which the given angle is a right angle "is done in a round-about way," whereas, as Clavius showed, the problem can be more easily solved by merely bisecting AB and describing a semicircle on it. A glance at Euclid's figure and proof will however show a more curious fact, namely that he does not, in the proof of the second case, use the *angle in the alternate segment*, as he does in the other two cases. He might have done so after proving that AD touches the circle; this would only have required his point E to be placed on the side of AB opposite to D . Instead of this, he uses III. 31, and proves that the angle AEB is equal to the angle C , because the former is an *angle in a semicircle*, and is therefore a right angle as C is.

The difference of procedure is no doubt owing to the fact that he has not, in III. 32, distinguished the case in which the cutting and touching straight lines are at right angles, i.e. in which the two alternate segments are semicircles. To prove this case would also have required III. 31, so that nothing would have been gained by stating it separately in III. 32 and then quoting the result as part of III. 32, instead of referring directly to III. 31.

It is assumed in Euclid's proof of the first and third cases that AE and FG will meet; but of course there is no difficulty in satisfying ourselves of this.

PROPOSITION 34.

From a given circle to cut off a segment admitting an angle equal to a given rectilineal angle.

Let ABC be the given circle, and the angle at D the given rectilineal angle; thus it is required to cut off from the circle ABC a segment admitting an angle equal to the given rectilineal angle, the angle at D .

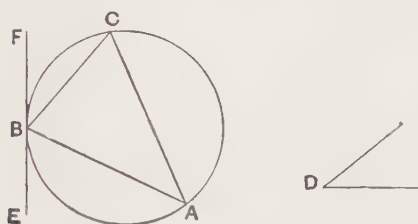
Let EF be drawn touching ABC at the point B , and on the straight line FB , and at the point B on it, let the angle FBC be constructed equal to the angle at D . [I. 23]

Then, since a straight line EF touches the circle ABC ,

and BC has been drawn across from the point of contact at B ,

the angle FBC is equal to the angle constructed in the alternate segment BAC .

[III. 32]



But the angle FBC is equal to the angle at D ;

therefore the angle in the segment BAC is equal to the angle at D .

Therefore from the given circle ABC the segment BAC has been cut off admitting an angle equal to the given rectilinear angle, the angle at D .

Q. E. F.

An alternative construction here would be to make an "angle at the centre" (in the extended sense, if necessary) double of the given angle; and, if the given angle is right, it is only necessary to draw a diameter of the circle.

PROPOSITION 35.

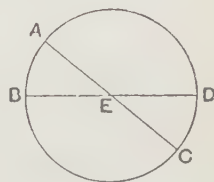
If in a circle two straight lines cut one another, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

For in the circle $ABCD$ let the two straight lines AC , BD cut one another at the point E ;

I say that the rectangle contained by AE , EC is equal to the rectangle contained by DE , EB .

If now AC , BD are through the centre, so that E is the centre of the circle $ABCD$, it is manifest that, AE , EC , DE , EB being equal,

the rectangle contained by AE , EC is also equal to the rectangle contained by DE , EB .



Next let AC , DB not be through the centre ;
let the centre of $ABCD$ be taken, and
let it be F ;

from F let FG , FH be drawn perpendicular to the straight lines AC , DB ,
and let FB , FC , FE be joined.

Then, since a straight line GF
through the centre cuts a straight line
 AC not through the centre at right
angles,

it also bisects it ; [III. 3]

therefore AG is equal to GC .

Since, then, the straight line AC has been cut into equal
parts at G and into unequal parts at E ,
the rectangle contained by AE , EC together with the square
on EG is equal to the square on GC ; [II. 5]

Let the square on GF be added ;

therefore the rectangle AE , EC together with the squares
on GE , GF is equal to the squares on CG , GF .

But the square on FE is equal to the squares on EG , GF ,
and the square on FC is equal to the squares on CG , GF ; [I. 47]

therefore the rectangle AE , EC together with the square
on FE is equal to the square on FC .

And FC is equal to FB ;

therefore the rectangle AE , EC together with the square on
 FE is equal to the square on FB .

For the same reason, also,

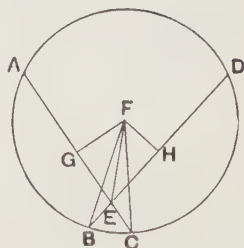
the rectangle DE , EB together with the square on FE is
equal to the square on FB .

But the rectangle AE , EC together with the square on
 FE was also proved equal to the square on FB ;
therefore the rectangle AE , EC together with the square on
 FE is equal to the rectangle DE , EB together with the
square on FE .

Let the square on FE be subtracted from each ;

therefore the rectangle contained by AE , EC which remains
is equal to the rectangle contained by DE , EB .

Therefore etc.



In addition to the two cases in Euclid's text, Simson (following Campanus) gives two intermediate cases, namely (1) that in which one chord passes through the centre and bisects the other which does not pass through the centre at right angles, and (2) that in which one passes through the centre and cuts the other which does not pass through the centre but not at right angles. Simson then reduces Euclid's second case, the most general one, to the second of the two intermediate cases by drawing the diameter through *E*. His note is as follows: "As the 25th and 33rd propositions are divided into more cases, so this 35th is divided into fewer cases than are necessary. Nor can it be supposed that Euclid omitted them because they are easy; as he has given the case which by far is the easiest of them all, viz. that in which both the straight lines pass through the centre: And in the following proposition he separately demonstrates the case in which the straight line passes through the centre, and that in which it does not pass through the centre: So that it seems Theon, or some other, has thought them too long to insert: But cases that require different demonstrations should not be left out in the Elements, as was before taken notice of: These cases are in the translation from the Arabic and are now put into the text." Notwithstanding the ingenuity of the argument based on the separate mention by Euclid of the simplest case of all, I think the conclusion that Euclid himself gave *four* cases is unsafe; in fact, in giving the simplest and most difficult cases only, he seems to be following quite consistently his habit of avoiding *too great* multiplicity of cases, while not ignoring their existence.

The deduction from the next proposition (III. 36) which Simson, following Clavius and others, gives as a corollary to it, namely that, *If from any point without a circle there be drawn two straight lines cutting it, the rectangles contained by the whole lines and the parts of them without the circle are equal to one another*, can of course be combined with III. 35 in one enunciation.

As remarked by Todhunter, a large portion of the proofs of III. 35, 36 amounts to proving the proposition, *If any point be taken on the base, or the base produced, of an isosceles triangle, the rectangle contained by the segments of the base (i.e. the respective distances of the ends of the base from the point) is equal to the difference between the square on the straight line joining the point to the vertex and the square on one of the equal sides of the triangle*. This is of course an immediate consequence of I. 47 combined with II. 5 or II. 6.

The converse of III. 35 and Simson's corollary to III. 36 may be stated thus. *If two straight lines AB, CD, produced if necessary, intersect at O, and if the rectangle AO, OB be equal to the rectangle CO, OD, the circumference of a circle will pass through the four points A, B, C, D*. The proof is indirect. We describe a circle through three of the points, as *A, B, C* (by the method used in Euclid's proofs of III. 9, 10), and then we prove, by the aid of III. 35 and the corollary to III. 36, that the circle cannot but pass through *D* also.

PROPOSITION 36.

If a point be taken outside a circle and from it there fall on the circle two straight lines, and if one of them cut the circle and the other touch it, the rectangle contained by the whole of the straight line which cuts the circle and the straight

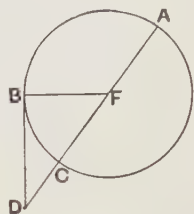
line intercepted on it outside between the point and the convex circumference will be equal to the square on the tangent.

For let a point D be taken outside the circle ABC , and from D let the two straight lines DCA , DB fall on the circle ABC ; let DCA cut the circle ABC and let DB touch it;

I say that the rectangle contained by AD , DC is equal to the square on DB .

Then DCA is either through the centre or not through the centre.

First let it be through the centre, and let F be the centre of the circle ABC ; let FB be joined;



therefore the angle FBD is right. [III. 18]

And, since AC has been bisected at F , and CD is added to it,

the rectangle AD , DC together with the square on FC is equal to the square on FD . [II. 6]

But FC is equal to FB ;

therefore the rectangle AD , DC together with the square on FB is equal to the square on FD .

And the squares on FB , BD are equal to the square on FD ; [I. 47]

therefore the rectangle AD , DC together with the square on FB is equal to the squares on FB , BD .

Let the square on FB be subtracted from each;

therefore the rectangle AD , DC which remains is equal to the square on the tangent DB .

Again, let DCA not be through the centre of the circle ABC ;

let the centre E be taken, and from E let EF be drawn perpendicular to AC ; let EB , EC , ED be joined.

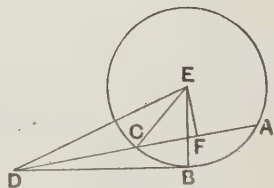
Then the angle EBD is right.

[III. 18]

And, since a straight line EF through the centre cuts a straight line AC not through the centre at right angles,

it also bisects it;

therefore AF is equal to FC .



[III. 3]

Now, since the straight line AC has been bisected at the point F , and CD is added to it, the rectangle contained by AD , DC together with the square on FC is equal to the square on FD . [II. 6]

Let the square on FE be added to each ; therefore the rectangle AD , DC together with the squares on CF , FE is equal to the squares on FD , FE .

But the square on EC is equal to the squares on CF , FE , for the angle EFC is right ; [I. 47]

and the square on ED is equal to the squares on DF , FE ; therefore the rectangle AD , DC together with the square on EC is equal to the square on ED .

And EC is equal to EB ; therefore the rectangle AD , DC together with the square on EB is equal to the square on ED .

But the squares on EB , BD are equal to the square on ED , for the angle EBD is right ; [I. 47]

therefore the rectangle AD , DC together with the square on EB is equal to the squares on EB , BD .

Let the square on EB be subtracted from each ; therefore the rectangle AD , DC which remains is equal to the square on DB .

Therefore etc.

Q. E. D.

Cf. note on the preceding proposition. Observe that, whereas it would be natural with us to prove first that, if A is an external point, and two straight lines AEB , AFC cut the circle in E , B and F , C respectively, the rectangle BA , AE is equal to the rectangle CA , AF , and thence that, the tangent from A being a straight line like AEB in its limiting position when E and B coincide, either rectangle is equal to the square on the tangent (cf. Mr H. M. Taylor, p. 253), Euclid and the Greek geometers generally did not allow themselves to infer the truth of a proposition in a limiting case directly from the general case including it, but preferred a separate proof of the limiting case (cf. *Apollonius of Perga*, p. 40, 139—140). This accounts for the form of III. 36.

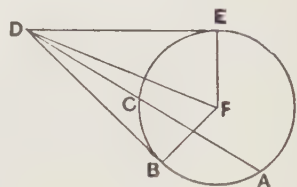
PROPOSITION 37.

If a point be taken outside a circle and from the point there fall on the circle two straight lines, if one of them cut the circle, and the other fall on it, and if further the rectangle contained by the whole of the straight line which cuts

the circle and the straight line intercepted on it outside between the point and the convex circumference be equal to the square on the straight line which falls on the circle, the straight line which falls on it will touch the circle.

For let a point D be taken outside the circle ABC ; from D let the two straight lines DCA , DB fall on the circle ACB ; let DCA cut the circle and DB fall on it; and let the rectangle AD , DC be equal to the square on DB .

I say that DB touches the circle ABC .



For let DE be drawn touching ABC ; let the centre of the circle ABC be taken, and let it be F ; let FE , FB , FD be joined.

Thus the angle FED is right. [III. 18]

Now, since DE touches the circle ABC , and DCA cuts it, the rectangle AD , DC is equal to the square on DE . [III. 36]

But the rectangle AD , DC was also equal to the square on DB ;

therefore the square on DE is equal to the square on DB ;
therefore DE is equal to DB .

And FE is equal to FB ;

therefore the two sides DE , EF are equal to the two sides DB , BF ;

and FD is the common base of the triangles;

therefore the angle DEF is equal to the angle DBF .

[I. 8]

But the angle DEF is right;

therefore the angle DBF is also right.

And FB produced is a diameter;

and the straight line drawn at right angles to the diameter of a circle, from its extremity, touches the circle; [III. 16, Por.]

therefore DB touches the circle.

Similarly this can be proved to be the case even if the centre be on AC .

Therefore etc.

Q. E. D.

De Morgan observes that there is here the same defect as in I. 48, i.e. an apparent avoidance of indirect demonstration by drawing the tangent DE on

the opposite side of DF from DB . The case is similar to the *apparently* direct proof which Campanus gave. He drew the straight line from D passing through the centre, and then (without drawing a second tangent) proved by the aid of II. 6 that the square on DF is equal to the sum of the squares on DB , BF ; whence (by I. 48) the angle DBF is a right angle. But this proof uses I. 48, the very proposition to which De Morgan's original remark relates.

The undisguised indirect proof is easy. If DB does not touch the circle, it must cut it if produced, and it follows that the square on DB must be equal to the rectangle contained by DB and a longer line: which is absurd.

BOOK IV.

DEFINITIONS.

1. A rectilineal figure is said to be **inscribed in a rectilineal figure** when the respective angles of the inscribed figure lie on the respective sides of that in which it is inscribed.

2. Similarly a figure is said to be **circumscribed about a figure** when the respective sides of the circumscribed figure pass through the respective angles of that about which it is circumscribed.

3. A rectilineal figure is said to be **inscribed in a circle** when each angle of the inscribed figure lies on the circumference of the circle.

4. A rectilineal figure is said to be **circumscribed about a circle**, when each side of the circumscribed figure touches the circumference of the circle.

5. Similarly a circle is said to be **inscribed in a figure** when the circumference of the circle touches each side of the figure in which it is inscribed.

6. A circle is said to be **circumscribed about a figure** when the circumference of the circle passes through each angle of the figure about which it is circumscribed.

7. A straight line is said to be **fitted into a circle** when its extremities are on the circumference of the circle.

DEFINITIONS 1—7.

I append, as usual, the Greek text of the definitions.

1. Σχήμα εὐθύγραμμον εἰς σχῆμα εὐθύγραμμον ἐγγράφεσθαι λέγεται, ὅταν ἐκάστη τῶν τοῦ ἐγγραφομένου σχήματος γωνιῶν ἐκάστης πλευρᾶς τοῦ, εἰς ὃ ἐγγράφεται, ᾗπτηται.

2. Σχήμα δὲ ὁμοίως περὶ σχῆμα περιγράφεσθαι λέγεται, ὅταν ἐκάστη πλευρὰ τοῦ περιγραφομένου ἐκάστης γωνίας τοῦ, περὶ ὃ περιγράφεται, ᾗπτηται.

3. Σχήμα εὐθύγραμμον εἰς κύκλον ἐγγράφεσθαι λέγεται, ὅταν ἐκάστη γωνία τοῦ ἐγγραφομένου ᾗπτηται τῆς τοῦ κύκλου περιφέρειας.

4. Σχήμα δὲ εὐθύγραμμον περὶ κύκλον περιγράφεσθαι λέγεται, ὅταν ἐκάστη πλευρὰ τοῦ περιγραφομένου ἐφάπτηται τῆς τοῦ κύκλου περιφέρειας.

5. Κύκλος δὲ εἰς σχῆμα ὁμοίως ἐγγράφεσθαι λέγεται, ὅταν ἡ τοῦ κύκλου περιφέρεια ἐκάστης πλευρᾶς τοῦ, εἰς ὃ ἐγγράφεται, ᾗπτηται.

6. Κύκλος δὲ περὶ σχῆμα περιγράφεσθαι λέγεται, ὅταν ἡ τοῦ κύκλου περιφέρεια ἐκάστης γωνίας τοῦ, περὶ ὃ περιγράφεται, ᾗπτηται.

7. Εὐθεΐα εἰς κύκλον ἐναρμόζεσθαι λέγεται, ὅταν τὰ πέρατα αὐτῆς ἐπὶ τῆς περιφέρειας ᾗ τοῦ κύκλου.

In the first two definitions an English translation, if it is to be clear, must depart slightly from the exact words used in the Greek, where "each side" of one figure is said to pass through "each angle" of another, or "each angle" (i.e. angular point) of one lies on "each side" of another (ἐκάστη πλευρὰ, ἐκάστη γωνία).

It is also necessary, in the five definitions 1, 2, 3, 5 and 6, to translate the same Greek word ᾗπτηται in three different ways. It was observed on III. Def. 2 that the usual meaning of ᾗπτεσθαι in Euclid is to *meet*, in contradistinction to ἐφάπτεσθαι, which means to *touch*. Exceptionally, as in Def. 5, ᾗπτεσθαι has the meaning of *touch*. But two new meanings of the word appear, the first being to *lie on*, as in Deff. 1 and 3, the second to *pass through*, as in Deff. 2 and 6; "each angle" lies on (ᾗπτεται) a side or on a circle, and "each side," or a circle, passes through (ᾗπτεται) an angle or "each angle." The first meaning of *lying on* is exemplified in the phrase of Pappus ἄψεται τὸ σημεῖον θέσει δεδομένης εὐθείας, "will lie on a straight line given in position"; the meaning of *passing through* seems to be much rarer (I have not seen it in Archimedes or Pappus), but, as pointed out on III. Def. 2, Aristotle uses the compound ἐφάπτεσθαι in this sense.

Simson proposed to read ἐφάπτηται in the case (Def. 5) where ᾗπτηται means *touches*. He made the like suggestion as regards the Greek text of III. 11, 12, 13, 18, 19; in the first four of these cases there seems to be MS. authority for the compound verb, and in the fifth Heiberg adopts Simson's correction.

BOOK IV. PROPOSITIONS

PROPOSITION I.

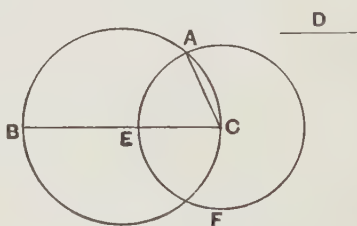
Into a given circle to fit a straight line equal to a given straight line which is not greater than the diameter of the circle.

Let ABC be the given circle, and D the given straight line not greater than the diameter of the circle ;

thus it is required to fit into the circle ABC a straight line equal to the straight line D .

Let a diameter BC of the circle ABC be drawn.

Then, if BC is equal to D , that which was enjoined will have been done ; for BC has been fitted into the circle ABC equal to the straight line D .



But, if BC is greater than D ,

let CE be made equal to D , and with centre C and distance CE let the circle EAF be described ;

let CA be joined.

Then, since the point C is the centre of the circle EAF ,

CA is equal to CE .

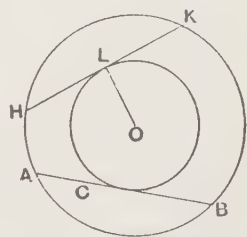
But CE is equal to D ;

therefore D is also equal to CA .

Therefore into the given circle ABC there has been fitted CA equal to the given straight line D .

Of this problem as it stands there are of course an infinite number of solutions; and, if a particular point be chosen as one extremity of the chord to be "fitted in," there are two solutions. More difficult cases of "fitting into" a circle a chord of given length are arrived at by adding some further condition, e.g. (1) that the chord is to be parallel to a given straight line, or (2) that the chord, produced if necessary, shall pass through a given point. The former problem is solved by Pappus (III. p. 132); instead of drawing the chord as a tangent to a circle concentric with the given circle and having as radius a straight line the square on which is equal to the difference between the squares on the radius of the given circle and on half the given length, he merely draws the diameter of the circle which is parallel to the given direction, measures from the centre along it in each direction a length equal to half the given length, and then draws, on one side of the diameter, perpendiculars to it through the two points so determined.

The second problem of drawing a chord of given length, being less than the diameter of the circle, and passing through a given point, is more important as having been one of the problems discussed by Apollonius in his work entitled *νέυσεις*, now lost. Pappus states the problem thus (VII. p. 670): "A circle being given in position, to fit into it a straight line given in magnitude and verging (*νέύουσαν*) towards a given (point)." To do this we have only to place any chord HK in the given circle (with centre O) equal to the given length, take L the middle point of it, with O as centre and OL as radius describe a circle, and lastly through the given point C draw a tangent to this circle meeting the given circle in A, B . AB is then one of *two* chords which can be drawn satisfying the given conditions, if C is outside the inner circle; if C is *on* the inner circle, there is one solution only; and, if C is within the inner circle, there is no solution. Thus, if C is within the outer (given) circle, besides the condition that the given length must not be greater than the diameter of the circle, there is another necessary condition of the possibility of a solution, viz. that the given length must not be *less* than double of the straight line the square on which is equal to the difference between the squares (1) on the radius of the given circle and (2) on the distance between its centre and the given point.



PROPOSITION 2.

In a given circle to inscribe a triangle equiangular with a given triangle.

Let ABC be the given circle, and DEF the given triangle;

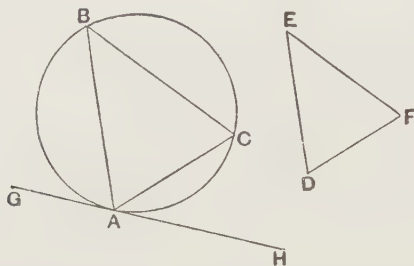
thus it is required to inscribe in the circle ABC a triangle equiangular with the triangle DEF .

Let GH be drawn touching the circle ABC at A [III. 16, Por.];

on the straight line AH , and at the point A on it, let the angle HAC be constructed equal to the angle DEF ,
and on the straight line AG , and at the point A on it, let the angle GAB be constructed equal to the angle DFE ;

[I. 23]

let BC be joined.



Then, since a straight line AH touches the circle ABC , and from the point of contact at A the straight line AC is drawn across in the circle, therefore the angle HAC is equal to the angle ABC in the alternate segment of the circle. [III. 32]

But the angle HAC is equal to the angle DEF ;
therefore the angle ABC is also equal to the angle DEF .

For the same reason

the angle ACB is also equal to the angle DFE ;
therefore the remaining angle BAC is also equal to the remaining angle EDF . [I. 32]

Therefore in the given circle there has been inscribed a triangle equiangular with the given triangle. Q. E. F.

Here again, since any point on the circle may be taken as an angular point of the triangle, there are an infinite number of solutions. Even when a particular point has been chosen to form one angular point, the required triangle may be constructed in six ways. For any one of the three angles may be placed at the point; and, whichever is placed there, the positions of the two others relatively to it may be interchanged. The sides of the triangle will, in all the different solutions, be of the same length respectively; only their relative positions will be different.

This problem can of course be reduced (as it was by Borelli) to III. 34, namely the problem of cutting off from a given circle a segment containing an angle equal to a given angle. It can also be solved by the alternative method applicable to III. 34 of drawing "angles at the centre" equal to double the angles of the given triangle respectively; and by this method we can easily solve this problem, or III. 34, with the further condition that one side of the

required triangle, or the base of the required segment, respectively, shall be parallel to a given straight line.

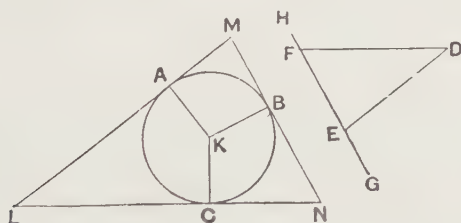
As a particular case, we can, by the method of this proposition, describe an *equilateral* triangle in any circle after we have first constructed any equilateral triangle by the aid of I. 1. The possibility of this is assumed in IV. 16. It is of course equivalent to dividing the circumference of a circle into *three equal parts*. As De Morgan says, the idea of dividing a revolution into equal parts should be kept prominent in considering Book IV.; this aspect of the construction of regular polygons is obvious enough, and the reason why the division of the circle into *three equal parts* is not given by Euclid is that it happens to be as easy to divide the circle into three parts which are in the ratio of the angles of any triangle as to divide it into three equal parts.

PROPOSITION 3.

About a given circle to circumscribe a triangle equiangular with a given triangle.

Let ABC be the given circle, and DEF the given triangle;

5 thus it is required to circumscribe about the circle ABC a triangle equiangular with the triangle DEF .



Let EF be produced in both directions to the points G, H ,

let the centre K of the circle ABC be taken [III. 1], and let
10 the straight line KB be drawn across at random;

on the straight line KB , and at the point K on it, let the angle BKA be constructed equal to the angle DEG ,

and the angle BKC equal to the angle DFH ; [I. 23]

and through the points A, B, C let LAM, MBN, NCL be
15 drawn touching the circle ABC . [III. 16, Por.]

Now, since LM, MN, NL touch the circle ABC at the points A, B, C ,

and KA, KB, KC have been joined from the centre K to the points A, B, C ,

20 therefore the angles at the points A, B, C are right. [III. 18]

And, since the four angles of the quadrilateral $AMBK$ are equal to four right angles, inasmuch as $AMBK$ is in fact divisible into two triangles,

and the angles KAM, KBM are right,

25 therefore the remaining angles AKB, AMB are equal to two right angles.

But the angles DEG, DEF are also equal to two right angles; [I. 13]

therefore the angles AKB, AMB are equal to the angles
30 DEG, DEF ,

of which the angle AKB is equal to the angle DEG ;

therefore the angle AMB which remains is equal to the angle DEF which remains.

Similarly it can be proved that the angle LNB is also
35 equal to the angle DFE ;

therefore the remaining angle MLN is equal to the angle EDF . [I. 32]

Therefore the triangle LMN is equiangular with the triangle DEF ; and it has been circumscribed about the
40 circle ABC .

Therefore about a given circle there has been circumscribed a triangle equiangular with the given triangle.

Q. E. F.

10. at random, literally "as it may chance," $\omega\varsigma \epsilon\tau\upsilon\chi\epsilon\nu$. The same expression is used in III. 1 and commonly.

22. is in fact divisible, $\kappa\alpha\iota \delta\iota\alpha\iota\rho\epsilon\acute{\iota}\tau\alpha\iota$, literally "is actually divided."

The remarks as to the number of ways in which Prop. 2 can be solved apply here also.

Euclid leaves us to satisfy ourselves that the three tangents *will* meet and form a triangle. This follows easily from the fact that each of the angles AKB, BKC, CKA is less than two right angles. The first two are so by construction, being the supplements of two angles of the given triangle respectively, and, since all three angles round K are together equal to four right angles, it follows that the third, the angle AKC , is equal to the sum of the two angles E, F of the triangle, i.e. to the supplement of the angle D , and is therefore less than two right angles.

Peletarius and Borelli gave an alternative solution, first inscribing a triangle equiangular to the given triangle, by IV. 2, and then drawing tangents to the circle parallel to the sides of the inscribed triangle respectively. This method will of course give two solutions, since two tangents can be drawn parallel to each of the sides of the inscribed triangle.

If the three pairs of parallel tangents be drawn and produced far enough,

they will form *eight* triangles, two of which are the triangles *circumscribed* to the circle in the manner required in the proposition. The other six triangles are so related to the circle that the circle touches two of the sides in each *produced*, i.e. the circle is an *escribed* circle to each of the six triangles.

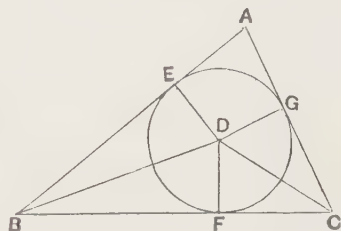
PROPOSITION 4.

In a given triangle to inscribe a circle.

Let ABC be the given triangle ;

thus it is required to inscribe a circle in the triangle ABC .

Let the angles ABC , ACB
 5 be bisected by the straight lines
 BD , CD [I. 9], and let these meet
 one another at the point D ;
 from D let DE , DF , DG be
 drawn perpendicular to the straight
 10 lines AB , BC , CA .



Now, since the angle ABD
 is equal to the angle CBD ,
 and the right angle BED is also equal to the right angle
 BFD ,
 15 EBD , FBD are two triangles having two angles equal to two
 angles and one side equal to one side, namely that subtending
 one of the equal angles, which is BD common to the
 triangles ;

therefore they will also have the remaining sides equal to
 20 the remaining sides ; [I. 26]

therefore DE is equal to DF .

For the same reason

DG is also equal to DF .

Therefore the three straight lines DE , DF , DG are equal
 25 to one another ;

therefore the circle described with centre D and distance
 one of the straight lines DE , DF , DG will pass also
 through the remaining points, and will touch the straight
 lines AB , BC , CA , because the angles at the points E , F , G
 30 are right.

For, if it cuts them, the straight line drawn at right angles
 to the diameter of the circle from its extremity will be found
 to fall within the circle : which was proved absurd ; [III. 16]

therefore the circle described with centre D and distance
 35 one of the straight lines DE , DF , DG will not cut the
 straight lines AB , BC , CA ;

therefore it will touch them, and will be the circle inscribed
 in the triangle ABC . [IV. Def. 5]

Let it be inscribed, as FGE .

40 Therefore in the given triangle ABC the circle EFG has
 been inscribed.

Q. E. F.

26, 34. and distance one of the (straight lines D) E , (D) F , (D) G . The words and letters here shown in brackets are put in to fill out the rather careless language of the Greek. Here and in several other places in Book IV. Euclid says literally "and with distance one of the (points) E , F , G " (καὶ διαστήματι ἐνὶ τῶν E , Z , H) and the like. In one case (IV. 13) he actually has "with distance one of the points G , H , K , L , M " (διαστήματι ἐνὶ τῶν H , Θ , K , A , M σημειῶν). Heiberg notes "Graecam locutionem satis miram et negligenter," but, in view of its frequent occurrence in good MSS., does not venture to correct it.

Euclid does not think it necessary to prove that BD , CD will meet; this is indeed obvious, for the angles DBC , DCB are together half of the angles ABC , ACB , which themselves are together less than two right angles, and therefore the two bisectors of the angles B , C must meet, by Post. 5.

It follows from the proof of this proposition that, if the bisectors of two angles B , C of a triangle meet in D , the line joining D to A also bisects the third angle A , or the bisectors of the three angles of a triangle meet in a point.

It will be observed that Euclid uses the *indirect* form of proof when showing that the circle touches the three sides of the triangle. Simson proves it directly, and points out that Euclid does the same in III. 17, 33 and 37, whereas in IV. 8 and 13 as well as here he uses the *indirect* form. The difference is unimportant, being one of form and not of substance; the indirect proof refers back to III. 16, whereas the direct refers back to the Porism to that proposition.

We may state this problem in the more general form: *To describe a circle touching three given straight lines which do not all meet in one point, and of which not more than two are parallel.*

In the case (1) where two of the straight lines are parallel and the third cuts them, two pairs of interior angles are formed, one on each side of the third straight line. If we bisect each of the interior angles on one side, the bisectors will meet in a point, and this point will be the centre of a circle which can be drawn touching each of the three straight lines, its radius being the perpendicular from the point on any one of the three. Since the *alternate* angles are equal, two equal circles can be drawn in this manner satisfying the given condition.

In the case (2) where the three straight lines form a triangle, suppose each straight line produced indefinitely. Then each straight line will make two pairs of interior angles with the other two, one pair forming two angles of the triangle, and the other pair being their supplements. By bisecting each angle of either pair we obtain, in the manner of the proposition, two circles satisfying the conditions, one of them being the inscribed circle of the triangle and the other being a circle *escribed* to it, i.e. touching one side and the other

two sides *produced*. Next, taking the pairs of interior angles formed by a second side with the other two produced indefinitely, we get two circles satisfying the conditions, one of which is the same inscribed circle that we had before, while the other is a second escribed circle. Similarly with the third side. Hence we have the inscribed circle, and three escribed circles (one opposite each angle of the triangle), i.e. four circles in all, satisfying the conditions of the problem.

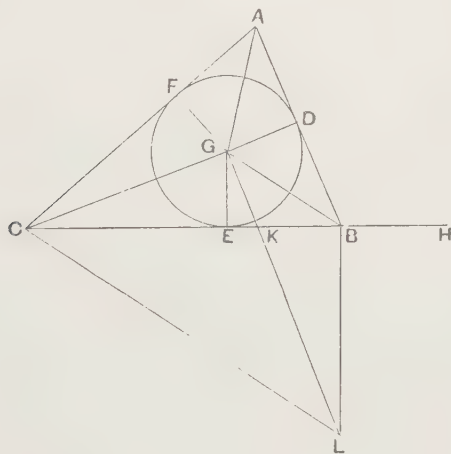
It may perhaps not be inappropriate to give at this point Heron's elegant proof of the formula for the area of a triangle in terms of the sides, which we usually write thus :

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)},$$

although it requires the theory of proportions and uses some ungeometrical expressions, e.g. the product of two areas and the "side" of such a product, where of course the areas are so many square units of length. The proof is given in the *Metrica*, i. 8, and in the *Dioptra*, 30 (Heron, Vol. III., Teubner, 1903, pp. 20—24 and pp. 280—4, or Heron, ed. Hultsch, pp. 235—7).

Suppose the sides of the triangle ABC to be given in length.

Inscribe the circle DEF , and let G be its centre.



Join AG, BG, CG, DG, EG, FG .

Then $BC \cdot EG = 2 \cdot \Delta BGC,$
 $CA \cdot FG = 2 \cdot \Delta ACG,$
 $AB \cdot DG = 2 \cdot \Delta ABG.$

Therefore, by addition,

$$p \cdot EG = 2 \cdot \Delta ABC,$$

where p is the perimeter.

Produce CB to H , so that $BH = AD$.

Then, since $AD = AF, DB = BE, FC = CE,$

$$CH = \frac{1}{2}p.$$

Hence

$$CH \cdot EG = \Delta ABC.$$

But $CH \cdot EG$ is the "side" of the product $CH^2 \cdot EG^2$, that is $\sqrt{CH^2 \cdot EG^2}$;

therefore $(\triangle ABC)^2 = CH^2 \cdot EG^2$.

Draw GL at right angles to CG , and BL at right angles to CB , meeting at L . Join CL .

Then, since each of the angles CGL , CBL is right, $CGBL$ is a quadrilateral in a circle.

Therefore the angles CGB , CLB are equal to two right angles.

Now the angles CGB , AGD are equal to two right angles, since AG , BG , CG bisect the angles at G , and the angles CGB , AGD are equal to the angles AGC , DGB , while the sum of all four is equal to four right angles.

Therefore the angles AGD , CLB are equal.

So are the right angles ADG , CBL .

Therefore the triangles AGD , CLB are similar.

Hence $BC : BL = AD : DG$

$$= BH : EG,$$

and, alternately, $CB : BH = BL : EG$

$$= BK : KE,$$

whence, *componendo*, $CH : HB = BE : EK$.

It follows that $CH^2 : CH \cdot HB = BE \cdot EC : CE \cdot EK$
 $= BE \cdot EC : EG^2$

Therefore

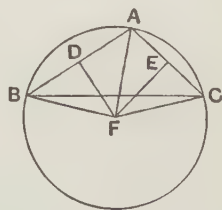
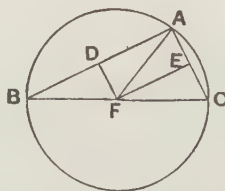
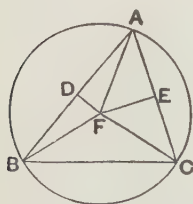
$$(\triangle ABC)^2 = CH^2 \cdot EG^2 = CH \cdot HB \cdot CE \cdot EB \\ = \frac{1}{2}p \left(\frac{1}{2}p - BC \right) \left(\frac{1}{2}p - AB \right) \left(\frac{1}{2}p - AC \right).$$

PROPOSITION 5.

About a given triangle to circumscribe a circle.

Let ABC be the given triangle;

thus it is required to circumscribe a circle about the given triangle ABC .



Let the straight lines AB , AC be bisected at the points D , E [I. 10], and from the points D , E let DF , EF be drawn at right angles to AB , AC ;

they will then meet within the triangle ABC , or on the straight line BC , or outside BC .

First let them meet within at F , and let FB , FC , FA be joined.

Then, since AD is equal to DB ,
and DF is common and at right angles,
therefore the base AF is equal to the base FB . [1. 4]

Similarly we can prove that

CF is also equal to AF ;

so that FB is also equal to FC ;

therefore the three straight lines FA , FB , FC are equal to one another.

Therefore the circle described with centre F and distance one of the straight lines FA , FB , FC will pass also through the remaining points, and the circle will have been circumscribed about the triangle ABC .

Let it be circumscribed, as ABC .

Next, let DF , EF meet on the straight line BC at F , as is the case in the second figure; and let AF be joined.

Then, similarly, we shall prove that the point F is the centre of the circle circumscribed about the triangle ABC .

Again, let DF , EF meet outside the triangle ABC at F , as is the case in the third figure, and let AF , BF , CF be joined.

Then again, since AD is equal to DB ,
and DF is common and at right angles,
therefore the base AF is equal to the base BF . [1. 4]

Similarly we can prove that

CF is also equal to AF ;

so that BF is also equal to FC ;

therefore the circle described with centre F and distance one of the straight lines FA , FB , FC will pass also through the remaining points, and will have been circumscribed about the triangle ABC .

Therefore about the given triangle a circle has been circumscribed.

Q. E. F.

And it is manifest that, when the centre of the circle falls within the triangle, the angle BAC , being in a segment greater than the semicircle, is less than a right angle;

when the centre falls on the straight line BC , the angle BAC , being in a semicircle, is right ;

and when the centre of the circle falls outside the triangle, the angle BAC , being in a segment less than the semicircle, is greater than a right angle. [III. 31]

Simson points out that Euclid does not prove that DF, EF will meet, and he inserts in the text the following argument to supply the omission.

" DF, EF produced meet one another. For, if they do not meet, they are parallel, wherefore AB, AC , which are at right angles to them, are parallel [or, he should have added, in a straight line] ; which is absurd."

This assumes, of course, that straight lines which are at right angles to two parallels are themselves parallel ; but this is an obvious deduction from I. 28.

On the assumption that DF, EF will meet Todhunter has this note : "It has been proposed to show this in the following way : join DE ; then the angles EDF and DEF are together less than the angles ADF and AEF , that is, they are together less than two right angles ; and therefore DF and EF will meet, by Axiom 12 [Post. 5]. This assumes that ADE and AED are acute angles ; it may, however, be easily shown that DE is parallel to BC , so that the triangle ADE is equiangular to the triangle ABC ; and we must therefore select the two sides AB and AC such that ABC and ACB may be acute angles."

This is, however, unsatisfactory. Euclid makes no such selection in III. 9 and III. 10, where the same assumption is tacitly made ; and it is unnecessary, because it is easy to prove that the straight lines DF, EF meet in *all* cases, by considering the different possibilities separately and drawing a separate figure for each case.

Simson thinks that Euclid's demonstration had been spoiled by some unskilful hand both because of the omission to prove that the perpendicular bisectors meet, and because "without any reason he divides the proposition into three cases, whereas one and the same construction and demonstration serves for them all, as Campanus has observed." However, up to the usual words $\delta\pi\epsilon\rho\ \epsilon\delta\epsilon\iota\ \pi\omega\iota\eta\sigma\alpha\iota$ there seems to be no doubt about the text. Heiberg suggests that Euclid gave separately the case where F falls on BC because, in that case, only AF needs to be drawn and not BF, CF as well.

The addition, though given in Simson and the text-books as a "corollary," has no heading $\pi\acute{o}\rho\iota\sigma\mu\alpha$ in the best mss. ; it is an explanation like that which is contained in the penultimate paragraph of III. 25.

The Greek text has a further addition, which is rejected by Heiberg as not genuine, "So that, further, when the given angle happens to be less than a right angle, DF, EF will fall within the triangle, when it is right, on BC , and, when it is greater than a right angle, outside BC . (being) what it was required to do." Simson had already observed that the text here is vitiated "where mention is made of a given angle, though there neither is, nor can be, anything in the proposition relating to a given angle."

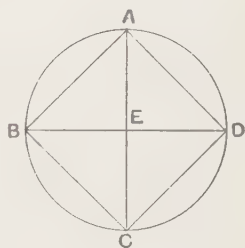
PROPOSITION 6.

In a given circle to inscribe a square.

Let $ABCD$ be the given circle ;
thus it is required to inscribe a square in the circle $ABCD$.

Let two diameters AC , BD of the circle $ABCD$ be drawn at right angles to one another, and let AB , BC , CD , DA be joined.

Then, since BE is equal to ED , for E is the centre,
and EA is common and at right angles,
therefore the base AB is equal to the base AD . [I. 4]



For the same reason
each of the straight lines BC , CD is also equal to each of the straight lines AB , AD ;

therefore the quadrilateral $ABCD$ is equilateral.

I say next that it is also right-angled.

For, since the straight line BD is a diameter of the circle $ABCD$,

therefore BAD is a semicircle ;

therefore the angle BAD is right. [III. 31]

For the same reason

each of the angles ABC , BCD , CDA is also right ;

therefore the quadrilateral $ABCD$ is right-angled.

But it was also proved equilateral ;

therefore it is a square ; [I. Def. 22]

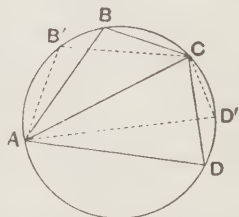
and it has been inscribed in the circle $ABCD$.

Therefore in the given circle the square $ABCD$ has been inscribed.

Q. E. F.

Euclid here proceeds to consider problems corresponding to those in Props. 2—5 with reference to figures of four or more sides, but with the difference that, whereas he dealt with triangles of any form, he confines himself henceforth to regular figures. It happened to be as easy to divide a circle into *three* parts which are in the ratio of the angles, or of the supplements of the angles, of a triangle as into three *equal* parts. But, when it is required to inscribe in a circle a figure equiangular to a given *quadrilateral*, this can only be

done provided that the quadrilateral has either pair of opposite angles equal to two right angles. Moreover, in this case, the problem *may* be solved in the same way as that of IV. 2, i.e. by simply inscribing a triangle equiangular to one of the triangles into which the quadrilateral is divided by either diagonal, and then drawing on the side corresponding to the diagonal as base another triangle equiangular to the other triangle contained in the quadrilateral. But this is not the *only* solution; there are an infinite number of other solutions in which the inscribed quadrilateral will, unlike that found by this particular method, not be of the same *form* as the given quadrilateral. For suppose $ABCD$ to be the quadrilateral inscribed in the circle by the method of IV. 2. Take *any* point B' on AB , join AB' , and then make the angle DAD' (measured towards AC) equal to the angle BAB' . Join $B'C$, CD' . Then $AB'CD'$ is also *equiangular* to the given quadrilateral, but not of the same *form*. Hence the problem is indeterminate in the case of the general quadrilateral. It is equally so if the given quadrilateral is a rectangle; and it is determinate only when the given quadrilateral is a *square*.



PROPOSITION 7.

About a given circle to circumscribe a square.

Let $ABCD$ be the given circle;

thus it is required to circumscribe a square about the circle $ABCD$.

Let two diameters AC , BD of the circle $ABCD$ be drawn at right angles to one another, and through the points A , B , C , D let FG , GH , HK , KF be drawn touching the circle $ABCD$.

[III. 16, Por.]

Then, since FG touches the circle $ABCD$,

and EA has been joined from the centre E to the point of contact at A ,

therefore the angles at A are right.

[III. 18]

For the same reason

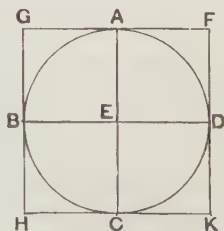
the angles at the points B , C , D are also right.

Now, since the angle AEB is right,

and the angle EBG is also right,

therefore GH is parallel to AC .

[I. 28]



For the same reason

AC is also parallel to FK ,

so that GH is also parallel to FK .

[I. 30]

Similarly we can prove that

each of the straight lines GF , HK is parallel to BED .

Therefore GK , GC , AK , FB , BK are parallelograms ;
therefore GF is equal to HK , and GH to FK .

[I. 34]

And, since AC is equal to BD ,

and AC is also equal to each of the straight lines GH , FK ,
while BD is equal to each of the straight lines GF , HK ,

[I. 34]

therefore the quadrilateral $FGHK$ is equilateral.

I say next that it is also right-angled.

For, since $GBEA$ is a parallelogram,
and the angle AEB is right,

therefore the angle AGB is also right.

[I. 34]

Similarly we can prove that

the angles at H , K , F are also right.

Therefore $FGHK$ is right-angled.

But it was also proved equilateral ;

therefore it is a square ;

and it has been circumscribed about the circle $ABCD$.

Therefore about the given circle a square has been circumscribed.

Q. E. F.

It is just as easy to describe about a given circle a polygon equiangular to any given polygon as it is to describe a square about a given circle. We have only to use the method of iv. 3, i.e. to take any radius of the circle, to measure round the centre successive angles in one and the same direction equal to the supplements of the successive angles of the given polygon and, lastly, to draw tangents to the circle at the extremities of the several radii so determined ; but again the polygon would in general not be of the same *form* as the given one ; it would only be so if the given polygon happened to be such that a circle could be inscribed in it. To take the case of a *quadrilateral* only : it is easy to prove that, if a quadrilateral be described about a circle, the sum of one pair of opposite sides must be equal to the sum of the other pair. It may be proved, *conversely*, that, if a quadrilateral has the sums of the pairs of opposite sides equal, a circle can be inscribed in it. If then a given quadrilateral has the sums of the pairs of opposite sides equal, a quadrilateral can be described about any given circle not only equiangular with it but having the same *form* or, in the words of Book vi., *similar* to it.

PROPOSITION 8.

In a given square to inscribe a circle.

Let $ABCD$ be the given square ;
thus it is required to inscribe a circle in the given square
 $ABCD$.

Let the straight lines AD , AB be
bisected at the points E , F respectively

[I. 10]

through E let EH be drawn parallel
to either AB or CD , and through
 F let FK be drawn parallel to either
 AD or BC ;

[I. 31]

therefore each of the figures AK , KB ,
 AH , HD , AG , GC , BG , GD is a parallelogram,
and their opposite sides are evidently equal.

[I. 34]

Now, since AD is equal to AB ,
and AE is half of AD , and AF half of AB ,

therefore AE is equal to AF ,

so that the opposite sides are also equal ;

therefore FG is equal to GE .

Similarly we can prove that each of the straight lines GH ,
 GK is equal to each of the straight lines FG , GE ;

therefore the four straight lines GE , GF , GH , GK are
equal to one another.

Therefore the circle described with centre G and distance
one of the straight lines GE , GF , GH , GK will pass also
through the remaining points.

And it will touch the straight lines AB , BC , CD , DA ,
because the angles at E , F , H , K are right.

For, if the circle cuts AB , BC , CD , DA , the straight
line drawn at right angles to the diameter of the circle from
its extremity will fall within the circle : which was proved
absurd ;

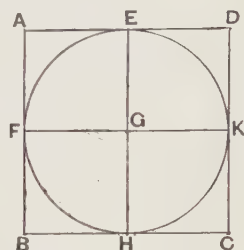
[III. 16]

therefore the circle described with centre G and distance
one of the straight lines GE , GF , GH , GK will not cut
the straight lines AB , BC , CD , DA .

Therefore it will touch them, and will have been inscribed
in the square $ABCD$.

Therefore in the given square a circle has been inscribed.

Q. E. F.



As was remarked in the last note, a circle can be inscribed in any *quadrilateral* which has the sum of one pair of opposite sides equal to the sum of the other pair. In particular, it follows that a circle can be inscribed in a *square* or a *rhombus*, but not in a rectangle or a rhomboid.

PROPOSITION 9.

About a given square to circumscribe a circle.

Let $ABCD$ be the given square ;
thus it is required to circumscribe a circle about the square $ABCD$.

For let AC , BD be joined, and let them cut one another at E .

Then, since DA is equal to AB ,
and AC is common,
therefore the two sides DA , AC are equal
to the two sides BA , AC ;
and the base DC is equal to the base BC ;

therefore the angle DAC is equal to
the angle BAC .

[I. 8]

Therefore the angle DAB is bisected by AC .

Similarly we can prove that each of the angles ABC , BCD , CDA is bisected by the straight lines AC , DB .

Now, since the angle DAB is equal to the angle ABC ,
and the angle EAB is half the angle DAB ,
and the angle EBA half the angle ABC ,
therefore the angle EAB is also equal to the angle EBA ;
so that the side EA is also equal to EB .

[I. 6]

Similarly we can prove that each of the straight lines EA , EB is equal to each of the straight lines EC , ED .

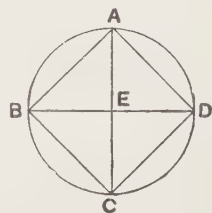
Therefore the four straight lines EA , EB , EC , ED are equal to one another.

Therefore the circle described with centre E and distance one of the straight lines EA , EB , EC , ED will pass also through the remaining points ;

and it will have been circumscribed about the square $ABCD$.

Let it be circumscribed, as $ABCD$.

Therefore about the given square a circle has been circumscribed.



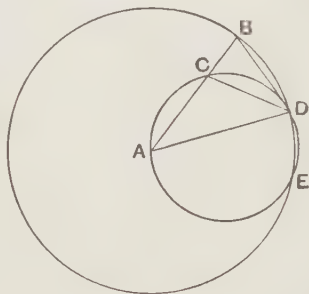
PROPOSITION 10.

To construct an isosceles triangle having each of the angles at the base double of the remaining one.

Let any straight line AB be set out, and let it be cut at the point C so that the rectangle contained by AB , BC is equal to the square on CA ; [II. 11]

with centre A and distance AB let the circle BDE be described,

and let there be fitted in the circle BDE the straight line BD equal to the straight line AC which is not greater than the diameter of the circle BDE . [IV. 1]



Let AD , DC be joined, and let the circle ACD be circumscribed about the triangle ACD . [IV. 5]

Then, since the rectangle AB , BC is equal to the square on AC ,

and AC is equal to BD ,

therefore the rectangle AB , BC is equal to the square on BD .

And, since a point B has been taken outside the circle ACD ,

and from B the two straight lines BA , BD have fallen on the circle ACD , and one of them cuts it, while the other falls on it,

and the rectangle AB , BC is equal to the square on BD ,

therefore BD touches the circle ACD . [III. 37]

Since, then, BD touches it, and DC is drawn across from the point of contact at D ,

therefore the angle BDC is equal to the angle DAC in the alternate segment of the circle. [III. 32]

Since, then, the angle BDC is equal to the angle DAC , let the angle CDA be added to each;

therefore the whole angle BDA is equal to the two angles CDA , DAC .

But the exterior angle BCD is equal to the angles CDA , DAC ;

[I. 32]

therefore the angle BDA is also equal to the angle BCD .

But the angle BDA is equal to the angle CBD , since the side AD is also equal to AB ;

[I. 5]

so that the angle DBA is also equal to the angle BCD .

Therefore the three angles BDA , DBA , BCD are equal to one another.

And, since the angle DBC is equal to the angle BCD ,

the side BD is also equal to the side DC .

[I. 6]

But BD is by hypothesis equal to CA ;

therefore CA is also equal to CD ,

so that the angle CDA is also equal to the angle DAC ;

[I. 5]

therefore the angles CDA , DAC are double of the angle DAC .

But the angle BCD is equal to the angles CDA , DAC ;

therefore the angle BCD is also double of the angle CAD .

But the angle BCD is equal to each of the angles BDA , DBA ;

therefore each of the angles BDA , DBA is also double of the angle DAB .

Therefore the isosceles triangle ABD has been constructed having each of the angles at the base DB double of the remaining one.

Q. E. F.

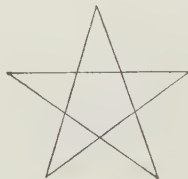
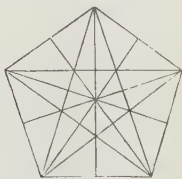
There is every reason to conclude that the connexion of the triangle constructed in this proposition with the regular pentagon, and the construction of the triangle itself, were the discovery of the Pythagoreans. In the first place the Scholium iv. No. 2 (Heiberg, Vol. v. p. 273) says "this Book is the discovery of the Pythagoreans." Secondly, the summary in Proclus (p. 65, 20) says that Pythagoras discovered "the construction of the cosmic figures," by which must be understood the five regular solids. Thirdly, Iamblichus (*Vit. Pyth.* c. 18, s. 88) quotes a story of Hippasus, "that he was one of the Pythagoreans but, owing to his being the first to publish and write down (the construction of) the sphere arising from the twelve pentagons (τὴν ἐκ τῶν δώδεκα πενταγώνων), perished by shipwreck for his impiety, having got credit for the discovery all the same, whereas everything belonged to HIM (ἐκείνου τοῦ ἀνδρός), for it is thus that they refer to Pythagoras, and they do not call him by his name." Cantor has (13, pp. 176 sqq.) collected notices which help us to form an idea how the discovery of the Euclidean construction for a regular pentagon may have been arrived at by the Pythagoreans.

Plato puts into the mouth of Timaeus a description of the formation from

right-angled triangles of the figures which are the faces of the first four regular solids. The face of the cube is the square which is formed from isosceles right-angled triangles by placing four of these triangles contiguously so that the four right angles are in contact at the centre. The equilateral triangle, however, which is the form of the faces of the tetrahedron, the octahedron and the icosahedron, cannot be constructed from isosceles right-angled triangles, but is constructed from a particular scalene right-angled triangle which Timaeus (54 A, B) regards as the most beautiful of all scalene right-angled triangles, namely that in which the square on one of the sides about the right angle is three times the square on the other. This is, of course, the triangle forming half of an equilateral triangle bisected by the perpendicular from one angular point on the opposite side. The Platonic Timaeus does not construct his equilateral triangle from two such triangles but from six, by placing the latter contiguously round a point so that the hypotenuses and the smaller of the sides about the right angles respectively adjoin, and all of them meet at the common centre, as shown in the figure (*Timaeus*, 54 D, E). The probability that this exposition was Pythagorean is confirmed by the independent testimony of Proclus (pp. 304—5), who attributes to the Pythagoreans the theorem that six equilateral triangles, or three hexagons, or four squares, placed contiguously with one angular point of each at a common point, will just fill up the four right angles round that point, and that no other regular polygons in any numbers have this property.



How then would it be proposed to split up into triangles, or to make up out of triangles, the face of the remaining solid, the dodecahedron? It would easily be seen that the pentagon could not be constructed by means of the two right-angled triangles which were used for constructing the square and the equilateral triangle respectively. But attempts would naturally be made to split up the pentagon into elementary triangles, and traces of such attempts are actually forthcoming. Plutarch has in two passages spoken of the division of the faces of the dodecahedron into triangles, remarking in one place (*Quaest. Platon.* v. 1) that each of the twelve faces is made up of 30 elemen-



tary scalene triangles, so that, taken together, they give 360 such triangles, and in another (*De defectu oraculorum*, c. 33) that the elementary triangle of the dodecahedron must be different from that of the tetrahedron, octahedron and icosahedron. Another writer of the 2nd cent., Alcinoüs, has, in his introduction to the study of Plato (*De doctrina Platonis*, c. 11), spoken similarly of the 360 elements which are produced when every one of the pentagons is divided into 5 isosceles triangles, and each of the latter into 6 scalene triangles. Now, if we proceed to draw lines in a pentagon separating it into this number of small triangles as shown in the above figure, the figure

which stands out most prominently in the mass of lines is the "star-pentagon," as drawn separately, which then (if the consecutive corners be joined) suggests the drawing, as part of a pentagon, of a triangle of a definite character. Now we are expressly told by Lucian and the scholiast to the *Clouds* of Aristophanes (see Bretschneider, pp. 85—86) that the triple interwoven triangle, the pentagram ($\tau\acute{o}$ $\tau\rho\iota\pi\lambda\omicron\upsilon\eta\nu$ $\tau\rho\acute{\iota}\gamma\omega\nu\omicron\nu$, $\tau\acute{o}$ $\delta\iota'$ $\alpha\lambda\lambda\acute{\eta}\lambda\omega\nu$, $\tau\acute{o}$ $\pi\epsilon\nu\acute{\tau}\alpha\gamma\rho\alpha\mu\mu\omicron\nu$), was used by the Pythagoreans as a symbol of recognition between the members of the same school ($\sigma\upsilon\nu\beta\acute{o}\lambda\eta$ $\pi\rho\acute{o}s$ $\tau\omicron\upsilon\varsigma$ $\delta\omicron\mu\omicron\delta\acute{o}\xi\omicron\upsilon\varsigma$ $\acute{\epsilon}\chi\rho\omega\hat{\nu}\tau\omicron$), and was called by them Health. There seems to be therefore no room for doubt that the construction of a pentagon by means of an isosceles triangle having each of its base angles double of the vertical angle was due to the Pythagoreans.

The construction of this triangle depends upon II. 11, or the problem of dividing a straight line so that the rectangle contained by the whole and one of the parts is equal to the square on the other part. This problem of course appears again in Eucl. VI. 30 as the problem of cutting a given straight line *in extreme and mean ratio*, i.e. the problem of the *golden section*, which is no doubt "the section" referred to in the passage of the summary given by Proclus (p. 67, 6) which says that Eudoxus "greatly added to the number of the theorems which Plato originated regarding the section." This idea that Plato began the study of the "golden section" as a subject in itself is not in the least inconsistent with the supposition that the problem of Eucl. II. 11 was solved by the Pythagoreans. The very fact that Euclid places it among other propositions which are clearly Pythagorean in origin is significant, as is also the fact that its solution is effected by "applying to a straight line a rectangle equal to a given square and exceeding by a square," while Proclus says plainly (p. 419, 15) that, according to Eudemus, "the application of areas, their *exceeding* and their *falling short*, are ancient and discoveries of the Muse of the Pythagoreans."

We may suppose the construction of IV. 10 to have been arrived at by analysis somewhat as follows (Todhunter's *Euclid*, p. 325).

Suppose the problem solved, i.e. let ABD be an isosceles triangle having each of its base angles double of the vertical angle.

Bisect the angle ADB by the straight line DC meeting AB in C . [I. 9]

Therefore the angle BDC is equal to the angle BAD ; and the angle CDA is also equal to the angle BAD ,

so that DC is equal to CA .

Again, since, in the triangles BCD , BDA ,

the angle BDC is equal to the angle BAD ,

and the angle B is common,

the third angle BCD is equal to the third angle BDA , and therefore to the angle DBC .

Therefore DC is equal to DB .

Now, if a circle be described about the triangle ACD [IV. 5], since the angle BDC is equal to the angle in the segment CAD ,

BD must touch the circle [by the converse of III. 32 easily proved from it by *reductio ad absurdum*].

Hence [III. 36] the square on BD and therefore the square on CD , or AC , is equal to the rectangle AB , BC .

Thus the problem is reduced to that of cutting AB at C so that the rectangle AB , BC is equal to the square on AC . [II. 11]

When this is done, we have only to draw a circle with centre A and radius AB and place in it a chord BD equal in length to AC . [IV. 1]

Since each of the angles ABD , ADB is double of the angle BAD , the latter is equal to one-fifth of the sum of all three, i.e. is one-fifth of two right angles, or two-fifths of a right angle, and each of the base angles is four-fifths of a right angle.

If we bisect the angle BAD , we obtain an angle equal to one-fifth of a right angle, so that the proposition enables us to *divide a right angle into five equal parts*.

It will be observed that BD is the side of a regular *decagon* inscribed in the larger circle.

Proclus, as remarked above (Vol. I. p. 130), gives IV. 10 as an instance in which two of the six formal divisions of a proposition, the *setting-out* and the "*definition*," are left out, and explains that they are unnecessary because there is no *datum* in the enunciation. This is however no more than formally true, because Euclid does begin his proposition by *setting out* "any straight line AB ," and he constructs an isosceles triangle having AB for one of its equal sides, i.e. he does practically imply a *datum* in the enunciation, and a corresponding *setting-out* and "*definition*" in the proposition itself.

PROPOSITION 11.

In a given circle to inscribe an equilateral and equiangular pentagon.

Let $ABCDE$ be the given circle ;
thus it is required to inscribe in the circle $ABCDE$ an equilateral and equiangular pentagon.

Let the isosceles triangle FGH be set out having each of the angles at G , H double of the angle at F ;

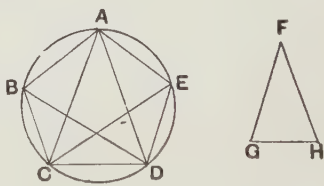
[IV. 10]

let there be inscribed in the circle $ABCDE$ the triangle ACD equiangular with the triangle FGH , so

that the angle CAD is equal to the angle at F and the angles at G , H respectively equal to the angles ACD , CDA ; [IV. 2]
therefore each of the angles ACD , CDA is also double of the angle CAD .

Now let the angles ACD , CDA be bisected respectively by the straight lines CE , DB [I. 9], and let AB , BC , DE , EA be joined.

Then, since each of the angles ACD , CDA is double of the angle CAD ,
and they have been bisected by the straight lines CE , DB ,



therefore the five angles DAC , ACE , ECD , CDB , BDA are equal to one another.

But equal angles stand on equal circumferences; [III. 26]

therefore the five circumferences AB , BC , CD , DE , EA are equal to one another.

But equal circumferences are subtended by equal straight lines; [III. 29]

therefore the five straight lines AB , BC , CD , DE , EA are equal to one another;

therefore the pentagon $ABCDE$ is equilateral.

I say next that it is also equiangular.

For, since the circumference AB is equal to the circumference DE , let BCD be added to each;

therefore the whole circumference $ABCD$ is equal to the whole circumference $EDCB$.

And the angle AED stands on the circumference $ABCD$, and the angle BAE on the circumference $EDCB$;

therefore the angle BAE is also equal to the angle AED . [III. 27]

For the same reason

each of the angles ABC , BCD , CDE is also equal to each of the angles BAE , AED ;

therefore the pentagon $ABCDE$ is equiangular.

But it was also proved equilateral;

therefore in the given circle an equilateral and equiangular pentagon has been inscribed.

Q. E. F.

De Morgan remarks that "the method of IV. 11 is not so natural as making a direct use of the angle obtained in the last." On the other hand, if we look at the figure and notice that it shows the whole of the *pentagram-star* except one line (that connecting B and E), I think we shall conclude that the method is nearer to that used by the Pythagoreans, and therefore of much more historical interest.

Another method would of course be to use IV. 10 to describe a *decagon* in the circle, and then to join any vertex to the next alternate one, the latter to the next alternate one, and so on.

Mr H. M. Taylor gives "a complete geometrical construction for inscribing a regular decagon or pentagon in a given circle," as follows.

"Find O the centre.

Draw two diameters AOC , BOD at right angles to one another.

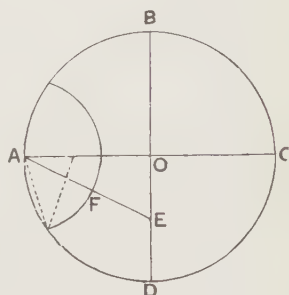
Bisect OD in E .

Draw AE and cut off EF equal to OE .

Place round the circle ten chords equal to AF .

These chords will be the sides of a regular decagon. Draw the chords joining five alternate vertices of the decagon; they will be the sides of a regular pentagon."

The construction is of course only a combination of those in II. 11 and IV. 1; and the proof would have to follow that in IV. 10.



PROPOSITION 12.

About a given circle to circumscribe an equilateral and equiangular pentagon.

Let $ABCDE$ be the given circle; thus it is required to circumscribe an equilateral and equiangular pentagon about the circle $ABCDE$.

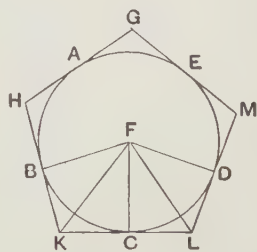
Let A, B, C, D, E be conceived to be the angular points of the inscribed pentagon, so that the circumferences AB, BC, CD, DE, EA are equal;

[IV. 11]

through A, B, C, D, E let GH, HK, KL, LM, MG be drawn touching the circle;

[III. 16, Por.]

let the centre F of the circle $ABCDE$ be taken [III. 1], and let FB, FK, FC, FL, FD be joined.



Then, since the straight line KL touches the circle $ABCDE$ at C ,

and FC has been joined from the centre F to the point of contact at C ,

therefore FC is perpendicular to KL ;

[III. 18]

therefore each of the angles at C is right.

For the same reason

the angles at the points B, D are also right.

And, since the angle FCK is right,
therefore the square on FK is equal to the squares on FC, CK .

For the same reason [I. 47]

the square on FK is also equal to the squares on FB, BK ;
so that the squares on FC, CK are equal to the squares
on FB, BK ,

of which the square on FC is equal to the square on FB ;
therefore the square on CK which remains is equal to the
square on BK .

Therefore BK is equal to CK .

And, since FB is equal to FC ,
and FK common,

the two sides BF, FK are equal to the two sides CF, FK ;
and the base BK equal to the base CK ;

therefore the angle BFK is equal to the angle KFC , [I. 8]
and the angle BKF to the angle FKC .

Therefore the angle BFC is double of the angle KFC ,
and the angle BKC of the angle FKC .

For the same reason

the angle CFD is also double of the angle CFL ,
and the angle DLC of the angle FLC .

Now, since the circumference BC is equal to CD ,
the angle BFC is also equal to the angle CFD . [III. 27]

And the angle BFC is double of the angle KFC , and the
angle DFC of the angle LFC ;

therefore the angle KFC is also equal to the angle LFC .

But the angle FCK is also equal to the angle FCL ;
therefore FKC, FLC are two triangles having two angles
equal to two angles and one side equal to one side, namely
 FC which is common to them;

therefore they will also have the remaining sides equal to the
remaining sides, and the remaining angle to the remaining
angle; [I. 26]

therefore the straight line KC is equal to CL ,
and the angle FKC to the angle FLC ,

And, since KC is equal to CL ,
therefore KL is double of KC .

For the same reason it can be proved that

HK is also double of BK .

And BK is equal to KC ;

therefore HK is also equal to KL .

Similarly each of the straight lines HG , GM , ML can also be proved equal to each of the straight lines HK , KL ;

therefore the pentagon $GHKLM$ is equilateral.

I say next that it is also equiangular.

For, since the angle FKC is equal to the angle FLC , and the angle HKL was proved double of the angle FKC ,

and the angle KLM double of the angle FLC ,

therefore the angle HKL is also equal to the angle KLM .

Similarly each of the angles KHG , HGM , GML can also be proved equal to each of the angles HKL , KLM ;

therefore the five angles GHK , HKL , KLM , LMG , MGH are equal to one another.

Therefore the pentagon $GHKLM$ is equiangular.

And it was also proved equilateral; and it has been circumscribed about the circle $ABCDE$.

Q. E. F.

De Morgan remarks that IV. 12, 13, 14 supply the place of the following : *Having given a regular polygon of any number of sides inscribed in a circle, to describe the same about the circle; and, having given the polygon, to inscribe and circumscribe a circle.* For the method can be applied generally, as indeed Euclid practically says in the Porism to IV. 15 about the regular hexagon and in the remark appended to IV. 16 about the regular fifteen-angled figure.

The *conclusion* of this proposition, "therefore about the given circle an equilateral and equiangular pentagon has been circumscribed," is omitted in the MSS.

PROPOSITION 13.

In a given pentagon, which is equilateral and equiangular, to inscribe a circle.

Let $ABCDE$ be the given equilateral and equiangular pentagon;

thus it is required to inscribe a circle in the pentagon $ABCDE$.

For let the angles BCD , CDE be bisected by the straight lines CF , DF respectively; and from the point F , at

which the straight lines CF , DF meet one another, let the straight lines FB , FA , FE be joined.

Then, since BC is equal to CD ,
and CF common,
the two sides BC , CF are equal to the
two sides DC , CF ;
and the angle BCF is equal to the
angle DCF ;

therefore the base BF is equal
to the base DF ,
and the triangle BCF is equal to the
triangle DCF ,

and the remaining angles will be equal to the remaining angles,
namely those which the equal sides subtend. [I. 4]

Therefore the angle CBF is equal to the angle CDF .

And, since the angle CDE is double of the angle CDF ,
and the angle CDE is equal to the angle ABC ,
while the angle CDF is equal to the angle CBF ;

therefore the angle CBA is also double of the angle CBF ;

therefore the angle ABF is equal to the angle FBC ;
therefore the angle ABC has been bisected by the straight
line BF .

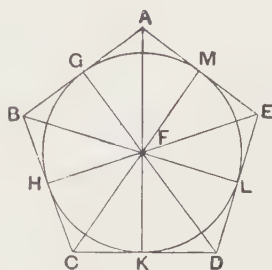
Similarly it can be proved that
the angles BAE , AED have also been bisected by the straight
lines FA , FE respectively.

Now let FG , FH , FK , FL , FM be drawn from the point
 F perpendicular to the straight lines AB , BC , CD , DE , EA .

Then, since the angle HCF is equal to the angle KCF ,
and the right angle FHC is also equal to the angle FKC ,
 FHC , FKC are two triangles having two angles equal to two
angles and one side equal to one side, namely FC which is
common to them and subtends one of the equal angles;
therefore they will also have the remaining sides equal to the
remaining sides; [I. 26]

therefore the perpendicular FH is equal to the perpendicular
 FK .

Similarly it can be proved that
each of the straight lines FL , FM , FG is also equal to each
of the straight lines FH , FK ;



therefore the five straight lines FG , FH , FK , FL , FM are equal to one another.

Therefore the circle described with centre F and distance one of the straight lines FG , FH , FK , FL , FM will pass also through the remaining points ;

and it will touch the straight lines AB , BC , CD , DE , EA , because the angles at the points G , H , K , L , M are right.

For, if it does not touch them, but cuts them, it will result that the straight line drawn at right angles to the diameter of the circle from its extremity falls within the circle : which was proved absurd. [III. 16]

Therefore the circle described with centre F and distance one of the straight lines FG , FH , FK , FL , FM will not cut the straight lines AB , BC , CD , DE , EA ;

therefore it will touch them.

Let it be described, as $GHLKM$.

Therefore in the given pentagon, which is equilateral and equiangular, a circle has been inscribed.

Q. E. F.

PROPOSITION 14.

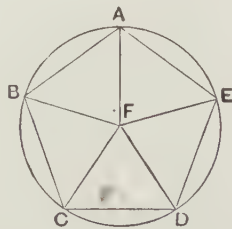
About a given pentagon, which is equilateral and equiangular, to circumscribe a circle.

Let $ABCDE$ be the given pentagon, which is equilateral and equiangular ;

thus it is required to circumscribe a circle about the pentagon $ABCDE$.

Let the angles BCD , CDE be bisected by the straight lines CF , DF respectively, and from the point F , at which the straight lines meet, let the straight lines FB , FA , FE be joined to the points B , A , E .

Then in manner similar to the preceding it can be proved that the angles CBA , BAE , AED have also been bisected by the straight lines FB , FA , FE respectively.



Now, since the angle BCD is equal to the angle CDE ,
 and the angle FCD is half of the angle BCD ,
 and the angle CDF half of the angle CDE ,
 therefore the angle FCD is also equal to the angle CDF ,
 so that the side FC is also equal to the side FD . [1. 6]

Similarly it can be proved that
 each of the straight lines FB , FA , FE is also equal to each
 of the straight lines FC , FD ;
 therefore the five straight lines FA , FB , FC , FD , FE are
 equal to one another.

Therefore the circle described with centre F and distance
 one of the straight lines FA , FB , FC , FD , FE will pass
 also through the remaining points, and will have been
 circumscribed.

Let it be circumscribed, and let it be $ABCDE$.

Therefore about the given pentagon, which is equilateral
 and equiangular, a circle has been circumscribed.

Q. E. F.

PROPOSITION 15.

*In a given circle to inscribe an equilateral and equiangular
 hexagon.*

Let $ABCDEF$ be the given circle;
 thus it is required to inscribe an equilateral and equiangular
 hexagon in the circle $ABCDEF$.

Let the diameter AD of the circle
 $ABCDEF$ be drawn;

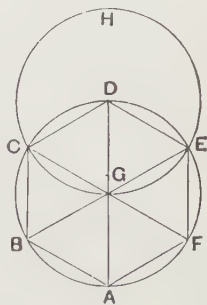
let the centre G of the circle be taken, and
 with centre D and distance DG let the
 circle $EGCH$ be described;

let EG , CG be joined and carried through
 to the points B , F ,

and let AB , BC , CD , DE , EF , FA be
 joined.

I say that the hexagon $ABCDEF$ is
 equilateral and equiangular.

For, since the point G is the centre of the circle $ABCDEF$,
 GE is equal to GD .



Again, since the point D is the centre of the circle GCH ,
 DE is equal to DG .

But GE was proved equal to GD ;

therefore GE is also equal to ED ;

therefore the triangle EGD is equilateral;

and therefore its three angles EGD , GDE , DEG are equal to one another, inasmuch as, in isosceles triangles, the angles at the base are equal to one another. [I. 5]

And the three angles of the triangle are equal to two right angles; [I. 32]

therefore the angle EGD is one-third of two right angles.

Similarly, the angle DGC can also be proved to be one-third of two right angles.

And, since the straight line CG standing on EB makes the adjacent angles EGC , CGB equal to two right angles, therefore the remaining angle CGB is also one-third of two right angles.

Therefore the angles EGD , DGC , CGB are equal to one another;

so that the angles vertical to them, the angles BGA , AGF , FGE are equal. [I. 15]

Therefore the six angles EGD , DGC , CGB , BGA , AGF , FGE are equal to one another.

But equal angles stand on equal circumferences; [III. 26]
 therefore the six circumferences AB , BC , CD , DE , EF , FA are equal to one another.

And equal circumferences are subtended by equal straight lines; [III. 29]

therefore the six straight lines are equal to one another;

therefore the hexagon $ABCDEF$ is equilateral.

I say next that it is also equiangular.

For, since the circumference FA is equal to the circumference ED ,

let the circumference $ABCD$ be added to each;

therefore the whole $FABCD$ is equal to the whole $EDCBA$;

and the angle FED stands on the circumference $FABCD$,
and the angle AFE on the circumference $EDCBA$;

therefore the angle AFE is equal to the angle DEF .

[III. 27]

Similarly it can be proved that the remaining angles of the hexagon $ABCDEF$ are also severally equal to each of the angles AFE , FED ;

therefore the hexagon $ABCDEF$ is equiangular.

But it was also proved equilateral;

and it has been inscribed in the circle $ABCDEF$.

Therefore in the given circle an equilateral and equiangular hexagon has been inscribed.

Q. E. F.

PORISM. From this it is manifest that the side of the hexagon is equal to the radius of the circle.

And, in like manner as in the case of the pentagon, if through the points of division on the circle we draw tangents to the circle, there will be circumscribed about the circle an equilateral and equiangular hexagon in conformity with what was explained in the case of the pentagon.

And further by means similar to those explained in the case of the pentagon we can both inscribe a circle in a given hexagon and circumscribe one about it.

Q. E. F.

Heiberg, I think with good reason, considers the Porism to this proposition to be referred to in the instance which Proclus (p. 304, 2) gives of a porism following a *problem*. As the text of Proclus stands, "the (porism) found in the second Book ($\tau\acute{o}\ \delta\grave{\epsilon}\ \epsilon\nu\ \tau\hat{\omega}\ \delta\epsilon\upsilon\tau\acute{\epsilon}\rho\omega\ \beta\iota\beta\lambda\acute{\iota}\omega\ \kappa\epsilon\acute{\iota}\mu\epsilon\nu\omicron\nu$) is a porism to a problem"; but this is not true of the only porism that we find in the second Book, namely the porism to II. 4. Hence Heiberg thinks that for $\tau\hat{\omega}\ \delta\epsilon\upsilon\tau\acute{\epsilon}\rho\omega\ \beta\iota\beta\lambda\acute{\iota}\omega$ should be read $\tau\hat{\omega}\ \delta'\ \beta\iota\beta\lambda\acute{\iota}\omega$, i.e. the fourth Book. Moreover Proclus speaks of the porism in the particular Book, from which we gather that there was only *one* porism in Book IV. as he knew it, and therefore that he did not regard as a *porism* the addition to IV. 5. Cf. note on that proposition.

It appears that Theon substituted for the first words of the Porism to IV. 15 "And in like manner as in the case of the pentagon" ($\delta\omicron\mu\omicron\iota\omega\varsigma\ \delta\grave{\epsilon}\ \tau\omicron\iota\varsigma\ \epsilon\pi\acute{\iota}\ \tau\omicron\upsilon\ \pi\epsilon\upsilon\tau\alpha\gamma\acute{\omega}\nu\omicron\nu$) the simple word "and" or "also" ($\kappa\alpha\acute{\iota}$), apparently thinking that the words had the same meaning as the similar words lower down. This is however not the case, the meaning being that "if, as in the case of the pentagon, we draw tangents, we can prove, *also* as was done in the case of the pentagon, that the figure so formed is a circumscribed regular hexagon."

PROPOSITION 16.

In a given circle to inscribe a fifteen-angled figure which shall be both equilateral and equiangular.

Let $ABCD$ be the given circle ;

thus it is required to inscribe in the circle $ABCD$ a fifteen-angled figure which shall be both equilateral and equiangular.

In the circle $ABCD$ let there be inscribed a side AC of the equilateral triangle inscribed in it, and a side AB of an equilateral pentagon ; therefore, of the equal segments of which there are fifteen in the circle $ABCD$, there will be five in the circumference ABC which is one-third of the circle, and there will be three in the circumference AB which is one-fifth of the circle ;

therefore in the remainder BC there will be two of the equal segments.

Let BC be bisected at E ;

[III. 30]

therefore each of the circumferences BE , EC is a fifteenth of the circle $ABCD$.

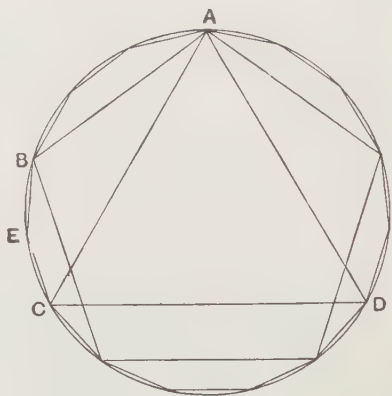
If therefore we join BE , EC and fit into the circle $ABCD$ straight lines equal to them and in contiguity, a fifteen-angled figure which is both equilateral and equiangular will have been inscribed in it.

Q. E. F.

And, in like manner as in the case of the pentagon, if through the points of division on the circle we draw tangents to the circle, there will be circumscribed about the circle a fifteen-angled figure which is equilateral and equiangular.

And further, by proofs similar to those in the case of the pentagon, we can both inscribe a circle in the given fifteen-angled figure and circumscribe one about it.

Q. E. F.



Here, as in III. 10, we have the term "circle" used by Euclid in its exceptional sense of the *circumference* of a circle, instead of the "*plane figure* contained by one line" of I. Def. 15. Cf. the note on that definition (Vol. I. pp. 184—5).

Proclus (p. 269) refers to this proposition in illustration of his statement that Euclid gave proofs of a number of propositions with an eye to their use in astronomy. "With regard to the last proposition in the fourth Book in which he inscribes the side of the fifteen-angled figure in a circle, for what object does anyone assert that he propounds it except for the reference of this problem to astronomy? For, when we have inscribed the fifteen-angled figure in the circle through the poles, we have the distance from the poles both of the equator and the zodiac, since they are distant from one another by the side of the fifteen-angled figure." This agrees with what we know from other sources, namely that up to the time of Eratosthenes (*circa* 284 -- 204 B.C.) 24' was generally accepted as the correct measurement of the obliquity of the ecliptic. This measurement, and the construction of the fifteen-angled figure, were probably due to the Pythagoreans, though it would appear that the former was not known to Oenopides of Chios (fl. *circa* 460 B.C.), as we learn from Theon of Smyrna (pp. 198—9, ed. Hiller), who gives Dercyllides as his authority, that Eudemus (fl. *circa* 320 B.C.) stated in his *ἀστρολογίαι* that, while Oenopides discovered certain things, and Thales, Anaximander and Anaximenes others, it was the rest (*οἱ λοιποὶ*) who added other discoveries to these and, among them, that "the axes of the fixed stars and of the planets respectively are distant from one another by the side of a fifteen-angled figure." Eratosthenes evaluated the angle to $\frac{1}{3}\frac{1}{3}$ rds of 180° , i.e. about $23^\circ 51' 20''$, which measurement was apparently not improved upon in antiquity (cf. Ptolemy, *Syntaxis*, ed. Heiberg, p. 68).

Euclid has now shown how to describe regular polygons with 3, 4, 5, 6 and 15 sides. Now, when any regular polygon is given, we can construct a regular polygon with twice the number of sides by first describing a circle about the given polygon and then bisecting all the smaller arcs subtended by the sides. Applying this process any number of times, we see that we can by Euclid's methods construct regular polygons with $3 \cdot 2^n$, $4 \cdot 2^n$, $5 \cdot 2^n$, $15 \cdot 2^n$ sides, where n is zero or any positive integer.

BOOK VI.

INTRODUCTORY NOTE.

The theory of proportions has been established in Book v. in a perfectly general form applicable to all kinds of magnitudes (although the representation of magnitudes by straight lines gives it a *geometrical* appearance); it is now necessary to apply the theory to the particular case of *geometrical* investigation. The only thing still required in order that this may be done is a proof of the existence of such a magnitude as bears to any given finite magnitude any given finite ratio; and this proof is supplied, so far as regards the subject matter of geometry, by vi. 12 which shows how to construct a fourth proportional to three given straight lines.

A few remarks on the enormous usefulness of the theory of proportions to geometry will not be out of place. We have already in Books i. and ii. made acquaintance with one important part of what has been well called geometrical algebra, the method, namely, of *application of areas*. We have seen that this method, working by the representation of products of two quantities as rectangles, enables us to solve some particular quadratic equations. But the limitations of such a method are obvious. So long as general quantities are represented by *straight lines* only, we cannot, if our geometry is *plane*, deal with products of more than two such quantities; and, even by the use of three dimensions, we cannot work with products of more than three quantities, since no geometrical meaning could be attached to such a product. This limitation disappears so soon as we can represent any general quantity, corresponding to what we denote by a letter in algebra, by a *ratio*; and this we can do because, on the general theory of proportion established in Book v., a ratio may be a ratio of two incommensurable quantities as well as of commensurables. Ratios can be *compounded ad infinitum*, and the division of one ratio by another is equally easy, since it is the same thing as compounding the first ratio with the inverse of the second. Thus e.g. it is seen at once that the *coefficients* in a quadratic of the most general form can be represented by ratios between straight lines, and the solution by means of Books i. and ii. of problems corresponding to quadratic equations with particular coefficients can now be extended to cover *any* quadratic with real roots. As indicated, we can perform, by composition of ratios, the operation corresponding to multiplying algebraical quantities, and this to any extent. We can divide quantities by compounding a ratio with the inverse of the ratio representing the divisor. For the addition and subtraction of quantities we have only to use the geometrical equivalent of bringing to a common denominator, which is effected by means of the fourth proportional.

DEFINITIONS.

1. **Similar rectilineal figures** are such as have their angles severally equal and the sides about the equal angles proportional.

[2. **Reciprocally related figures.** *See note.*]

3. A straight line is said to have been **cut in extreme and mean ratio** when, as the whole line is to the greater segment, so is the greater to the less.

4. The **height** of any figure is the perpendicular drawn from the vertex to the base.

DEFINITION 1.

Ὅμοια σχήματα εὐθύγραμμά ἐστιν, ὅσα τὰς τε γωνίας ἴσας ἔχει κατὰ μίαν καὶ τὰς περὶ τὰς ἴσας γωνίας πλευρὰς ἀνάλογον.

This definition is quoted by Aristotle, *Anal. post.* II. 17, 99 a 13, where he says that *similarity* (τὸ ὁμοιον) in the case of figures "consists, let us say (*ἴσως*), in their having their sides proportional and their angles equal." The use of the word *ἴσως* may suggest that, in Aristotle's time, this definition had not quite established itself in the text-books (Heiberg, *Mathematisches zu Aristoteles*, p. 9).

It was pointed out in Van Swinden's *Elements of Geometry* (Jacobi's edition, 1834, pp. 114—5) that Euclid omits to state an essential part of the definition, namely that "the corresponding sides must be opposite to equal angles," which is necessary in order that the corresponding sides may follow in the same order in both figures.

At the same time the definition states more than is absolutely necessary, for it is true to say that *two polygons are similar when, if the sides and angles are taken in the same order, the angles are equal and the sides about the equal angles are proportional, omitting*

(1) *three consecutive angles,*

or (2) *two consecutive angles and the side common to them,*

or (3) *two consecutive sides and the angle included by them,*

and making no assumption with regard to the omitted sides and angles.

Austin objected to this definition on the ground that it is not obvious that the properties (1) of having their angles respectively equal and (2) of having the sides about the equal angles proportional can *co-exist* in two figures; but, a definition not being concerned to prove the *existence* of the thing defined, the objection falls to the ground. We are properly left to satisfy ourselves as to the existence of similar figures in the course of the exposition in Book VI., where we learn how to construct on any given straight line a rectilineal figure similar to a given one (VI. 18).

DEFINITION 2.

The Greek text gives here a definition of *reciprocally related figures* (ἀντιπεπονθότα σχήματα). “[Two] figures are *reciprocally related* when there are in each of the two figures antecedent and consequent ratios” (Ἀντιπεπονθότα δὲ σχήματά ἐστιν, ὅταν ἐν ἑκατέρῳ τῶν σχημάτων ἡγούμενοί τε καὶ ἐπόμενοι λόγοι ᾖσιν). No intelligible meaning can be attached to “antecedent and consequent ratios” here; the sense would require rather “an antecedent and a consequent of (two equal) ratios in each figure.” Hence Candalla and Peyrard read λόγων ὅροι (“terms of ratios”) instead of λόγοι. Camerer reads λόγων without ὅροι. But the objection to the definition lies deeper. It is never used; when we come, in VI. 14, 15, XI. 34 etc. to parallelograms, triangles etc. having the property indicated, they are not called “reciprocal” parallelograms etc., but parallelograms etc. “the sides of which are reciprocally proportional,” ὧν ἀντιπεπόνθασιν αἱ πλευραί. Hence Simson appears to be right in condemning the definition; it may have been interpolated from Heron, who has it.

Simson proposes in his note to substitute the following definition. “Two magnitudes are said to be reciprocally proportional to two others when one of the first is to one of the other magnitudes as the remaining one of the last two is to the remaining one of the first.” This definition requires that the magnitudes shall be all of the same kind.

DEFINITION 3.

Ἄκρον καὶ μέσον λόγον εὐθείᾳ τετμηθῆναι λέγεται, ὅταν ἡ ὡς ἡ ὅλη πρὸς τὸ μείζον τμήμα, οὕτως τὸ μείζον πρὸς τὸ ἔλαττον.

DEFINITION 4.

Ὑψος ἐστὶ παντὸς σχήματος ἢ ἀπὸ τῆς κορυφῆς ἐπὶ τὴν βάσιν κάθετος ἀγομένη.

The definition of “height” is not found in Campanus and is perhaps rightly suspected, since it does not apply in terms to parallelograms, parallelepipeds, cylinders and prisms, though it is used in the *Elements* with reference to these latter figures. Aristotle does not appear to know altitude (ὕψος) in the mathematical sense; he uses κάθετος of triangles (*Meteorologica* III. 3, 373 a 11). The term is however readily understood, and scarcely requires definition.

[DEFINITION 5.

Λόγος ἐκ λόγων συγκείσθαι λέγεται, ὅταν αἱ τῶν λόγων πηλικότητες ἐφ’ ἑαυτὰς πολλαπλασιασθείσαι ποιῶσιν τινα.

“A ratio is said to be compounded of ratios when the sizes (πηλικότητες) of the ratios multiplied together make some (? ratio, or size).”]

As already remarked (pp. 116, 132), it is beyond doubt that this definition of ratio is interpolated. It has little MS. authority. The best MS. (P) only has it in the margin; it is omitted altogether in Campanus’ translation from the

Arabic; and the other mss. which contain it do not agree in the position which they give to it. There is no reference to the definition in the place where compound ratio is mentioned for the first time (vi. 23), nor anywhere else in Euclid; neither is it ever referred to by the other great geometers, Archimedes, Apollonius and the rest. It appears to be only twice mentioned at all, (1) in the passage of Eutocius referred to above (p. 116) and (2) by Theon in his commentary on Ptolemy's *σύνταξις*. Moreover the content of the definition is in itself suspicious. It speaks of the "sizes of ratios being multiplied together (literally, into themselves)," an operation unknown to geometry. There is no wonder that Eutocius, and apparently Theon also, in their efforts to explain it, had to give the word *πηλικότης* a meaning which has no application except in the case of such ratios as can be expressed by numbers (Eutocius e.g. making it the "number by which the ratio is called"). Nor is it surprising that Wallis should have found it necessary to substitute for the "quantitas" of Commandinus a different translation, "quantuplicity," which he said was represented by the "*exponent of the ratio*" (*rationis exponens*), what Peletarius had described as "*denominatio ipsae proportionis*" and Clavius as "denominator." The fact is that the definition is ungeometrical and useless, as was already seen by Savile, in whose view it was one of the two blemishes in the body of geometry (the other being of course Postulate 5).

It is right to add that Hultsch (art. "Eukleides" in Pauly-Wissowa's *Real-Encyclopädie der classischen Altertumswissenschaft*) thought the definition genuine. His grounds are (1) that it stood in the *παλαιὰ ἔκδοσις* represented by P (though P has it in the margin only) and (2) that some explanation on the subject must have been given by way of preparation for vi. 23, while there is nothing in the definition which is *inconsistent* with the mode of statement of vi. 23. If the definition is after all genuine, I should be inclined to regard it as a mere survival from earlier textbooks, like the first of the two alternative definitions of a solid angle (xi. Def. 11); for its form seems to suit the old theory of proportion, applicable to commensurable magnitudes only, better than the generalised theory of Eudoxus.

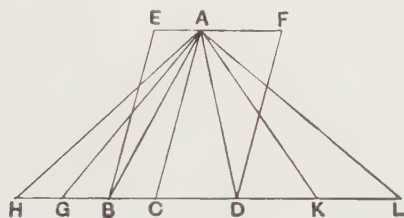
BOOK VI. PROPOSITIONS.

PROPOSITION I.

Triangles and parallelograms which are under the same height are to one another as their bases.

Let ABC , ACD be triangles and EC , CF parallelograms under the same height ;

I say that, as the base BC is to the base CD , so is the triangle ABC to the triangle ACD , and the parallelogram EC to the parallelogram CF .



For let BD be produced in both directions to the points H , L and let [any number of straight lines] BG , GH be made equal to the base BC , and any number of straight lines DK , KL equal to the base CD ;

let AG , AH , AK , AL be joined.

Then, since CB , BG , GH are equal to one another, the triangles ABC , AGB , AHG are also equal to one another. [1. 38]

Therefore, whatever multiple the base HC is of the base BC , that multiple also is the triangle AHC of the triangle ABC .

For the same reason, whatever multiple the base LC is of the base CD , that multiple also is the triangle ALC of the triangle ACD ; and, if the base HC is equal to the base CL , the triangle AHC is also equal to the triangle ACL , [1. 38]

if the base HC is in excess of the base CL , the triangle AHC
 25 is also in excess of the triangle ACL ,
 and, if less, less.

Thus, there being four magnitudes, two bases BC , CD
 and two triangles ABC , ACD ,
 equimultiples have been taken of the base BC and the
 30 triangle ABC , namely the base HC and the triangle AHC ,
 and of the base CD and the triangle ADC other, chance, equi-
 multiples, namely the base LC and the triangle ALC ;

and it has been proved that,

if the base HC is in excess of the base CL , the triangle AHC
 35 is also in excess of the triangle ALC ;
 if equal, equal; and, if less, less.

Therefore, as the base BC is to the base CD , so is the
 triangle ABC to the triangle ACD . [v. Def. 5]

Next, since the parallelogram EC is double of the triangle
 40 ABC , [I. 41]

and the parallelogram FC is double of the triangle ACD ,
 while parts have the same ratio as the same multiples of
 them, [v. 15]

therefore, as the triangle ABC is to the triangle ACD , so is
 45 the parallelogram EC to the parallelogram FC .

Since, then, it was proved that, as the base BC is to CD ,
 so is the triangle ABC to the triangle ACD ,

and, as the triangle ABC is to the triangle ACD , so is the
 parallelogram EC to the parallelogram FC ,

50 therefore also, as the base BC is to the base CD , so is the
 parallelogram EC to the parallelogram FC . [v. 11]

Therefore etc.

Q. E. D.

4. Under the same height. The Greek text has "under the same height AC ," with a figure in which the side AC common to the two triangles is perpendicular to the base and is therefore itself the "height." But, even if the two triangles are placed contiguously so as to have a common side AC , it is quite gratuitous to require it to be perpendicular to the base. Theon, on this occasion making an improvement, altered to "which are ($\delta\upsilon\tau\alpha$) under the same height, (namely) the perpendicular drawn from A to BD ." I have ventured to alter so far as to omit " AC " and to draw the figure in the usual way.

14. ABC, AGB, AHG . Euclid, indifferent to exact order, writes " AHG, AGB, ABC ."

46. Since then it was proved that, as the base BC is to CD , so is the triangle ABC to the triangle ACD . Here again words have to be supplied in translating the extremely terse Greek $\epsilon\pi\epsilon\iota\ \delta\upsilon\upsilon\iota\ \epsilon\delta\epsilon\lambda\chi\theta\eta$, $\omega\varsigma\ \mu\acute{\epsilon}\nu\ \eta\ \beta\acute{\alpha}\sigma\iota\varsigma\ B\Gamma\ \pi\rho\acute{o}\varsigma\ \tau\eta\upsilon\iota\ \Gamma\Delta$, $\delta\upsilon\tau\omega\varsigma\ \tau\acute{o}\ A\beta\Gamma\ \tau\rho\acute{\iota}\gamma\omega\upsilon\iota\ \pi\rho\acute{o}\varsigma\ \tau\acute{o}\ A\Gamma\Delta\ \tau\rho\acute{\iota}\gamma\omega\upsilon\iota$, literally "since was proved, as the base BC to CD , so the triangle ABC to the triangle ACD ." Cf. note on v. 16, p. 165.

The proof assumes—what is however an obvious deduction from I. 38—that, of triangles or parallelograms on *unequal* bases and between the same parallels, the greater is that which has the greater base.

It is of course not necessary that the two given triangles should have a common side, as in the figure; the proof is just as easy if they have not. The proposition being equally true of triangles and parallelograms of *equal* heights, Simson states this fact in a corollary thus:

“From this it is plain that triangles and parallelograms that have equal altitudes are to one another as their bases.

Let the figures be so placed as to have their bases in the same straight line; and, if we draw perpendiculars from the vertices of the triangles to the bases, the straight line which joins the vertices is parallel to that in which their bases are, because the perpendiculars are both equal and parallel to one another [I. 33]. Then, if the same construction be made as in the proposition, the demonstration will be the same.”

The object of placing the bases in one straight line is to get the triangles and parallelograms *within the same parallels*. Cf. Proclus' remark on I. 38 (p. 405, 17) that having the same height is the same thing as being in the same parallels.

Rectangles, or right-angled triangles, which have one of the sides about the right angle of the same length can be placed so that the equal sides coincide and the others are in a straight line. If then we call the common side the base, the rectangles or the right-angled triangles are to one another as their heights, by VI. 1. Now, instead of each right-angled triangle or rectangle, we can take any other triangle or parallelogram respectively with an equal base and between the same parallels. Thus

Triangles and parallelograms having equal bases are to one another as their heights.

Legendre and those authors of modern text-books who follow him in basing their treatment of proportion on the algebraical definition are obliged to divide their proofs of propositions like this into two parts, the first of which proves the particular theorem in the case where the magnitudes are commensurable, and the second extends it to the case where they are incommensurable.

Legendre (*Éléments de Géométrie*, III. 3) uses for this extension a rigorous method by *reductio ad absurdum* similar to that used by Archimedes in his treatise *On the equilibrium of planes*, I. 7. The following is Legendre's proof of the extension of VI. 1 to incommensurable parallelograms and bases.

The proposition having been proved for commensurable bases, let there be two rectangles $ABCD$, $AEFD$ as in the figure, on bases AB , AE which are incommensurable with one another.

To prove that $\text{rect. } ABCD : \text{rect. } AEFD = AB : AE$.

For, if not, let $\text{rect. } ABCD : \text{rect. } AEFD = AB : AO$, ... (1)
where AO is (for instance) greater than AE .

Divide AB into equal parts each of which is less than EO , and mark off on AO lengths equal to one of the parts; then there will be at least one point of division between E and O .

Let it be I , and draw IK parallel to EF .



Then the rectangles $ABCD$, $AIKD$ are in the ratio of the bases AB , AI , since the latter are commensurable.

Therefore, inverting the proportion,

$$\text{rect. } AIKD : \text{rect. } ABCD = AI : AB \dots \dots \dots (2).$$

From this and (1), *ex aequali*,

$$\text{rect. } AIKD : \text{rect. } AEFD = AI : AO.$$

But $AO > AI$; therefore $\text{rect. } AEFD > \text{rect. } AIKD$.

But this is impossible, for the rectangle $AEFD$ is less than the rectangle $AIKD$.

Similarly an impossibility can be proved if $AO < AE$.

Therefore $\text{rect. } ABCD : \text{rect. } AEFD = AB : AE$.

Some modern American and German text-books adopt the less rigorous method of appealing to the theory of *limits*.

PROPOSITION 2.

If a straight line be drawn parallel to one of the sides of a triangle, it will cut the sides of the triangle proportionally; and, if the sides of the triangle be cut proportionally, the line joining the points of section will be parallel to the remaining side of the triangle.

For let DE be drawn parallel to BC , one of the sides of the triangle ABC ;

I say that, as BD is to DA , so is CE to EA .

For let BE , CD be joined.

Therefore the triangle BDE is equal to the triangle CDE ;

for they are on the same base DE and in the same parallels DE , BC . [I. 38]

And the triangle ADE is another area.

But equals have the same ratio to the same; [v. 7]

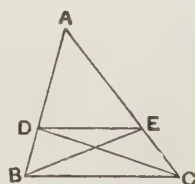
therefore, as the triangle BDE is to the triangle ADE , so is the triangle CDE to the triangle ADE .

But, as the triangle BDE is to ADE , so is BD to DA ; for, being under the same height, the perpendicular drawn from E to AB , they are to one another as their bases. [vi. 1]

For the same reason also,

as the triangle CDE is to ADE , so is CE to EA .

Therefore also, as BD is to DA , so is CE to EA . [v. 11]



Again, let the sides AB , AC of the triangle ABC be cut proportionally, so that, as BD is to DA , so is CE to EA ; and let DE be joined.

I say that DE is parallel to BC .

For, with the same construction,
since, as BD is to DA , so is CE to EA ,
but, as BD is to DA , so is the triangle BDE to the triangle ADE ,
and, as CE is to EA , so is the triangle CDE to the triangle ADE ,
[VI. 1]
therefore also,

as the triangle BDE is to the triangle ADE , so is the triangle CDE to the triangle ADE . [v. 11]

Therefore each of the triangles BDE , CDE has the same ratio to ADE .

Therefore the triangle BDE is equal to the triangle CDE ;
[v. 9]

and they are on the same base DE .

But equal triangles which are on the same base are also in the same parallels. [I. 39]

Therefore DE is parallel to BC .

Therefore etc.

Q. E. D.

Euclid evidently did not think it worth while to distinguish in the enunciation, or in the figure, the cases in which the parallel to the base cuts the other two sides produced (*a*) beyond the point in which they intersect, (*b*) in the other direction. Simson gives the three figures and inserts words in the enunciation, reading "it shall cut the other sides, *or those sides produced*, proportionally" and "if the sides, *or the sides produced*, be cut proportionally."

Todhunter observes that the second part of the enunciation ought to make it clear which segments in the proportion correspond to which. Thus e.g., if AD were double of DB , and CE double of EA , the sides would be cut proportionally, but DE would not be parallel to BC . The omission could be supplied by saying "and if the sides of the triangle be cut proportionally *so that the segments adjacent to the third side are corresponding terms in the proportion.*"

PROPOSITION 3.

If an angle of a triangle be bisected and the straight line cutting the angle cut the base also, the segments of the base will have the same ratio as the remaining sides of the triangle; and, if the segments of the base have the same ratio as the

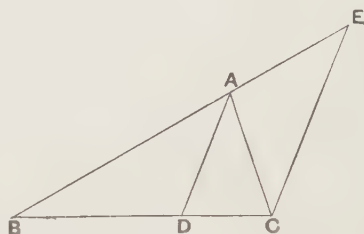
remaining sides of the triangle, the straight line joined from the vertex to the point of section will bisect the angle of the triangle.

Let ABC be a triangle, and let the angle BAC be bisected by the straight line AD ;

I say that, as BD is to CD , so is BA to AC .

For let CE be drawn through C parallel to DA , and let BA be carried through and meet it at E .

Then, since the straight line AC falls upon the parallels AD , EC ,



the angle ACE is equal to the angle CAD . [I. 29]

But the angle CAD is by hypothesis equal to the angle BAD ;

therefore the angle BAD is also equal to the angle ACE .

Again, since the straight line BAE falls upon the parallels AD , EC ,

the exterior angle BAD is equal to the interior angle AEC . [I. 29]

But the angle ACE was also proved equal to the angle BAD ;

therefore the angle ACE is also equal to the angle AEC ,
so that the side AE is also equal to the side AC . [I. 6]

And, since AD has been drawn parallel to EC , one of the sides of the triangle BCE ,

therefore, proportionally, as BD is to DC , so is BA to AE .

But AE is equal to AC ; [VI. 2]

therefore, as BD is to DC , so is BA to AC .

Again, let BA be to AC as BD to DC , and let AD be joined;

I say that the angle BAC has been bisected by the straight line AD .

For, with the same construction,
since, as BD is to DC , so is BA to AC ,

and also, as BD is to DC , so is BA to AE : for AD has been drawn parallel to EC , one of the sides of the triangle BCE ;

[VI. 2]

therefore also, as BA is to AC , so is BA to AE .

[v. 11]

Therefore AC is equal to AE ,

[v. 9]

so that the angle AEC is also equal to the angle ACE .

[I. 5]

But the angle AEC is equal to the exterior angle BAD ,

[I. 29]

and the angle ACE is equal to the alternate angle CAD ; [id.]

therefore the angle BAD is also equal to the angle CAD .

Therefore the angle BAC has been bisected by the straight line AD .

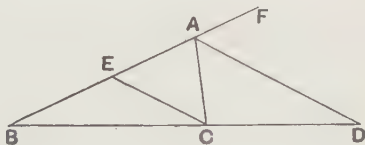
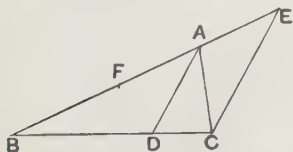
Therefore etc.

Q. E. D.

The demonstration assumes that CE will meet BA produced in some point E . This is proved in the same way as it is proved in VI. 4 that BA, ED will meet if produced. The angles ABD, BDA in the figure of VI. 3 are together less than two right angles, and the angle BDA is equal to the angle BCE , since DA, CE are parallel. Therefore the angles ABC, BCE are together less than two right angles; and BA, CE must meet, by I. Post. 5.

The corresponding proposition about the segments into which BC is divided *externally* by the bisector of the *external angle* at A when that bisector meets BC produced (i.e. when the sides AB, AC are not equal) is important. Simson gives it as a separate proposition, A, noting the fact that Pappus assumes the result without proof (Pappus, VII. p. 730, 24).

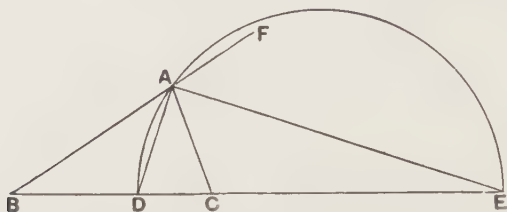
The best plan is however, as De Morgan says, to combine Props. 3 and A in one proposition, which may be enunciated thus: *If an angle of a triangle be bisected internally or externally by a straight line which cuts the opposite side or the opposite side produced, the segments of that side will have the same ratio as the other sides of the triangle; and, if a side of a triangle be divided internally or externally so that its segments have the same ratio as the other sides of the triangle, the straight line drawn from the point of section to the angular point which is opposite to the first mentioned side will bisect the interior or exterior angle at that angular point.*



Let AC be the smaller of the two sides AB, AC , so that the bisector AD of the exterior angle at A may meet BC produced beyond C . Draw CE through C parallel to DA , meeting BA in E .

Then, if FAC is the exterior angle bisected by AD in the case of external bisection, and if a point F is taken on AB in the figure of VI. 3, the proof of

VI. 3 can be used almost word for word for the other case. We have only to speak of the angle " FAC " for the angle " BAC ," and of the angle " FAD " for the angle " BAD " wherever they occur, to say "let BA , or BA produced, meet CE in E ," and to substitute " BA or BA produced" for " BAE " lower down.



If AD , AE be the internal and external bisectors of the angle A in a triangle of which the sides AB , AC are unequal, AC being the smaller, and if AD , AE meet BC and BC produced in D , E respectively,

the ratios of BD to DC and of BE to EC are alike equal to the ratio of BA to AC .

Therefore BE is to EC as BD to DC ,

that is, BE is to EC as the difference between BE and ED is to the difference between ED and EC ,

whence BE , ED , EC are in *harmonic progression*, or DE is a *harmonic mean* between BE and EC , or again B , D , C , E is a *harmonic range*.

Since the angle DAC is half of the angle BAC ,

and the angle CAE half of the angle CAF ,

while the angles BAC , CAF are equal to two right angles,

the angle DAE is a right angle.

Hence the circle described on DE as diameter passes through A .

Now, if the ratio of BA to AC is given, and if BC is given, the points D , E on BC and BC produced are given, and therefore so is the circle on DE as diameter. Hence *the locus of a point such that its distances from two given points are in a given ratio (not being a ratio of equality) is a circle*.

This locus was discussed by Apollonius in his *Plane Loci*, Book II., as we know from Pappus (VII. p. 666), who says that the book contained the theorem that, if from two given points straight lines inflected to another point are in a given ratio, the point in which they meet will lie on either a straight line or a circumference of a circle. The straight line is of course the locus when the ratio is one of equality. The other case is quoted in the following form by Eutocius (Apollonius, ed. Heiberg, II. pp. 180—4).

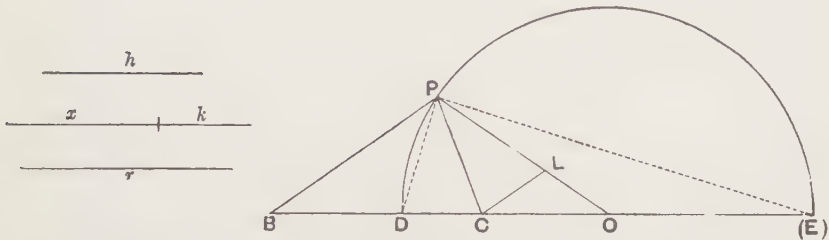
Given two points in a plane and a proportion between unequal straight lines, it is possible to describe a circle in the plane so that the straight lines inflected from the given points to the circumference of the circle shall have a ratio the same as the given one.

Apollonius' construction, as given by Eutocius, is remarkable because he makes no use of either of the points D , E . He finds O , the centre of the required circle, and the length of its radius directly from the data BC and the given ratio which we will call $h:k$. But the construction was not discovered by Apollonius; it belongs to a much earlier date, since it appears in exactly

the same form in Aristotle, *Meteorologica* III. 5, 376 a 3 sqq. The analysis leading up to the construction is, as usual, not given either by Aristotle or Eutocius. We are told to take three straight lines x , CO (a length measured along BC produced beyond C , where B , C are the points at which the greater and smaller of the inflected lines respectively terminate), and r , such that, if $h:k$ be the given ratio and $h > k$,

$$k : h = h : k + x, \dots\dots\dots(\alpha)$$

$$x : BC = k : CO = h : r. \dots\dots\dots(\beta)$$



This determines the position of O , and the length of r , the radius of the required circle. The circle is then drawn, any point P is taken on it and joined to B , C respectively, and it is proved that

$$PB : PC = h : k.$$

We may conjecture that the analysis proceeded somewhat as follows.

It would be seen that B , C are "conjugate points" with reference to the circle on DE as diameter. (Cf. Apollonius, *Conics*, I. 36, where it is proved, in terms, for a circle as well as for an ellipse and a hyperbola, that, if the polar of B meets the diameter DE in C , then $EC : CD = EB : BD$.)

If O be the middle point of DE , and therefore the centre of the circle, D , E may be eliminated, as in the *Conics*, I. 37, thus.

$$\text{Since} \quad EC : CD = EB : BD,$$

it follows that $EC + CD : EC \sim CD = EB + BD : EB \sim BD$,

$$\text{or} \quad 2OD : 2OC = 2OB : 2OD,$$

$$\text{that is,} \quad BO \cdot OC = OD^2 = r^2, \text{ say.}$$

If therefore P be any point on the circle with centre O and radius r ,

$$BO : OP = OP : OC,$$

so that BOP , POC are similar triangles.

$$\text{In addition, } h : k = BD : DC = BE : EC$$

$$= BD + BE : DE = BO : r.$$

Hence we require that

$$BO : r = r : OC = BP : PC = h : k. \dots\dots\dots(\delta)$$

Therefore, alternately,

$$k : CO = h : r,$$

which is the second relation in (β) above.

Now assume a length x such that each of the last ratios is equal to $x : BC$, as in (β) .

Then

$$x : BC = k : CO = h : r.$$

Therefore

$$x + k : BO = h : r,$$

and, alternately,

$$x + k : h = BO : r$$

$$= h : k, \text{ from } (\delta) \text{ above ;}$$

and this is the relation (a) which remained to be found.

Apollonius' proof of the construction is given by Eutocius, who begins by saying that it is manifest that r is a mean proportional between BO and OC . This is seen as follows :

From (β) we derive

$$x : BC = k : CO = h : r = (k + x) : BO,$$

whence

$$BO : r = (k + x) : h$$

$$= h : k, \text{ by } (a),$$

$$= r : CO, \text{ by } (\beta),$$

and therefore

$$r^2 = BO \cdot CO.$$

But the triangles BOP , POC have the angle at O common, and, since $BO : OP = OP : OC$, the triangles are similar and the angles OPC , OBP are equal.

[Up to this point Aristotle's proof is exactly the same ; from this point it diverges slightly.]

If now CL be drawn parallel to BP meeting OP in L , the angles BPC LCP are equal also.

Therefore the triangles BPC , PCL are similar, and

$$BP : PC = PC : CL,$$

whence

$$BP^2 : PC^2 = BP : CL$$

$$= BO : OC, \text{ by parallels,}$$

$$= BO^2 : OP^2 \text{ (since } BO : OP = OP : OC \text{).}$$

Therefore

$$BP : PC = BO : OP$$

$$= h : k \text{ (for } OP = r \text{).}$$

[Aristotle infers this more directly from the similar triangles POB , COP . Since these triangles are similar,

$$OP : CP = OB : BP,$$

whence

$$BP : PC = BO : OP$$

$$= h : k.]$$

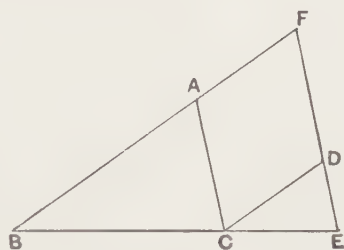
Apollonius proves lastly, by *reductio ad absurdum*, that the last equation cannot be true with reference to any point P which is not on the circle so described.

PROPOSITION 4.

In equiangular triangles the sides about the equal angles are proportional, and those are corresponding sides which subtend the equal angles.

Let ABC , DCE be equiangular triangles having the angle ABC equal to the angle DCE , the angle BAC to the angle CDE , and further the angle ACB to the angle CED ;

I say that in the triangles ABC , DCE the sides about the equal angles are proportional, and those are corresponding sides which subtend the equal angles.



For let BC be placed in a straight line with CE .

Then, since the angles ABC , ACB are less than two right angles, [I. 17]

and the angle ACB is equal to the angle DEC , therefore the angles ABC , DEC are less than two right angles;

therefore BA , ED , when produced, will meet. [I. Post. 5]

Let them be produced and meet at F .

Now, since the angle DCE is equal to the angle ABC ,
 BF is parallel to CD . [I. 28]

Again, since the angle ACB is equal to the angle DEC ,
 AC is parallel to FE . [I. 28]

Therefore $FACD$ is a parallelogram;
 therefore FA is equal to DC , and AC to FD . [I. 34]

And, since AC has been drawn parallel to FE , one side of the triangle FBE ,
 therefore, as BA is to AF , so is BC to CE . [VI. 2]

But AF is equal to CD ;

therefore, as BA is to CD , so is BC to CE ,
 and alternately, as AB is to BC , so is DC to CE . [v. 16]

Again, since CD is parallel to BF ,
 therefore, as BC is to CE , so is FD to DE . [VI. 2]

But FD is equal to AC ;

therefore, as BC is to CE , so is AC to DE ,
 and alternately, as BC is to CA , so is CE to ED . [v. 16]

Since then it was proved that,
 as AB is to BC , so is DC to CE ,
 and, as BC is to CA , so is CE to ED ;
 therefore, *ex aequali*, as BA is to AC , so is CD to DE . [v. 22]
 Therefore etc.

Q. E. D.

Todhunter remarks that "the manner in which the two triangles are to be placed is very imperfectly described; their bases are to be in the same straight line and contiguous, their vertices are to be on the same side of the base, and each of the two angles which have a common vertex is to be equal to the remote angle of the other triangle." But surely Euclid's description is sufficient, except for not saying that B and D must be on the same side of BCE .

VI. 4 can be immediately deduced from VI. 2 if we superpose one triangle on the other three times in succession, so that each angle successively coincides with its equal, the triangles being similarly situated, e.g. if (A, B, C and D, E, F being the equal angles respectively) we apply the angle DEF to the angle ABC so that D lies on AB (produced if necessary) and F on BC (produced if necessary). De Morgan prefers this method. "Abandon," he says, "the peculiar mode of construction by which Euclid proves two cases at once; make an angle coincide with its equal, and suppose this process repeated three times, one for each angle."

PROPOSITION 5.

If two triangles have their sides proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend.

Let ABC, DEF be two triangles having their sides proportional, so that,

 as AB is to BC , so is DE to EF ,

 as BC is to CA , so is EF to FD ,

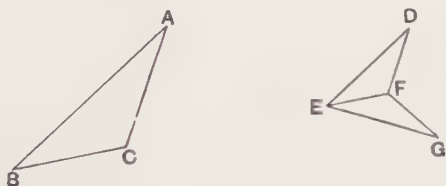
and further, as BA is to AC , so is ED to DF ;

I say that the triangle ABC is equiangular with the triangle DEF , and they will have those angles equal which the corresponding sides subtend, namely the angle ABC to the angle DEF , the angle BCA to the angle EFD , and further the angle BAC to the angle EDF .

For on the straight line EF , and at the points E, F on it, let there be constructed the angle FEG equal to the angle ABC , and the angle EFG equal to the angle ACB ; [I. 23]

therefore the remaining angle at A is equal to the remaining angle at G . [I. 32]

Therefore the triangle ABC is equiangular with the triangle GEF .



Therefore in the triangles ABC , GEF the sides about the equal angles are proportional, and those are corresponding sides which subtend the equal angles; [vi. 4]

therefore, as AB is to BC , so is GE to EF .

But, as AB is to BC , so by hypothesis is DE to EF ;

therefore, as DE is to EF , so is GE to EF . [v. 11]

Therefore each of the straight lines DE , GE has the same ratio to EF ;

therefore DE is equal to GE . [v. 9]

For the same reason

DF is also equal to GF .

Since then DE is equal to EG ,

and EF is common,

the two sides DE , EF are equal to the two sides GE , EF ;

and the base DF is equal to the base FG ;

therefore the angle DEF is equal to the angle GEF , [I. 8]

and the triangle DEF is equal to the triangle GEF ,

and the remaining angles are equal to the remaining angles, namely those which the equal sides subtend. [I. 4]

Therefore the angle DFE is also equal to the angle GFE , and the angle EDF to the angle EGF .

And, since the angle FED is equal to the angle GEF , while the angle GEF is equal to the angle ABC , therefore the angle ABC is also equal to the angle DEF .

For the same reason

the angle ACB is also equal to the angle DFE ,
and further, the angle at A to the angle at D ;

therefore the triangle ABC is equiangular with the triangle DEF .

Therefore etc.

Q. E. D.

This proposition is the complete converse, VI. 6 a partial converse, of VI. 4.

Todhunter, after Walker, remarks that the enunciation should make it clear that the sides of the triangles *taken in order* are proportional. It is quite possible that there should be two triangles ABC , DEF such that

AB is to BC as DE to EF ,

and BC is to CA as DF is to ED (instead of EF to FD),

so that AB is to AC as DF to EF

(*ex aequali* in *perturbed proportion*);

in this case the sides of the triangles are proportional, but not in the same order, and the triangles are not necessarily equiangular to one another. For a numerical illustration we may suppose the sides of one triangle to be 3, 4 and 5 feet respectively, and those of another to be 12, 15 and 20 feet respectively.

In VI. 5 there is the same *apparent* avoidance of indirect demonstration which has been noticed on I. 48.

PROPOSITION 6.

If two triangles have one angle equal to one angle and the sides about the equal angles proportional, the triangles will be equiangular and will have those angles equal which the corresponding sides subtend.

Let ABC , DEF be two triangles having one angle BAC equal to one angle EDF and the sides about the equal angles proportional, so that,

as BA is to AC , so is ED to DF ;

I say that the triangle ABC is equiangular with the triangle DEF ; and will have the angle ABC equal to the angle DEF , and the angle ACB to the angle DFE .

For on the straight line DF , and at the points D , F on it, let there be constructed the angle FDG equal to either of the angles BAC , EDF , and the angle DFG equal to the angle ACB ;

[I. 23]

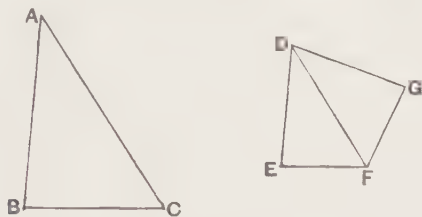
therefore the remaining angle at B is equal to the remaining angle at G .

[I. 32]

Therefore the triangle ABC is equiangular with the triangle DGF .

Therefore, proportionally, as BA is to AC , so is GD to DF . [VI. 4]

But, by hypothesis, as BA is to AC , so also is ED to DF ; therefore also, as ED is to DF , so is GD to DF . [v. 11]



Therefore ED is equal to DG ; [v. 9]
and DF is common;

therefore the two sides ED , DF are equal to the two sides GD , DF ; and the angle EDF is equal to the angle GDF ;

therefore the base EF is equal to the base GF ,

and the triangle DEF is equal to the triangle DGF ,

and the remaining angles will be equal to the remaining angles, namely those which the equal sides subtend. [I. 4]

Therefore the angle DFG is equal to the angle DFE ,
and the angle DGF to the angle DEF .

But the angle DFG is equal to the angle ACB ;
therefore the angle ACB is also equal to the angle DFE .

And, by hypothesis, the angle BAC is also equal to the angle EDF ;

therefore the remaining angle at B is also equal to the remaining angle at E ; [I. 32]

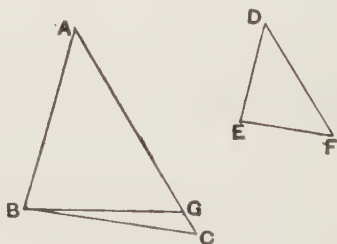
therefore the triangle ABC is equiangular with the triangle DEF .

Therefore etc.

PROPOSITION 7.

If two triangles have one angle equal to one angle, the sides about other angles proportional, and the remaining angles either both less or both not less than a right angle, the triangles will be equiangular and will have those angles equal, the sides about which are proportional.

Let ABC , DEF be two triangles having one angle equal to one angle, the angle BAC to the angle EDF , the sides about other angles ABC , DEF proportional, so that, as AB is to BC , so is DE to EF , and, first, each of the remaining angles at C , F less than a right angle;



I say that the triangle ABC is equiangular with the triangle DEF , the angle ABC will be equal to the angle DEF , and the remaining angle, namely the angle at C , equal to the remaining angle, the angle at F .

For, if the angle ABC is unequal to the angle DEF , one of them is greater.

Let the angle ABC be greater;

and on the straight line AB , and at the point B on it, let the angle ABG be constructed equal to the angle DEF . [I. 23]

Then, since the angle A is equal to D , and the angle ABG to the angle DEF , therefore the remaining angle AGB is equal to the remaining angle DFE . [I. 32]

Therefore the triangle ABG is equiangular with the triangle DEF .

Therefore, as AB is to BG , so is DE to EF . [VI. 4]

But, as DE is to EF , so by hypothesis is AB to BC ; therefore AB has the same ratio to each of the straight lines BC , BG ; [V. 11]

therefore BC is equal to BG , [V. 9]

so that the angle at C is also equal to the angle BGC . [I. 5]

But, by hypothesis, the angle at C is less than a right angle ;

therefore the angle BGC is also less than a right angle ;

so that the angle AGB adjacent to it is greater than a right angle. [I. 13]

And it was proved equal to the angle at F ;

therefore the angle at F is also greater than a right angle.

But it is by hypothesis less than a right angle : which is absurd.

Therefore the angle ABC is not unequal to the angle DEF ;

therefore it is equal to it.

But the angle at A is also equal to the angle at D ;

therefore the remaining angle at C is equal to the remaining angle at F . [I. 32]

Therefore the triangle ABC is equiangular with the triangle DEF .

But, again, let each of the angles at C, F be supposed not less than a right angle ;

I say again that, in this case too, the triangle ABC is equiangular with the triangle DEF .

For, with the same construction, we can prove similarly that

BC is equal to BG ;

so that the angle at C is also equal to the angle BGC . [I. 5]

But the angle at C is not less than a right angle ;

therefore neither is the angle BGC less than a right angle.

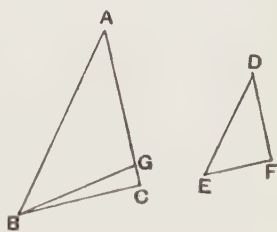
Thus in the triangle BGC the two angles are not less than two right angles : which is impossible. [I. 17]

Therefore, once more, the angle ABC is not unequal to the angle DEF ;

therefore it is equal to it.

But the angle at A is also equal to the angle at D ;

therefore the remaining angle at C is equal to the remaining angle at F . [I. 32]



Therefore the triangle ABC is equiangular with the triangle DEF .

Therefore etc.

Q. E. D.

Todhunter points out, after Walker, that some more words are necessary to make the enunciation precise: "If two triangles have one angle equal to one angle, the sides about other angles proportional *<so that the sides subtending the equal angles are homologous>....*"

This proposition is the extension to similar triangles of the *ambiguous case* already mentioned as omitted by Euclid in relation to *equality* of triangles in all respects (cf. note following I. 26, Vol. I. p. 306). The enunciation of VI. 7 has suggested the ordinary method of enunciating the *ambiguous case* where *equality* and not *similarity* is in question. Cf. Todhunter's note on I. 26.

Another possible way of presenting this proposition is given by Todhunter. The essential theorem to prove is:

If two triangles have two sides of the one proportional to two sides of the other, and the angles opposite to one pair of corresponding sides equal, the angles which are opposite to the other pair of corresponding sides shall either be equal or be together equal to two right angles.

For the angles included by the proportional sides must be either equal or unequal.

If they are equal, then, since the triangles have two angles of the one equal to two angles of the other, respectively, they are equiangular to one another.

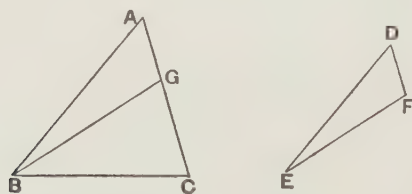
We have therefore only to consider the case in which the angles included by the proportional sides are unequal.

The proof is, except at the end, like that of VI. 7.

Let the triangles ABC , DEF have the angle at A equal to the angle at D ;

let AB be to BC as DE to EF ,

but let the angle ABC be *not* equal to the angle DEF .



The angles ACB , DFE shall be together equal to two right angles.

For one of the angles ABC , DEF must be the greater.

Let ABC be the greater; and make the angle ABG equal to the angle DEF .

Then we prove, as in VI. 7, that the triangles ABG , DEF are equiangular, whence

AB is to BG as DE is to EF .

But AB is to BC as DE is to EF , by hypothesis.

Therefore BG is equal to BC ,

and the angle BGC is equal to the angle BCA .

Now, since the triangles ABG , DEF are equiangular,
the angle BGA is equal to the angle EFD .

Add to them respectively the equal angles BGC , BCA ; therefore the angles BCA , EFD are together equal to the angles BGA , BGC , i.e. to two right angles.

It follows therefore that the angles BCA , EFD must be either equal or supplementary.

But (1), if each of them is less than a right angle, they cannot be supplementary, and they must therefore be equal;

(2) if each of them is greater than a right angle, they cannot be supplementary and must therefore be equal;

(3) if one of them is a right angle, they are supplementary and also equal.

Simson distinguishes the last case (3) in his enunciation: "then, if each of the remaining angles be either less or not less than a right angle, or if one of them be a right angle...."

The change is right, on the principle of restricting the conditions to the minimum necessary to enable the conclusion to be inferred. Simson adds a separate proof of the case in which one of the remaining angles is a right angle.

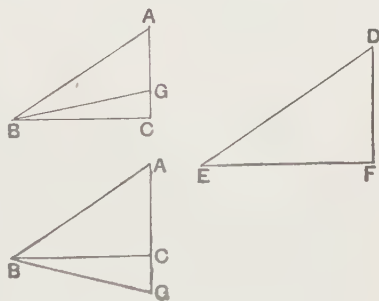
"Lastly, let one of the angles at C , F , viz. the angle at C , be a right angle; in this case likewise the triangle ABC is equiangular to the triangle DEF .

For, if they be not equiangular, make, at the point B of the straight line AB , the angle ABG equal to the angle DEF ; then it may be proved, as in the first case, that BG is equal to BC .

But the angle BCG is a right angle; therefore the angle BGC is also a right angle;

whence two of the angles of the triangle BGC are together not less than two right angles: which is impossible.

Therefore the triangle ABC is equiangular to the triangle DEF ."



PROPOSITION 8.

If in a right-angled triangle a perpendicular be drawn from the right angle to the base, the triangles adjoining the perpendicular are similar both to the whole and to one another.

Let ABC be a right-angled triangle having the angle BAC right, and let AD be drawn from A perpendicular to BC ;

I say that each of the triangles ABD , ADC is similar to the whole ABC and, further, they are similar to one another.

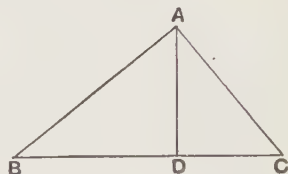
For, since the angle BAC is equal to the angle ADB , for each is right,

and the angle at B is common to the two triangles ABC and ABD ,

therefore the remaining angle ACB is equal to the remaining angle BAD ;

[I. 32]

therefore the triangle ABC is equiangular with the triangle ABD .



Therefore, as BC which subtends the right angle in the triangle ABC is to BA which subtends the right angle in the triangle ABD , so is AB itself which subtends the angle at C in the triangle ABC to BD which subtends the equal angle BAD in the triangle ABD , and so also is AC to AD which subtends the angle at B common to the two triangles.

[vi. 4]

Therefore the triangle ABC is both equiangular to the triangle ABD and has the sides about the equal angles proportional.

Therefore the triangle ABC is similar to the triangle ABD .

[vi. Def. 1]

Similarly we can prove that

the triangle ABC is also similar to the triangle ADC ;

therefore each of the triangles ABD , ADC is similar to the whole ABC .

I say next that the triangles ABD , ADC are also similar to one another.

For, since the right angle BDA is equal to the right angle ADC ,

and moreover the angle BAD was also proved equal to the angle at C ,

therefore the remaining angle at B is also equal to the remaining angle DAC ;

[I. 32]

therefore the triangle ABD is equiangular with the triangle ADC .

Therefore, as BD which subtends the angle BAD in the triangle ABD is to DA which subtends the angle at C in the triangle ADC equal to the angle BAD , so is AD itself which subtends the angle at B in the triangle ABD to DC which subtends the angle DAC in the triangle ADC equal

to the angle at B , and so also is BA to AC , these sides subtending the right angles; [VI. 4]

therefore the triangle ABD is similar to the triangle ADC . [VI. Def. 1]

Therefore etc.

PORISM. From this it is clear that, if in a right-angled triangle a perpendicular be drawn from the right angle to the base, the straight line so drawn is a mean proportional between the segments of the base. Q. E. D.

Simson remarks on this proposition: "It seems plain that some editor has changed the demonstration that Euclid gave of this proposition: For, after he has demonstrated that the triangles are equiangular to one another, he particularly shows that their sides about the equal angles are proportionals, as if this had not been done in the demonstration of prop. 4 of this book: this superfluous part is not found in the translation from the Arabic, and is now left out."

This seems a little hypercritical, for the "particular showing" that the sides about the equal angles are proportionals is really nothing more than a somewhat full citation of VI. 4. Moreover to shorten his proof still more, Simson says, after proving that each of the triangles ABD , ADC is similar to the whole triangle ABC , "And the triangles ABD , ADC being both equiangular and similar to ABC are equiangular and similar to one another," thus assuming a particular case of VI. 21, which might well be proved here, as Euclid proves it, with somewhat more detail.

We observe that, here as generally, Euclid seems to disdain to give the reader such small help as might be afforded by arranging the letters used to denote the triangles so as to show the corresponding angular points in the same order for each pair of triangles; A is the first letter throughout, and the other two for each triangle are in the order of the figure from left to right. It may be in compensation for this that he states at such length which side corresponds to which when he comes to the proportions.

In the Greek texts there is an addition to the Porism inserted after "(Being) what it was required to prove," viz. "and further that between the base and any one of the segments the side adjacent to the segment is a mean proportional." Heiberg concludes that these words are an interpolation (1) because they come after the words ὅπερ ἔδει δεῖξαι which as a rule follow the Porism, (2) they are absent from the best Theonine mss., though P and Campanus have them without the ὅπερ ἔδει δεῖξαι . Heiberg's view seems to be confirmed by the fact noted by Austin, that, whereas the first part of the Porism is quoted later in VI. 13, in the lemma before X. 33 and in the lemma after XIII. 13, the second part is *proved* in the former lemma, and elsewhere, as also in Pappus (III. p. 72, 9—23).

PROPOSITION 9.

From a given straight line to cut off a prescribed part.

Let AB be the given straight line;

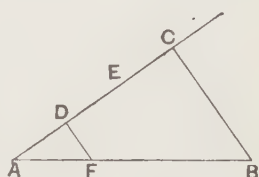
thus it is required to cut off from AB a prescribed part.

Let the third part be that prescribed.

- 5 Let a straight line AC be drawn through from A containing with AB any angle;

let a point D be taken at random on AC , and let DE , EC be made equal to AD . [I. 3]

- 10 Let BC be joined, and through D let DF be drawn parallel to it. [I. 31]



Then, since FD has been drawn parallel to BC , one of the sides of the triangle ABC , therefore, proportionally, as CD is to DA , so is BF to FA .

[VI. 2]

- 15 But CD is double of DA ;
therefore BF is also double of FA ;
therefore BA is triple of AF .

Therefore from the given straight line AB the prescribed third part AF has been cut off.

Q. E. F.

6. any angle. The expression here and in the two following propositions is *τυχούσα γωνία*, corresponding exactly to *τυχὸν σημεῖον* which I have translated as "a point (taken) at random"; but "an angle (taken) at random" would not be so appropriate where it is a question, not of *taking* any angle at all, but of drawing a straight line casually so as to make any angle with another straight line.

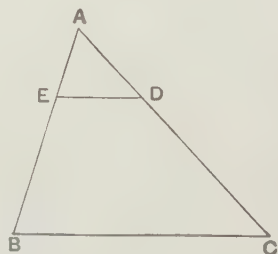
Simson observes that "this is demonstrated in a particular case, viz. that in which the third part of a straight line is required to be cut off; which is not at all like Euclid's manner. Besides, the author of that demonstration, from four magnitudes being proportionals, concludes that the third of them is the same multiple of the fourth which the first is of the second; now this is nowhere demonstrated in the 5th book, as we now have it; but the editor assumes it from the confused notion which the vulgar have of proportionals."

The truth of the assumption referred to is proved by Simson in his proposition D given above (p. 128); hence he is able to supply a general and legitimate proof of the present proposition.

"Let AB be the given straight line; it is required to cut off any part from it.

From the point A draw a straight line AC making any angle with AB ; in AC take any point D , and take AC the same multiple of AD that AB is of the part which is to be cut off from it;

join BC , and draw DE parallel to it:



then AE is the part required to be cut off.

Because ED is parallel to one of the sides of the triangle ABC , viz. to BC ,
as CD is to DA , so is BE to EA , [VI. 2]

and, *componendo*,

CA is to AD , as BA to AE . [v. 18]

But CA is a multiple of AD ;

therefore BA is the same multiple of AE . [Prop. D]

Whatever part therefore AD is of AC , AE is the same part of AB ;

wherefore from the straight line AB the part required is cut off."

The use of Simson's Prop. D can be avoided, as noted by Camerer after Baermann, in the following way. We first prove, as above, that

CA is to AD as BA is to AE .

Then we infer that, alternately,

CA is to BA as AD to AE . [v. 16]

But AD is to AE as $n \cdot AD$ to $n \cdot AE$

(where n is the number of times that AD is contained in AC); [v. 15]

whence AC is to AB as $n \cdot AD$ is to $n \cdot AE$. [v. 11]

In this proportion the first term is equal to the third; therefore [v. 14]
the second is equal to the fourth,

so that AB is equal to n times AE .

Prop. 9 is of course only a particular case of Prop. 10.

PROPOSITION 10.

To cut a given uncut straight line similarly to a given cut straight line.

Let AB be the given uncut straight line, and AC the straight line cut at the points D , E ; and let them be so placed as to contain any angle;

let CB be joined, and through D , E let DF , EG be drawn parallel to BC , and through D let DHK be drawn parallel to AB . [I. 31]



Therefore each of the figures FH , HB is a parallelogram;

therefore DH is equal to FG and HK to GB . [I. 34]

Now, since the straight line HE has been drawn parallel to KC , one of the sides of the triangle DKC ,

therefore, proportionally, as CE is to ED , so is KH to HD [VI. 2]

But KH is equal to BG , and HD to GF ;
therefore, as CE is to ED , so is BG to GF .

Again, since FD has been drawn parallel to GE , one of the sides of the triangle AGE ,

therefore, proportionally, as ED is to DA , so is GF to FA .
[VI. 2]

But it was also proved that,

as CE is to ED , so is BG to GF ;

therefore, as CE is to ED , so is BG to GF ,

and, as ED is to DA , so is GF to FA .

Therefore the given uncut straight line AB has been cut similarly to the given cut straight line AC .

Q. E. F.

PROPOSITION 11.

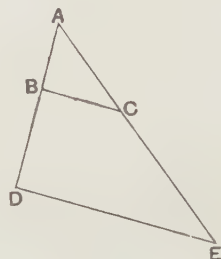
To two given straight lines to find a third proportional.

Let BA , AC be the two given straight lines, and let them be placed so as to contain any angle;

thus it is required to find a third proportional to BA , AC .

For let them be produced to the points D , E , and let BD be made equal to AC ;

let BC be joined, and through D let DE be drawn parallel to it.



Since, then, BC has been drawn parallel to DE , one of the sides of the triangle ADE , proportionally, as AB is to BD , so is AC to CE .
[VI. 2]

But BD is equal to AC ;
therefore, as AB is to AC , so is AC to CE .

Therefore to two given straight lines AB , AC a third proportional to them, CE , has been found.

Q. E. F.

1. to find. The Greek word, here and in the next two propositions, is *προσευρεῖν*, literally "to find in addition."

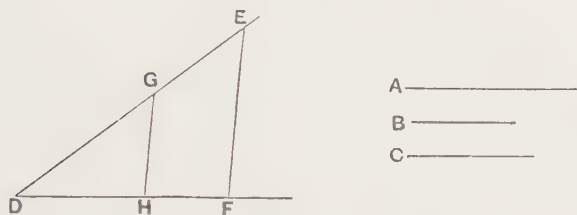
This proposition is again a particular case of the succeeding Prop. 12.

Given a ratio between straight lines, VI. 11 enables us to find the ratio which is its duplicate.

PROPOSITION 12.

To three given straight lines to find a fourth proportional.

Let A, B, C be the three given straight lines ;
thus it is required to find a fourth proportional to A, B, C .



Let two straight lines DE, DF be set out containing any angle EDF ;

let DG be made equal to A , GE equal to B , and further DH equal to C ;

let GH be joined, and let EF be drawn through E parallel to it. [I. 31]

Since, then, GH has been drawn parallel to EF , one of the sides of the triangle DEF ,

therefore, as DG is to GE , so is DH to HF . [VI. 2]

But DG is equal to A , GE to B , and DH to C ;

therefore, as A is to B , so is C to HF .

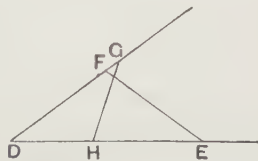
Therefore to the three given straight lines A, B, C a fourth proportional HF has been found.

Q. E. F.

We have here the geometrical equivalent of the "rule of three."

It is of course immaterial whether, as in Euclid's proof, the first and second straight lines are measured on one of the lines forming the angle and the third on the other, or the first and third are measured on one and the second on the other.

If it should be desired that the first and the required fourth be measured on one of the lines, and the second and third on the other, we can use the following construction. Measure DE on one straight line equal to A , and on any other straight line making an angle with the first at the point D measure DF equal to B , and DG equal to C . Join EF , and through G draw GH anti-parallel to EF , i.e. make the angle DGH equal to the angle DEF ; let GH meet DE (produced if necessary) in H .



DH is then the fourth proportional.

For the triangles EDF , GDH are similar, and the sides about the equal angles are proportional, so that

DE is to DF as DG to DH ,

or

A is to B as C to DH .

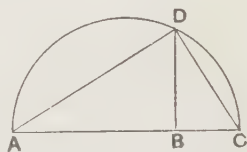
PROPOSITION 13.

To two given straight lines to find a mean proportional.

Let AB , BC be the two given straight lines ;
thus it is required to find a mean
proportional to AB , BC .

Let them be placed in a straight
line, and let the semicircle ADC be
described on AC ;

let BD be drawn from the point B at
right angles to the straight line AC ,
and let AD , DC be joined.



Since the angle ADC is an angle in a semicircle, it is
right. [III. 31]

And, since, in the right-angled triangle ADC , DB has
been drawn from the right angle perpendicular to the base,
therefore DB is a mean proportional between the segments of
the base, AB , BC . [VI. 8, Por.]

Therefore to the two given straight lines AB , BC a mean
proportional DB has been found.

Q. E. F.

This proposition, the Book VI. version of II. 14, is equivalent to the
extraction of the square root. It further enables us, given a ratio between
straight lines, to find the ratio which is its *sub-duplicate*, or the ratio of which
it is duplicate.

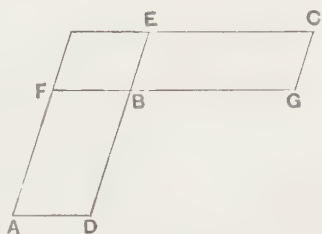
PROPOSITION 14.

*In equal and equiangular parallelograms the sides about
the equal angles are reciprocally proportional; and equiangular
parallelograms in which the sides about the equal angles are
reciprocally proportional are equal.*

Let AB , BC be equal and equiangular parallelograms having the angles at B equal, and let DB , BE be placed in a straight line;

therefore FB , BG are also in a straight line. [I. 14]

I say that, in AB , BC , the sides about the equal angles are reciprocally proportional, that is to say, that, as DB is to BE , so is GB to BF .



For let the parallelogram FE be completed.

Since, then, the parallelogram AB is equal to the parallelogram BC ,

and FE is another area,

therefore, as AB is to FE , so is BC to FE . [v. 7]

But, as AB is to FE , so is DB to BE , [vi. 1]

and, as BC is to FE , so is GB to BF . [id.]

therefore also, as DB is to BE , so is GB to BF . [v. 11]

Therefore in the parallelograms AB , BC the sides about the equal angles are reciprocally proportional.

Next, let GB be to BF as DB to BE ;

I say that the parallelogram AB is equal to the parallelogram BC .

For since, as DB is to BE , so is GB to BF ,

while, as DB is to BE , so is the parallelogram AB to the parallelogram FE , [vi. 1]

and, as GB is to BF , so is the parallelogram BC to the parallelogram FE , [vi. 1]

therefore also, as AB is to FE , so is BC to FE ; [v. 11]

therefore the parallelogram AB is equal to the parallelogram BC . [v. 9]

Therefore etc.

Q. E. D.

De Morgan says upon this proposition: "Owing to the disjointed manner in which Euclid treats comp. and ratio, this proposition is strangely out of place. It is a particular case of vi. 23, being that in which the ratio of the sides, compounded, gives a ratio of equality. The proper definition of four magnitudes being reciprocally proportional is: that the ratio compounded of their ratios is that of equality."

It is true that VI. 14 is a particular case of VI. 23, but, if either is out of place, it is rather the latter that should be placed before VI. 14, since most of the propositions between VI. 15 and VI. 23 depend upon VI. 14 and 15. But it is perfectly consistent with Euclid's manner to give a particular case first and its extension later, and such an arrangement often has great advantages in that it enables the more difficult parts of a subject to be led up to more easily and gradually. Now, if De Morgan's view were here followed, we should, as it seems to me, be committing the mistake of explaining what is relatively easy to understand, viz. two ratios of which one is the inverse of the other, by a more complicated conception, that of compound ratio. In other words, it is easier for a learner to realise the relation indicated by the statement that the sides of equal and equiangular parallelograms are "reciprocally proportional" than to form a conception of parallelograms such that "the ratio compounded of the ratio of their sides is one of equality." For this reason I would adhere to Euclid's arrangement.

The conclusion that, since DB , BE are placed in a straight line, FB , BG are also in a straight line is referred to I. 14. The deduction is made clearer by the following steps.

The angle DBF is equal to the angle GBE ;

add to each the angle FBE ;

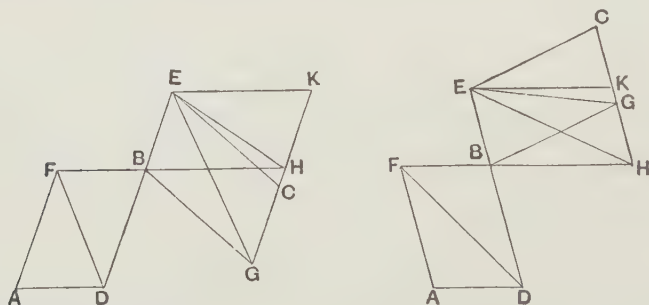
therefore the angles DBF , FBE are together equal to the angles GBE , FBE .
[C. N. 2]

But the angles DBF , FBE are together equal to two right angles, [I. 13]
therefore the angles GBE , FBE are together equal to two right angles,
[C. N. 1]

and hence FB , BG are in one straight line. [I. 14]

The result is also obvious from the converse of I. 15 given by Proclus (see note on I. 15).

The proposition VI. 14 contains a theorem and one partial converse of it; so also does VI. 15. To each proposition may be added the other partial converse, which may be enunciated as follows, the words in square brackets applying to the case of triangles (VI. 15).



Equal parallelograms [triangles] which have the sides about one angle in each reciprocally proportional are equiangular [have the angles included by those sides either equal or supplementary.]

Let AB , BC be equal parallelograms, or let FBD , EBG be equal

triangles, such that the sides about the angles at B are reciprocally proportional, i.e. such that

$$DB : BE = GB : BF.$$

We shall prove that the angles FBD , EBG are either equal or supplementary.

Place the figures so that DB , BE are in one straight line.

Then FB , BG are either in a straight line, or not in a straight line.

(1) If FB , BG are in a straight line, the figure of the proposition (with the diagonals FD , EG drawn) represents the facts, and

the angle FBD is equal to the angle EBG . [1. 15]

(2) If FB , BG are not in a straight line,
produce FB to H so that BH may be equal to BG .

Join EH , and complete the parallelogram $EBHK$.

Now, since $DB : BE = GB : BF$

and $GB = HB$,

$$DB : BE = HB : BF,$$

and therefore, by VI. 14 or 15,

the parallelograms AB , BK are equal, or the triangles FBD , EBH are equal.

But the parallelograms AB , BC are equal, and the triangles FBD , EBG are equal;

therefore the parallelograms BC , BK are equal, and the triangles EBH , EBG are equal.

Therefore these parallelograms or triangles are within the same parallels: that is, G , C , H , K are in a straight line which is parallel to DE . [1. 39]

Now, since BG , BH are equal,

the angles BGH , BHG are equal.

By parallels, it follows that

the angle EBG is equal to the angle DBH ,

whence the angle EBG is supplementary to the angle FBD .

PROPOSITION 15.

In equal triangles which have one angle equal to one angle the sides about the equal angles are reciprocally proportional; and those triangles which have one angle equal to one angle, and in which the sides about the equal angles are reciprocally proportional, are equal.

Let ABC , ADE be equal triangles having one angle equal to one angle, namely the angle BAC to the angle DAE ;

I say that in the triangles ABC , ADE the sides about the equal angles are reciprocally proportional, that is to say, that,
as CA is to AD , so is EA to AB .

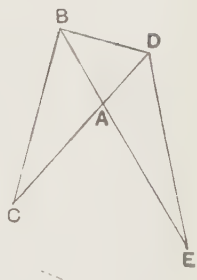
For let them be placed so that CA is in a straight line with AD ;

therefore EA is also in a straight line with AB . [I. 14]

Let BD be joined.

Since then the triangle ABC is equal to the triangle ADE , and BAD is another area,

therefore, as the triangle CAB is to the triangle BAD , so is the triangle EAD to the triangle BAD . [v. 7]



But, as CAB is to BAD , so is CA to AD , [vi. 1]
and, as EAD is to BAD , so is EA to AB . [id.]

Therefore also, as CA is to AD , so is EA to AB . [v. 11]

Therefore in the triangles ABC , ADE the sides about the equal angles are reciprocally proportional.

Next, let the sides of the triangles ABC , ADE be reciprocally proportional, that is to say, let EA be to AB as CA to AD ;

I say that the triangle ABC is equal to the triangle ADE .

For, if BD be again joined,

since, as CA is to AD , so is EA to AB ,

while, as CA is to AD , so is the triangle ABC to the triangle BAD ,

and, as EA is to AB , so is the triangle EAD to the triangle BAD , [vi. 1]

therefore, as the triangle ABC is to the triangle BAD , so is the triangle EAD to the triangle BAD . [v. 11]

Therefore each of the triangles ABC , EAD has the same ratio to BAD .

Therefore the triangle ABC is equal to the triangle EAD . [v. 9]

Therefore etc.

Q. E. D.

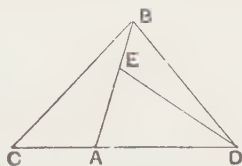
As indicated in the partial converse given in the last note, this proposition is equally true if the angle included by the two sides in one triangle is supplementary, instead of being equal, to the angle included by the two sides in the other.

Let ABC , ADE be two triangles such that the angles BAC , DAE are supplementary, and also

$$CA : AD = EA : AB.$$

In this case we can place the triangles so that CA is in a straight line with AD , and AB lies along AE (since the angle EAC , being supplementary to the angle EAD , is equal to the angle BAC).

If we join BD , the proof given by Euclid applies to this case also.



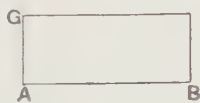
It is true that VI. 15 can be immediately inferred from VI. 14, since a triangle is half of a parallelogram with the same base and height. But, Euclid's object being to give the student a grasp of *methods* rather than results, there seems to be no advantage in deducing one proposition from the other instead of using the same method on each.

PROPOSITION 16.

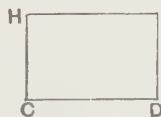
If four straight lines be proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means; and, if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines will be proportional.

Let the four straight lines AB , CD , E , F be proportional, so that, as AB is to CD , so is E to F ;

I say that the rectangle contained by AB , F is equal to the rectangle contained by CD , E .



E ———



F ———

Let AG , CH be drawn from the points A , C at right angles to the straight lines AB , CD , and let AG be made equal to F , and CH equal to E .

Let the parallelograms BG , DH be completed.

Then since, as AB is to CD , so is E to F , while E is equal to CH , and F to AG , therefore, as AB is to CD , so is CH to AG .

Therefore in the parallelograms BG , DH the sides about the equal angles are reciprocally proportional.

But those equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal ;
[VI. 14]

therefore the parallelogram BG is equal to the parallelogram DH .

And BG is the rectangle AB, F , for AG is equal to F ;
and DH is the rectangle CD, E , for E is equal to CH ;
therefore the rectangle contained by AB, F is equal to the rectangle contained by CD, E .

Next, let the rectangle contained by AB, F be equal to the rectangle contained by CD, E ;

I say that the four straight lines will be proportional, so that, as AB is to CD , so is E to F .

For, with the same construction,
since the rectangle AB, F is equal to the rectangle CD, E ,
and the rectangle AB, F is BG , for AG is equal to F ,
and the rectangle CD, E is DH , for CH is equal to E ,
therefore BG is equal to DH .

And they are equiangular

But in equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional. [VI. 14]

Therefore, as AB is to CD , so is CH to AG .

But CH is equal to E , and AG to F ;
therefore, as AB is to CD , so is E to F .

Therefore etc.

Q. E. D.

This proposition is a particular case of VI. 14, but one which is on all accounts worth separate statement. It may also be enunciated in the following form :

Rectangles which have their bases reciprocally proportional to their heights are equal in area ; and equal rectangles have their bases reciprocally proportional to their heights.

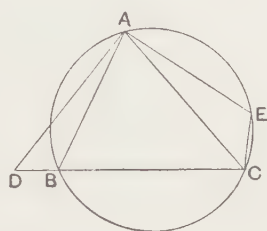
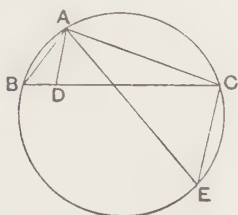
Since any parallelogram is equal to a rectangle of the same height and on the same base, and any triangle with the same height and on the same base is equal to half the parallelogram or rectangle, it follows that *Equal parallelograms or triangles have their bases reciprocally proportional to their heights and vice versa.*

The present place is suitable for giving certain important propositions, including those which Simson adds to Book VI. as Props. B, C and D, which are proved directly by means of VI. 16.

1. Proposition B is a particular case of the following theorem.

If a circle be circumscribed about a triangle ABC and there be drawn through A any two straight lines either both within or both without the angle BAC, viz.

AD meeting BC (produced if necessary) in D and AE meeting the circle again in E, such that the angles DAB, EAC are equal, then the rectangle AD, AE is equal to the rectangle BA, AC.



Join *CE*.

The angles *BAD*, *EAC* are equal, by hypothesis ;
and the angles *ABD*, *AEC* are equal.

[III. 21, 22]

Therefore the triangles *ABD*, *AEC* are equiangular.

Hence *BA* is to *AD* as *EA* is to *AC*,

and therefore the rectangle *BA*, *AC* is equal to the rectangle *AD*, *AE*.

[VI. 16]

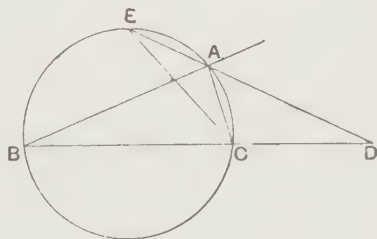
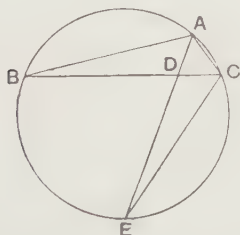
There are now two particular cases to be considered.

(a) Suppose that *AD*, *AE* coincide ;

ADE will then bisect the angle *BAC*.

(b) Suppose that *AD*, *AE* are in one straight line but that *D*, *E* are on opposite sides of *A* ;

AD will then bisect the external angle at *A*.



In the first case (a) we have

the rectangle *BA*, *AC* equal to the rectangle *EA*, *AD* ;

and the rectangle *EA*, *AD* is equal to the rectangle *ED*, *DA* together with the square on *AD*, [II. 3]

i.e. to the rectangle *BD*, *DC* together with the square on *AD*. [III. 35]

Therefore the rectangle *BA*, *AC* is equal to the rectangle *BD*, *DC* together with the square on *AD*. [This is Simson's Prop. B]

In case (b) the rectangle *EA*, *AD* is equal to the excess of the rectangle *ED*, *DA* over the square on *AD* ;

therefore the rectangle *BA*, *AC* is equal to the excess of the rectangle *BD*, *DC* over the square on *AD*.

The following converse of Simson's Prop. B may be given: *If a straight line AD be drawn from the vertex A of a triangle to meet the base, so that the square on AD together with the rectangle BD, DC is equal to the rectangle BA, AC, the line AD will bisect the angle BAC except when the sides AB, AC are equal, in which case every line drawn to the base will have the property mentioned.*

Let the circumscribed circle be drawn, and let AD produced meet it in E; join CE.

The rectangle BD, DC is equal to the rectangle ED, DA. [III. 35]

Add to each the square on AD;

therefore the rectangle BA, AC is equal to the rectangle EA, AD.

[hyp. and II. 3]

Hence AB is to AD as AE to AC.

[VI. 16]

But the angle ABD is equal to the angle AEC.

[III. 21]

Therefore the angles BDA, ECA are either equal or supplementary.

[VI. 7 and note]

(a) If they are equal, the angles BAD, EAC are also equal, and AD bisects the angle BAC.

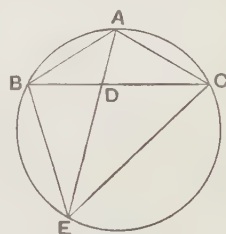
(b) If they are supplementary, the angle ADC must be equal to the angle ACE.

Therefore the angles BAD, ABD are together equal to the angles ACB, BCE, i.e. to the angles ACD, BAD.

Take away the common angle BAD, and

the angles ABD, ACD are equal, or

AB is equal to AC.

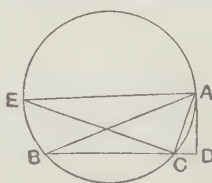
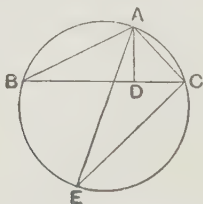


Euclid himself assumes, in Prop. 67 of the *Data*, the result of so much of this proposition as relates to the case where $BA = AC$. He assumes namely, without proof, that, if $BA = AC$, and if D be any point on BC, the rectangle BD, DC together with the square on AD is equal to the square on AB.

PROPOSITION C.

If from any angle of a triangle a straight line be drawn perpendicular to the opposite side, the rectangle contained by the other two sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle circumscribed about the triangle.

Let ABC be a triangle and AD the perpendicular on AB. Draw the diameter AE of the circle circumscribed about the triangle ABC.



Then shall the rectangle BA, AC be equal to the rectangle EA, AD.
Join EC.

Since the right angle BDA is equal to the right angle ECA in a semi-circle, [III. 31]
 and the angles ABD , AEC in the same segment are equal, [III. 21]
 the triangles ABD , AEC are equiangular.

Therefore, as BA is to AD , so is EA to AC , [VI. 4]
 whence the rectangle BA , AC is equal to the rectangle EA , AD . [VI. 16]

This result corresponds to the trigonometrical formula for R , the radius of the circumscribed circle,

$$R = \frac{abc}{4\Delta}.$$

PROPOSITION D.

This is the highly important lemma given by Ptolemy (ed. Heiberg, Vol. 1, pp. 36—7) which is the basis of his calculation of the table of chords in the section of Book 1. of the *μεγάλη σύνταξις* entitled “concerning the size of the straight lines [i.e. chords] in the circle” (*περὶ τῆς πηλικότητος τῶν ἐν τῷ κύκλῳ εὐθειῶν*).

The theorem may be enunciated thus.

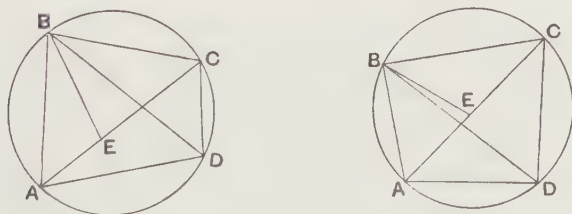
The rectangle contained by the diagonals of any quadrilateral inscribed in a circle is equal to the sum of the rectangles contained by the pairs of opposite sides.

I shall give the proof in Ptolemy's words, with the addition only, in brackets, of two words applying to a second figure not given by Ptolemy.

“Let there be a circle with any quadrilateral $ABCD$ inscribed in it, and let AC , BD be joined.

It is to be proved that the rectangle contained by AC and BD is equal to the sum of the rectangles AB , DC and AD , BC .

For let the angle ABE be made equal to the angle contained by DB , BC .



If then we add [or subtract] the angle EBD ,
 the angle ABD will also be equal to the angle EBC .

But the angle BDA is also equal to the angle BCE , [III. 21]
 for they subtend the same segment;
 therefore the triangle ABD is equiangular with the triangle EBC .

Hence, proportionally,

as BC is to CE , so is BD to DA . [VI. 4]

Therefore the rectangle BC , AD is equal to the rectangle BD , CE . [VI. 16]

Again, since the angle ABE is equal to the angle DBC ,
 and the angle BAE is also equal to the angle BDC , [III. 21]
 the triangle ABE is equiangular with the triangle DBC .

Therefore, proportionally,

as BA is to AE , so is BD to DC ; [vi. 4]

therefore the rectangle BA, DC is equal to the rectangle BD, AE . [vi. 16]

But it was also proved that

the rectangle BC, AD is equal to the rectangle BD, CE ;

therefore the rectangle AC, BD as a whole is equal to the sum of the rectangles AB, DC and AD, BC :

(being) what it was required to prove."

Another proof of this proposition, and of its converse, is indicated by Dr Lachlan (*Elements of Euclid*, pp. 273—4). It depends on two preliminary propositions.

(1) *If two circles be divided, by a chord in each, into segments which are similar respectively, the chords are proportional to the corresponding diameters.*

The proof is instantaneous if we join the ends of each chord to the centre of the circle which it divides, when we obtain two similar triangles.

(2) *If D be any point on the circle circumscribed about a triangle ABC , and DX, DY, DZ be perpendicular to the sides BC, CA, AB of the triangle respectively, then X, Y, Z lie in one straight line; and, conversely, if the feet of the perpendiculars from any point D on the sides of a triangle lie in one straight line, D lies on the circle circumscribed about the triangle.*

The proof depending on III. 21, 22 is well known.

Now suppose that D is any point in the plane of a triangle ABC , and that DX, DY, DZ are perpendicular to the sides BC, CA, AB respectively.

Join YZ, DA .

Then, since the angles at Y, Z are right, A, Y, D, Z lie on a circle of which DA is the diameter.

And YZ divides this circle into segments which are similar respectively to the segments into which BC divides the circle circumscribing ABC , since the angles ZAY, BAC coincide, and their supplements are equal.

Therefore, if d be the diameter of the circle circumscribing ABC ,

BC is to d as YZ is to DA ;

and therefore the rectangle AD, BC is equal to the rectangle d, YZ .

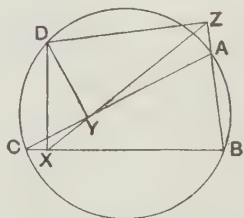
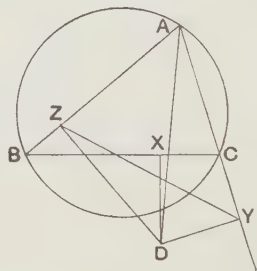
Similarly the rectangle BD, CA is equal to the rectangle d, ZX , and the rectangle CD, AB is equal to the rectangle d, XY .

Hence, in a quadrilateral in general, the rectangle contained by the diagonals is less than the sum of the rectangles contained by the pairs of opposite sides.

Next, suppose that D lies on the circle circumscribed about ABC , but so that A, B, C, D follow each other on the circle in this order, as in the figure annexed.

Let DX, DY, DZ be perpendicular to BC, CA, AB respectively, so that X, Y, Z are in a straight line.

Then, since the rectangles AD, BC ; BD, CA ; CD, AB are equal to the rectangles d, YZ ; d, ZX ; d, XY respectively, and XZ is equal to the sum of



XY, YZ , so that the rectangle d, XZ is equal to the sum of the rectangles d, XY and d, YZ , it follows that

the rectangle AC, BD is equal to the sum of the rectangles AD, BC and AB, CD .

Conversely, if the latter statement is true, while we are supposed to know nothing about the position of D , it follows that

XZ must be equal to the sum of XY, YZ ,

so that X, Y, Z must be in a straight line.

Hence, from the theorem (2) above, it follows that D must lie on the circle circumscribed about ABC , i.e. that $ABCD$ is a quadrilateral about which a circle can be described.

All the above propositions can be proved on the basis of Book III. and without using Book VI., since it is possible by the aid of III. 21 and 35 alone to prove that *in equiangular triangles the rectangles contained by the non-corresponding sides about equal angles are equal to one another* (a result arrived at by combining VI. 4 and VI. 16). This is the method adopted by Casey, H. M. Taylor, and Lachlan; but I fail to see any particular advantage in it.

Lastly, the following proposition may be given which Playfair added as VI. E. It appears in the *Data* of Euclid, Prop. 93, and may be thus enunciated.

If the angle BAC of a triangle ABC be bisected by the straight line AD meeting the circle circumscribed about the triangle in D, and if BD be joined, then

the sum of BA, AC is to AD as BC is to BD.

Join CD . Then, since AD bisects the angle BAC , the subtended arcs BD, DC , and therefore the chords BD, DC , are equal.

(1) The result can now be easily deduced from Ptolemy's theorem.

For the rectangle AD, BC is equal to the sum of the rectangles AB, DC and AC, BD , i.e. (since BD, CD are equal) to the rectangle contained by $BA + AC$ and BD .

Therefore the sum of BA, AC is to AD as BC is to BD . [VI. 16]

(2) Euclid proves it differently in *Data*, Prop. 93. Let AD meet BC in E .

Then, since AE bisects the angle BAC ,

BA is to AC as BE to EC , [VI. 3]

or, alternately,

AB is to BE as AC to CE . [V. 16]

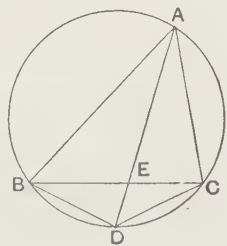
Therefore also

$BA + AC$ is to BC as AC to CE . [V. 12]

Again, since the angles BAD, EAC are equal, and the angles ADB, ACE are also equal, [III. 21]

the triangles ABD, AEC are equiangular.

Therefore AC is to CE as AD to BD . [VI. 4]



Hence $BA + AC$ is to BC as AD to BD , [v. 11]
and, alternately,

$BA + AC$ is to AD as BC is to BD . [v. 16]

Euclid concludes that, if the circle ABC is given in magnitude, and the chord BC cuts off a segment of it containing a given angle (so that, by *Data* Prop. 87, BC and also BD are given in magnitude),

the ratio of $BA + AC$ to AD is given,

and further that (since, by similar triangles, BD is to DE as AC is to CE , while $BA + AC$ is to BC as AC is to CE),

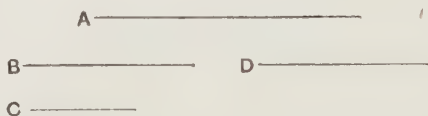
the rectangle $(BA + AC)$, DE , being equal to the rectangle BC , BD , is also given.

PROPOSITION 17

If three straight lines be proportional, the rectangle contained by the extremes is equal to the square on the mean; and, if the rectangle contained by the extremes be equal to the square on the mean, the three straight lines will be proportional.

Let the three straight lines A , B , C be proportional, so that, as A is to B , so is B to C ;

I say that the rectangle contained by A , C is equal to the square on B .



Let D be made equal to B .

Then, since, as A is to B , so is B to C ,
and B is equal to D ,
therefore, as A is to B , so is D to C .

But, if four straight lines be proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means. [vi. 16]

Therefore the rectangle A , C is equal to the rectangle B , D .

But the rectangle B , D is the square on B , for B is equal to D ;

therefore the rectangle contained by A , C is equal to the square on B .

Next, let the rectangle A , C be equal to the square on B ;
I say that, as A is to B , so is B to C .

For, with the same construction,
 since the rectangle A, C is equal to the square on B ,
 while the square on B is the rectangle B, D , for B is equal
 to D ,
 therefore the rectangle A, C is equal to the rectangle B, D .

But, if the rectangle contained by the extremes be equal
 to that contained by the means, the four straight lines are
 proportional. [vi. 16]

Therefore, as A is to B , so is D to C .

But B is equal to D ;

therefore, as A is to B , so is B to C .

Therefore etc.

Q. E. D.

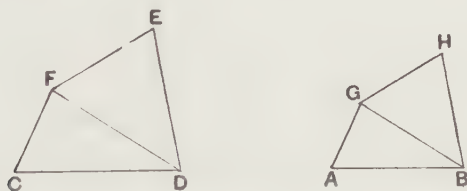
vi. 17 is, of course, a particular case of vi. 16.

PROPOSITION 18.

*On a given straight line to describe a rectilinear figure
 similar and similarly situated to a given rectilinear figure.*

Let AB be the given straight line and CE the given
 rectilinear figure;

thus it is required to describe on the straight line AB a
 rectilinear figure similar and similarly situated to the recti-
 lineal figure CE .



Let DF be joined, and on the straight line AB , and at
 the points A, B on it, let the angle GAB be constructed
 equal to the angle at C , and the angle ABG equal to the
 angle CDF . [1. 23]

Therefore the remaining angle CFD is equal to the angle
 AGB ; [1. 32]

therefore the triangle FCD is equiangular with the triangle
 GAB .

Therefore, proportionally, as FD is to GB , so is FC to
 GA , and CD to AB .

Again, on the straight line BG , and at the points B , G on it, let the angle BGH be constructed equal to the angle DFE , and the angle GBH equal to the angle FDE . [I. 23]

Therefore the remaining angle at E is equal to the remaining angle at H ; [I. 32]

therefore the triangle FDE is equiangular with the triangle GBH ;

therefore, proportionally, as FD is to GB , so is FE to GH , and ED to HB . [VI. 4]

But it was also proved that, as FD is to GB , so is FC to GA , and CD to AB ;

therefore also, as FC is to AG , so is CD to AB , and FE to GH , and further ED to HB .

And, since the angle CFD is equal to the angle AGB , and the angle DFE to the angle BGH , therefore the whole angle CFE is equal to the whole angle AGH .

For the same reason

the angle CDE is also equal to the angle ABH .

And the angle at C is also equal to the angle at A ,

and the angle at E to the angle at H .

Therefore AH is equiangular with CE ;

and they have the sides about their equal angles proportional;

therefore the rectilineal figure AH is similar to the rectilineal figure CE . [VI. Def. 1]

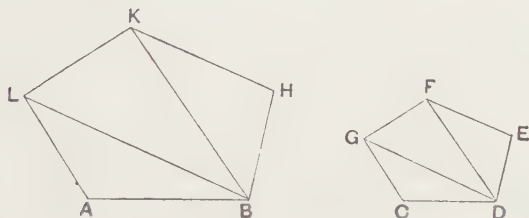
Therefore on the given straight line AB the rectilineal figure AH has been described similar and similarly situated to the given rectilineal figure CE .

Q. E. F.

Simson thinks the proof of this proposition has been vitiated, his grounds for this view being (1) that it is demonstrated only with reference to quadrilaterals, and does not show how it may be extended to figures of five or more sides, (2) that Euclid infers, from the fact of two triangles being equiangular, that a side of the one is to the corresponding side of the other as another side of the first is to the side corresponding to it in the other, i.e. he permutes, without mentioning the fact that he does so, the proportions obtained in VI. 4, whereas the proof of the very next proposition gives, in a similar case, the intermediate step of permutation. I think this is hypercriticism. As regards (2) it should be noted that the permuted form of the proportion is arrived at first in the *proof* of VI. 4; and the omission of the

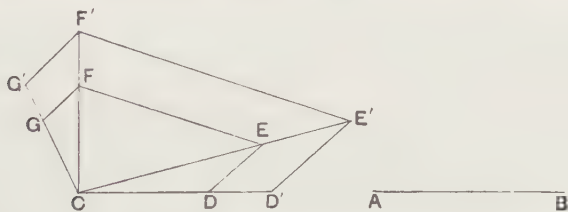
intermediate step of *alternando*, whether accidental or not, is of no importance. On the other hand, the use of this form of the proportion certainly simplifies the proof of the proposition, since it makes unnecessary the subsequent *ex aequali* steps of Simson's proof, their place being taken by the inference [v. 11] that ratios which are the same with a third ratio are the same with one another.

Nor is the first objection of any importance. We have only to take as the



given polygon a polygon of five sides at least, as $CDEFG$, join one extremity of CD , say D , to each of the angular points other than C and E , and then use the same mode of construction as Euclid's for any number of successive triangles as ABL , LBK , etc., that may have to be made. Euclid's construction and proof for a quadrilateral are quite sufficient to show how to deal with the case of a figure of five or any greater number of sides.

Clavius has a construction which, given the power of moving a figure



bodily from one position to any other, is easier. $CDEFG$ being the given polygon, join CE , CF . Place AB on CD so that A falls on C , and let B fall on D' , which may either lie on CD or on CD produced.

Now draw $D'E'$ parallel to DE , meeting CE , produced if necessary, in E' , $E'F'$ parallel to EF , meeting CF , produced if necessary, in F' , and so on.

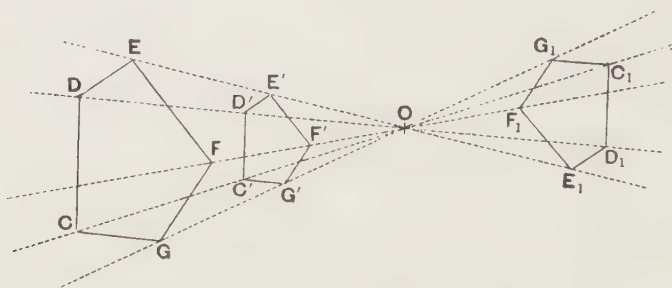
Let the parallel to the last side but one, FG , meet CG , produced if necessary, in G' .

Then $CDE'F'G'$ is similar and similarly situated to $CDEFG$, and it is constructed on CD' , a straight line equal to AB .

The proof of this is obvious.

A more general construction is indicated in the subjoined figure. If $CDEFG$ be the given polygon, suppose its angular points all joined to any point O and the connecting straight lines produced both ways. Then, if CD' , a straight line equal to AB , be placed so that it is parallel to CD , and C' , D' lie respectively on OC , OD (this can of course be done by finding fourth proportionals), we have only to draw $D'E'$, $E'F'$, etc., parallel to the corresponding sides of the original polygon in the manner shown.

De Morgan would rearrange Props. 18 and 20 in the following manner. He would combine Prop. 18 and the first part of Prop. 20 into one, with the enunciation :



Pairs of similar triangles, similarly put together, give similar figures ; and every pair of similar figures is composed of pairs of similar triangles similarly put together.

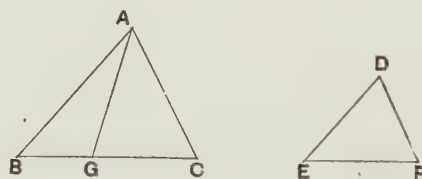
He would then make the *problem* of vi. 18 an application of the first part. In form this would certainly appear to be an improvement ; but, provided that the relation of the propositions is understood, the matter of form is perhaps not of great importance.

PROPOSITION 19.

Similar triangles are to one another in the duplicate ratio of the corresponding sides.

Let ABC , DEF be similar triangles having the angle at B equal to the angle at E , and such that, as AB is to BC , so is DE to EF , so that BC corresponds to EF ; [v. Def. 11]

I say that the triangle ABC has to the triangle DEF a ratio duplicate of that which BC has to EF .



For let a third proportional BG be taken to BC , EF , so that, as BC is to EF , so is EF to BG ; [vi. 11]
 10 and let AG be joined.

Since then, as AB is to BC , so is DE to EF , therefore, alternately, as AB is to DE , so is BC to EF . [v. 16]

But, as BC is to EF , so is EF to BG ;

therefore also, as AB is to DE , so is EF to BG . [v. 11]

15 Therefore in the triangles ABG , DEF the sides about the equal angles are reciprocally proportional.

But those triangles which have one angle equal to one angle, and in which the sides about the equal angles are reciprocally proportional, are equal; [vi. 15]

20 therefore the triangle ABG is equal to the triangle DEF .

Now since, as BC is to EF , so is EF to BG ,

and, if three straight lines be proportional, the first has to the third a ratio duplicate of that which it has to the second, [v. Def. 9]

therefore BC has to BG a ratio duplicate of that which CB 25 has to EF .

But, as CB is to BG , so is the triangle ABC to the triangle ABG ; [vi. 1]

therefore the triangle ABC also has to the triangle ABG a ratio duplicate of that which BC has to EF .

30 But the triangle ABG is equal to the triangle DEF ;

therefore the triangle ABC also has to the triangle DEF a ratio duplicate of that which BC has to EF .

Therefore etc.

PORISM. From this it is manifest that, if three straight 35 lines be proportional, then, as the first is to the third, so is the figure described on the first to that which is similar and similarly described on the second.

Q. E. D.

4. and such that, as AB is to BC , so is DE to EF , literally "(triangles) having the angle at B equal to the angle at E , and (having), as AB to BC , so DE to EF ."

Having combined Prop. 18 and the first part of Prop. 20 as just indicated, De Morgan would tack on to Prop. 19 the second part of Prop. 20, which asserts that, if similar polygons be divided into the same number of similar triangles, the triangles are "*homologous* to the wholes" (in the sense that the polygons have the same ratio as the corresponding triangles have), and that the polygons are to one another in the duplicate ratio of corresponding sides. This again, though no doubt an improvement of form, would necessitate the drawing over again of the figure of the altered Proposition 18 and a certain amount of repetition.

Agreeably to his suggestion that Prop. 23 should come before Prop. 14 which is a particular case of it, De Morgan would prove Prop. 19 for *parallelograms* by means of Prop. 23, and thence infer the truth of it for

triangles or the halves of the parallelograms. He adds: "The method of Euclid is an elegant application of the *operation* requisite to compound equal ratios, by which the conception of the process is lost sight of." For the general reason given in the note on VI. 14 above, I think that Euclid showed the sounder discretion in the arrangement which he adopted. Moreover it is not easy to see how performing the actual operation of compounding two equal ratios can obscure the process, or the fact that two equal ratios are being compounded. On the definition of compounded ratios and duplicate ratio, De Morgan has himself acutely pointed out that "composition" is here used for the process of detecting the single alteration which produces the effect of two or more, the duplicate ratio being the result of compounding two equal ratios. The proof of VI. 19 does in fact exhibit the single alteration which produces the effect of two. And the *operation* was of the essence of the Greek geometry, because it was the manipulation of ratios in this manner, by simplification and transformation, that gave it so much power, as every one knows who has read, say, Archimedes or Apollonius. Hence the introduction of the necessary *operation*, as well as the theoretical proof, in this proposition seems to me to have been distinctly worth while, and, as it is somewhat simpler in this case than in the more general case of VI. 23, it was in accordance with the plan of enabling the difficulties of Book VI. to be more easily and gradually surmounted to give the simpler case first.

That Euclid wished to emphasise the importance of the *method* adopted, as well as of the result obtained, in VI. 19 seems to me clearly indicated by the Porism which follows the proposition. It is as if he should say: "I have shown you that similar triangles are to one another in the duplicate ratio of corresponding sides; but I have also shown you incidentally how it is possible to work conveniently with duplicate ratios, viz. by transforming them into simple ratios between straight lines. I shall have occasion to illustrate the use of this method in the proof of VI. 22."

The Porism to VI. 19 presents one difficulty. It will be observed that it speaks of the *figure* (εἶδος) described on the first straight line and of that which is similar and similarly described on the second. If "figure" could be regarded as loosely used for the figure of the *proposition*, i.e. for a *triangle*, there would be no difficulty. If on the other hand "the figure" means any rectilinear figure, i.e. any polygon, the Porism is not really established until the next proposition, VI. 20, has been proved, and therefore it is out of place here. Yet the correction *τρίγωνον*, *triangle*, for *εἶδος*, *figure*, is due to Theon alone; P and Campanus have "figure," and the reading of Philoponus and Psellus, *τετράγωνον*, *square*, partly supports *εἶδος*, since it can be reconciled with *εἶδος* but not with *τρίγωνον*. Again the second Porism to VI. 20, in which this Porism is reasserted for any rectilinear figure, and which is omitted by Campanus and only given by P in the margin, was probably interpolated by Theon. Heiberg concludes that Euclid wrote "figure" (εἶδος), and Theon, seeing the difficulty, changed the word into "triangle" here and added Por. 2 to VI. 20 in order to make the matter clear. If one may hazard a guess as to how Euclid made the slip, may it be that he first put it after VI. 20 and then, observing that the expression of the duplicate ratio by a single ratio between two straight lines does not come in VI. 20 but in VI. 19, moved the Porism to the end of VI. 19 in order to make the connexion clearer, without noticing that, if this were done, *εἶδος* would need correction?

The following explanation at the end of the Porism is bracketed by Heiberg, viz. "Since it was proved that, as *CB* is to *BG*, so is the triangle

ABC to the triangle ABG , that is DEF ." Such explanations in Porisms are not in Euclid's manner, and the words are not in Campanus, though they date from a time earlier than Theon.

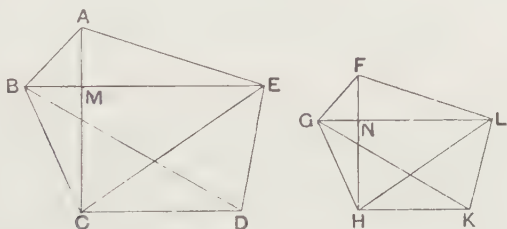
PROPOSITION 20.

Similar polygons are divided into similar triangles, and into triangles equal in multitude and in the same ratio as the wholes, and the polygon has to the polygon a ratio duplicate of that which the corresponding side has to the corresponding
5 *side.*

Let $ABCDE$, $FGHKL$ be similar polygons, and let AB correspond to FG ;

I say that the polygons $ABCDE$, $FGHKL$ are divided into similar triangles, and into triangles equal in multitude and in the same ratio as the wholes, and the polygon $ABCDE$ has
10 to the polygon $FGHKL$ a ratio duplicate of that which AB has to FG .

Let BE , EC , GL , LH be joined.



Now, since the polygon $ABCDE$ is similar to the polygon
15 $FGHKL$,

the angle BAE is equal to the angle GFL ;

and, as BA is to AE , so is GF to FL . [vi. Def. 1]

Since then ABE , FGL are two triangles having one angle equal to one angle and the sides about the equal angles
20 proportional,

therefore the triangle ABE is equiangular with the triangle FGL ;
[vi. 6]

so that it is also similar; [vi. 4 and Def. 1]

therefore the angle ABE is equal to the angle FGL .

25 But the whole angle ABC is also equal to the whole angle FGH because of the similarity of the polygons ;
therefore the remaining angle EBC is equal to the angle LGH .

And, since, because of the similarity of the triangles ABE ,
30 FGL ,

as EB is to BA , so is LG to GF ,

and moreover also, because of the similarity of the polygons,

as AB is to BC , so is FG to GH ,

therefore, *ex aequali*, as EB is to BC , so is LG to GH ; [v. 22]

35 that is, the sides about the equal angles EBC , LGH are proportional ;

therefore the triangle EBC is equiangular with the triangle LGH , [vi. 6]

so that the triangle EBC is also similar to the triangle
40 LGH . [vi. 4 and Def. 1]

For the same reason

the triangle ECD is also similar to the triangle LHK .

Therefore the similar polygons $ABCDE$, $FGHKL$ have
45 been divided into similar triangles, and into triangles equal in multitude.

I say that they are also in the same ratio as the wholes,
that is, in such manner that the triangles are proportional,
and ABE , EBC , ECD are antecedents, while FGL , LGH ,
 LHK are their consequents, and that the polygon $ABCDE$
50 has to the polygon $FGHKL$ a ratio duplicate of that which
the corresponding side has to the corresponding side, that is
 AB to FG .

For let AC , FH be joined.

Then since, because of the similarity of the polygons,

55 the angle ABC is equal to the angle FGH ,

and, as AB is to BC , so is FG to GH ,

the triangle ABC is equiangular with the triangle FGH ; [vi. 6]

therefore the angle BAC is equal to the angle GFH ,

and the angle BCA to the angle GHF .

60 And, since the angle BAM is equal to the angle GFN ,
and the angle ABM is also equal to the angle FGN ,

therefore the remaining angle AMB is also equal to the remaining angle FNG ; [I. 32]

therefore the triangle ABM is equiangular with the triangle
65 FGN .

Similarly we can prove that

the triangle BMC is also equiangular with the triangle GNH .

Therefore, proportionally, as AM is to MB , so is FN to
 NG ,

70 and, as BM is to MC , so is GN to NH ;

so that, in addition, *ex aequali*,

as AM is to MC , so is FN to NH .

But, as AM is to MC , so is the triangle ABM to MBC ,
and AME to EMC ; for they are to one another as their
75 bases. [VI. 1]

Therefore also, as one of the antecedents is to one of the
consequents, so are all the antecedents to all the consequents;

[V. 12]

therefore, as the triangle AMB is to BMC , so is ABE to
 CBE .

80 But, as AMB is to BMC , so is AM to MC ;

therefore also, as AM is to MC , so is the triangle ABE to
the triangle EBC .

For the same reason also,

as FN is to NH , so is the triangle FGL to the triangle
85 GLH .

And, as AM is to MC , so is FN to NH ;

therefore also, as the triangle ABE is to the triangle BEC ,
so is the triangle FGL to the triangle GLH ;

and, alternately, as the triangle ABE is to the triangle FGL ,
90 so is the triangle BEC to the triangle GLH .

Similarly we can prove, if BD , GK be joined, that, as the
triangle BEC is to the triangle LGH , so also is the triangle
 ECD to the triangle LHK .

And since, as the triangle ABE is to the triangle FGL ,
95 so is EBC to LGH , and further ECD to LHK ,

therefore also, as one of the antecedents is to one of the
consequents, so are all the antecedents to all the consequents;

[V. 12]

therefore, as the triangle ABE is to the triangle FGL ,

so is the polygon $ABCDE$ to the polygon $FGHKL$.

100 But the triangle ABE has to the triangle FGL a ratio duplicate of that which the corresponding side AB has to the corresponding side FG ; for similar triangles are in the duplicate ratio of the corresponding sides. [VI. 19]

Therefore the polygon $ABCDE$ also has to the polygon
105 $FGHKL$ a ratio duplicate of that which the corresponding side AB has to the corresponding side FG .

Therefore etc.

PORISM. Similarly also it can be proved in the case of
110 quadrilaterals that they are in the duplicate ratio of the corresponding sides. And it was also proved in the case of triangles; therefore also, generally, similar rectilineal figures are to one another in the duplicate ratio of the corresponding sides.

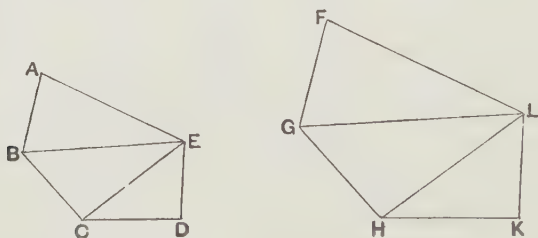
Q. E. D.

2. in the same ratio as the wholes. The same word $\acute{\alpha}\mu\acute{\omicron}\lambda\omicron\gamma\omicron\varsigma$ is used which I have generally translated by "corresponding." But here it is followed by a dative, $\acute{\alpha}\mu\acute{\omicron}\lambda\omicron\gamma\omicron\alpha\ \tau\omicron\iota\varsigma\ \acute{\alpha}\lambda\omicron\upsilon\varsigma$ "*homologous with the wholes*," instead of being used absolutely. The meaning can therefore here be nothing else but "in the same ratio with" or "proportional to the wholes"; and Euclid seems to recognise that he is making a special use of the word, because he explains it lower down (I. 46): "the triangles are homologous to the wholes, that is, in such manner that the triangles are proportional, and ABE , EBC , ECD are antecedents, while FGL , LGH , LHK are their consequents."

49. $\acute{\epsilon}\pi\acute{\omicron}\mu\epsilon\nu\alpha\ \alpha\upsilon\tau\acute{\omega}\nu$, "*their consequents*," is a little awkward, but may be supposed to indicate which triangles correspond to which as consequent to antecedent.

An alternative proof of the second part of this proposition given after the Porisms is relegated by August and Heiberg to an Appendix as an interpolation. It is shorter than the proof in the text, and is the only one given by many editors, including Clavius, Billingsley, Barrow and Simson. It runs as follows:

"We will now also prove that the triangles are homologous in another and an easier manner.



Again, let the polygons $ABCDE$, $FGHKL$ be set out, and let BE , EC , GL , LH be joined.

I say that, as the triangle ABE is to FGL , so is EBC to LGH and CDE to HKL .

For, since the triangle ABE is similar to the triangle FGL , the triangle ABE has to the triangle FGL a ratio duplicate of that which BE has to GL .

For the same reason also

the triangle BEC has to the triangle GLH a ratio duplicate of that which BE has to GL .

Therefore, as the triangle ABE is to the triangle FGL , so is BEC to GLH .

Again, since the triangle EBC is similar to the triangle LGH , EBC has to LGH a ratio duplicate of that which the straight line CE has to HL .

For the same reason also

the triangle ECD has to the triangle LHK a ratio duplicate of that which CE has to HL .

Therefore, as the triangle EBC is to LGH , so is ECD to LHK .

But it was proved that,

as EBC is to LGH , so also is ABE to FGL .

Therefore also, as ABE is to FGL , so is BEC to GLH and ECD to LHK .

Q. E. D."

Now Euclid cannot fail to have noticed that the second part of his proposition could be proved in this way. It seems therefore that, in giving the other and longer method, he deliberately wished to avoid using the result of VI. 19, preferring to prove the first two parts of the theorem, as they can be proved, independently of any relation between the areas of similar triangles.

The first part of the Porism, stating that the theorem is true of *quadrilaterals*, would be superfluous but for the fact that technically, according to Book I. Def. 19, the term "polygon" (or figure of many sides, *πολύπλευρον*) used in the enunciation of the proposition is confined to rectilinear figures of *more than four* sides, so that a quadrilateral might seem to be excluded. The mention of the triangle in addition fills up the tale of "similar rectilinear figures."

The second Porism, Theon's interpolation, given in the text by the editors, but bracketed by Heiberg, is as follows:

"And, if we take O a third proportional to AB , FG , then BA has to O a ratio duplicate of that which AB has to FG .

But the polygon has also to the polygon, or the quadrilateral to the quadrilateral, a ratio duplicate of that which the corresponding side has to the corresponding side, that is AB to FG ;

and this was proved in the case of triangles also;

so that it is also manifest generally that, if three straight lines be proportional, as the first is to the third, so will the figure described on the first be to the similar and similarly described figure on the second."

PROPOSITION 21.

Figures which are similar to the same rectilinear figure are also similar to one another.

For let each of the rectilinear figures A , B be similar to C ;
I say that A is also similar to B .

For, since A is similar to C ,
it is equiangular with it and has the sides about the equal
angles proportional. [VI. Def. 1]



Again, since B is similar to C ,
it is equiangular with it and has the sides about the equal
angles proportional.

Therefore each of the figures A , B is equiangular with C
and with C has the sides about the equal angles proportional;
therefore A is similar to B .

Q. E. D.

It will be observed that the text above omits a step which the editions generally have before the final inference " A is similar to B ." The words omitted are " A is also equiangular with B and [with B] has the sides about the equal angles proportional." Heiberg follows P in leaving them out, conjecturing that they may be an addition of Theon's.

PROPOSITION 22.

If four straight lines be proportional, the rectilineal figures similar and similarly described upon them will also be proportional; and, if the rectilineal figures similar and similarly described upon them be proportional, the straight lines will themselves also be proportional.

Let the four straight lines AB , CD , EF , GH be proportional,

so that, as AB is to CD , so is EF to GH ,

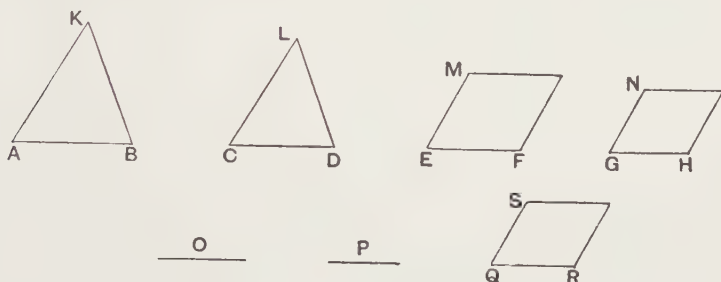
and let there be described on AB , CD the similar and similarly situated rectilineal figures KAB , LCD ,

and on EF , GH the similar and similarly situated rectilineal figures MF , NH ;

I say that, as KAB is to LCD , so is MF to NH .

For let there be taken a third proportional O to AB , CD ,
and a third proportional P to EF , GH . [VI. 11]

Then since, as AB is to CD , so is EF to GH ,
 and, as CD is to O , so is GH to P ,
 therefore, *ex aequali*, as AB is to O , so is EF to P . [v. 22]
 But, as AB is to O , so is KAB to LCD , [vi. 19, Por.]
 and, as EF is to P , so is MF to NH ;
 therefore also, as KAB is to LCD , so is MF to NH . [v. 11]



Next, let MF be to NH as KAB is to LCD ;
 I say also that, as AB is to CD , so is EF to GH .
 For, if EF is not to GH as AB to CD ,
 let EF be to QR as AB to CD , [vi. 12]
 and on QR let the rectilinear figure SR be described similar
 and similarly situated to either of the two MF , NH . [vi. 18]
 Since then, as AB is to CD , so is EF to QR ,
 and there have been described on AB , CD the similar and
 similarly situated figures KAB , LCD ,
 and on EF , QR the similar and similarly situated figures
 MF , SR ,
 therefore, as KAB is to LCD , so is MF to SR .

But also, by hypothesis,

as KAB is to LCD , so is MF to NH ;

therefore also, as MF is to SR , so is MF to NH . [v. 11]

Therefore MF has the same ratio to each of the figures
 NH , SR ;

therefore NH is equal to SR . [v. 9]

But it is also similar and similarly situated to it;

therefore GH is equal to QR .

And, since, as AB is to CD , so is EF to QR ,
 while QR is equal to GH ,
 therefore, as AB is to CD , so is EF to GH .

Therefore etc.

Q. E. D.

The second assumption in the first step of the first part of the proof, viz. that, as CD is to O , so GH to P , should perhaps be explained. It is a deduction [by v. 11] from the facts that

AB is to CD as CD to O ,

EF is to GH as GH to P ,

and

AB is to CD as EF to GH .

The defect in the proof of this proposition is well known, namely the assumption, without proof, that, because the figures NH , SR are equal, besides being similar and similarly situated, their corresponding sides GH , QR are equal. Hence the minimum addition necessary to make the proof complete is a proof of a lemma to the effect that, *if two similar figures are also equal, any pair of corresponding sides are equal*.

To supply this lemma is one alternative; another is to prove, as a preliminary proposition, a much more general theorem, viz. that, *if the duplicate ratios of two ratios are equal, the two ratios are themselves equal*. When this is proved, the second part of VI. 22 is an immediate inference from it, and the effect is, of course, to substitute a new proof instead of supplementing Euclid's.

I. It is to be noticed that the lemma required as a minimum is very like what is needed to supplement vi. 28 and 29, in the proofs of which Euclid assumes that, *if two similar parallelograms are unequal, any side in the greater is greater than the corresponding side in the smaller*. Therefore, on the whole, it seems preferable to adopt the alternative of proving the simpler lemma which will serve to supplement all three proofs, viz. that, *if of two similar rectilinear figures the first is greater than, equal to, or less than, the second, any side of the first is greater than, equal to, or less than, the corresponding side of the second respectively*.

The case of *equality* of the figures is the case required for VI. 22; and the proof of it is given in the Greek text after the proposition. Since to give such a "lemma" after the proposition in which it is required is contrary to Euclid's manner, Heiberg concludes that it is an interpolation, though it is earlier than Theon. The lemma runs thus:

"But that, if rectilinear figures be equal and similar, their corresponding sides are equal to one another we will prove thus.

Let NH , SR be equal and similar rectilinear figures, and suppose that,

as HG is to GN , so is RQ to QS ;

I say that RQ is equal to HG .

For, if they are unequal, one of them is greater;

let RQ be greater than HG .

Then, since, as RQ is to QS , so is HG to GN ,
alternately also, as RQ is to HG , so is QS to GN ;
and QR is greater than HG ;

therefore QS is also greater than GN ;
so that RS is also greater than HN^* .

But it is also equal : which is impossible.

Therefore QR is not unequal to GH ;
therefore it is equal to it."

[The step marked * is easy to see if it is remembered that it is only necessary to prove its truth in the case of *triangles* (since similar polygons are divisible into the same number of similar and similarly situated triangles having the same ratio to each other respectively as the polygons have). If the triangles be applied to each other so that the two corresponding sides of each, which are used in the question, and the angles included by them coincide, the truth of the inference is obvious.]

The lemma might also be arrived at by proving that, *if a ratio is greater than a ratio of equality, the ratio which is its duplicate is also greater than a ratio of equality; and if the ratio which is duplicate of another ratio is greater than a ratio of equality, the ratio of which it is the duplicate is also greater than a ratio of equality.* It is not difficult to prove this from the particular case of v. 25 in which the second magnitude is equal to the third, i.e. from the fact that in this case the sum of the extreme terms is greater than double the middle term.

II. We now come to the alternative which substitutes a new proof for the second part of the proposition, making the whole proposition an immediate inference from one to which it is practically equivalent, viz. that

(1) *If two ratios be equal, their duplicate ratios are equal, and (2) conversely, if the duplicate ratios of two ratios be equal, the ratios are equal.*

The proof of part (1) is after the manner of Euclid's own proof of the first part of VI. 22.

Let A be to B as C to D ,

and let X be a third proportional to A , B , and Y a third proportional to C , D ,
so that

A is to B as B to X ,

and

C is to D as D to Y ;

whence

A is to X in the duplicate ratio of A to B ,

and

C is to Y in the duplicate ratio of C to D .

Since

A is to B as C is to D ,

and

B is to X as A is to B ,

i.e. as C is to D ,

i.e. as D is to Y ,

[v. 11]

therefore, *ex aequali*, A is to X as C is to Y .

Part (2) is much more difficult and is the crux of the whole thing.

Most of the proofs depend on the assumption that, B being any magnitude and P and Q two magnitudes of the same kind, there does exist a magnitude A which is to B in the same ratio as P to Q . It is this same assumption

which makes Euclid's proof of v. 18 illegitimate, since it is nowhere proved in Book v. Hence any proof of the proposition now in question which involves this assumption even in the case where B, P, Q are all straight lines should not properly be given as an addition to Book v.; it should at least be postponed until we have learnt, by means of vi. 12, giving the actual construction of a fourth proportional, that such a fourth proportional exists.

Two proofs which are given of the proposition depend upon the following lemma.

If A, B, C be three magnitudes of one kind, and D, E, F three magnitudes of one kind, then, if

the ratio of A to B is greater than that of D to E ,

and the ratio of B to C greater than that of E to F ,

ex aequali, the ratio of A to C is greater than that of D to F .

One proof of this does not depend upon the assumption referred to, and therefore, if this proof is used, the theorem can be added to Book v. The proof is that of Hauber (Camerer's Euclid, p. 358 of Vol. II.) and is reproduced by Mr H. M. Taylor. For brevity we will use symbols.

Take equimultiples mA, mD of A, D and nB, nE of B, E such that

$$mA > nB, \text{ but } mD \nless nE.$$

Also let pB, pE be equimultiples of B, E and qC, qF equimultiples of C, F such that

$$pB > qC, \text{ but } pE \nless qF.$$

Therefore, multiplying the first line by p and the second by n , we have

$$p mA > p n B, \text{ } p m D \nless p n E,$$

and

$$n p B > n q C, \text{ } n p E \nless n q F,$$

whence

$$p mA > n q C, \text{ } p m D \nless n q F.$$

Now

$$p mA, p m D \text{ are equimultiples of } mA, mD,$$

and

$$n q C, n q F \text{ equimultiples of } qC, qF.$$

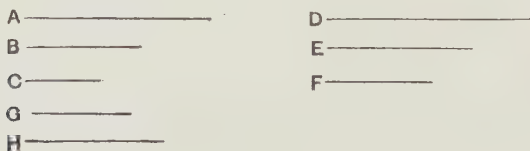
Therefore [v. 3] they are respectively equimultiples of A, D and of C, F .

Hence [v. Def. 7] $A : C > D : F$.

Another proof given by Clavius, though depending on the assumption referred to, is neat.

Take G such that

$$G : C = E : F.$$



Therefore

$$B : C > G : C,$$

[v. 13]

and

$$B > G.$$

[v. 10]

Therefore

$$A : G > A : B.$$

[v. 8]

But	$A : B > D : E.$	
Therefore, <i>a fortiori</i> ,	$A : G > D : E.$	
Suppose H taken such that	$H : G = D : E.$	
Therefore	$A > H.$	[v. 13, 10]
Hence	$A : C > H : C.$	[v. 8]
But	$H : G = D : E,$ $G : C = E : F.$	
Therefore, <i>ex aequali</i> ,	$H : C = D : F.$	[v. 22]
Hence	$A : C > D : F.$	[v. 13]

Now we can prove that

Ratios of which equal ratios are duplicate are equal.

Suppose that	$A : B = B : C,$
and	$D : E = E : F,$
and further that	$A : C = D : F.$
it is required to prove that	$A : B = D : E.$

For, if not, one of the ratios must be greater than the other.
Let $A : B$ be the greater.

Then, since	$A : B = B : C,$	
and	$D : E = E : F,$	
while	$A : B > D : E,$	
it follows that	$B : C > E : F.$	[v. 13]
Hence, by the lemma, <i>ex aequali</i> ,	$A : C > D : F,$	

which contradicts the hypothesis.

Thus the ratios $A : B$ and $D : E$ cannot be unequal; that is, they are equal.

Another proof, given by Dr Lachlan, also assumes the existence of a fourth proportional, but depends upon a simpler lemma to the effect that

It is impossible that two different ratios can have the same duplicate ratio:

For, if possible, let the ratio $A : B$ be duplicate both of $A : X$ and $A : Y$, so that

	$A : X = X : B,$	
and	$A : Y = Y : B.$	
Let X be greater than Y .		
Then	$A : X < A : Y;$	[v. 8]
that is,	$X : B < Y : B,$	[v. 11, 13]
or	$X < Y.$	[v. 10]

But X is greater than Y : which is absurd, etc.

Hence $X = Y.$

Now suppose that $A : B = B : C$,
 $D : E = E : F$,
 and $A : C = D : F$.
 To prove that $A : B = D : E$.

If this is not so, suppose that
 $A : B = D : Z$.
 Since $A : C = D : F$,
 therefore, inversely, $C : A = F : D$.

Therefore, *ex aequali*,
 $C : B = F : Z$, [v. 22]

or, inversely,
 $B : C = Z : F$.
 Therefore $A : B = Z : F$. [v. 11]

But $A : B = D : Z$, by hypothesis.
 Therefore $D : Z = Z : F$. [v. 11]

Also, by hypothesis, $D : E = E : F$;
 whence, by the lemma, $E = Z$.

Therefore $A : B = D : E$.

De Morgan remarks that the best way of remedying the defect in Euclid is to insert the proposition (the lemma to the last proof) that *it is impossible that two different ratios can have the same duplicate ratio*, "which," he says, "immediately proves the second (or defective) case of the theorem." But this seems to be either too much or too little: too much, if we choose to make the *minimum* addition to Euclid (for that addition is a lemma which shall prove that, if a duplicate ratio is a ratio of equality, the ratio of which it is duplicate is also one of equality), and too little if the proof is to be altered in the more fundamental manner explained above.

I think that, if Euclid's attention had been drawn to the defect in his proof of VI. 22 and he had been asked to remedy it, he would have done so by supplying what I have called the minimum lemma and not by making the more fundamental alteration. This I infer from Prop. 24 of the *Data*, where he gives a theorem corresponding to the proposition that *ratios of which equal ratios are duplicate are equal*. The proposition in the *Data* is enunciated thus: *If three straight lines be proportional, and the first have to the third a given ratio, it will also have to the second a given ratio*.

A , B , C being the three straight lines, so that

$$A : B = B : C,$$

and $A : C$ being a given ratio, it is required to prove that $A : B$ is also a given ratio.

Euclid takes any straight line D , and first finds another, F , such that

$$D : F = A : C,$$

whence $D : F$ must be a given ratio, and, as D is given, F is therefore given.

Then he takes E a mean proportional between D , F , so that

$$D : E = E : F.$$

It follows [vi. 17] that

the rectangle D, F is equal to the square on E .

But D, F are both given ;

therefore the square on E is given, so that E is also given.

[Observe that De Morgan's lemma is here assumed without proof. It may be proved (1) as it is by De Morgan, whose proof is that given above, p. 245, (2) in the manner of the "minimum lemma," pp. 242—3 above, or (3) as it is by Proclus on i. 46 (see note on that proposition).]

Hence the ratio $D : E$ is given.

Now, since

$$A : C = D : F,$$

and $A : C = (\text{square on } A) : (\text{rect. } A, C),$

while $D : F = (\text{square on } D) : (\text{rect. } D, F),$

[vi. 1]

therefore $(\text{square on } A) : (\text{rect. } A, C) = (\text{square on } D) : (\text{rect. } D, F).$

[v. 11]

But, since $A : B = B : C, (\text{rect. } A, C) = (\text{sq. on } B) ;$

[vi. 17]

and $(\text{rect. } D, F) = (\text{sq. on } E),$ from above ;

therefore $(\text{square on } A) : (\text{square on } B) = (\text{sq. on } D) : (\text{sq. on } E).$

Therefore, says Euclid,

$$A : B = D : E,$$

that is, *he assumes the truth of vi. 22 for squares.*

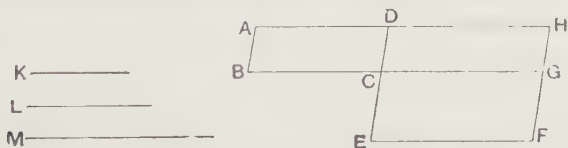
Thus he deduces his proposition from vi. 22, instead of proving vi. 22 by means of it (or the corresponding proposition used by Mr Taylor and Dr Lachlan).

PROPOSITION 23.

Equiangular parallelograms have to one another the ratio compounded of the ratios of their sides.

Let AC, CF be equiangular parallelograms having the angle BCD equal to the angle ECG ;

5 I say that the parallelogram AC has to the parallelogram CF the ratio compounded of the ratios of the sides.



For let them be placed so that BC is in a straight line with CG ;

therefore DC is also in a straight line with CE .

10 Let the parallelogram DG be completed ;

let a straight line K be set out, and let it be contrived that,

as BC is to CG , so is K to L ,

and, as DC is to CE , so is L to M .

[vi. 12]

Then the ratios of K to L and of L to M are the same
 15 as the ratios of the sides, namely of BC to CG and of DC to CE .

But the ratio of K to M is compounded of the ratio of K to L and of that of L to M ;

so that K has also to M the ratio compounded of the ratios
 20 of the sides.

Now since, as BC is to CG , so is the parallelogram AC to the parallelogram CH , [VI. 1]

while, as BC is to CG , so is K to L ,

therefore also, as K is to L , so is AC to CH . [V. 11]

25 Again, since, as DC is to CE , so is the parallelogram CH to CF , [VI. 1]

while, as DC is to CE , so is L to M ,

therefore also, as L is to M , so is the parallelogram CH to the parallelogram CF . [V. 11]

30 Since then it was proved that, as K is to L , so is the parallelogram AC to the parallelogram CH ,

and, as L is to M , so is the parallelogram CH to the parallelogram CF ,

therefore, *ex aequali*, as K is to M , so is AC to the parallelo-
 35 gram CF .

But K has to M the ratio compounded of the ratios of the sides;

therefore AC also has to CF the ratio compounded of the ratios of the sides.

40 Therefore etc.

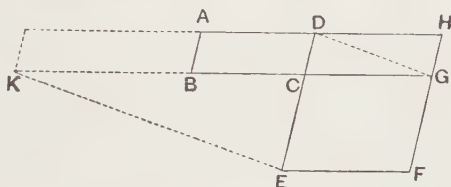
Q. E. D.

1, 6, 19, 36. the ratio compounded of the ratios of the sides, λόγον τὸν συγκελμένον ἐκ τῶν πλευρῶν which, meaning literally "the ratio compounded of the sides," is negligently written here and commonly for λόγον τὸν συγκελμένον ἐκ τῶν τῶν πλευρῶν (sc. λόγων).

11. let it be contrived that, as BC is to CG , so is K to L . The Greek phrase is of the usual terse kind, untranslatable literally: καὶ γερονέτω ὡς μὲν ἡ $BΓ$ πρὸς τὴν $ΓΗ$, οὕτως ἡ K πρὸς τὸ $Λ$, the words meaning "and let (there) be made, as BC to CG , so K to L ," where L is the straight line which has to be constructed.

The second definition of the *Data* says that *A ratio is said to be given if we can find (πορίσασθαι) [another ratio that is] the same with it.* Accordingly VI. 23 not only proves that equiangular parallelograms have to one another a ratio which is compounded of two others, but shows that that ratio is "given" when its component ratios are given, or that it can be represented as a simple ratio between straight lines.

Just as VI. 23 exhibits the operation necessary for *compounding* two ratios, a proposition (8) of the *Data* indicates the operation by which we may divide one ratio by another. The proposition proves that *Things which have a given ratio to the same thing have also a given ratio to one another*. Euclid's procedure is of course to *compound* one ratio with the *inverse* of the other; but, when this is once done and the result of Prop. 8 obtained, he uses the result in the later propositions as a substitute for the method of composition. Thus he uses the *division* of ratios, instead of composition, in the propositions of the *Data* which deal with the same subject-matter as VI. 23. The effect is to represent the ratio of two equiangular parallelograms as a ratio between straight lines one of which is *one side of one of the parallelograms*. Prop. 56 of the *Data* shows us that, if we want to express the ratio of the parallelogram AC to the parallelogram CF in the figure



of VI. 23 in the form of a ratio in which, for example, the side BC is the antecedent term, the required ratio of the parallelograms is $BC : X$, where

$$DC : CE = CG : X,$$

or X is a fourth proportional to DC and the two sides of the parallelogram CF .

Measure CK along CB , produced if necessary, so that

$$DC : CE = CG : CK$$

(whence CK is equal to X).

[This may be simply done by joining DG and then drawing EK parallel to it meeting CB in K .]

Complete the parallelogram AK .

Then, since $DC : CE = CG : CK$,

the parallelograms DK , CF are equal.

[VI. 14]

Therefore $(AC) : (CF) = (AC) : (DK)$

[V. 7]

$$= BC : CK$$

[VI. 1]

$$= BC : X.$$

Prop. 68 of the *Data* uses the same construction to prove that, *If two equiangular parallelograms have to one another a given ratio, and one side have to one side a given ratio, the remaining side will also have to the remaining side a given ratio*.

I do not use the figure of the *Data* but, for convenience' sake, I adhere to the figure given above. Suppose that the ratio of the parallelograms is given, and also that of CD to CE .

Apply to CD the parallelogram DK equal to CF and such that CK , CB coincide in direction.

[I. 45]

Then the ratio of AC to KD is given, being equal to that of AC to CF .

And $(AC) : (KD) = CB : CK$;

therefore the ratio of CB to CK is given.

But, since

$$KD = CF, \\ CD : CE = CG : CK.$$

[VI. 14]

Hence $CG : CK$ is a given ratio.

And $CB : CK$ was proved to be a given ratio.

Therefore the ratio of CB to CG is given.

[Data, Prop. 8]

Lastly we may refer to Prop. 70 of the *Data*, the first part of which proves what corresponds exactly to VI. 23, namely that, *If in two equiangular parallelograms the sides containing the equal angles have a given ratio to one another [i.e. one side in one to one side in the other], the parallelograms themselves will also have a given ratio to one another.* [Here the ratios of BC to CG and of CD to CE are given.]

The construction is the same as in the last case, and we have KD equal to CF , so that

$$CD : CE = CG : CK.$$

[VI. 14]

But the ratio of CD to CE is given;

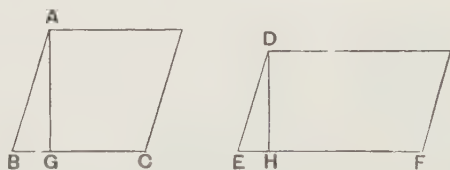
therefore the ratio of CG to CK is given.

And, by hypothesis, the ratio of CG to CB is given.

Therefore, by *dividing* the ratios [Data, Prop. 8], we see that the ratio of CB to CK , and therefore [VI. 1] the ratio of AC to DK , or of AC to CF , is given.

Euclid extends these propositions to the case of two parallelograms which have *given* but not equal angles.

Pappus (VII. p. 928) exhibits the result of VI. 23 in a different way, which throws new light on compounded ratios. He proves, namely, that a *parallelogram is to an equiangular parallelogram as the rectangle contained by the adjacent sides of the first is to the rectangle contained by the adjacent sides of the second.*



Let AC, DF be equiangular parallelograms on the bases BC, EF , and let the angles at B, E be equal.

Draw perpendiculars AG, DH to BC, EF respectively.

Since the angles at B, G are equal to those at E, H ,

the triangles ABG, DEH are equiangular.

Therefore

$$BA : AG = ED : DH.$$

[VI. 4]

But $BA : AG = (\text{rect. } BA, BC) : (\text{rect. } AG, BC),$

and $ED : DH = (\text{rect. } ED, EF) : (\text{rect. } DH, EF).$

[VI. 1]

Therefore [v. 11 and v. 16]

$$(\text{rect. } AB, BC) : (\text{rect. } DE, EF) = (\text{rect. } AG, BC) : (\text{rect. } DH, EF) \\ = (AC) : (DF).$$

Thus it is proved that the ratio compounded of the ratios $AB : DE$ and $BC : EF$ is equal to the ratio of the rectangle AB, BC to the rectangle DE, EF .

Since each parallelogram in the figure of the proposition can be divided into pairs of equal triangles, and all the triangles which are the halves of either parallelogram have two sides respectively equal and the angles included by them equal or supplementary, it can be at once deduced from VI. 23 (or it can be independently proved by the same method) that *triangles which have one angle of the one equal or supplementary to one angle of the other are in the ratio compounded of the ratios of the sides about the equal or supplementary angles*. Cf. Pappus VII. pp. 894—6.

VI. 23 also shows that *rectangles, and therefore parallelograms or triangles, are to one another in the ratio compounded of the ratios of their bases and heights*.

The converse of VI. 23 is also true, as is easily proved by *reductio ad absurdum*. More generally, *if two parallelograms or triangles are in the ratio compounded of the ratios of two adjacent sides, the angles included by those sides are either equal or supplementary*.

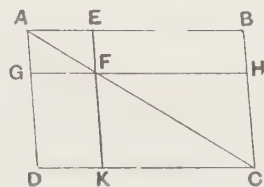
PROPOSITION 24.

In any parallelogram the parallelograms about the diameter are similar both to the whole and to one another.

Let $ABCD$ be a parallelogram, and AC its diameter, and let EG , HK be parallelograms about AC ;

I say that each of the parallelograms EG , HK is similar both to the whole $ABCD$ and to the other.

For, since EF has been drawn parallel to BC , one of the sides of the triangle ABC ,



proportionally, as BE is to EA , so is CF to FA . [VI. 2]

Again, since FG has been drawn parallel to CD , one of the sides of the triangle ACD ,

proportionally, as CF is to FA , so is DG to GA . [VI. 2]

But it was proved that,

as CF is to FA , so also is BE to EA ;

therefore also, as BE is to EA , so is DG to GA ,

and therefore, *componendo*,

as BA is to AE , so is DA to AG , [V. 18]

and, alternately,

as BA is to AD , so is EA to AG . [V. 16]

Therefore in the parallelograms $ABCD$, EG , the sides about the common angle BAD are proportional.

And, since GF is parallel to DC ,

the angle AFG is equal to the angle DCA ;
and the angle DAC is common to the two triangles ADC ,
 AGF ;

therefore the triangle ADC is equiangular with the triangle
 AGF .

For the same reason

the triangle ACB is also equiangular with the triangle
 AFE ,

and the whole parallelogram $ABCD$ is equiangular with the
parallelogram EG .

Therefore, proportionally,

as AD is to DC , so is AG to GF ,

as DC is to CA , so is GF to FA ,

as AC is to CB , so is AF to FE ,

and further, as CB is to BA , so is FE to EA .

And, since it was proved that,

as DC is to CA , so is GF to FA ,

and, as AC is to CB , so is AF to FE ,

therefore, *ex aequali*, as DC is to CB , so is GF to FE . [v. 22]

Therefore in the parallelograms $ABCD$, EG the sides
about the equal angles are proportional ;

therefore the parallelogram $ABCD$ is similar to the parallelo-
gram EG . [vi. Def. 1]

For the same reason

the parallelogram $ABCD$ is also similar to the parallelogram
 KH ;

therefore each of the parallelograms EG , HK is similar to
 $ABCD$.

But figures similar to the same rectilineal figure are also
similar to one another ; [vi. 21]

therefore the parallelogram EG is also similar to the parallelo-
gram HK .

Therefore etc.

Q. E. D.

Simson was of opinion that this proof was made up by some unskilful editor out of two others, the first of which proved by parallels (vi. 2) that the sides about the common angle in the parallelograms are proportional, while the other used the similarity of triangles (vi. 4). It is of course true

that, when we have proved by VI. 2 the fact that the sides about the common angle are proportional, we can infer the proportionality of the other sides directly from I. 34 combined with V. 7. But it does not seem to me unnatural that Euclid should (1) deliberately refrain from making any use of I. 34 and (2) determine beforehand that he would prove the sides proportional *in a definite order* beginning with the sides EA, AG and BA, AD about the common angle and then taking the remaining sides in the order indicated by the order of the letters A, G, F, E . Given that Euclid started the proof with such a fixed intention in his mind, the course taken presents no difficulty, nor is the proof unsystematic or unduly drawn out. And its genuineness seems to me supported by the fact that the proof, when once the first two sides about the common angle have been disposed of, follows closely the order and method of VI. 18. Moreover, it could readily be adapted to the more general case of two polygons having a common angle and the other corresponding sides respectively parallel.

The parallelograms in the proposition are of course similarly situated as well as similar; and those "about the diameter" may be "about" the diameter *produced* as well as about the diameter itself.

From the first part of the proof it follows that parallelograms which have one angle equal to one angle and the sides about those angles proportional are similar.

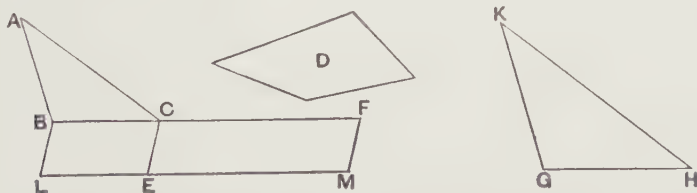
Prop. 26 is the converse of Prop. 24, and there seems to be no reason why they should be separated as they are in the text by the interposition of VI. 25. Campanus has VI. 24 and 26 as VI. 22 and 23 respectively, VI. 23 as VI. 24, and VI. 25 as we have it.

PROPOSITION 25.

To construct one and the same figure similar to a given rectilineal figure and equal to another given rectilineal figure.

Let ABC be the given rectilineal figure to which the figure to be constructed must be similar, and D that to which it must be equal;

thus it is required to construct one and the same figure similar to ABC and equal to D .



Let there be applied to BC the parallelogram BE equal to the triangle ABC [I. 44], and to CE the parallelogram CM equal to D in the angle FCE which is equal to the angle CBL . [I. 45]

Therefore BC is in a straight line with CF , and LE with EM .

Now let GH be taken a mean proportional to BC , CF [vi. 13], and on GH let KGH be described similar and similarly situated to ABC . [vi. 18]

Then, since, as BC is to GH , so is GH to CF , and, if three straight lines be proportional, as the first is to the third, so is the figure on the first to the similar and similarly situated figure described on the second, [vi. 19, Por.] therefore, as BC is to CF , so is the triangle ABC to the triangle KGH .

But, as BC is to CF , so also is the parallelogram BE to the parallelogram EF . [vi. 1]

Therefore also, as the triangle ABC is to the triangle KGH , so is the parallelogram BE to the parallelogram EF ; therefore, alternately, as the triangle ABC is to the parallelogram BE , so is the triangle KGH to the parallelogram EF . [v. 16]

But the triangle ABC is equal to the parallelogram BE ; therefore the triangle KGH is also equal to the parallelogram EF .

But the parallelogram EF is equal to D ; therefore KGH is also equal to D .

And KGH is also similar to ABC .

Therefore one and the same figure KGH has been constructed similar to the given rectilinear figure ABC and equal to the other given figure D .

Q. E. D.

3. to which the figure to be constructed must be similar, literally "to which it is required to construct (one) similar," $\psi \delta\epsilon\iota \delta\mu\iota\omicron\nu \sigma\upsilon\sigma\tau\eta\sigma\alpha\sigma\theta\alpha\iota$.

This is the highly important problem which Pythagoras is credited with having solved. Compare the passage from Plutarch (*Symp.* viii. 2, 4) quoted in the note on I. 44 above, Vol. I. pp. 343—4.

We are bidden to construct a rectilinear figure which shall have the *form* of one and the *size* of another rectilinear figure. The corresponding proposition of the *Data*, Prop. 55, asserts that, "if an area ($\chi\omega\rho\acute{\iota}\omicron\nu$) be given in form ($\epsilon\acute{\iota}\delta\epsilon\iota$) and in magnitude, its sides will also be given in magnitude."

Simson sees signs of corruption in the text of this proposition also. In the first place, the proof speaks of the *triangle* ABC , though, according to the enunciation, the figure for which ABC is taken may be *any* rectilinear figure, $\epsilon\upsilon\theta\acute{\upsilon}\gamma\gamma\alpha\mu\mu\omicron\nu$ "rectilinear figure" would be more correct, or $\epsilon\acute{\iota}\delta\omicron\varsigma$, "figure"; the mistake, however, of using $\tau\rho\acute{\iota}\gamma\omega\nu\omicron\nu$ is not one of great importance, being no

doubt due to the accident by which the figure was drawn as a triangle in the diagram.

The other observation is more important. After Euclid has proved that

$$(\text{fig. } ABC) : (\text{fig. } KGH) = (BE) : (EF),$$

he might have inferred *directly* from v. 14 that, since ABC is equal to BE , KGH is equal to EF . For v. 14 includes the proof of the fact that, if A is to B as C is to D , and A is equal to C , then B is equal to D , or that of four proportional magnitudes, if the first is equal to the third, the second is equal to the fourth. Instead of proceeding in this way, Euclid first permutes the proportion by v. 16 into

$$(\text{fig. } ABC) : (BE) = (\text{fig. } KGH) : (EF),$$

and then infers, as if the inference were easier in this form, that, since the *first* is equal to the *second*, the *third* is equal to the *fourth*. Yet there is no proposition to this effect in Euclid. The same unnecessary step of permutation is also found in the Greek text of xi. 23 and xii. 2, 5, 11, 12 and 18. In reproducing the proofs we may simply leave out the steps and refer to v. 14.

PROPOSITION 26.

If from a parallelogram there be taken away a parallelogram similar and similarly situated to the whole and having a common angle with it, it is about the same diameter with the whole

For from the parallelogram $ABCD$ let there be taken away the parallelogram AF similar and similarly situated to $ABCD$, and having the angle DAB common with it;

I say that $ABCD$ is about the same diameter with AF .

For suppose it is not, but, if possible, let AHC be the diameter < of $ABCD$ >, let GF be produced and carried through to H , and let HK be drawn through H parallel to either of the straight lines AD , BC . [I. 31]

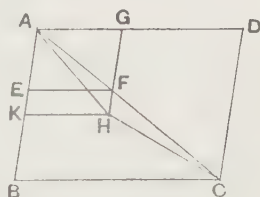
Since, then, $ABCD$ is about the same diameter with KG , therefore, as DA is to AB , so is GA to AK . [VI. 24]

But also, because of the similarity of $ABCD$, EG ,

as DA is to AB , so is GA to AE ;

therefore also, as GA is to AK , so is GA to AE . [v. 11]

Therefore GA has the same ratio to each of the straight lines AK , AE .



Therefore AE is equal to AK [v. 9], the less to the greater: which is impossible.

Therefore $ABCD$ cannot but be about the same diameter with AF ;

therefore the parallelogram $ABCD$ is about the same diameter with the parallelogram AF .

Therefore etc.

Q. E. D.

"For suppose it is not, but, if possible, let AHC be the diameter." What is meant is "For, if AFC is not the diameter of the parallelogram AC , let AHC be its diameter." The Greek text has $\epsilon\sigma\tau\omega\ \alpha\upsilon\tau\omega\acute{\nu}\ \delta\iota\acute{\alpha}\mu\epsilon\tau\rho\omicron\varsigma\ \eta\ \Lambda\Theta\Gamma$; but clearly $\alpha\upsilon\tau\omega\acute{\nu}$ is wrong, as we cannot assume that one straight line is the diameter of both parallelograms, which is just what we have to prove. F and V omit the $\alpha\upsilon\tau\omega\acute{\nu}$, and Heiberg prefers this correction to substituting $\alpha\iota\tau\omicron\upsilon$ after Peyrard. I have inserted "< of $ABCD$ >" to make the meaning clear.

If the straight line AHC does not pass through F , it must meet either GF or GF produced in some point H . The reading in the text "and let GF be produced and carried through to H " ($\kappa\alpha\iota\ \epsilon\kappa\beta\lambda\eta\theta\epsilon\iota\sigma\alpha\ \eta\ HZ\ \delta\iota\eta\chi\theta\omega\ \epsilon\pi\iota\ \tau\omicron\ \Theta$) corresponds to the supposition that H is on GF produced. The words were left out by Theon, evidently because in the figure of the mss. the letters E, Z and K, Θ were interchanged. Heiberg therefore, following August, has preferred to retain the words and to correct the figure, as well as the passage in the text where AE , AK were interchanged to be in accord with the ms. figure.

It is of course possible to prove the proposition directly, as is done by Dr Lachlan. Let AF , AC be the diagonals, and let us make no assumption as to how they fall.

Then, since EF is parallel to AG and therefore to BC ,

the angles AEF , ABC are equal.

And, since the parallelograms are similar,

$$AE : EF = AB : BC.$$

[vi. Def. 1]

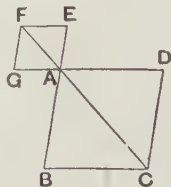
Hence the triangles AEF , ABC are similar,

[vi. 6]

and therefore the angle FAE is equal to the angle CAB .

Therefore AF falls on AC .

The proposition is equally true if the parallelogram which is similar and similarly situated to the given parallelogram is not "taken away" from it, but is so placed that it is entirely outside the other, while two sides form an angle vertically opposite to an angle of the other. In this case the diameters are not "the same," in the words of the enunciation, but are in a straight line with one another. This extension of the proposition is, as will be seen, necessary for obtaining, according to the method adopted by Euclid in his solution of the problem in VI. 28, the second solution of that problem.

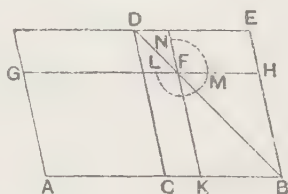


PROPOSITION 27.

Of all the parallelograms applied to the same straight line and deficient by parallelogrammic figures similar and similarly situated to that described on the half of the straight line, that parallelogram is greatest which is applied to the half of the straight line and is similar to the defect.

Let AB be a straight line and let it be bisected at C ; let there be applied to the straight line AB the parallelogram AD deficient by the parallelogrammic figure DB described on the half of AB , that is, CB ;

I say that, of all the parallelograms applied to AB and deficient by parallelogrammic figures similar and similarly situated to DB , AD is greatest.



For let there be applied to the straight line AB the parallelogram AF deficient by the parallelogrammic figure FB similar and similarly situated to DB ;

I say that AD is greater than AF .

For, since the parallelogram DB is similar to the parallelogram FB ,

they are about the same diameter. [VI. 26]

Let their diameter DB be drawn, and let the figure be described.

Then, since CF is equal to FE , [I. 43]
and FB is common,

therefore the whole CH is equal to the whole KE .

But CH is equal to CG , since AC is also equal to CB . [I. 36]

Therefore GC is also equal to EK .

Let CF be added to each;

therefore the whole AF is equal to the gnomon LMN ;

so that the parallelogram DB , that is, AD , is greater than the parallelogram AF .

Therefore etc.

Q. E. D.

We have already (note on I. 44) seen the significance, in Greek geometry, of the theory of "the application of areas, their exceeding and their falling-short." In I. 44 it was a question of "applying to a given straight line (exactly, without 'excess' or 'defect') a parallelogram equal to a given rectilineal figure, in a given angle." Here, in VI. 27—29, it is a question of parallelograms applied to a straight line but "*deficient (or exceeding) by parallelograms similar and similarly situated to a given parallelogram.*"

Apart from size, it is easy to construct any number of parallelograms "deficient" or "exceeding" in the manner described. Given the straight line AB to which the parallelogram has to be applied, we describe on the base CB , where C is on AB , or on BA produced beyond A , any parallelogram "similarly situated" and either equal or similar to the given parallelogram (Euclid takes the similar and similarly situated parallelogram on half the line), draw the diagonal BD , take on it (produced if necessary) any points as E , K , draw EF , or KL , parallel to CD to meet AB or AB produced and complete the parallelograms, as AH , ML .

If the point E is taken on BD or BD produced beyond D , it must be so taken that EF meets AB between A and B . Otherwise the parallelogram AE would not be applied to AB itself, as it is required to be.

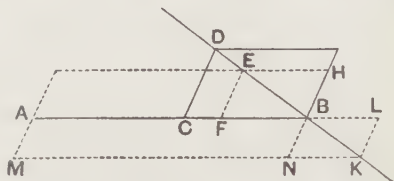
The parallelograms BD , BE , being about the same diameter, are similar [VI. 24], and BE is the defect of the parallelogram AE relatively to AB . AE is then a parallelogram applied to AB but deficient by a parallelogram similar and similarly situated to BD .

If K is on DE produced, the parallelogram BK is similar to BD , but it is the *excess* of the parallelogram AK relatively to the base AB . AK is a parallelogram applied to AB but exceeding by a parallelogram similar and similarly situated to BD .

Thus it is seen that BD produced both ways is the *locus* of points, such as E or K , which determine, with the direction of CD , the position of A , and the direction of AB , parallelograms applied to AB and *deficient* or *exceeding* by parallelograms similar and similarly situated to the given parallelogram.

The importance of VI. 27—29 from a historical point of view cannot be overrated. They give the geometrical equivalent of the algebraical solution of the most general form of quadratic equation when that equation has a real and positive root. It will also enable us to find a real *negative* root of a quadratic equation; for such an equation can, by altering the sign of x , be turned into another with a real *positive* root, when the geometrical method again becomes applicable. It will also, as we shall see, enable us to represent *both* roots when both are real and positive, and therefore to represent both roots when both are real but either positive or negative.

The method of these propositions was constantly used by the Greek geometers in the solution of problems, and they constitute the foundation of Book x. of the *Elements* and of Apollonius' treatment of the conic sections. Simson's observation on the subject is entirely justified. He says namely on VI. 28, 29: "These two problems, to the first of which the 27th Prop. is necessary, are the most general and useful of all in the *Elements*, and are most frequently made use of by the ancient geometers in the solution of other problems; and therefore are very ignorantly left out by Tacquet and



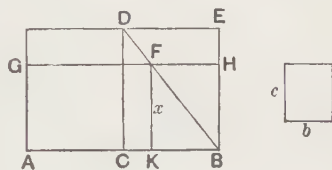
Dechales in their editions of the Elements, who pretend that they are scarce of any use."

It is strange that, with this observation before him, even Todhunter should have written as follows. "We have omitted in the sixth Book Propositions 27, 28, 29 and the first solution which Euclid gives of Proposition 30, as they appear now to be never required, and have been condemned as useless by various modern commentators; see Austin, Walker and Lardner."

VI. 27 contains the *διορισμός*, the condition for a real solution, of the problem contained in the proposition following it. The maximum of all the parallelograms having the given property which can be applied to a given straight line is that which is described upon half the line (*τὸ ἀπὸ τῆς ἡμισείας ἀναγραφόμενον*). This corresponds to the condition that an equation of the form

$$ax - px^2 = A$$

may have a real root. The correctness of the result may be seen by taking the case in which the parallelograms are rectangles, which enables us to leave out of account the *sine of the angle* of the parallelograms without any real loss of generality. Suppose the sides of the rectangle to which the *defect* is to be similar to be as b to c , b corresponding to the side of the defect which lies along AB . Suppose that $AKFG$ is any parallelogram applied to AB having the given property, that $AB = a$, and that $FK = x$. Then



$$KB = \frac{b}{c} x, \text{ and therefore } AK = a - \frac{b}{c} x.$$

Hence $\left(a - \frac{b}{c} x\right) x = S$, where S is the area of the rectangle $AKFG$.

Thus, given the equation

$$ax - \frac{b}{c} x^2 = S,$$

where S is undetermined, VI. 27 tells us that, if x is to have a real value, S cannot be greater than the rectangle CE .

$$\text{Now } CB = \frac{a}{2}, \text{ and therefore } CD = \frac{c}{b} \cdot \frac{a}{2};$$

whence

$$S \nless \frac{c}{b} \cdot \frac{a^2}{4},$$

which is just the same result as we obtain by the algebraical method.

In the particular case where the *defect* of the parallelogram is to be a *square*, the condition becomes the statement of the fact that, *if a straight line be divided into two parts, the rectangle contained by the parts cannot exceed the square on half the line*.

Now suppose that, instead of taking F on BD as in the figure of the proposition, we take F on BD produced beyond D but so that DF is less than BD .

Complete the figure, as shown, after the manner of the construction in the proposition.

Then the parallelogram $FKBH$ is similar to the given parallelogram to which the defect is to be similar. Hence the parallelogram $GAKF$ is also a parallelogram applied to AB and satisfying the given condition.

We can now prove that $GAKF$ is less than CE or AD .

Let ED produced meet AG in O .

Now, since BF is the diagonal of the parallelogram KH , the complements KD , DH are equal.

But

$DH = DG$, and DG is greater than OF .

Therefore $KD > OF$.

Add OK to each;

and

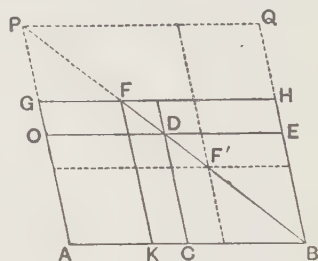
AD , or CE , $> AF$.

This other "case" of the proposition is found in all the mss., but Heiberg relegates it to the Appendix as being very obviously interpolated. The reasons for this course are that it is not in Euclid's manner to give a separate demonstration of such a "case"; it is rather his habit to give one case only and to leave the student to satisfy himself about any others (cf. 1. 7). Internal evidence is also against the genuineness of the separate proof. It is put after the *conclusion* of the proposition instead of before it, and, if Euclid had intended to discuss two cases, he would have distinguished them at the beginning of the proposition, as it was his invariable practice to do. Moreover the second "case" is the less worth giving because it can be so easily reduced to the first. For suppose F' to be taken on BD so that $FD = F'D$. Produce BF to meet AG produced in P . Complete the parallelogram $BAPQ$, and draw through F' straight lines parallel to and meeting its opposite sides.

Then the complement $F'Q$ is equal to the complement AF' .

And it is at once seen that AF , $F'Q$ are equal and similar. Hence the solution of the problem represented by AF or $F'Q$ gives a parallelogram of the same size as AF' arrived at as in the first "case."

It is worth noting that the actual difference between the parallelogram AF and the maximum area AD that it can possibly have is represented in the figure. The difference is the small parallelogram DF .



PROPOSITION 28.

To a given straight line to apply a parallelogram equal to a given rectilineal figure and deficient by a parallelogrammic figure similar to a given one: thus the given rectilineal figure must not be greater than the parallelogram described on the half of the straight line and similar to the defect.

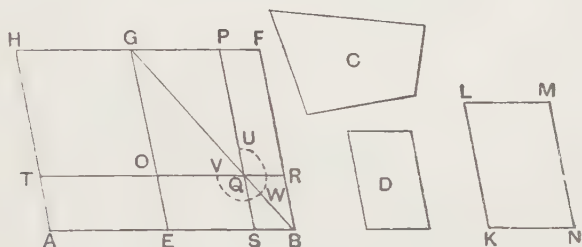
Let AB be the given straight line, C the given rectilineal figure to which the figure to be applied to AB is required to be equal, not being greater than the parallelogram described on the half of AB and similar to the defect, and D the parallelogram to which the defect is required to be similar;

thus it is required to apply to the given straight line AB a parallelogram equal to the given rectilinear figure C and deficient by a parallelogrammic figure which is similar to D .

Let AB be bisected at the point E , and on EB let $EBFG$ be described similar and similarly situated to D ; [vi. 18] let the parallelogram AG be completed.

If then AG is equal to C , that which was enjoined will have been done;

for there has been applied to the given straight line AB the parallelogram AG equal to the given rectilinear figure C and deficient by a parallelogrammic figure GB which is similar to D .



But, if not, let HE be greater than C .

Now HE is equal to GB ;

therefore GB is also greater than C .

Let $KLMN$ be constructed at once equal to the excess by which GB is greater than C and similar and similarly situated to D . [vi. 25]

But D is similar to GB ;

therefore KM is also similar to GB . [vi. 21]

Let, then, KL correspond to GE , and LM to GF .

Now, since GB is equal to C , KM ,

therefore GB is greater than KM ;

therefore also GE is greater than KL , and GF than LM .

Let GO be made equal to KL , and GP equal to LM ; and let the parallelogram $OGPQ$ be completed;

therefore it is equal and similar to KM .

Therefore GQ is also similar to GB ; [vi. 21]

therefore GQ is about the same diameter with GB . [vi. 26]

Let GQB be their diameter, and let the figure be described.
 Then, since BG is equal to C , KM ,
 and in them GQ is equal to KM ,
 therefore the remainder, the gnomon UWV , is equal to the remainder C .

And, since PR is equal to OS ,

let QB be added to each ;

therefore the whole PB is equal to the whole OB .

But OB is equal to TE , since the side AE is also equal to the side EB ; [1. 36]

therefore TE is also equal to PB .

Let OS be added to each ;

therefore the whole TS is equal to the whole, the gnomon VWU .

But the gnomon VWU was proved equal to C ;

therefore TS is also equal to C .

Therefore to the given straight line AB there has been applied the parallelogram ST equal to the given rectilinear figure C and deficient by a parallelogrammic figure QB which is similar to D .

Q. E. F.

The second part of the enunciation of this proposition which states the *διορισμός* appears to have been considerably amplified, but not improved in the process, by Theon. His version would read as follows. "But the given rectilinear figure, that namely to which the applied parallelogram must be equal (ψ δὲ ἴσον παραβαλεῖν), must not be greater than that applied to the half (*παραβαλλομένου* instead of *ἀναγεγομένου*), the defects being similar, (namely) that (of the parallelogram applied) to the half and that (of the required parallelogram) which must have a similar defect" (*ὁμοίων ὄντων τῶν ἐλλειμμάτων τοῦ τε ἀπὸ τῆς ἡμισείας καὶ ὃ δὲ ὅμοιον ἐλλείπειν*). The first amplification "that to which the applied parallelogram must be equal" is quite unnecessary, since "the given rectilinear figure" could mean nothing else. The above attempt at a translation will show how difficult it is to make sense of the words at the end ; they speak of *two* defects apparently and, while one may well be the "defect on the half," the other can hardly be *the given parallelogram* "to which the defect (of the required parallelogram) must be similar." Clearly the reading given above (from P) is by far the better.

In this proposition and the next there occurs the tacit assumption (already alluded to in the note on VI. 22) that *if, of two similar parallelograms, one is greater than the other, either side of the greater is greater than the corresponding side of the less.*

As already remarked, vi. 28 is the geometrical equivalent of the solution of the quadratic equation

$$ax - \frac{b}{c}x^2 = S,$$

subject to the condition necessary to admit of a real solution, namely that

$$S \nlessgtr \frac{c}{b} \cdot \frac{a^2}{4}.$$

The corresponding proposition in the *Data* is (Prop. 58), *If a given (area) be applied (i.e. in the form of a parallelogram) to a given straight line and be deficient by a figure (i.e. a parallelogram) given in species, the breadths of the defect are given.*

To exhibit the exact correspondence between Euclid's geometrical and the ordinary algebraical method of solving the equation we will, as before (in order to avoid bringing in a constant dependent on the sine of the angle of the parallelograms), suppose the parallelograms to be rectangles. To solve the equation algebraically we change the signs and write it

$$\frac{b}{c}x^2 - ax = -S.$$

We may now complete the square by adding $\frac{c}{b} \cdot \frac{a^2}{4}$.

Thus
$$\frac{b}{c}x^2 - ax + \frac{c}{b} \cdot \frac{a^2}{4} = \frac{c}{b} \cdot \frac{a^2}{4} - S;$$

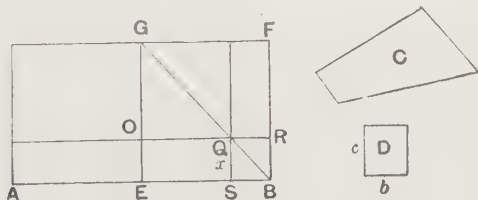
and, extracting the square root, we have

$$\sqrt{\frac{b}{c}} \cdot x - \sqrt{\frac{c}{b}} \cdot \frac{a}{2} = \pm \sqrt{\frac{c}{b} \cdot \frac{a^2}{4} - S},$$

and

$$x = \frac{c}{b} \cdot \frac{a}{2} \pm \sqrt{\frac{c}{b} \left(\frac{c}{b} \cdot \frac{a^2}{4} - S \right)}.$$

Now let us observe Euclid's method.



He first describes $GEBF$ on EB (half of AB) similar to the given parallelogram D .

He then places in one angle FGE of $GEBF$ a similar and similarly situated parallelogram GQ , equal to the difference between the parallelogram GB and the area C .

With our notation, $GO : OQ = c : b$,

whence
$$OQ = GO \cdot \frac{b}{c}.$$

gram applied to CD falling short by a rhombus (AD) with a given angle EDB . The case is thus a particular case of Prop. 58 of the *Data* quoted above (p. 263) as corresponding to VI. 28.

A particular case of the last, that namely in which the defect is a *square*, corresponding to the equation

$$ax - x^2 = b^2,$$

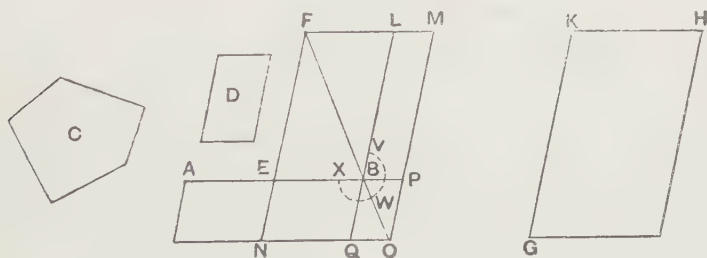
is important. This is the problem of *applying to a given straight line a rectangle equal to a given area and falling short by a square*; and it can be solved, without the aid of Book VI., as shown above under II. 5 (Vol. I. pp. 383—4).

PROPOSITION 29.

To a given straight line to apply a parallelogram equal to a given rectilineal figure and exceeding by a parallelogrammic figure similar to a given one.

Let AB be the given straight line, C the given rectilineal figure to which the figure to be applied to AB is required to be equal, and D that to which the excess is required to be similar;

thus it is required to apply to the straight line AB a parallelogram equal to the rectilineal figure C and exceeding by a parallelogrammic figure similar to D .



Let AB be bisected at E ;

let there be described on EB the parallelogram BF similar and similarly situated to D ;

and let GH be constructed at once equal to the sum of BF , C and similar and similarly situated to D . [VI. 25]

Let KH correspond to FL and KG to FE .

Now, since GH is greater than FB ,

therefore KH is also greater than FL , and KG than FE .

Let FL , FE be produced,
let FLM be equal to KH , and FEN to KG ,
and let MN be completed ;

therefore MN is both equal and similar to GH .

But GH is similar to EL ;

therefore MN is also similar to EL ; [vi. 21]

therefore EL is about the same diameter with MN . [vi. 26]

Let their diameter FO be drawn, and let the figure be described.

Since GH is equal to EL , C ,
while GH is equal to MN ,
therefore MN is also equal to EL , C .

Let EL be subtracted from each ;

therefore the remainder, the gnomon XWV , is equal to C .

Now, since AE is equal to EB ,

AN is also equal to NB [I. 36], that is, to LP [I. 43].

Let EO be added to each ;

therefore the whole AO is equal to the gnomon VWX .

But the gnomon VWX is equal to C ;

therefore AO is also equal to C .

Therefore to the given straight line AB there has been applied the parallelogram AO equal to the given rectilineal figure C and exceeding by a parallelogrammic figure QP which is similar to D , since PQ is also similar to EL [vi. 24].

Q. E. F.

The corresponding proposition in the *Data* is (Prop. 59), *If a given (area) be applied (i.e. in the form of a parallelogram) to a given straight line exceeding by a figure given in species, the breadths of the excess are given.*

The problem of vi. 29 corresponds of course to the solution of the quadratic equation

$$ax + \frac{b}{c} x^2 = S.$$

The algebraical solution of this equation gives

$$x = -\frac{c}{b} \cdot \frac{a}{2} \pm \sqrt{\frac{c}{b} \left(\frac{c}{b} \cdot \frac{a^2}{4} + S \right)}.$$

The exact correspondence of Euclid's method to the algebraical solution may be seen, as in the case of vi. 28, by supposing the parallelograms to be *rectangles*. In this case Euclid's construction on EB of the parallelogram EL similar to D is equivalent to finding that

$$FE = \frac{c}{b} \cdot \frac{a}{2}, \text{ and } EL = \frac{c}{b} \cdot \frac{a^2}{4}.$$

His determination of the similar parallelogram MN equal to the sum of EL and S corresponds to proving that

$$FN^2 \cdot \frac{b}{c} = \frac{c}{b} \cdot \frac{a^2}{4} + S,$$

or

$$FN = \sqrt{\frac{c}{b} \left(\frac{c}{b} \cdot \frac{a^2}{4} + S \right)},$$

whence x is found as

$$x = FN - FE = \sqrt{\frac{c}{b} \left(\frac{c}{b} \cdot \frac{a^2}{4} + S \right)} - \frac{c}{b} \cdot \frac{a}{2}.$$

Euclid takes, in this case, the solution corresponding to the *positive* sign before the radical because, from his point of view, that would be the *only* solution.

No διορισμός is necessary because a real geometrical solution is always possible whatever be the size of S .

Again the *Data* has a proposition indicating the particular case in which the *excess* is a *rhombus* with a given angle. Prop. 84 proves that, *If two straight lines contain a given area in a given angle, and one of the straight lines is greater than the other by a given straight line, each of the two straight lines is given also.* The proof reduces the proposition to a particular case of *Data*, Prop. 59, quoted above as corresponding to VI. 29.

Again there is an important particular case which can be solved by means of Book II. only, as shown under II. 6 above (Vol. I. pp. 386—8), the case namely in which the excess is a *square*, corresponding to the solution of the equation

$$ax + x^2 = b^2.$$

This is the problem of *applying to a given straight line a rectangle equal to a given area and exceeding by a square.*

PROPOSITION 30.

To cut a given finite straight line in extreme and mean ratio.

Let AB be the given finite straight line ;

thus it is required to cut AB in extreme and mean ratio.

On AB let the square BC be described ;

and let there be applied to AC the parallelogram CD equal to BC and exceeding by the figure AD similar to BC . [VI. 29]

Now BC is a square ;

therefore AD is also a square.

And, since BC is equal to CD ,

let CE be subtracted from each ;

therefore the remainder BF is equal to the remainder AD .



But it is also equiangular with it ;
 therefore in BF , AD the sides about the equal angles are reciprocally proportional ; [vi. 14]

therefore, as FE is to ED , so is AE to EB .

But FE is equal to AB , and ED to AE .

Therefore, as BA is to AE , so is AE to EB .

And AB is greater than AE ;

therefore AE is also greater than EB .

Therefore the straight line AB has been cut in extreme and mean ratio at E , and the greater segment of it is AE .

Q. E. F.

It will be observed that the construction in the text is a direct application of the preceding Prop. 29 in the particular case where the *excess* of the parallelogram which is applied is a *square*. This fact coupled with the position of vi. 30 is a sufficient indication that the construction is Euclid's.

In one place Theon appears to have amplified the argument. The text above says "But FE is equal to AB ," while the mss. B, F, V and p have "But FE is equal to AC , that is, to AB ."

The mss. give after $\delta\pi\epsilon\rho\ \epsilon\delta\epsilon\iota\ \pi\omicron\iota\eta\sigma\alpha\iota$ an alternative construction which Heiberg relegates to the Appendix. The text-books give this construction alone and leave out the other. It will be remembered that the alternative proof does no more than refer to the equivalent construction in II. 11.

"Let AB be cut at C so that the rectangle AB , BC is equal to the square on CA . [II. 11]

Since then the rectangle AB , BC is equal to the square on CA ,
 therefore, as BA is to AC , so is AC to CB . [VI. 17]

Therefore AB has been cut in extreme and mean ratio at C ."

It is intrinsically improbable that this alternative construction was added to the other by Euclid himself. It is however just the kind of interpolation that might be expected from an editor. If Euclid had preferred the alternative construction, he would have been more likely to give it alone.

PROPOSITION 31.

In right-angled triangles the figure on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle.

Let ABC be a right-angled triangle having the angle BAC right ;

I say that the figure on BC is equal to the similar and similarly described figures on BA , AC .

Let AD be drawn perpendicular.

Then since, in the right-angled triangle ABC , AD has

been drawn from the right angle at A perpendicular to the base BC ,

the triangles ABD , ADC adjoining the perpendicular are similar both to the whole ABC and to one another. [vi. 8]

And, since ABC is similar to ABD ,

therefore, as CB is to BA , so is AB to BD . [vi. Def. 1]

And, since three straight lines are proportional,

as the first is to the third, so is the figure on the first to the similar and similarly described figure on the second. [vi. 19, Por.]

Therefore, as CB is to BD , so is the figure on CB to the similar and similarly described figure on BA .

For the same reason also,

as BC is to CD , so is the figure on BC to that on CA ;
so that, in addition,

as BC is to BD , DC , so is the figure on BC to the similar and similarly described figures on BA , AC .

But BC is equal to BD , DC ;

therefore the figure on BC is also equal to the similar and similarly described figures on BA , AC .

Therefore etc.

Q. E. D.

As we have seen (note on I. 47), this extension of I. 47 is credited by Proclus to Euclid personally.

There is one inference in the proof which requires examination. Euclid proves that

$$CB : BD = (\text{figure on } CB) : (\text{figure on } BA),$$

$$\text{and that } BC : CD = (\text{figure on } BC) : (\text{figure on } CA),$$

and then infers directly that

$$BC : (BD + CD) = (\text{fig. on } BC) : (\text{sum of figs. on } BA \text{ and } AC).$$

Apparently v. 24 must be relied on as justifying this inference. But it is not directly applicable ; for what it proves is that, if

$$a : b = c : d,$$

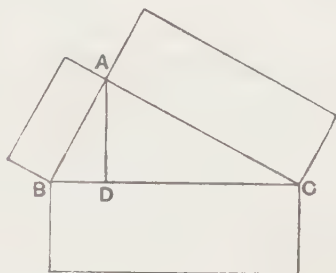
and

$$e : b = f : d,$$

then

$$(a + e) : b = (c + f) : d.$$

Thus we should *invert* the first two proportions given above (by Simson's



Prop. B which, as we have seen, is a direct consequence of the definition of proportion), and thence infer by v. 24 that

$$(BD + CD) : BC = (\text{sum of figs. on } BA, AC) : (\text{fig. on } BC).$$

But $BD + CD$ is equal to BC ;

therefore (by Simson's Prop. A, which again is an immediate consequence of the definition of proportion) the sum of the figures on BA, AC is equal to the figure on BC .

The mss. again give an alternative proof which Heiberg places in the Appendix. It first shows that the similar figures on the three sides have the same ratios to one another as the *squares* on the sides respectively. Whence, by using I. 47 and the same argument based on v. 24 as that explained above, the result is obtained.

If it is considered essential to have a proof which does not use Simson's Props. B and A or any proposition but those actually given by Euclid, no method occurs to me except the following.

Eucl. v. 22 proves that, if a, b, c are three magnitudes, and d, e, f three others, such that

$$a : b = d : e,$$

$$b : c = e : f,$$

then, *ex aequali*, $a : c = d : f$.

If now in addition $a : b = b : c$,

so that, also, $d : e = e : f$,

the ratio $a : c$ is duplicate of the ratio $a : b$, and the ratio $d : f$ duplicate of the ratio $d : e$, whence *the ratios which are duplicate of equal ratios are equal*.

$$\begin{aligned} \text{Now } (\text{fig. on } AC) : (\text{fig. on } AB) &= \text{the ratio duplicate of } AC : AB \\ &= \text{the ratio duplicate of } CD : DA \\ &= CD : BD. \end{aligned}$$

Hence $(\text{sum of figs. on } AC, AB) : (\text{fig. on } AB) = BC : BD$. [v. 18]

But $(\text{fig. on } BC) : (\text{fig. on } AB) = BC : BD$
(as in Euclid's proof).

Therefore the sum of the figures on AC, AB has to the figure on AB the same ratio as the figure on BC has to the figure on AB , whence

the figures on AC, AB are together equal to the figure on BC . [v. 9]

PROPOSITION 32.

If two triangles having two sides proportional to two sides be placed together at one angle so that their corresponding sides are also parallel, the remaining sides of the triangles will be in a straight line.

Let ABC, DCE be two triangles having the two sides BA, AC proportional to the two sides DC, DE , so that, as AB is to AC , so is DC to DE , and AB parallel to DC , and AC to DE ;

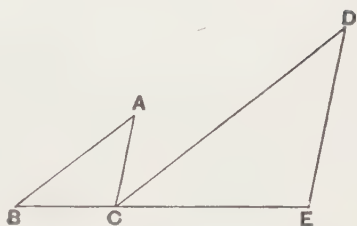
I say that BC is in a straight line with CE .

For, since AB is parallel to DC ,
and the straight line AC has fallen upon them,
the alternate angles BAC , ACD
are equal to one another. [I. 29]

For the same reason

the angle CDE is also
equal to the angle ACD ;
so that the angle BAC is equal
to the angle CDE .

And, since ABC , DCE are
two triangles having one angle, the angle at A , equal to one
angle, the angle at D ,



and the sides about the equal angles proportional,
so that, as BA is to AC , so is CD to DE ,

therefore the triangle ABC is equiangular with the
triangle DCE ; [vi. 6]

therefore the angle ABC is equal to the angle DCE .

But the angle ACD was also proved equal to the angle
 BAC ;

therefore the whole angle ACE is equal to the two angles
 ABC , BAC .

Let the angle ACB be added to each ;

therefore the angles ACE , ACB are equal to the angles BAC ,
 ACB , CBA .

But the angles BAC , ABC , ACB are equal to two right
angles; [I. 32]

therefore the angles ACE , ACB are also equal to two
right angles.

Therefore with a straight line AC , and at the point C on
it, the two straight lines BC , CE not lying on the same side
make the adjacent angles ACE , ACB equal to two right
angles ;

therefore BC is in a straight line with CE . [I. 14]

Therefore etc.

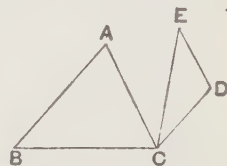
Q. E. D.

It has often been pointed out (e.g. by Clavius, Lardner and Todhunter)
that the enunciation of this proposition is not precise enough. Suppose that

ABC is a triangle. From C draw CD parallel to BA and of any length. From D draw DE parallel to CA and of such length that

$$CD : DE = BA : AC.$$

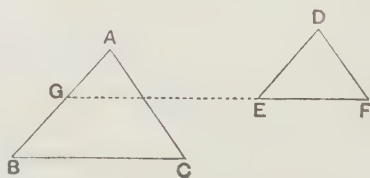
Then the triangles ABC , ECD , which have the angular point C common literally satisfy Euclid's enunciation; but by no possibility can CE be in a straight line with CB if, as in the case supposed, the angles included by the corresponding sides are supplementary (unless both are right angles). Hence the included angles must be *equal*, so that the triangles must be *similar*. That being so, if they are to have nothing more than one angular point common, and two pairs of corresponding sides are to be *parallel* as distinguished from one or both being in the *same straight line*, the triangles can only be placed so that the corresponding sides in both are on the same side of the third side of either, and the sides (other than the third sides) which meet at the common angular point are not corresponding sides.



Todhunter remarks that the proposition seems of no use. Presumably he did not know that it *is* used by Euclid himself in XIII. 17. This is so however, and therefore it was not necessary, as several writers have thought, to do away with the proposition and find a substitute which should be more useful.

1. De Morgan proposes this theorem: "If two similar triangles be placed with their bases parallel, and the equal angles at the bases towards the same parts, the other sides are parallel, each to each; or one pair of sides are in the same straight line and the other pair are parallel."

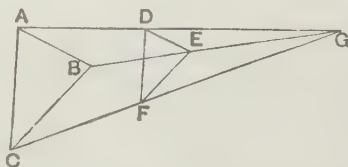
2. Dr Lachlan substitutes the somewhat similar theorem, "If two similar triangles be placed so that two sides of the one are parallel to the corresponding sides of the other, the third sides are parallel."



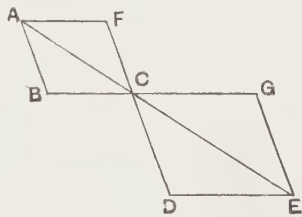
But it is to be observed that these propositions can be proved without using Book VI. at all; they can be proved from Book I., and the triangles may as well be called "equiangular" simply. It is true that Book VI. is no more than formally necessary to Euclid's proposition. He merely uses VI. 6 because his enunciation does not *say* that the triangles are similar; and he only proves them to be similar in order to conclude that they are equiangular. From this point of view Mr Taylor's substitute seems the best, viz.

3. "If two triangles have sides parallel in pairs, the straight lines joining the corresponding vertices meet in a point, or are parallel."

Simon has a theory (unnecessary in the circumstances) as to the possible object of VI. 32 as it stands. He points out that the enunciation of VI. 26 might be more general so as to cover the case of similar and similarly situated parallelograms with equal angles not coincident but vertically opposite. It can then be proved that the diagonals drawn



through the common angular point are in one straight line. If $ABCF$, $CDEG$ be similar and similarly situated parallelograms, so that BCG , DCF are straight lines, and if the diagonals AC , CE be drawn, the triangles ABC , CDE are similar and are placed exactly as described in VI. 32, so that AC , CE are in a straight line. Hence Simson suggests that there may have been, in addition to the indirect demonstration in VI. 26, a *direct* proof covering the case just given which may have used the result of VI. 32. I think however that the place given to the latter proposition in Book VI. is against this view.

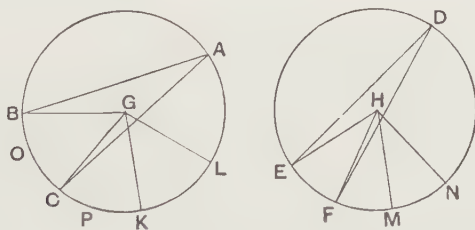


PROPOSITION 33.

In equal circles angles have the same ratio as the circumferences on which they stand, whether they stand at the centres or at the circumferences.

Let ABC , DEF be equal circles, and let the angles BGC , EHF be angles at their centres G , H , and the angles BAC , EDF angles at the circumferences;

I say that, as the circumference BC is to the circumference EF , so is the angle BGC to the angle EHF , and the angle BAC to the angle EDF .



For let any number of consecutive circumferences CK , KL be made equal to the circumference BC ,

and any number of consecutive circumferences FM , MN equal to the circumference EF ;

and let GK , GL , HM , HN be joined.

Then, since the circumferences BC , CK , KL are equal to one another,

the angles BGC , CGK , KGL are also equal to one another;

[III. 27]

therefore, whatever multiple the circumference BL is of BC , that multiple also is the angle BGL of the angle BGC .

For the same reason also, whatever multiple the circumference NE is of EF , that multiple also is the angle NHE of the angle EHF .

If then the circumference BL is equal to the circumference EN , the angle BGL is also equal to the angle EHN ; [III. 27] if the circumference BL is greater than the circumference EN , the angle BGL is also greater than the angle EHN ; and, if less, less.

There being then four magnitudes, two circumferences BC , EF , and two angles BGC , EHF , there have been taken, of the circumference BC and the angle BGC equimultiples, namely the circumference BL and the angle BGL ,

and of the circumference EF and the angle EHF equimultiples, namely the circumference EN and the angle EHN .

And it has been proved that, if the circumference BL is in excess of the circumference EN , the angle BGL is also in excess of the angle EHN ; if equal, equal; and if less, less.

Therefore, as the circumference BC is to EF , so is the angle BGC to the angle EHF . [v. Def. 5]

But, as the angle BGC is to the angle EHF , so is the angle BAC to the angle EDF ; for they are doubles respectively.

Therefore also, as the circumference BC is to the circumference EF , so is the angle BGC to the angle EHF , and the angle BAC to the angle EDF .

Therefore etc.

Q. E. D.

This proposition as generally given includes a second part relating to *sectors* of circles, corresponding to the following words added to the enunciation: "and further the sectors, as constructed at the centres" (ἐτι δὲ καὶ οἱ τομεῖς ἅτε [or οὔτε] πρὸς τοῖς κέντροις συνιστάμενοι). There is of course a corresponding addition to the "definition" or "particular statement," "and further the sector $GBOC$ to the sector $HEQF$." These additions are clearly due to Theon, as may be gathered from his own statement in his commentary on the *μαθηματικὴ σύνταξις* of Ptolemy, "But that sectors in equal circles are to one another as the angles on which they stand, has been proved by me in my edition of the

Elements at the end of the sixth book." Campanus omits them, and P has them only in a later hand in the margin or between the lines. Theon's proof scarcely needs to be given here in full, as it can easily be supplied. From the equality of the arcs BC , CK he infers [III. 29] the equality of the chords BC , CK . Hence, the radii being equal, the triangles GBC , GCK are equal in all respects [I. 8, 4]. Next, since the arcs BC , CK are equal, so are the arcs BAC , CAK . Therefore the angles at the circumference subtended by the latter, i.e. the angles in the segments BOC , CPK , are equal [III. 27], and the segments are therefore similar [III. Def. 11] and equal [III. 24].

Adding to the equal segments the equal triangles GBC , GCK respectively, we see that

the sectors GBC , GCK are equal.

Thus, in equal circles, sectors standing on equal arcs are equal; and the rest of the proof proceeds as in Euclid's proposition.

As regards Euclid's proposition itself, it will be noted that (1), besides quoting the theorem in III. 27 that in equal circles angles which stand on equal arcs are equal, the proof assumes that the angle standing on a greater arc is greater and that standing on a less arc is less. This is indeed a sufficiently obvious deduction from III. 27.

(2) Any equimultiples *whatever* are taken of the angle BGC and the arc BC , and any equimultiples *whatever* of the angle EHF and the arc EF . (Accordingly the words "*any equimultiples whatever*" should have been used in the step immediately preceding the inference that the angles are proportional to the arcs, where the text merely states that there have been taken of the circumference BC and the angle BGC *equimultiples* BL and BGL .) But, if *any multiple* of an angle is regarded as being itself an angle, it follows that the restriction in I. Def. 8, 10, 11, 12 of the term *angle* to an angle *less than two right angles* is implicitly given up; as De Morgan says, "the angle breaks prison." Mr Dodgson (*Euclid and his Modern Rivals*, p. 193) argues that Euclid conceived of the multiple of an angle as so many separate angles not added together into one, and that, when it is inferred that, where two such multiples of an angle are equal, the arcs subtended are also equal, the argument is that the sum total of the first set of angles is equal to the sum total of the second set, and hence the second set can be broken up and put together again in such amounts as to make a set equal, each to each, to the first set, and then the sum total of the arcs will evidently be equal also. If on the other hand the multiples of the angles are regarded as single angular magnitudes, the equality of the subtending arcs is not inferrible directly from Euclid, because *his* proof of III. 26 only applies to cases where the angle is less than the sum of two right angles. (As a matter of fact, it is a question of inferring equality of angles or multiples of angles from equality of arcs, and not the converse, so that the reference should have been to III. 27, but this does not affect the question at issue.) Of course it is against this view of Mr Dodgson that Euclid speaks throughout of "*the angle* BGL " and "*the angle* EHN " ($\eta \dot{\upsilon} \pi \acute{o} B\Lambda\Delta \gamma \omega \nu \iota \alpha$, $\eta \dot{\upsilon} \pi \acute{o} E\Theta\Lambda \gamma \omega \nu \iota \alpha$). I think the probable explanation is that here, as in III. 20, 21, 26 and 27, Euclid deliberately took no cognisance of the case in which the multiples of the angles in question would be greater than two right angles. If his attention had been called to the fact that III. 20 takes no account of the case where the segment is less than a semicircle, so that the angle in the segment is obtuse, and therefore the "angle at the centre" in that case (if the term were still applicable) would be

greater than two right angles, Euclid would no doubt have refused to regard the latter as an angle, and would have represented it otherwise, e.g. as the sum of two angles or as what is left when an *angle* in the true sense is subtracted from four right angles. Here then, if Euclid had been asked what course he would take if the multiples of the angles in question should be greater than two right angles, he would probably have represented them, I think, as being *equal to so many right angles plus an angle less than a right angle*, or *so many times two right angles plus an angle, acute or obtuse*. Then the equality of the arcs would be the equality of the sums of so many circumferences, semi-circumferences or quadrants plus arcs less than a semicircle or a quadrant. Hence I agree with Mr Dodgson that vi. 33 affords no evidence of a recognition by Euclid of "angles" greater than two right angles.

Theon adds to his theorem about sectors the Porism that, *As the sector is to the sector, so also is the angle to the angle*. This corollary was used by Zenodorus in his tract *περὶ ἰσομέτρων σχημάτων* preserved by Theon in his commentary on Ptolemy's *σύνταξις*, unless indeed Theon himself interpolated the words (ὡς δ' ὁ τομεὺς πρὸς τὸν τομέα, ἢ ὑπὸ ΕΘΛ γωνία πρὸς τὴν ὑπὸ ΜΘΛ).

BOOK XI.

DEFINITIONS.

1. A **solid** is that which has length, breadth, and depth.
2. An extremity of a solid is a surface.
3. A **straight line** is **at right angles to a plane**, when it makes right angles with all the straight lines which meet it and are in the plane.
4. A **plane** is **at right angles to a plane** when the straight lines drawn, in one of the planes, at right angles to the common section of the planes are at right angles to the remaining plane.
5. The **inclination of a straight line to a plane** is, assuming a perpendicular drawn from the extremity of the straight line which is elevated above the plane to the plane, and a straight line joined from the point thus arising to the extremity of the straight line which is in the plane, the angle contained by the straight line so drawn and the straight line standing up.
6. The **inclination of a plane to a plane** is the acute angle contained by the straight lines drawn at right angles to the common section at the same point, one in each of the planes.
7. A plane is said to be **similarly inclined** to a plane as another is to another when the said angles of the inclinations are equal to one another.
8. **Parallel planes** are those which do not meet.

9. **Similar solid figures** are those contained by similar planes equal in multitude.

10. **Equal and similar solid figures** are those contained by similar planes equal in multitude and in magnitude.

11. A **solid angle** is the inclination constituted by more than two lines which meet one another and are not in the same surface, towards all the lines.

Otherwise: A **solid angle** is that which is contained by more than two plane angles which are not in the same plane and are constructed to one point.

12. A **pyramid** is a solid figure, contained by planes, which is constructed from one plane to one point.

13. A **prism** is a solid figure contained by planes two of which, namely those which are opposite, are equal, similar and parallel, while the rest are parallelograms.

14. When, the diameter of a semicircle remaining fixed, the semicircle is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a **sphere**.

15. The **axis of the sphere** is the straight line which remains fixed and about which the semicircle is turned.

16. The **centre of the sphere** is the same as that of the semicircle.

17. A **diameter of the sphere** is any straight line drawn through the centre and terminated in both directions by the surface of the sphere.

18. When, one side of those about the right angle in a right-angled triangle remaining fixed, the triangle is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a **cone**.

And, if the straight line which remains fixed be equal to the remaining side about the right angle which is carried round, the cone will be **right-angled**; if less, **obtuse-angled**; and if greater, **acute-angled**.

19. The **axis of the cone** is the straight line which remains fixed and about which the triangle is turned.

20. And the **base** is the circle described by the straight line which is carried round.

21. When, one side of those about the right angle in a rectangular parallelogram remaining fixed, the parallelogram is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a **cylinder**.

22. The **axis of the cylinder** is the straight line which remains fixed and about which the parallelogram is turned.

23. And the **bases** are the circles described by the two sides opposite to one another which are carried round.

24. **Similar cones and cylinders** are those in which the axes and the diameters of the bases are proportional.

25. A **cube** is a solid figure contained by six equal squares.

26. An **octahedron** is a solid figure contained by eight equal and equilateral triangles.

27. An **icosahedron** is a solid figure contained by twenty equal and equilateral triangles.

28. A **dodecahedron** is a solid figure contained by twelve equal, equilateral, and equiangular pentagons.

DEFINITION I.

Στερεόν ἐστι τὸ μῆκος καὶ πλάτος καὶ βάθος ἔχον.

This definition was evidently traditional, as may be inferred from a number of passages in Plato and Aristotle. Thus Plato speaks (*Sophist*, 235 D) of making an imitation of a model (*παράδειγμα*) "in length and breadth and depth" and (*Laws*, 817 E) of "the art of measuring length, surface and depth" as one of three *μαθήματα*. *Depth*, the third dimension, is used alone as a description of "body" by Aristotle, the term being regarded as connoting the other two dimensions; thus (*Metaph.* 1020 a 13, 11) "length is a line, breadth a surface, and depth body"; "that which is continuous in one direction is length, in two directions breadth, and in three depth." Similarly Plato (*Rep.* 528 B, D), when reconsidering his classification of astronomy as next to (plane) geometry: "although the science dealing with the additional dimension of depth is next in order, yet, owing to the fact that it is studied absurdly, I passed it over and put next to geometry astronomy, the *motion* of (bodies having) depth." In Aristotle (*Tòpics* VI. 5, 142 b 24) we find "the definition of body, that which has three dimensions (*διαστάσεις*)"; elsewhere he speaks of it as "that which has all the dimensions" (*De caelo* I. 1, 268 b 6), "that which has dimension every way" (*τὸ πάντῃ διάστασιν ἔχον*, *Metaph.* 1066 b 32) etc. In the *Physics*

(IV. 1, 208 b 13 sqq.) he speaks of the "dimensions" as *μέτρα*, dividing each of the three into two opposites, "up and down, before and behind, right and left," though of course, as he explains, these terms are relative.

Heron, as might be expected, combines the two forms of the definition. "A solid body is that which has length, breadth, and depth: or that which possesses the three dimensions." (Def. 11.)

Similarly Theon of Smyrna (p. 111, 19, ed. Hiller): "that which is extended (*διαστατόν*) and divisible in three directions is solid, having length, breadth and depth."

DEFINITION 2.

Στερεοῦ δὲ πέρασ ἐπιφάνεια.

In like manner Aristotle says (*Metaph.* 1066 b 23) that the notion (*λόγος*) of body is "that which is bounded by surfaces" (*ἐπιπέδοις* in this case) and (*Metaph.* 1060 b 15) "surfaces (*ἐπιφάνειαι*) are divisions of bodies."

So Heron (Def. 11): "Every solid is bounded (*περατοῦται*) by surfaces, and is produced when a surface is moved from a forward position in a backward direction."

DEFINITION 3.

Εὐθεῖα πρὸς ἐπίπεδον ὀρθή ἐστιν, ὅταν πρὸς πάσας τὰς ἀπτομένας αὐτῆς εὐθείας καὶ οὐσας ἐν τῷ ἐπιπέδῳ ὀρθὰς ποιῇ γωνίας.

This definition and the next are given almost word for word by Heron (Def. 115).

That a straight line *can* be so related to a plane as described in Def. 3 is established in XI. 4. The fact has been made the basis of a definition of a *plane* which is attributed by Crelle to Fourier, and is as follows. "A plane is formed by the totality of all the straight lines which, passing through one and the same point of a straight line in space, stand perpendicular to it." Stated in this form, the definition is open to the objection that the conception of a right angle, involving the measurement of angles, presupposes a plane, inasmuch as the measurement of angles depends ultimately upon the superposition of two planes and their coincidence throughout when two lines in one coincide with two lines in the other respectively. Cf. my note on I. Def. 7, Vol. I. pp. 173—5.

DEFINITION 4.

Ἐπίπεδον πρὸς ἐπίπεδον ὀρθόν ἐστιν, ὅταν αἱ τῇ κοινῇ τομῇ τῶν ἐπιπέδων πρὸς ὀρθὰς ἀγόμεναι εὐθεῖαι ἐν ἐνὶ τῶν ἐπιπέδων τῷ λοιπῷ ἐπιπέδῳ πρὸς ὀρθὰς ὦσιν.

Both this definition and Def. 6 use the *common section* of two planes, though it is not till XI. 3 that this common section is proved to be a straight line. The definition however, just like Def. 3, is legitimate, because the object is to explain the meaning of terms, not to prove anything.

The definition of perpendicular planes is made by Legendre a particular case of Def. 6, the limiting case, namely, where the angle representing the "inclination of a plane to a plane" is a right angle.

DEFINITION 5.

Εὐθείας πρὸς ἐπίπεδον κλίσις ἐστίν, ὅταν ἀπὸ τοῦ μετεώρου πέρατος τῆς εὐθείας ἐπὶ τὸ ἐπίπεδον κάθετος ἀχθῇ, καὶ ἀπὸ τοῦ γενομένου σημείου ἐπὶ τὸ ἐν τῷ ἐπιπέδῳ πέρασ τῆς εὐθείας εὐθεῖα ἐπιζευχθῇ, ἡ περιεχομένη γωνία ὑπὸ τῆς ἀχθείσης καὶ τῆς ἐφεστώσης.

In other words, the inclination of a straight line to a plane is the angle between the straight line and its *projection* on the plane. This angle is of course less than the angle between the straight line and any other straight line in the plane through the intersection of the straight line and plane; and the fact is sometimes made the subject of a proposition in modern text-books. It is easily proved by means of the propositions XI. 4, I. 19 and 18.

DEFINITION 6.

Ἐπιπέδου πρὸς ἐπίπεδον κλίσις ἐστὶν ἡ περιεχομένη ὀξεία γωνία ὑπὸ τῶν πρὸς ὀρθὰς τῇ κοινῇ τομῇ ἀγομένων πρὸς τῷ αὐτῷ σημείῳ ἐν ἑκατέρῳ τῶν ἐπιπέδων.

When two planes meet in a straight line, they form what is called in modern text-books a *dihedral angle*, which is defined as the *opening* or *angular opening* between the two planes. This *dihedral angle* is an "angle" altogether different in kind from a plane angle, as again it is different from a *solid angle* as defined by Euclid (i.e. a trihedral, tetrahedral, etc. angle). Adopting for the moment Apollonius' conception of an angle as the "bringing together of a surface or solid towards one point under a broken line or surface" (Proclus, p. 123, 16), we may regard a dihedral angle as the bringing together of the broken surface formed by two intersecting planes not to a *point* but to a *straight line*, namely the intersection of the planes. Legendre, in a proposition on the subject, applied provisionally the term *corner* to describe the dihedral angle between two planes; and this would be a better word, I think, than *opening* to use in the definition.

The distinct species of "angle" which we call dihedral is, however, *measured* by a certain plane angle, namely that which Euclid describes in the present definition and calls the *inclination of a plane to a plane*, and which in some modern text-books is called the *plane angle of the dihedral angle*.

It is necessary to show that this plane angle is a proper measure of the dihedral angle, and accordingly Legendre has a proposition to this effect. In order to prove it, it is necessary to show that, given two planes meeting in a straight line,

- (1) the plane angle in question is the same at all points of the straight line forming the common section;
- (2) if the dihedral angle between two planes increases or diminishes in a certain ratio, the plane angle in question will increase or diminish in the same ratio.

(1) If MAN , MAP be two planes intersecting in MA , and if AN , AP be drawn in the planes respectively and at right angles to MA , the angle NAP is the *inclination of the plane to the plane* or the *plane angle of the dihedral angle*.

Let MC , MB be also drawn in the respective planes at right angles to MA .

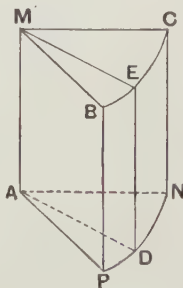
Then since, in the plane MAN , MC and AN are drawn at right angles to the same straight line MA ,

MC , AN are parallel.

For the same reason, MB , AP are parallel.

Therefore [XI. 10] the angle BMC is equal to the angle PAN .

And M may be any point on MA . Therefore the plane angle described in the definition is the same at all points of AM .



(2) In the plane NAP draw the arc NDP of any circle with centre A , and draw the radius AD .

Now the planes NAP , CMB , being both at right angles to the straight line MA , are parallel; [xi. 14]

therefore the intersections AD , ME of these planes with the plane MAD are parallel, [xi. 16]

and consequently the angles BME , PAD are equal. [xi. 10]

If now the plane angle NAD were equal to the plane angle DAP , the dihedral angle $NAMD$ would be equal to the dihedral angle $DAMP$;

for, if the angle PAD were applied to the angle DAN , AM remaining the same, the corresponding dihedral angles would coincide.

Successive applications of this result show that, if the angles NAD , DAP each contain a certain angle a certain number of times, the dihedral angles $NAMD$, $DAMP$ will contain the corresponding dihedral angle the same number of times respectively.

Hence, where the angles NAD , DAP are *commensurable*, the dihedral angles corresponding to them are in the same ratio.

Legendre then extends the proof to the case where the plane angles are incommensurable by reference to an exactly similar extension in his proposition corresponding to Euclid vi. 1, for which see the note on that proposition.

Modern text-books make the extension by an appeal to *limits*.

DEFINITION 7.

Ἐπίπεδον πρὸς ἐπίπεδον ὁμοίως κεκλίωθαι λέγεται καὶ ἕτερον πρὸς ἕτερον, ὅταν αἱ εἰρημέναι τῶν κλίσεων γωνίαι ἴσαι ἀλλήλαις ᾖσιν.

DEFINITION 8.

Παράλληλα ἐπίπεδά ἐστι τὰ ἀσύμπτωτα.

Heron has the same definition of parallel planes (Def. 115). The Greek word which is translated "which do not meet" is ἀσύμπτωτα, the term which has been adopted for the *asymptotes* of a curve.

DEFINITION 9.

Ὅμοια στερεὰ σχήματά ἐστι τὰ ὑπὸ ὁμοίων ἐπιπέδων περιεχόμενα ἴσων τῷ πλήθος.

DEFINITION 10.

Ἰσα δὲ καὶ ὅμοια στερεὰ σχήματά ἐστι τὰ ὑπὸ ὁμοίων ἐπιπέδων περιεχόμενα ἴσων τῷ πλήθει καὶ τῷ μεγέθει.

These definitions, the second of which practically only substitutes the words "equal and similar" for the word "similar" in the first, have been the mark of much criticism.

Simson holds that the equality of solid figures is a thing which ought to be *proved*, by the method of superposition, or otherwise, and hence that Def. 10 is not a definition but a *theorem* which ought not to have been placed among the definitions. Secondly, he gives an example to show that the definition or theorem is not universally true. He takes a pyramid and then erects on the base, on opposite sides of it, two equal pyramids smaller than the first. The addition and subtraction of these pyramids respectively from the first give two

solid figures which satisfy the definition but are clearly not equal (the smaller having a re-entrant angle); whence it also appears that two unequal solid angles may be contained by the same number of equal plane angles.

Maintaining then that Def. 10 is an interpolation by "an unskilful hand," Simson transfers to a place before Def. 9 the definition of a solid angle, and then defines similar solid figures as follows:

Similar solid figures are such as have all their solid angles equal, each to each, and which are contained by the same number of similar planes.

Legendre has an invaluable discussion of the whole subject of these definitions (Note XII., pp. 323—336, of the 14th edition of his *Éléments de Géométrie*). He remarks in the first place that, as Simson said, Def. 10 is not properly a definition, but a theorem which it is necessary to prove; for it is not evident that two solids are equal for the sole reason that they have an equal number of equal faces, and, if true, the fact should be proved by superposition or otherwise. The fault of Def. 10 is also common to Def. 9. For, if Def. 10 is not proved, one might suppose that there exist two unequal and dissimilar solids with equal faces; but, in that case, according to Definition 9, a solid having faces similar to those of the two first would be similar to both of them, i.e. to two solids of different form: a conclusion implying a contradiction or at least not according with the natural meaning of the word "similar."

What then is to be said in defence of the two definitions as given by Euclid? It is to be observed that the figures which Euclid actually proves equal or similar by reference to Deff. 9, 10 are such that their solid angles do not consist of more than *three* plane angles; and he proves sufficiently clearly that, if three plane angles forming one solid angle be respectively equal to three plane angles forming another solid angle, the two solid angles are equal. If now two polyhedra have their faces equal respectively, the corresponding solid angles will be made up of the same number of plane angles, and the plane angles forming each solid angle in one polyhedron will be respectively equal to the plane angles forming the corresponding solid angle in the other. Therefore, if the plane angles in each solid angle are not more than three in number, the corresponding solid angles will be equal. But if the corresponding faces are equal, and the corresponding solid angles equal, the solids must be equal; for they can be superposed, or at least they will be symmetrical with one another. Hence the statement of Deff. 9, 10 is true and admissible at all events in the case of figures with trihedral angles, which is the only case taken by Euclid.

Again, the example given by Simson to prove the incorrectness of Def. 10 introduces a solid with a re-entrant angle. But it is more than probable that Euclid deliberately intended to exclude such solids and to take cognizance of *convex* polyhedra only; hence Simson's example is not conclusive against the definition.

Legendre observes that Simson's own definition, though true, has the disadvantage that it contains a number of superfluous conditions. To get over the difficulties, Legendre himself divides the definition of similar solids into two, the first of which defines similar *triangular pyramids* only, and the second (which defines similar polyhedra in general) is based on the first.

Two triangular pyramids are similar when they have pairs of faces respectively similar, similarly placed and equally inclined to one another.

Then, having formed a triangle with the vertices of three angles taken on the same face or base of a polyhedron, we may imagine the vertices of the

different solid angles of the polyhedron situated outside of the plane of this base to be the vertices of as many triangular pyramids which have the triangle for common base, and each of these pyramids will determine the position of one solid angle of the polyhedron. This being so,

Two polyhedra are similar when they have similar bases, and the vertices of their corresponding solid angles outside the bases are determined by triangular pyramids similar each to each.

As a matter of fact, Cauchy proved that two *convex* solid figures are equal if they are contained by equal plane figures similarly arranged. Legendre gives a proof which, he says, is nearly the same as Cauchy's, depending on two lemmas which lead to the theorem that, *Given a convex polyhedron in which all the solid angles are made up of more than three plane angles, it is impossible to vary the inclinations of the planes of this solid so as to produce a second polyhedron formed by the same planes arranged in the same manner as in the given polyhedron.* The convex polyhedron in which all the solid angles are made up of more than three plane angles is obtained by cutting off from any given polyhedron all the triangular pyramids forming trihedral angles (if one and the same edge is common to two trihedral angles, only one of these angles is suppressed in the first operation). This is legitimate because trihedral angles are invariable from their nature.

Hence it would appear that Heron's definition of equal solid figures, which adds "similarly situated" to Euclid's "similar" is correct, if it be understood to apply to *convex* polyhedra only: *Equal solid figures are those which are contained by equal and similarly situated planes, equal in number and magnitude:* where, however, the words "equal and" before "similarly situated" might be dispensed with.

Heron (Def. 118) defines *similar solid figures* as *those which are contained by planes similar and similarly situated.* If understood of *convex* polyhedra, there would not appear to be any objection to this, in view of the truth of Cauchy's proposition about equal solid figures.

DEFINITION 11.

Στερεὰ γωνία ἐστὶν ἡ ὑπὸ πλειόνων ἢ δύο γραμμῶν ἀπτομένων ἀλλήλων καὶ μὴ ἐν τῇ αὐτῇ ἐπιφανείᾳ οὐσῶν πρὸς πάσαις ταῖς γραμμαῖς κλίσις. Ἄλλως· στερεὰ γωνία ἐστὶν ἡ ὑπὸ πλειόνων ἢ δύο γωνιῶν ἐπιπέδων περιεχομένη μὴ οὐσῶν ἐν τῷ αὐτῷ ἐπιπέδῳ πρὸς ἐνὶ σημείῳ συνισταμένων.

Heiberg conjectures that the first of these two definitions, which is not in Euclid's manner, was perhaps taken by him from some earlier *Elements*.

The phraseology of the second definition is exactly that of Plato when he is speaking of solid angles in the *Timaeus* (p. 55). Thus he speaks (1) of four equilateral triangles so put together (ξυνιστάμενα) that each set of three plane angles makes one solid angle, (2) of eight equilateral triangles put together so that each set of four plane angles makes one solid angle, and (3) of six squares making eight solid angles, each composed of three plane right angles.

As we know, Apollonius defined an angle as the "bringing together of a surface or solid to one point under a broken line or surface." Heron (Def. 22) even omits the word "broken" and says that *A solid angle is in general (κοινῶς) the bringing together of a surface which has its concavity in one and the same direction to one point.* It is clear from an allusion in Proclus (p. 123, 1—6) to the half of a cone cut off by a triangle through the axis, and from a scholium to

this definition, that there was controversy as to the correctness of describing as a solid angle the "angle" enclosed by fewer than three surfaces (including curved surfaces). Thus the scholiast says that Euclid's definition of a solid angle as made up of three or more plane angles is deficient because it does not e.g. cover the case of the angle of a "fourth part of a sphere," which is contained by more than two surfaces, though not all plane. But he declines to admit that the half-cone forms a solid angle at the vertex, for in that case the vertex of the cone would itself be an angle, and a solid angle would then be formed both by two surfaces and by one surface: "which is not true." Heron on the other hand (Def. 22) distinctly speaks of solid angles which are not contained by plane rectilinear angles, "e.g. the angles of cones." The conception of the latter "angles" as the *limit* of solid angles with an infinite number of infinitely small constituent plane angles does not appear in the Greek geometers so far as I know.

In modern text-books a polyhedral angle is usually spoken of as *formed* (or *bounded*) by *three or more planes meeting at a point*, or it is *the angular opening between such planes at the point where they meet*.

DEFINITION 12.

Πυραμὶς ἐστὶ σχῆμα στερεὸν ἐπιπέδοις περιεχόμενον ἀπὸ ἐνὸς ἐπιπέδου πρὸς ἐνὶ σημείῳ συνεστῶς.

This definition is by no means too clear, nor is the slightly amplified definition added to it by Heron (Def. 99). *A pyramid is the figure brought together to one point, by putting together triangles, from a triangular, quadrilateral or polygonal, that is, any rectilinear, base.*

As we might expect, there is great variety in the definitions given in modern text-books. Legendre says *a pyramid is the solid formed when several triangular planes start from one point and are terminated at the different sides of one polygonal plane.*

Mr H. M. Taylor and Smith and Bryant call it *a polyhedron all but one of whose faces meet in a point.*

Mehler reverses Legendre's form and gives the content of Euclid's in clearer language. "*An n-sided pyramid is bounded by an n-sided polygon as base and n triangles which connect its sides with one and the same point outside it.*"

Rausenberger points out that a pyramid is the figure cut off from a solid angle formed of any number of plane angles by a plane which intersects the solid angle.

DEFINITION 13.

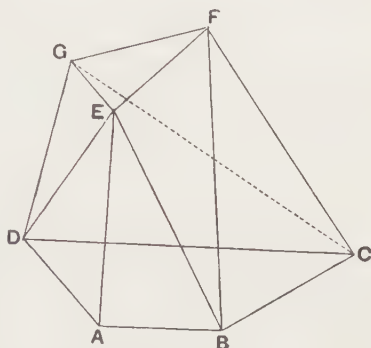
Πρίσμα ἐστὶ σχῆμα στερεὸν ἐπιπέδοις περιεχόμενον, ὧν δύο τὰ ἀπεναντίον ἴσα τε καὶ ὁμοῖά ἐστι καὶ παράλληλα, τὰ δὲ λοιπὰ παράλληλόγραμμα.

Mr H. M. Taylor, followed by Smith and Bryant, defines a prism as a *polyhedron all but two of the faces of which are parallel to one straight line.*

Mehler calls an *n-sided prism a body contained between two parallel planes and enclosed by n other planes with parallel lines of intersection.*

Heron's definition of a prism is much wider (Def. 105). *Prisms are those figures which are connected (συνάπτοντα) from a rectilinear base to a rectilinear area by rectilinear collocation (κατ' εὐθύγραμμον σύνθεσιν).* By this Heron must

apparently mean any convex solid formed by connecting the sides and angles of two polygons in different planes, and each having any number of sides, by straight lines forming triangular faces (where of course two adjacent triangles may be in one plane and so form one quadrilateral face) in the manner shown in the annexed figure, where $ABCD$, EFG represent the base and its opposite.



Heron goes on to explain that, if the face opposite to the base reduces to a straight line, and a solid is formed by connecting the base to its extremities by straight lines, as in the other case, the resulting figure is neither a pyramid nor a prism.

Further, he defines *parallelogrammic* (in the body of the definition *parallel-sided*) *prisms* as being those prisms which have six faces and have their opposite planes parallel.

DEFINITION 14.

Σφαῖρά ἐστιν, ὅταν ἡμικυκλίου μενούσης τῆς διαμέτρου περιενεχθὲν τὸ ἡμικύκλιον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, τὸ περιληφθὲν σχῆμα.

The scholiast observes that this definition is not properly a definition of a sphere but a description of the mode of generating it. But it will be seen, in the last propositions of Book XIII., why Euclid put the definition in this form. It is because it is this particular view of a sphere which he uses to prove that the vertices of the regular solids which he wishes to "comprehend" in certain spheres do lie on the surfaces of those spheres. He proves in fact that the said vertices lie on *semicircles described on certain diameters of the spheres*. For the real definition the scholiast refers to Theodosius' *Sphaerica*. But of course the proper definition was given much earlier. In Aristotle the characteristic of a sphere is that *its extremity is equally distant from its centre* (τὸ ἴσον ἀπέχειν τοῦ μέσου τὸ ἔσχατον, *De caelo* II. 14, 297 a 24). Heron (Def. 76) uses the same form as that in which Euclid defines the circle: *A sphere is a solid figure bounded by one surface, such that all the straight lines falling on it from one point of those which lie within the figure are equal to one another*. So the usual definition in the text-books: *A sphere is a closed surface such that all points of it are equidistant from a fixed point within it*.

DEFINITION 15.

*Ἄξων δὲ τῆς σφαίρας ἐστὶν ἡ μένουσα εὐθεῖα, περὶ ἣν τὸ ἡμικύκλιον στρέφεται.

That any diameter of a sphere may be called an axis is made clear by Heron (Def. 78). *The diameter of the sphere is called an axis, and is any straight line drawn through the centre and bounded in both directions by the sphere, immovable, about which the sphere is moved and turned*. Cf. Euclid's Def. 17.

DEFINITION 16.

Κέντρον δὲ τῆς σφαίρας ἐστὶ τὸ αὐτό, ὃ καὶ τοῦ ἡμικυκλίου.

Heron, Def. 77. *The middle (point) of the sphere is called its centre ; and this same point is also the centre of the hemisphere.*

DEFINITION 17.

Διάμετρος δὲ τῆς σφαίρας ἐστὶν εὐθεῖα τις διὰ τοῦ κέντρου ἡγμένη καὶ περατουμένη ἐφ' ἑκάτερα τὰ μέρη ὑπὸ τῆς ἐπιφανείας τῆς σφαίρας.

DEFINITION 18.

Κῶνός ἐστιν, ὅταν ὀρθογωνίου τριγώνου μενούσης μιᾶς πλευρᾶς τῶν περὶ τὴν ὀρθὴν γωνίαν περιεχθὲν τὸ τρίγωνον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, τὸ περιληφθὲν σχῆμα. καὶ μὲν ἡ μένουσα εὐθεῖα ἴση ἢ τῇ λοιπῇ [τῇ] περὶ τὴν ὀρθὴν περιφερομένη, ὀρθογωνίος ἐστὶ ὁ κῶνος, ἐὰν δὲ ἐλάττων, ἀμβλυγωνίος, ἐὰν δὲ μείζων, ὀξυγωνίος.

This definition, or rather description of the genesis, of a (right) cone is interesting on account of the second sentence distinguishing between *right-angled*, *obtuse-angled* and *acute-angled* cones. This distinction is quite unnecessary for Euclid's purpose and is not used by him in Book XII.; it is no doubt a relic of the method, still in use in Euclid's time, by which the earlier Greek geometers produced conic sections, namely, by cutting right cones only by sections always perpendicular to an edge. With this system the parabola was a *section of a right-angled cone*, the hyperbola a *section of an obtuse-angled cone*, and the ellipse a *section of an acute-angled cone*. The conic sections were so called by Archimedes, and generally until Apollonius, who was the first to give the complete theory of their generation by means of sections not perpendicular to an edge, and from cones which are in general *oblique circular* cones. Thus Apollonius begins his *Conics* with the more scientific definition of a cone. If, he says, a straight line infinite in length, and passing always through a fixed point, be made to move round the circumference of a circle which is not in the same plane with the point, so as to pass successively through every point of that circumference, the moving straight line will trace out the surface of a *double cone*, or two similar cones lying in opposite directions and meeting in the fixed point, which is the *apex* of each cone. The circle about which the straight line moves is called the *base* of the cone lying between the said circle and the fixed point, and the *axis* is defined as the straight line drawn from the fixed point, or the apex, to the centre of the circle forming the base. Apollonius goes on to say that the cone is a *scalene* or *oblique* cone except in the particular case where the axis is perpendicular to the base. In this latter case it is a *right* cone.

Archimedes called the right cone an *isosceles* cone. This fact, coupled with the appearance in his treatise *On Conoids and Spheroids* (7, 8, 9) of *sections of acute-angled cones* (ellipses) as sections of conical surfaces which are proved to be oblique circular cones by finding their circular sections, makes it sufficiently clear that Archimedes, if he had defined a cone, would have defined it in the same way as Apollonius does.

DEFINITION 19.

*Αξων δὲ τοῦ κώνου ἐστὶν ἡ μένουσα εὐθεῖα, περὶ ἣν τὸ τρίγωνον στρέφεται.

DEFINITION 20.

Βάσις δὲ ὁ κύκλος ὁ ὑπὸ τῆς περιφερομένης εὐθείας γραφόμενος.

DEFINITION 21.

Κύλινδρός ἐστιν, ὅταν ὀρθογωνίου παραλληλογράμμου μενούσης μιᾶς πλευρᾶς τῶν περὶ τὴν ὀρθὴν γωνίαν περιενεχθὲν τὸ παραλληλόγραμμον εἰς τὸ αὐτὸ πάλιν ἀποκατασταθῇ, ὅθεν ἤρξατο φέρεσθαι, τὸ περιληφθὲν σχῆμα.

DEFINITION 22.

*Αξων δὲ τοῦ κυλίνδρου ἐστὶν ἡ μένουσα εὐθεῖα, περὶ ἣν τὸ παραλληλόγραμμον στρέφεται.

DEFINITION 23.

Βάσεις δὲ οἱ κύκλοι οἱ ὑπὸ τῶν ἀπεναντίον περιεγομένων δύο πλευρῶν γραφόμενοι.

DEFINITION 24.

*Ομοιοὶ κῶνοι καὶ κύλινδροί εἰσιν, ὧν οἱ τε ἄξονες καὶ αἱ διαμέτροι τῶν βάσεων ἀνάλογόν εἰσιν.

DEFINITION 25.

Κύβος ἐστὶ σχῆμα στερεὸν ὑπὸ ἑξ τετραγώνων ἴσων περιεχόμενον.

DEFINITION 26.

*Οκτάεδρόν ἐστι σχῆμα στερεὸν ὑπὸ ὀκτὼ τριγώνων ἴσων καὶ ἰσοπλεύρων περιεχόμενον.

DEFINITION 27.

Εἰκοσάεδρόν ἐστι σχῆμα στερεὸν ὑπὸ εἴκοσι τριγώνων ἴσων καὶ ἰσοπλεύρων περιεχόμενον.

DEFINITION 28.

Δωδεκάεδρόν ἐστι σχῆμα στερεὸν ὑπὸ δώδεκα πενταγώνων ἴσων καὶ ἰσοπλεύρων καὶ ἰσογωνίων περιεχόμενον.

BOOK XI. PROPOSITIONS.

PROPOSITION I.

A part of a straight line cannot be in the plane of reference and a part in a plane more elevated.

For, if possible, let a part AB of the straight line ABC be in the plane of reference, and a part BC in a plane more elevated.

There will then be in the plane of reference some straight line continuous with AB in a straight line.

Let it be BD ;

therefore AB is a common segment of the two straight lines ABC , ABD :

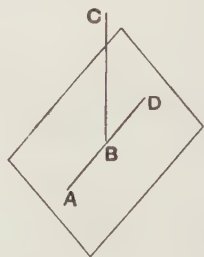
which is impossible, inasmuch as, if we describe a circle with centre B and distance AB , the diameters will cut off unequal circumferences of the circle.

Therefore a part of a straight line cannot be in the plane of reference, and a part in a plane more elevated.

Q. E. D.

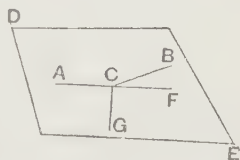
1. the plane of reference, τὸ ὑποκείμενον ἐπίπεδον, the plane laid down or assumed.
2. more elevated, μετεωροτέρῳ.

There is no doubt that the proofs of the first three propositions are unsatisfactory owing to the fact that Euclid is not able to make any use of his definition of a plane for the purpose of these proofs, and they really depend upon truths which can only be assumed as axiomatic. The definition of a plane as *that surface which lies evenly with the straight lines on itself*, whatever its exact meaning may be, is nowhere appealed to as a criterion to show whether a particular surface is or is not a plane. If the meaning of it is what I conjecture in the note on Book I., Def. 7 (Vol. I. p. 171), if, namely, it only tries to express without an appeal to sight what Plato meant by the "middle covering the extremities" (i.e. apparently, in the case of a plane, the fact that a plane looked at edgewise takes the form of a straight line), then it is perhaps possible to connect the definition with a method of generating a plane which



has commended itself to many writers as giving a better definition. Thus, if we conceive a straight line in space and a point outside it placed so that, in Plato's words, the line "covers" the point as we look at them, the line will also "cover" every straight line which passes through the given point and some one point on the given straight line. Hence, if a straight line passing always through a fixed point moves in such a way as to pass successively through every point of a given straight line which does not contain the given point, the moving straight line describes a surface which satisfies the Euclidean definition of a plane as I have interpreted it. But if we adopt the definition of a plane as *the surface described by a straight line which, passing through a given point, turns about it in such a way as always to intersect a given straight line not passing through the given point*, this definition, though it would help us to prove Eucl. XI. 2, does not give us the fundamental properties of a plane; some postulate is necessary in addition. The same is true even if we take a definition which gives more than is required to determine a plane, the definition known as Simson's, though it is at least as early as the time of Theon of Smyrna, who says (p. 112, 5) that *a plane is a surface such that, if a straight line meet it in two points, the straight line lies wholly in it* ($\delta\lambda\eta \alpha\upsilon\tau\omega \epsilon\phi\alpha\rho\mu\acute{o}\zeta\epsilon\tau\alpha\iota$). This is also called the *axiom of the plane*. (For some attempts to prove this on the basis of other definitions of a plane see my note on the definition of a *plane surface*, I. Def. 7.) If this definition or axiom be assumed, Prop. 1 becomes evident, for, as Legendre says, "In accordance with the definition of the plane, when a straight line has two points common with a plane, it lies wholly in the plane."

Euclid practically assumes the axiom when he says in this proposition "there will be in the plane of reference some straight line continuous with AB ." Clavius tries, unsuccessfully, to deduce this from Euclid's own definition of a plane; and he seems to admit his failure, because he proceeds to try another tack. Draw, he says, in the plane DE , the straight line CG at right angles to AC , and, again in the plane DE , CF at right angles to CG [I. 11]. Then AC , CF make right angles with CG in the same plane; therefore (I. 14) ACF is a straight line. But this does not really help, because Euclid assumes tacitly, in Book I. as well as Book XI., that a straight line joining two points in a plane lies wholly in that plane.



A curious point in Euclid's proof is the reason given why two straight lines cannot have a common segment. The argument is precisely that of the "proof" of the same thing given by Proclus on I. 1 (see note on Book I. Post. 2, Vol. I. p. 197) and is of course inconclusive. The fact that two straight lines cannot have a common segment must be taken to be involved in the definition of, and the postulates relating to, the straight line; and the "proof" given here can hardly, I should say, be Euclid's, though the interpolation, if it be such, must have been made very early.

The proof assumes too that a circle *can* be described so as to cut BA , BC and BD , or, in other words, it assumes that AD , BC are in one plane; that is, Prop. 1 as we have it really assumes the result of Prop. 2. There is therefore ground for Simson's alteration of the proof (after the point where BD has been taken in the given plane in a straight line with AB) to the following:

"Let any plane pass through the straight line AD and be turned about it until it pass through the point C ."

And, because the points B, C are in this plane, the straight line BC is in it. [Simson's def.]

Therefore there are two straight lines ABC, ABD in the same plane that have a common segment AB : which is impossible."

Simson, of course, justifies the last inference by reference to his Corollary to I. 11, which, however, as we have seen, is not a valid proof of the assumption, which is really implied in I. Post. 2.

An alternative reading, perhaps due to Theon, says, after the words "which is impossible" in the Greek text, "for a straight line does not meet a straight line in more points than one; otherwise the straight lines will coincide." Simson (who however does not seem to have had the second clause beginning "otherwise" in the text which he used) attacks this alternative reading in a rather confused note chiefly directed against a criticism by Thomas Simpson, without (as it seems to me) sufficient reason. It contains surely a legitimate argument. The supposed straight lines ABC, ABD meet in more than two points, namely in all the points between A and B . But two straight lines cannot have two points common without coinciding altogether; therefore ABC must coincide with ABD .

PROPOSITION 2.

If two straight lines cut one another, they are in one plane, and every triangle is in one plane.

For let the two straight lines AB, CD cut one another at the point E ;

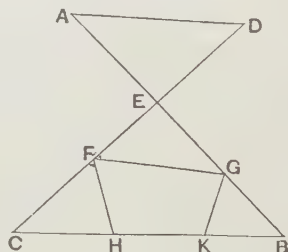
I say that AB, CD are in one plane, and every triangle is in one plane.

For let points F, G be taken at random on EC, EB ,

let CB, FG be joined,

and let FH, GK be drawn across;

I say first that the triangle ECB is in one plane.



For, if part of the triangle ECB , either FHC or GBK , is in the plane of reference, and the rest in another,

a part also of one of the straight lines EC, EB will be in the plane of reference, and a part in another.

But, if the part $FCBG$ of the triangle ECB be in the plane of reference, and the rest in another,

a part also of both the straight lines EC, EB will be in the plane of reference and a part in another:

which was proved absurd.

[XI. 1]

Therefore the triangle ECB is in one plane.

But, in whatever plane the triangle ECB is, in that plane also is each of the straight lines EC , EB ,

and, in whatever plane each of the straight lines EC , EB is, in that plane are AB , CD also. [x1. 1]

Therefore the straight lines AB , CD are in one plane, and every triangle is in one plane.

Q. E. D.

It must be admitted that the "proof" of this proposition is not of any value. For one thing, Euclid only takes certain triangles and a certain quadrilateral respectively forming part of the original triangle, and argues about these. But, for anything we are supposed to know, there may be some part of the triangle bounded (let us say) by some curve which is not in the same plane with the triangle.

We may agree with Simson that it would be preferable to enunciate the proposition as follows.

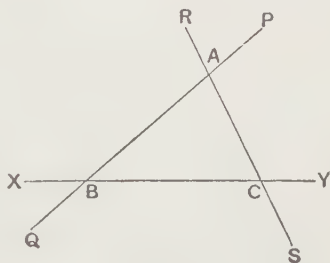
Two straight lines which intersect are in one plane, and three straight lines which intersect two and two are in one plane.

Adopting Smith and Bryant's figure in preference to Simson's, we suppose three straight lines PQ , RS , XY to intersect two and two in A , B , C .

Then Simson's proof (adopted by Legendre also) proceeds thus.

Let any plane pass through the straight line PQ , and let this plane be turned about PQ (produced indefinitely) as axis until it passes through the point C .

Then, since the points A , C are in this plane, the straight line AC (and therefore the straight line RS produced indefinitely) lies wholly in the plane. [Simson's def.]



For the same reason, since the points B , C are in the plane, the straight line XY lies wholly in the plane.

Hence all three straight lines PQ , RS , XY (and of course any pair of them) lie in one plane.

But it has still to be proved that there is *only* one plane passing through the three straight lines.

This may be done, as in Mr Taylor's Euclid, thus.

Suppose, if possible, that there are *two* different planes through A , B , C .

The straight lines BC , CA , AB then lie wholly in each of the two planes.

Now any straight line in one of the two planes must intersect at least two of the straight lines (produced if necessary);

let it intersect two of them in K , L .

Then, since K , L are also in the second plane; the line KL lies wholly in that plane.

Hence every straight line in either of the planes lies wholly in the other also; and therefore the planes are coincident throughout their whole surface.

It follows from the above that

A plane is determined (i.e. uniquely determined) by any of the following data :

- (1) *by three straight lines meeting one another two and two,*
- (2) *by three points not in a straight line,*
- (3) *by two straight lines meeting one another,*
- (4) *by a straight line and a point without it.*

PROPOSITION 3.

If two planes cut one another, their common section is a straight line.

For let the two planes AB , BC cut one another, and let the line DB be their common section ;

I say that the line DB is a straight line.

For, if not, from D to B let the straight line DEB be joined in the plane AB ; and in the plane BC the straight line DFB .

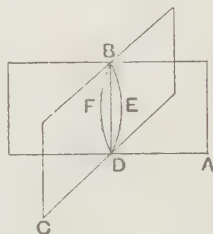
Then the two straight lines DEB , DFB will have the same extremities, and will clearly enclose an area :

which is absurd.

Therefore DEB , DFB are not straight lines.

Similarly we can prove that neither will there be any other straight line joined from D to B except DB the common section of the planes AB , BC .

Therefore etc.



Q. E. D.

I think Simson is right in objecting to the words after "which is absurd," to the effect that DEB , DFB are not straight lines, and that neither can there be any other straight line joined from D to B except DB , as being unnecessary. It is right to conclude at once from the absurdity that BD cannot *but* be a straight line.

Legendre makes his proof depend on Prop. 2. "For, if, among the points common to the two planes, three should be found which are not in a straight line, the two planes in question, each passing through three points, would only amount to one and the same plane." [This of course assumes that three points determine one and *only* one plane, which, strictly speaking, involves more than Prop. 2 itself, as shown in the last note.]

A favourite proposition in modern text-books is the following. The proof seems to be due to von Staudt (Killing, *Grundlagen der Geometrie*, Vol. II. p. 43).

If two planes meet in a point, they meet in a straight line.

Let ABC , ADE be two given planes meeting at A .

Take any points B , C lying on the plane ABC , and not on the plane ADE but on the same side of it.

Join AB , AC , and produce BA to F .

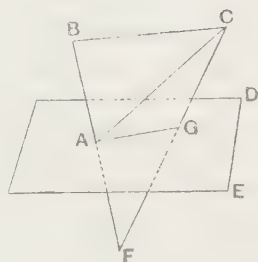
Join CF .

Then, since B , F are on opposite sides of the plane ADE ,

C , F are also on opposite sides of it.

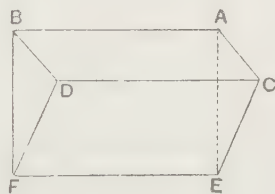
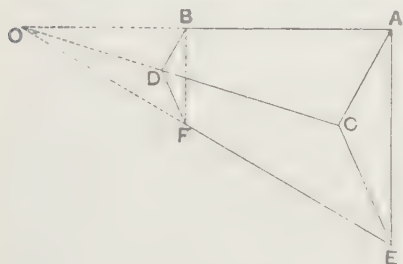
Therefore CF must meet the plane ADE in some point, say G .

Then, since A , G are both in each of the planes ABC , ADE , the straight line AG is in both planes. [Simson's def.]



This is also the place to insert the proposition that, *If three planes intersect two and two, their lines of intersection either meet in a point or are parallel two and two.*

Let there be three planes intersecting in the straight lines AB , CD , EF .



Now AB , EF are in a plane; therefore they either meet in a point or are parallel.

(1) Let them meet in O .

Then O , being a point in AB , lies in the plane AD , and, being also a point in EF , lies also in the plane ED .

Therefore O , being common to the planes AD , DE , must lie on CD , the line of their intersection;

i.e. CD , if produced, passes through O .

(2) Let AB , EF not meet, but let them be parallel.

Then CD cannot meet AB ; for, if it did, it must necessarily meet EF , by the first case.

Therefore CD , AB , being in one plane, are parallel.

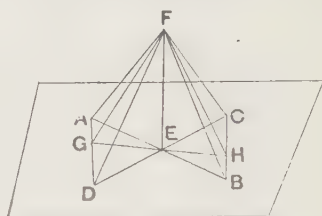
Similarly CD , EF are parallel.

PROPOSITION 4.

If a straight line be set up at right angles to two straight lines which cut one another, at their common point of section, it will also be at right angles to the plane through them.

For let a straight line EF be set up at right angles to the two straight lines AB , CD , which cut one another at the point E , from E ;

I say that EF is also at right angles to the plane through AB , CD .



For let AE , EB , CE , ED be cut off equal to one another,

and let any straight line GEH be drawn across through E , at random;

let AD , CB be joined,

and further let FA , FG , FD , FC , FH , FB be joined from the point F taken at random <on EF >.

Now, since the two straight lines AE , ED are equal to the two straight lines CE , EB , and contain equal angles, [I. 15] therefore the base AD is equal to the base CB , and the triangle AED will be equal to the triangle CEB ; [I. 4] so that the angle DAE is also equal to the angle EBC .

But the angle AEG is also equal to the angle BEH ; [I. 15] therefore AGE , BEH are two triangles which have two angles equal to two angles respectively, and one side equal to one side, namely that adjacent to the equal angles, that is to say, AE to EB ;

therefore they will also have the remaining sides equal to the remaining sides. [I. 26]

Therefore GE is equal to EH , and AG to BH .

And, since AE is equal to EB ,

while FE is common and at right angles,

therefore the base FA is equal to the base FB . [I. 4]

For the same reason

FC is also equal to FD .

And, since AD is equal to CB ,

and FA is also equal to FB ,

the two sides FA , AD are equal to the two sides FB , BC respectively;

and the base FD was proved equal to the base FC ;

therefore the angle FAD is also equal to the angle FBC . [I. 8]

And since, again, AG was proved equal to BH ,
and further FA also equal to FB ,
the two sides FA , AG are equal to the two sides FB , BH .

And the angle FAG was proved equal to the angle FBH ;
therefore the base FG is equal to the base FH . [1. 4]

Now since, again, GE was proved equal to EH ;
and EF is common,
the two sides GE , EF are equal to the two sides HE , EF ;
and the base FG is equal to the base FH ;
therefore the angle GEF is equal to the angle HEF . [1. 8]

Therefore each of the angles GEF , HEF is right.

Therefore FE is at right angles to GH drawn at random
through E .

Similarly we can prove that FE will also make right
angles with all the straight lines which meet it and are in the
plane of reference.

But a straight line is at right angles to a plane when it
makes right angles with all the straight lines which meet it
and are in that same plane; [xi. Def. 3]

therefore FE is at right angles to the plane of reference.

But the plane of reference is the plane through the straight
lines AB , CD .

Therefore FE is at right angles to the plane through
 AB , CD .

Therefore etc.

Q. E. D.

The steps to be successively proved in order to establish this proposition
by Euclid's method are

(1) triangles AED , BEC equal in all respects, [by 1. 4]

(2) triangles AEG , BEH equal in all respects, [by 1. 26]

so that AG is equal to BH , and GE to EH ,

(3) triangles AEF , BEF equal in all respects, [1. 4]

so that AF is equal to BF ,

(4) likewise triangles CEF , DEF ,

so that CF is equal to DF ,

(5) triangles FAD , FBC equal in all respects, [1. 8]

so that the angles FAG , FBH are equal,

(6) triangles FAG , FBH equal in all respects, [by '2), (3), (5) and 1. 4]

so that FG is equal to FH ,

(7) triangles FEG , FEH equal in all respects, [by (2), (6) and 1. 8]
 so that the angles FEG , FEH are equal,
 and therefore FE is at right angles to GH .

In consequence of the length of the above proof others have been suggested, and the proof which now finds most general acceptance is that of Cauchy, which is as follows.

Let AB be perpendicular to two straight lines BC , BD in the plane MN at their point of intersection B .

In the plane MN draw BE , any straight line through B .

Join CD , and let CD meet BE in E .

Produce AB to F so that BF is equal to AB .

Join AC , AE , AD , CF , EF , DF .

Since BC is perpendicular to AF at its middle point B ,

AC is equal to CF .

Similarly AD is equal to DF .

Since in the triangles ACD , FCD the two sides AC , CD are respectively equal to the two sides FC , CD , and the third sides AD , FD are also equal,

the angles ACD , FCD are equal. [1. 8]

The triangles ACE , FCE thus have two sides and the included angle equal, whence

EA is equal to EF . [1. 4]

The triangles ABE , FBE have now all their sides equal respectively ;
 therefore the angles ABE , FBE are equal, [1. 8]

and AB is perpendicular to BE .

And BE is in any straight line through B in the plane MN .

Legendre's proof is not so easy, but it is interesting. We are first required to draw through any point E within the angle CBD a straight line CD bisected at E .

To do this we draw EK parallel to DB meeting BC in K , and then mark off KC equal to BK .

CE is then joined and produced to D ; and CD is the straight line required.

Now, joining AC , AE , AD in the figure above, we have, since CD is bisected at E ,

(1) in the triangle ACD ,

$$AC^2 + AD^2 = 2AE^2 + 2ED^2,$$

and also (2) in the triangle BCD ,

$$BC^2 + BD^2 = 2BE^2 + 2ED^2.$$

Subtracting, and remembering that the triangles ABC , ABD are right-angled, so that

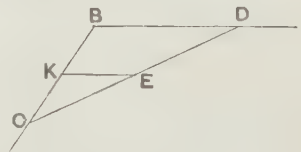
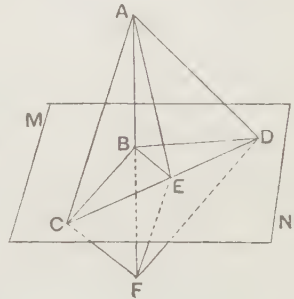
$$AC^2 - BC^2 = AB^2,$$

$$AD^2 - BD^2 = AB^2,$$

$$2AB^2 = 2AE^2 - 2BE^2,$$

$$AE^2 = AB^2 + BE^2,$$

and
 we have
 or



whence [I. 48] the angle ABE is a right angle, and AB is perpendicular to BE .

It follows of course from this proposition that the perpendicular AB is the *shortest distance from A to the plane MN*.

And it can readily be proved that,

If from a point without a plane oblique straight lines be drawn to the plane, (1) those meeting the plane at equal distances from the foot of the perpendicular are equal, and

(2) of two straight lines meeting the plane at unequal distances from the foot of the perpendicular, the more remote is the greater.

Lastly, it is easily seen that

From a point outside a plane only one perpendicular can be drawn to that plane.

For, if possible, let there be two perpendiculars. Then a plane can be drawn through them, and this will cut the original plane in a straight line.

This straight line and the two perpendiculars will form a plane triangle which has two right angles: which is impossible.

PROPOSITION 5.

If a straight line be set up at right angles to three straight lines which meet one another, at their common point of section, the three straight lines are in one plane.

For let a straight line AB be set up at right angles to the three straight lines BC , BD , BE , at their point of meeting at B ;

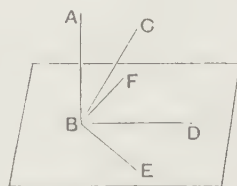
I say that BC , BD , BE are in one plane.

For suppose they are not, but, if possible, let BD , BE be in the plane of reference and BC in one more elevated; let the plane through AB , BC be produced;

it will thus make, as common section in the plane of reference, a straight line. [XI. 3]

Let it make BF .

Therefore the three straight lines AB , BC , BF are in one plane, namely that drawn through AB , BC .



Now, since AB is at right angles to each of the straight lines BD , BE ,

therefore AB is also at right angles to the plane through BD , BE . [XI. 4]

But the plane through BD , BE is the plane of reference ; therefore AB is at right angles to the plane of reference.

Thus AB will also make right angles with all the straight lines which meet it and are in the plane of reference.

[XI. Def. 3]

But BF which is in the plane of reference meets it ; therefore the angle ABF is right.

But, by hypothesis, the angle ABC is also right ; therefore the angle ABF is equal to the angle ABC .

And they are in one plane : which is impossible.

Therefore the straight line BC is not in a more elevated plane ;

therefore the three straight lines BC , BD , BE are in one plane.

Therefore, if a straight line be set up at right angles to three straight lines, at their point of meeting, the three straight lines are in one plane.

Q. E. D.

It follows that, if a right angle be turned about one of the straight lines containing it the other will describe a plane.

At any point in a straight line it is possible to draw *only one* plane which is at right angles to the straight line.

One such plane can be found by taking any two planes through the given straight line, drawing perpendiculars to the straight line in the respective planes, e.g. BO , CO in the planes AOB , AOC , each perpendicular to AO , and then drawing a plane (BOC) through the perpendiculars.

If there were another plane through O perpendicular to AO , it must meet the plane through AO and some perpendicular to it as OC in a straight line OC' different from OC .

Then, by XI. 4, AOC' is a right angle, and in the same plane with the right angle AOC : which is impossible.

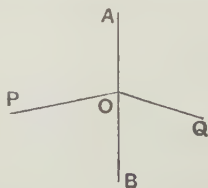
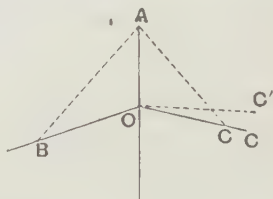
Next, *one plane and only one can be drawn through a point outside a straight line at right angles to that line.*

Let P be the given point, AB the given straight line.

In the plane through P and AB , draw PO perpendicular to AB , and through O draw another straight line OQ at right angles to AB .

Then the plane through OP , OQ is perpendicular to AB .

If there were another plane through P perpendicular to AB , either



- (1) it would intersect AB at O but not pass through OQ , or
 (2) it would intersect AB at a point different from O .

In either case, an absurdity would result.

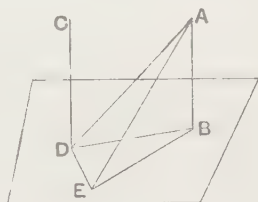
PROPOSITION 6.

If two straight lines be at right angles to the same plane, the straight lines will be parallel.

For let the two straight lines AB , CD be at right angles to the plane of reference ;

I say that AB is parallel to CD .

For let them meet the plane of reference at the points B , D , let the straight line BD be joined, let DE be drawn, in the plane of reference, at right angles to BD , let DE be made equal to AB , and let BE , AE , AD be joined.



Now, since AB is at right angles to the plane of reference, it will also make right angles with all the straight lines which meet it and are in the plane of reference. [xi. Def. 3]

But each of the straight lines BD , BE is in the plane of reference and meets AB ;

therefore each of the angles ABD , ABE is right.

For the same reason

each of the angles CDB , CDE is also right.

And, since AB is equal to DE ,

and BD is common,

the two sides AB , BD are equal to the two sides ED , DB ;

and they include right angles ;

therefore the base AD is equal to the base BE . [I. 4]

And, since AB is equal to DE ,

while AD is also equal to BE ,

the two sides AB , BE are equal to the two sides ED , DA ;

and AE is their common base ;

therefore the angle ABE is equal to the angle EDA . [I. 8]

But the angle ABE is right ;
 therefore the angle EDA is also right ;
 therefore ED is at right angles to DA .

But it is also at right angles to each of the straight lines BD, DC ;

therefore ED is set up at right angles to the three straight lines BD, DA, DC at their point of meeting ;

therefore the three straight lines BD, DA, DC are in one plane. [xi. 5]

But, in whatever plane DB, DA are, in that plane is AB also,

for every triangle is in one plane ; [xi. 2]

therefore the straight lines AB, BD, DC are in one plane.

And each of the angles ABD, BDC is right ;

therefore AB is parallel to CD . [i. 28]

Therefore etc.

Q. E. D.

If anyone wishes to convince himself of the real necessity for some general agreement as to the order in which propositions in elementary geometry should be taken, let him contemplate the hopeless result of too much independence on the part of editors in the matter of this proposition and its converse, xi. 8.

Legendre adopts a different, and elegant, method of proof ; but he applies it to xi. 8, which he gives first, and then deduces xi. 6 from it by *reductio ad absurdum*. Dr Mehler uses Legendre's method of proof but applies it to xi. 6, and then gives xi. 8 as a deduction from it. Lardner follows Legendre. Holgate, the editor of a recent American book, gives Euclid's proof of xi. 6 and deduces xi. 8 by *reductio ad absurdum*. His countrymen, Schultze and Sevenoak, give xi. 8 first, but put it after, and deduce it from, Eucl. xi. 10 ; they then give xi. 6, practically as a deduction from xi. 8 by *reductio ad absurdum*, after a proposition corresponding to Eucl. xi. 11 and 12, and a corollary to the effect that through a given point one and only one perpendicular can be drawn to a given plane.

We will now give the proof of xi. 6 by Legendre's method (adopted by Smith and Bryant as well as by Mehler).

Let AB, CD be both perpendicular to the same plane MN .

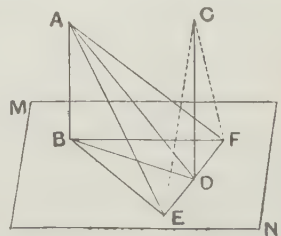
Join BD .

Now, since BD meets AB, CD , both of which are perpendicular to the plane MN in which BD is,

the angles ABD, CDB are right angles.

AB, CD will therefore be parallel *provided that they are in the same plane*.

Through D draw EDF , in the plane MN , at right angles to BD , and make ED equal to DF .



Join BE , BF , AE , AD , AF .

Then the triangles BDE , BDF are equal in all respects (by i. 4), so that BE is equal to BF .

It follows, since the angles ABE , ABF are right, that the triangles ABE , ABF are equal in all respects, and

AE is equal to AF .

[Mehler now argues elegantly thus. If CE , CF be also joined, it is clear that

CE is equal to CF .

Hence each of the four points A , B , C , D is equidistant from the two points E , F .

Therefore the points A , B , C , D are in one plane, so that AB , CD are parallel.

If, however, we do not use the locus of points equidistant from two fixed points, we proceed as follows.]

The triangles AED , AFD have their sides equal respectively; hence [i. 8] the angles ADE , ADF are equal, so that ED is at right angles to AD .

Thus ED is at right angles to BD , AD , CD ; therefore CD is in the plane through AD , BD .

[xi. 5]

But AB is in that same plane; therefore AB , CD are in the same plane.

[xi. 2]

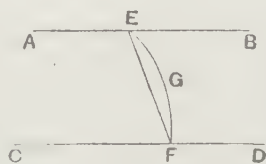
And the angles ABD , CDB are right; therefore AB , CD are parallel.

PROPOSITION 7.

If two straight lines be parallel and points be taken at random on each of them, the straight line joining the points is in the same plane with the parallel straight lines.

Let AB , CD be two parallel straight lines, and let points E , F be taken at random on them respectively;

I say that the straight line joining the points E , F is in the same plane with the parallel straight lines.



For suppose it is not, but, if possible, let it be in a more elevated plane as EGF ,

and let a plane be drawn through EGF ;

it will then make, as section in the plane of reference, a straight line.

[xi. 3]

Let it make it, as EF ;

therefore the two straight lines EGF , EF will enclose an area :

which is impossible.

Therefore the straight line joined from E to F is not in a plane more elevated ;

therefore the straight line joined from E to F is in the plane through the parallel straight lines AB , CD .

Therefore etc.

Q. E. D.

It is true that this proposition, in the form in which Euclid enunciates it, is hardly necessary if the plane is defined as a surface such that, if any two points be taken in it, the straight line joining them lies wholly in the surface. But Euclid did not give this definition ; and, moreover, Prop. 2 would be usefully supplemented by a proposition which should prove that *two parallel straight lines determine a plane* (i.e. one plane and one only) *which also contains all the straight lines which join a point on one of the parallels to a point on the other*. That there cannot be two planes through a pair of parallels would be proved in the same way as we prove that two or three intersecting straight lines cannot be in two different planes, inasmuch as each transversal lying in one of the two supposed planes through the parallels would lie wholly in the other also, so that the two supposed planes must coincide throughout (cf. note on Prop. 2 above).

But, whatever be the value of the proposition as it is, Simson seems to have spoilt it completely. He leaves out the construction of a plane through EGF , which, as Euclid says, must cut the plane containing the parallels in a straight line ; and, instead, he says, "In the plane $ABCD$ in which the parallels are draw the straight line EHF from E to F ." Now, although we can easily draw a straight line from E to F , to claim that we can draw it *in the plane in which the parallels are* is surely to assume the very result which is to be proved. All that we could properly say is that the straight line joining E to F is in *some* plane which contains the parallels ; we do not know that there is no more than *one* such plane, or that the parallels determine a plane *uniquely*, without some such argument as that which Euclid gives.

Nor can I subscribe to the remarks in Simson's note on the proposition. He says (1) "This proposition has been put into this book by some unskilful editor, as is evident from this, that straight lines which are drawn from one point to another in a plane are, in the preceding books, supposed to be in that plane ; and if they were not, some demonstrations in which one straight line is supposed to meet another would not be conclusive. For instance, in Prop. 30, Book 1, the straight line GK would not meet EF , if GK were not in the plane in which are the parallels AB , CD , and in which, by hypothesis, the straight line EF is." But the subject-matter of Book 1. and Book XI. is quite different ; in Book 1. everything is in one plane, and when Euclid, in defining parallels, says they are straight lines *in the same plane* etc., he only does so because he must, in order to exclude non-intersecting straight lines which are *not* parallel. Thus in 1. 30 there is nothing wrong in assuming that there may be three parallels in one plane, and that the straight line GHK cuts all three.

But in Book xi. it becomes a question whether there can be *more* than one plane through parallel straight lines.

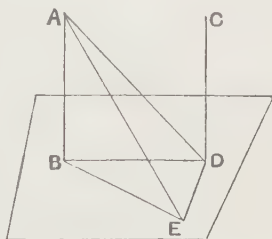
Simson goes on to say (2) "Besides, this 7th Proposition is demonstrated by the preceding 3rd: in which the very same thing which is proposed to be demonstrated in the 7th is twice assumed, viz., that the straight line drawn from one point to another in a plane is in that plane." But there is nothing in Prop. 3 about a plane in which two parallel straight lines are; therefore there is no assumption of the result of Prop. 7. What is assumed is that, given two points in a *plane*, they can be joined by a straight line in the plane: a legitimate assumption.

Lastly, says Simson, "And the same thing is assumed in the preceding 6th Prop. in which the straight line which joins the points *B, D* that are in the plane to which *AB* and *CD* are at right angles is supposed to be in that plane." Here again there is no question of a *plane in which two parallels are*; so that the criticism here, as with reference to Prop. 3, appears to rest on a misapprehension.

PROPOSITION 8.

If two straight lines be parallel, and one of them be at right angles to any plane, the remaining one will also be at right angles to the same plane.

Let *AB, CD* be two parallel straight lines, and let one of them, *AB*, be at right angles to the plane of reference; I say that the remaining one, *CD*, will also be at right angles to the same plane.



For let *AB, CD* meet the plane of reference at the points *B, D*, and let *BD* be joined; therefore *AB, CD, BD* are in one plane. [xi. 7]

Let *DE* be drawn, in the plane of reference, at right angles to *BD*, let *DE* be made equal to *AB*, and let *BE, AE, AD* be joined.

Now, since *AB* is at right angles to the plane of reference, therefore *AB* is also at right angles to all the straight lines which meet it and are in the plane of reference; [xi. Def. 3] therefore each of the angles *ABD, ABE* is right.

And, since the straight line *BD* has fallen on the parallels *AB, CD*,

therefore the angles ABD , CDB are equal to two right angles. [I. 29]

But the angle ABD is right ;
therefore the angle CDB is also right ;
therefore CD is at right angles to BD .

And, since AB is equal to DE ,
and BD is common,
the two sides AB , BD are equal to the two sides ED , DB ;
and the angle ABD is equal to the angle EDB ,
for each is right ;
therefore the base AD is equal to the base BE .

And, since AB is equal to DE ,
and BE to AD ,
the two sides AB , BE are equal to the two sides ED , DA respectively,
and AE is their common base ;
therefore the angle ABE is equal to the angle EDA .

But the angle ABE is right ;
therefore the angle EDA is also right ;
therefore ED is at right angles to AD .

But it is also at right angles to DB ;
therefore ED is also at right angles to the plane through BD , DA . [xi. 4]

Therefore ED will also make right angles with all the straight lines which meet it and are in the plane through BD , DA .

But DC is in the plane through BD , DA , inasmuch as AB , BD are in the plane through BD , DA , [xi. 2]
and DC is also in the plane in which AB , BD are.

Therefore ED is at right angles to DC ,
so that CD is also at right angles to DE .

But CD is also at right angles to BD .

Therefore CD is set up at right angles to the two straight lines DE , DB which cut one another, from the point of section at D ;

so that CD is also at right angles to the plane through DE, DB . [xi. 4]

But the plane through DE, DB is the plane of reference; therefore CD is at right angles to the plane of reference.

Therefore etc.

Q. E. D.

Simson objects to the words which explain why DC is in the plane through BD, DA , viz. "inasmuch as AB, BD are in the plane through BD, DA , and DC is also in the plane in which AB, BD are," as being too roundabout. He concludes that they are corrupt or interpolated, and that we ought only to have the words "because all three are in the plane in which are the parallels AB, CD " (by Prop. 7 preceding). But I think Euclid's words can be defended. Prop. 7 says nothing of a plane determined by *two* transversals as BD, DA are. Hence it is natural to say that DC is in the same plane in which AB, BD are [Prop. 7], and AB, BD are in the same plane as BD, DA [Prop. 2], so that DC is in the plane through BD, DA .

Legendre's alternative proof is split by him into two propositions.

(1) Let AB be a perpendicular to the plane MN and EF a line situated in that plane; if from B , the foot of the perpendicular, BD be drawn perpendicular to EF , and AD be joined, I say that AD will be perpendicular to EF .

(2) If AB is perpendicular to the plane MN , every straight line CD parallel to AB will be perpendicular to the same plane.

To prove both propositions together we suppose CD given, join BD , and draw EF perpendicular to BD in the plane MN .

(1) As before, we make DE equal to DF and join BE, BF, AE, AF .

Then, since the angles BDE, BDF are right, and DE, DF equal,

BE is equal to BF . [I. 4]

And, since AB is perpendicular to the plane,

the angles ABE, ABF are both right.

Therefore, in the triangles ABE, ABF ,

AE is equal to AF . [I. 4]

Lastly, in the triangles ADE, ADF , since AE is equal to AF , and DE to DF , while AD is common,

the angle ADE is equal to the angle ADF , [I. 8]

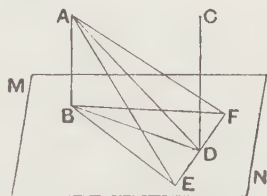
so that AD is perpendicular to EF .

(2) ED being thus perpendicular to DA , and also (by construction) perpendicular to DB ,

ED is perpendicular to the plane ADB . [xi. 4]

But CD , being parallel to AB , is in the plane ADB ;

therefore ED is perpendicular to CD . [xi. Def. 3]



Also, since AB , CD are parallel,
and ABD is a right angle,
 CDB is also a right angle.

Thus CD is perpendicular to both DE and DB , and therefore to the plane MN through DE , DB .

PROPOSITION 9.

Straight lines which are parallel to the same straight line and are not in the same plane with it are also parallel to one another.

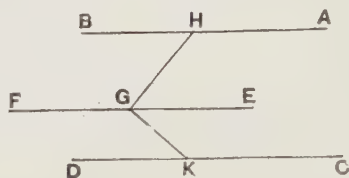
For let each of the straight lines AB , CD be parallel to EF , not being in the same plane with it;

I say that AB is parallel to CD .

For let a point G be taken at random on EF ,

and from it let there be drawn GH , in the plane through EF ,

AB , at right angles to EF , and GK in the plane through FE , CD again at right angles to EF .



Now, since EF is at right angles to each of the straight lines GH , GK ,

therefore EF is also at right angles to the plane through GH , GK . [xi. 4]

And EF is parallel to AB ;

therefore AB is also at right angles to the plane through HG , GK . [xi. 8]

For the same reason

CD is also at right angles to the plane through HG , GK ;
therefore each of the straight lines AB , CD is at right angles to the plane through HG , GK .

But, if two straight lines be at right angles to the same plane, the straight lines are parallel; [xi. 6]
therefore AB is parallel to CD .

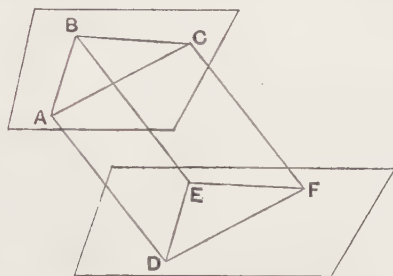
Q. E. D.

PROPOSITION 10.

If two straight lines meeting one another be parallel to two straight lines meeting one another not in the same plane, they will contain equal angles.

For let the two straight lines AB , BC meeting one another be parallel to the two straight lines DE , EF meeting one another, not in the same plane;

I say that the angle ABC is equal to the angle DEF .



For let BA , BC , ED , EF be cut off equal to one another, and let AD , CF , BE , AC , DF be joined.

Now, since BA is equal and parallel to ED ,
therefore AD is also equal and parallel to BE . [I. 33]

For the same reason
 CF is also equal and parallel to BE .

Therefore each of the straight lines AD , CF is equal and parallel to BE .

But straight lines which are parallel to the same straight line and are not in the same plane with it are parallel to one another; [XI. 9]

therefore AD is parallel and equal to CF .

And AC , DF join them;
therefore AC is also equal and parallel to DF . [I. 33]

Now, since the two sides AB , BC are equal to the two sides DE , EF ,

and the base AC is equal to the base DF ,
therefore the angle ABC is equal to the angle DEF . [I. 8]

Therefore etc.

Q. E. D.

The result of this proposition does not appear to be quoted in Euclid until XII. 3; but Euclid no doubt inserted it here advisedly, because it has the effect of incidentally proving that the "inclination of two planes to one another," as defined in XI. Def. 6, is one and the same angle at whatever point of the common section the plane angle measuring it is drawn.

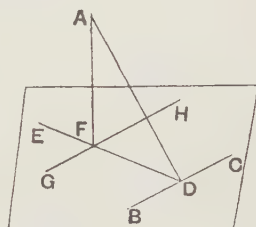
PROPOSITION 11.

From a given elevated point to draw a straight line perpendicular to a given plane.

Let A be the given elevated point, and the plane of reference the given plane; thus it is required to draw from the point A a straight line perpendicular to the plane of reference.

Let any straight line BC be drawn, at random, in the plane of reference, and let AD be drawn from the point A perpendicular to BC . [I. 12]

If then AD is also perpendicular to the plane of reference, that which was enjoined will have been done.



But, if not, let DE be drawn from the point D at right angles to BC and in the plane of reference, [I. 11]

let AF be drawn from A perpendicular to DE , [I. 12]

and let GH be drawn through the point F parallel to BC . [I. 31]

Now, since BC is at right angles to each of the straight lines DA , DE , therefore BC is also at right angles to the plane through ED , DA . [XI. 4]

And GH is parallel to it;

but, if two straight lines be parallel, and one of them be at right angles to any plane, the remaining one will also be at right angles to the same plane; [XI. 8]

therefore GH is also at right angles to the plane through ED , DA .

Therefore GH is also at right angles to all the straight lines which meet it and are in the plane through ED, DA .

[XI. Def. 3]

But AF meets it and is in the plane through ED , DA ; therefore GH is at right angles to FA , so that FA is also at right angles to GH .

But AF is also at right angles to DE ;
therefore AF is at right angles to each of the straight lines
 GH, DE .

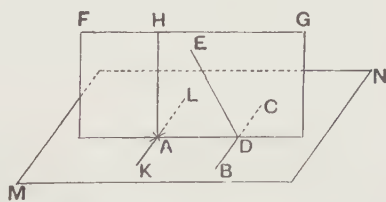
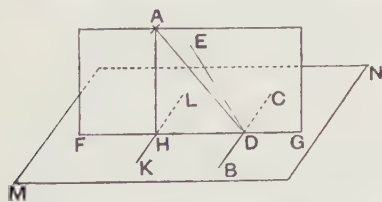
But, if a straight line be set up at right angles to two straight lines which cut one another, at the point of section, it will also be at right angles to the plane through them ; [xi. 4] therefore FA is at right angles to the plane through ED, GH .

But the plane through ED , GH is the plane of reference ; therefore AF is at right angles to the plane of reference.

Therefore from the given elevated point A the straight line AF has been drawn perpendicular to the plane of reference.

Q. E. F.

The text-books differ in the *form* which they give to this proposition rather than in substance. They commonly assume the construction of a *plane* through the point *A* at right angles to any straight line *BC* in the given plane (the construction being effected in the manner shown at the end of the note on xi. 5 above). The advantage of this method is that it enables a perpendicular to be drawn from a point *in* the plane also, by the same construction. (Where the letters for the two figures differ, those referring to the second figure are put in brackets.)



We can include the construction of the plane through A perpendicular to BC , and make the whole into one proposition, thus.

BC being any straight line in the given plane MN , draw AD perpendicular to BC .

In any plane passing through BC but not through A draw DE at right angles to BC .

Through DA , DE draw a plane; this will intersect the given plane MN in a straight line, as FD (AD).

In the plane AG draw AH perpendicular to FG (AD).

Then AH is the perpendicular required.

In the plane MN , through H in the first figure and A in the second, draw KL parallel to BC .

Now, since BC is perpendicular to both DA and DE , BC is perpendicular to the plane AG . [XI. 4]

Therefore KL , being parallel to BC , is also perpendicular to the plane AG [XI. 8], and therefore to AH which meets it and is in that plane.

Therefore AH is perpendicular to both FD (AD) and KL at their point of intersection.

Therefore AH is perpendicular to the plane MN .

Thus we have solved the problem in XI. 12 as well as that in XI. 11; and this direct method of drawing a perpendicular to a plane from a point *in* it is obviously preferable to Euclid's method by which the construction of a perpendicular to a plane from a point *without* it is assumed, and a line is merely drawn from a point in the plane parallel to the perpendicular obtained in XI. 11.

PROPOSITION 12.

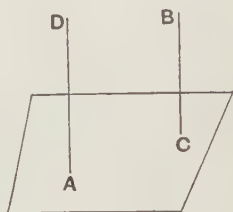
To set up a straight line at right angles to a given plane from a given point in it.

Let the plane of reference be the given plane,
and A the point in it;

thus it is required to set up from the point A a straight line at right angles to the plane of reference.

Let any elevated point B be conceived,
from B let BC be drawn perpendicular to
the plane of reference, [XI. 11]

and through the point A let AD be drawn
parallel to BC . [I. 31]



Then, since AD , CB are two parallel straight lines,
while one of them, BC , is at right angles to the plane of
reference,

therefore the remaining one, AD , is also at right angles to
the plane of reference. [XI. 8]

Therefore AD has been set up at right angles to the given
plane from the point A in it.

Q. E. F.

PROPOSITION 13.

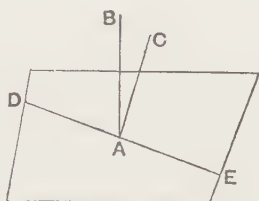
From the same point two straight lines cannot be set up at right angles to the same plane on the same side.

For, if possible, from the same point A let the two straight lines AB , AC be set up at right angles to the plane of reference and on the same side,

and let a plane be drawn through BA , AC ;

it will then make, as section through A in the plane of reference, a straight line.

[XI. 3]



Let it make DAE ;

therefore the straight lines AB , AC , DAE are in one plane.

And, since CA is at right angles to the plane of reference, it will also make right angles with all the straight lines which meet it and are in the plane of reference. [XI. Def. 3]

But DAE meets it and is in the plane of reference; therefore the angle CAE is right.

For the same reason

the angle BAE is also right;

therefore the angle CAE is equal to the angle BAE .

And they are in one plane:

which is impossible.

Therefore etc.

Q. E. D.

Simson added words to this as follows:

“Also, from a point above a plane there can be but one perpendicular to that plane; for, if there could be two, they would be parallel to one another [XI. 6], which is absurd.”

Euclid does not give this result, but we have already had it in the note above to XI. 4 (*ad fin.*).

PROPOSITION 14.

Planes to which the same straight line is at right angles will be parallel.

For let any straight line AB be at right angles to each of the planes CD , EF ;

I say that the planes are parallel.

For, if not, they will meet when produced.

Let them meet;

they will then make, as common section, a straight line.

Let them make GH ;

let a point K be taken at random on GH ,

and let AK , BK be joined.

Now, since AB is at right angles to the plane EF , therefore AB is also at right angles to BK which is a straight line in the plane EF produced; [xi. Def. 3]
therefore the angle ABK is right.

For the same reason the angle BAK is also right.

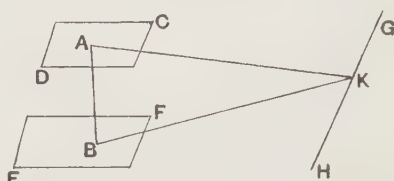
Thus, in the triangle ABK , the two angles ABK , BAK are equal to two right angles : which is impossible. [I. 17]

Therefore the planes CD , EF will not meet when produced ;

therefore the planes CD , EF are parallel. [xi. Def. 8]

Therefore planes to which the same straight line is at right angles are parallel.

Q. E. D.

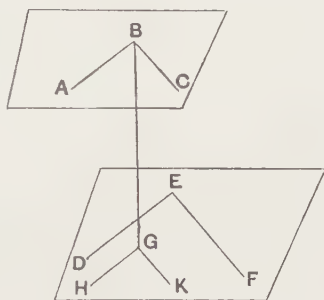


PROPOSITION 15.

If two straight lines meeting one another be parallel to two straight lines meeting one another, not being in the same plane, the planes through them are parallel.

For let the two straight lines AB, BC meeting one another be parallel to the two straight lines DE, EF meeting one another, not being in the same plane;

I say that the planes produced through AB, BC and DE, EF will not meet one another.



For let BG be drawn from the point B perpendicular to the plane through DE, EF [xi. 11], and let it meet the plane at the point G ;

through G let GH be drawn parallel to ED , and GK parallel to EF .

[i. 31]

Now, since BG is at right angles to the plane through DE, EF ,

therefore it will also make right angles with all the straight lines which meet it and are in the plane through DE, EF .

[xi. Def. 3]

But each of the straight lines GH, GK meets it and is in the plane through DE, EF ;

therefore each of the angles BGH, BGK is right.

And, since BA is parallel to GH ,

[xi. 9]

therefore the angles GBA, BGH are equal to two right angles.

[i. 29]

But the angle BGH is right;

therefore the angle GBA is also right;

therefore GB is at right angles to BA .

For the same reason

GB is also at right angles to BC .

Since then the straight line GB is set up at right angles to the two straight lines BA, BC which cut one another, therefore GB is also at right angles to the plane through BA, BC .

[xi. 4]

But planes to which the same straight line is at right angles are parallel; [xi. 14]

therefore the plane through AB , BC is parallel to the plane through DE , EF .

Therefore, if two straight lines meeting one another be parallel to two straight lines meeting one another, not in the same plane, the planes through them are parallel.

Q. E. D.

This result is arrived at in the American text-books already quoted by starting from the relation between a plane and a straight line parallel to it. The series of propositions is worth giving. A straight line and a plane being parallel if they do not meet however far they may be produced, we have the following propositions.

1. *Any plane containing one, and only one, of two parallel straight lines is parallel to the other.*

For suppose AB , CD to be parallel and CD to lie in the plane MN .

Then AB , CD determine a plane intersecting MN in the straight line CD .

Thus, if AB meets MN , it must meet it at some point in CD .

But this is impossible, since AB is parallel to CD .

Therefore AB will not meet the plane MN , and is therefore parallel to it.

[This proposition and the proof are in Legendre.]

The following theorems follow as corollaries.

2. *Through a given straight line a plane can be drawn parallel to any other given straight line; and, if the lines are not parallel, only one such plane can be drawn.*

We have simply to draw through any point on the first line a straight line parallel to the second line and then pass a plane through these two intersecting lines. This plane is then, by the above proposition, parallel to the second given straight line.

3. *Through a given point a plane can be drawn parallel to any two straight lines in space; and, if the latter are not parallel, only one such plane can be drawn.*

Here we draw through the point straight lines parallel respectively to the given straight lines and then draw a plane through the lines so drawn.

Next we have the partial converse of the first proposition above.

4. *If a straight line is parallel to a plane, it is also parallel to the intersection of any plane through it with the given plane.*

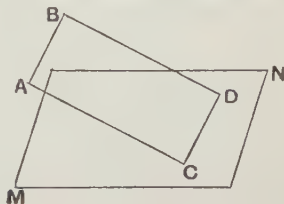
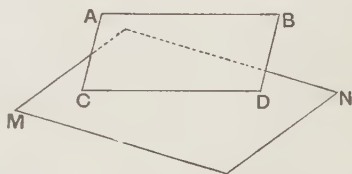
Let AB be parallel to the plane MN , and let any plane through AB intersect MN in CD .

Now AB and CD cannot meet, because, if they did, AB would meet the plane MN .

And AB , CD are in one plane.

Therefore AB , CD are parallel.

From this follows as a corollary:

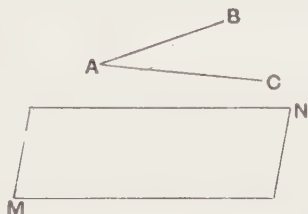


5. *If each of two intersecting straight lines is parallel to a given plane, the plane containing them is parallel to the given plane.*

Let AB, AC be parallel to the plane MN .

Then, if the plane ABC were to meet the plane MN , the intersection would be parallel both to AB and to AC : which is impossible.

Lastly, we have Euclid's proposition.



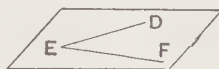
6. *If two straight lines forming an angle are respectively parallel to two other straight lines forming an angle, the plane of the first angle is parallel to the plane of the second.*

Let ABC, DEF be the angles formed by straight lines parallel to one another respectively.

Then, since AB is parallel to DE , the plane of DEF is parallel to AB [(1) above].

Similarly the plane of DEF is parallel to BC .

Hence the plane of DEF is parallel to the plane of ABC [(5)].



Legendre arrives at the result by yet another method. He first proves Eucl. xi. 16 to the effect that, *if two parallel planes are cut by a third, the lines of intersection are parallel*, and then deduces from this that, *if two parallel straight lines are terminated by two parallel planes, the straight lines are equal in length*.

(The latter inference is obvious because the plane through the parallels cuts the parallel planes in parallel lines; which therefore, with the given parallel lines, form a parallelogram.)

Legendre is now in a position to prove Euclid's proposition xi. 15.

If ABC, DEF be the angles, make AB equal to DE , and BC equal to EF , and join CA, FD, BE, CF, AD .

Then, as in Eucl. xi. 10, the triangles ABC, DEF are equal in all respects;

and AD, BE, CF are all equal.

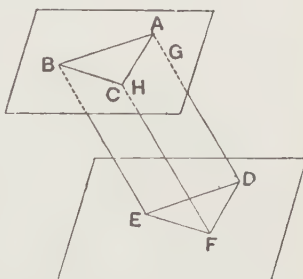
It is now proved that the planes are parallel by *reductio ad absurdum* from the last preceding result. For, if the plane ABC is not parallel to the plane DEF , let the plane drawn through B parallel to the plane DEF meet CF, AD in H, G respectively.

Then, by the last result BE, HF, GD will all be equal.

But BE, CF, AD are all equal:

which is impossible.

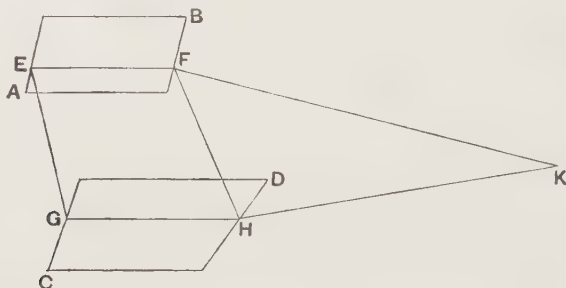
Therefore etc.



PROPOSITION 16.

If two parallel planes be cut by any plane, their common sections are parallel.

For let the two parallel planes AB , CD be cut by the plane $EFGH$,
and let EF , GH be their common sections ;
I say that EF is parallel to GH .



For, if not, EF , GH will, when produced, meet either in the direction of F , H or of E , G .

Let them be produced, as in the direction of F , H , and let them, first, meet at K .

Now, since EFK is in the plane AB ,
therefore all the points on EFK are also in the plane AB .

But K is one of the points on the straight line EFK ;
therefore K is in the plane AB . [XI. 1]

For the same reason
 K is also in the plane CD ;
therefore the planes AB , CD will meet when produced.

But they do not meet, because they are, by hypothesis, parallel ;

therefore the straight lines EF , GH will not meet when produced in the direction of F , H .

Similarly we can prove that neither will the straight lines EF , GH meet when produced in the direction of E , G .

But straight lines which do not meet in either direction are parallel.

Therefore EF is parallel to GH .

Therefore etc.

Q. E. D.

Simson points out that, in here quoting 1. Def. 23, Euclid should have said "But straight lines *in one plane* which do not meet in either direction are parallel."

From this proposition is deduced the converse of XI. 14.

If a straight line is perpendicular to one of two parallel planes, it is perpendicular to the other also.

For suppose that MN , PQ are two parallel planes, and that AB is perpendicular to MN .

Through AB draw any plane, and let it intersect the planes MN , PQ in AC , BD respectively.

Therefore AC , BD are parallel. [XI. 16]

But AC is perpendicular to AB ;

therefore AB is also perpendicular to BD .

That is, AB is perpendicular to any line in PQ passing through B ;

therefore AB is perpendicular to PQ .

It follows as a corollary that

Through a given point one plane, and only one, can be drawn parallel to a given plane.

In the above figure let A be the given point and PQ the given plane.

Draw AB perpendicular to PQ .

Through A draw a plane MN at right angles to AB (see note on XI. 5 above).

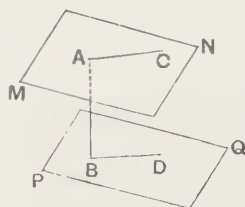
Then MN is parallel to PQ . [XI. 14]

If there could pass through A a second plane parallel to PQ , AB would also be perpendicular to it.

That is, AB would be perpendicular to two different planes through A : which is impossible (see the same note).

Also it is readily proved that,

If two planes are parallel to a third plane, they are parallel to one another.



PROPOSITION 17.

If two straight lines be cut by parallel planes, they will be cut in the same ratios.

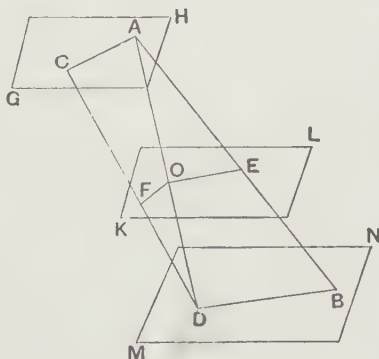
For let the two straight lines AB , CD be cut by the parallel planes GH , KL , MN at the points A , E , B and C , F , D ;

I say that, as the straight line AE is to EB , so is CF to FD .

For let AC , BD , AD be joined,

let AD meet the plane KL at the point O ,

and let EO , OF be joined.



Now, since the two parallel planes KL , MN are cut by the plane $EBDO$,
their common sections EO , BD are parallel. [xi. 16]

For the same reason, since the two parallel planes GH , KL are cut by the plane $AOFC$,
their common sections AC , OF are parallel. [id.]

And, since the straight line EO has been drawn parallel to BD , one of the sides of the triangle ABD ,
therefore, proportionally, as AE is to EB , so is AO to OD . [vi. 2]

Again, since the straight line OF has been drawn parallel to AC , one of the sides of the triangle ADC ,
proportionally, as AO is to OD , so is CF to FD . [id.]

But it was also proved that, as AO is to OD , so is AE to EB ;
therefore also, as AE is to EB , so is CF to FD . [v. 11]

Therefore etc.

Q. E. D.

PROPOSITION 18.

If a straight line be at right angles to any plane, all the planes through it will also be at right angles to the same plane.

For let any straight line AB be at right angles to the plane of reference;

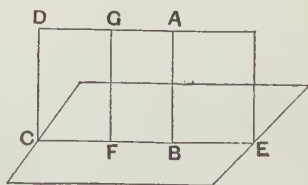
I say that all the planes through AB are also at right angles to the plane of reference.

For let the plane DE be drawn through AB ,

let CE be the common section of the plane DE and the plane of reference,

let a point F be taken at random on CE ,

and from F let FG be drawn in the plane DE at right angles to CE . [I. 11]



Now, since AB is at right angles to the plane of reference,

AB is also at right angles to all the straight lines which meet it and are in the plane of reference ; [xi. Def. 3]

so that it is also at right angles to CE ;
therefore the angle ABF is right.

But the angle GFB is also right ;
therefore AB is parallel to FG . [I. 28]

But AB is at right angles to the plane of reference ;
therefore FG is also at right angles to the plane of reference. [xi. 8]

Now a plane is at right angles to a plane, when the straight lines drawn, in one of the planes, at right angles to the common section of the planes are at right angles to the remaining plane. [xi. Def. 4]

And FG , drawn in one of the planes DE at right angles to CE , the common section of the planes, was proved to be at right angles to the plane of reference ;
therefore the plane DE is at right angles to the plane of reference.

Similarly also it can be proved that all the planes through AB are at right angles to the plane of reference.
Therefore etc.

Q. E. D.

Starting as Euclid does from the definition of perpendicular planes as planes such that all straight lines drawn in one of the planes at right angles to the common section are at right angles to the other plane, it is necessary for him to show that, if F be *any* point in CE , and FG be drawn in the plane DE at right angles to CE , FG will be perpendicular to the plane to which AB is perpendicular.

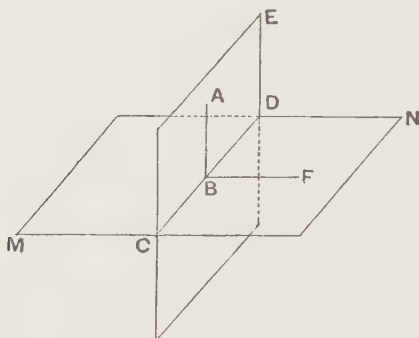
It is perhaps more scientific to make the definition, as Legendre makes it, a particular case of the definition of the *inclination of planes*. Perpendicular planes would thus be planes such that the angle which (when it is acute) Euclid calls the inclination of a plane to a plane is a right angle. When to this is added the fact incidentally proved in xi. 10 that the "inclination of a plane to a plane" is the same at whatever point in their common section it is drawn, it is sufficient to prove the perpendicularity of two planes if *one* straight line drawn, in one of them, perpendicular to their common section is perpendicular to the other.

If this point of view is taken, Props. 18, 19 are much simplified (cf. Legendre, H. M. Taylor, Smith and Bryant, Rausenberger, Schultze and Sevenoak, Holgate). The alternative proof is as follows.

Let AB be perpendicular to the plane MN , and CE any plane through AB , meeting the plane MN in the straight line CD .

In the plane MN draw BF at right angles to CD .

Then ABF is the angle which Euclid calls (in the case where it is acute) the "inclination of the plane to the plane."



But, since AB is perpendicular to the plane MN , it is perpendicular to BF in it.

Therefore the angle ABF is a right angle ;

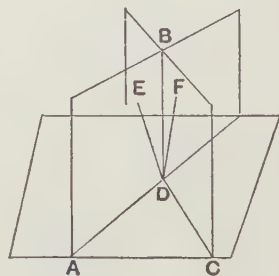
whence the plane CE is perpendicular to the plane MN .

PROPOSITION 19.

If two planes which cut one another be at right angles to any plane, their common section will also be at right angles to the same plane.

For let the two planes AB , BC be at right angles to the plane of reference, and let BD be their common section ; I say that BD is at right angles to the plane of reference.

For suppose it is not, and from the point D let DE be drawn in the plane AB at right angles to the straight line AD , and DF in the plane BC at right angles to CD .



Now, since the plane AB is at right angles to the plane of reference, and DE has been drawn in the plane AB at right angles to AD , their common section, therefore DE is at right angles to the plane of reference.

[XI. Def. 4]

Similarly we can prove that

DF is also at right angles to the plane of reference.

Therefore from the same point D two straight lines have been set up at right angles to the plane of reference on the same side :

which is impossible.

[XI. 13]

Therefore no straight line except the common section DB of the planes AB, BC can be set up from the point D at right angles to the plane of reference.

Therefore etc.

Q. E. D.

Legendre, followed by other writers already quoted, uses a preliminary proposition equivalent to Euclid's definition of planes at right angles to one another.

If two planes are perpendicular to one another, a straight line drawn in one of them perpendicular to their common section will be perpendicular to the other.

Let the perpendicular planes CE, MN (figure of last note) intersect in CD , and let AB be drawn in CE perpendicular to CD .

In the plane MN draw BF at right angles to CD .

Then, since the planes are perpendicular, the angle ABF (their inclination) is a right angle.

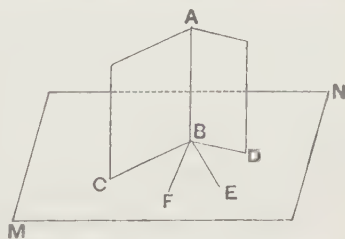
Therefore AB is perpendicular to both CD and BF , and therefore to the plane MN .

We are now in a position to prove XI. 19, viz. *If two planes be perpendicular to a third, their intersection is also perpendicular to that third plane.*

Let each of the two planes AC, AD intersecting in AB be perpendicular to the plane MN .

Let AC, AD intersect MN in BC, BD respectively.

In the plane MN draw BE at right angles to BC and BF at right angles to BD .



Now, since the planes AC, MN are at right angles, and BE is drawn in the latter perpendicular to BC , BE is perpendicular to the plane AC .

Hence AB is perpendicular to BE .

[XI. 4]

Similarly AB is perpendicular to BF .

Therefore AB is perpendicular to the plane through BE, BF , i.e. to the plane MN .

An useful problem is that of drawing a common perpendicular to two straight lines not in one plane, and in connexion with this the following proposition may be given.

Given a plane and a straight line not perpendicular to it, one plane, and only one, can be drawn through the straight line perpendicular to the plane.

Let AB be the given straight line, MN the given plane.

From any point C in AB draw CD perpendicular to the plane MN .

Through AB and CD draw a plane AE .

Then the plane AE is perpendicular to the plane MN . [XI. 18]

If any other plane could be drawn through AB perpendicular to MN , the intersection AB of the two planes perpendicular to MN would itself be perpendicular to MN :

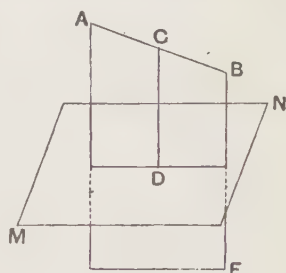
which contradicts the hypothesis.

To draw a common perpendicular to two straight lines not in the same plane.

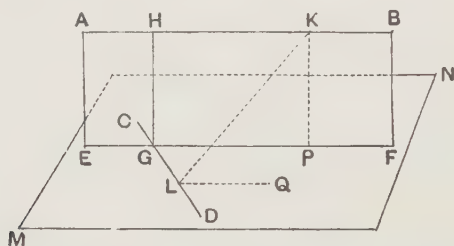
Let AB , CD be the given straight lines.

Through CD draw the plane MN parallel to AB (Prop. 2 in note to XI. 15).

Through AB draw the plane AF perpendicular to the plane MN (see the last preceding proposition).



[XI. 19]



Let the planes AF , MN intersect in EF , and let EF meet CD in G .

From G , in the plane AF , draw GH at right angles to EF , meeting AB in H . GH is then the required perpendicular.

For AB is parallel to EF (Prop. 4 in note to XI. 15); therefore GH , being perpendicular to EF , is also perpendicular to AB .

But, the plane AF being perpendicular to the plane MN , and GH being perpendicular to EF , their intersection,

GH is perpendicular to the plane MN , and therefore to CD .

Therefore GH is perpendicular to both AB and CD .

Only one common perpendicular can be drawn to two straight lines not in one plane.

For, if possible, let KL also be perpendicular to both AB and CD .

Let the plane through KL , AB meet the plane MN in LQ .

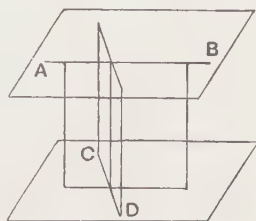
Then AB is parallel to LQ (Prop. 4 in note to XI. 15), so that KL , being perpendicular to AB , is also perpendicular to LQ .

Therefore KL is perpendicular to both CL and LQ , and consequently to the plane MN .

But, if KP be drawn in the plane AF perpendicular to EF , KP is also perpendicular to the plane MN .

Thus there are two perpendiculars from the point K to the plane MN : which is impossible.

Rausenberger's construction for the same problem is more elegant. Draw, he says, through each straight line a plane parallel to the other. Then draw through each straight line a plane perpendicular to the plane through the other. The two planes last drawn will intersect in a straight line, and this straight line is the common perpendicular required.



The form of the construction best suited for examination purposes, because the most self-contained, is doubtless that given by Smith and Bryant.

Let AB , CD be the two given straight lines.

Through any point E in CD draw EF parallel to AB .

From any point G in AB draw GH perpendicular to the plane CDF , meeting the plane in H .

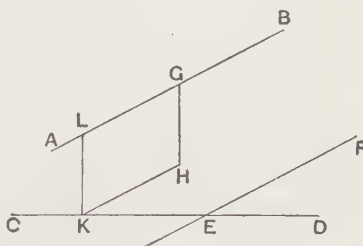
Through H in the plane CDF draw HK parallel to FE or AB , to cut CD in K .

Then, since AB , HK are parallel, $AGHK$ is a plane.

Complete the parallelogram $GHLK$.

Now, since LK , GH are parallel, and GH is perpendicular to the plane CDF ,

LK is perpendicular to the plane CDF .



Therefore LK is perpendicular to CD and KH , and therefore to AB which is parallel to KH .

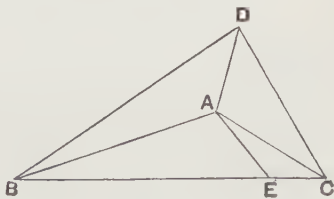
PROPOSITION 20.

If a solid angle be contained by three plane angles, any two, taken together in any manner, are greater than the remaining one.

For let the solid angle at A be contained by the three plane angles BAC , CAD , DAB ; I say that any two of the angles BAC , CAD , DAB , taken together in any manner, are greater than the remaining one.

If now the angles BAC , CAD , DAB are equal to one another, it is manifest that any two are greater than the remaining one.

But, if not, let BAC be greater, and on the straight line AB , and at the point A on it, let the



angle BAE be constructed, in the plane through BA , AC , equal to the angle DAB ;

let AE be made equal to AD ,

and let BEC , drawn across through the point E , cut the straight lines AB , AC at the points B , C ;

let DB , DC be joined.

Now, since DA is equal to AE ,

and AB is common,

two sides are equal to two sides;

and the angle DAB is equal to the angle BAE ;

therefore the base DB is equal to the base BE . [I. 4]

And, since the two sides BD , DC are greater than BC ,

[I. 20]

and of these DB was proved equal to BE ,

therefore the remainder DC is greater than the remainder EC .

Now, since DA is equal to AE ,

and AC is common,

and the base DC is greater than the base EC ,

therefore the angle DAC is greater than the angle EAC .

[I. 25]

But the angle DAB was also proved equal to the angle BAE ;

therefore the angles DAB , DAC are greater than the angle BAC .

Similarly we can prove that the remaining angles also, taken together two and two, are greater than the remaining one.

Therefore etc.

Q. E. D.

After excluding the obvious case in which all three angles are equal, Euclid goes on to say "If not, let the angle BAC be greater," without adding greater than *what*. Heiberg is clearly right in saying that he means greater than BAD , i.e. greater than *one* of the adjacent angles. This is proved by the words at the end "Similarly we can prove," etc. Euclid thus excludes as obvious the case where one of the three angles is not greater than either of the other two, but proves the remaining cases. This is scientific, but he might further have excluded as obvious the case in which one angle is greater than one of the others but equal to or less than the remaining one.

Simson remarks that the angle BAC may happen to be *equal* to one of the other two and writes accordingly "If they [all three angles] are not [equal], let BAC be that angle which is not less than either of the other two, and is greater than one of them DAB ." He then proves, in the same way as Euclid does, that the angles DAB , DAC are greater than the angle BAC , adding finally: "But BAC is not less than either of the angles DAB , DAC ; therefore BAC , with either of them, is greater than the other."

It would be better, as indicated by Legendre and Rausenberger, to begin by saying that, "If one of the three angles is either equal to or less than either of the other two, it is evident that the sum of those two is greater than the first. It is therefore only necessary to prove, *for the case in which one angle is greater than each of the others*, that the sum of the two latter is greater than the former.

Accordingly let BAC be greater than each of the other angles." We then proceed as in Euclid.

PROPOSITION 21.

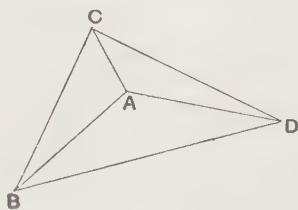
Any solid angle is contained by plane angles less than four right angles.

Let the angle at A be a solid angle contained by the plane angles BAC , CAD , DAB ;

I say that the angles BAC , CAD , DAB are less than four right angles.

For let points B , C , D be taken at random on the straight lines AB , AC , AD respectively.

and let BC , CD , DB be joined.



Now, since the solid angle at B is contained by the three plane angles CBA , ABD , CBD ,

any two are greater than the remaining one; [XI. 20]

therefore the angles CBA , ABD are greater than the angle CBD .

For the same reason

the angles BCA , ACD are also greater than the angle BCD , and the angles CDA , ADB are greater than the angle CDB ; therefore the six angles CBA , ABD , BCA , ACD , CDA , ADB are greater than the three angles CBD , BCD , CDB .

But the three angles CBD , BDC , BCD are equal to two right angles; [I. 32]

therefore the six angles CBA , ABD , BCA , ACD , CDA , ADB are greater than two right angles.

And, since the three angles of each of the triangles ABC , ACD , ADB are equal to two right angles, therefore the nine angles of the three triangles, the angles CBA , ACB , BAC , ACD , CDA , CAD , ADB , DBA , BAD are equal to six right angles ; and of them the six angles ABC , BCA , ACD , CDA , ADB , DBA are greater than two right angles ; therefore the remaining three angles BAC , CAD , DAB containing the solid angle are less than four right angles.

Therefore etc.

Q. E. D.

It will be observed that, although Euclid enunciates this proposition for *any* solid angle, he only proves it for the particular case of a *trihedral* angle. This is in accordance with his manner of proving one case and leaving the others to the reader. The omission of the convex polyhedral angle here corresponds to the omission, after 1. 32, of the proposition about the interior angles of a convex polygon given by Proclus and in most books. The proof of the present proposition for any convex polyhedral angle can of course be arranged so as not to assume the proposition that the interior angles of a convex polygon together with four right angles are equal to twice as many right angles as the figure has sides.

Let there be any convex polyhedral angle with V as vertex, and let it be cut by any plane meeting its faces in, say, the polygon $ABCDE$.

Take O any point within the polygon, and in its plane, and join OA , OB , OC , OD , OE .

Then all the angles of the triangles with vertex O are equal to twice as many right angles as the polygon has sides ;

[1. 32]

therefore the interior angles of the polygon together with all the angles round O are equal to twice as many right angles as the polygon has sides.

Also the sum of the angles of the triangles VAB , VBC , etc., with vertex V are equal to twice as many right angles as the polygon has sides ;

and all the said angles are equal to the sum of (1) the plane angles at V forming the polyhedral angle and (2) the base angles of the triangles with vertex V .

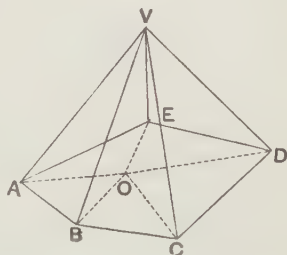
This latter sum is therefore equal to the sum of (3) all the angles round O and (4) all the interior angles of the polygon.

Now, by Euclid's proposition, of the three angles forming the solid angle at A , the angles VAE , VAB are together greater than the angle EAB .

Similarly, at B , the angles VBA , VBC are together greater than the angle ABC .

And so on.

Therefore, by addition, the base angles of the triangles with vertex V



[(2) above] are together greater than the sum of the angles of the polygon [(4) above].

Hence, by way of compensation, the sum of the plane angles at V [(1) above] is less than the sum of the angles round O [(3) above].

But the latter sum is equal to four right angles; therefore the plane angles forming the polyhedral angle are together less than four right angles.

The proposition is only true of *convex* polyhedral angles, i.e. those in which the plane of any face cannot, if produced, ever cut the solid angle.

There are certain propositions relating to equal (and symmetrical) trihedral angles which are necessary to the consideration of the polyhedra dealt with by Euclid, all of which (as before remarked) have trihedral angles only.

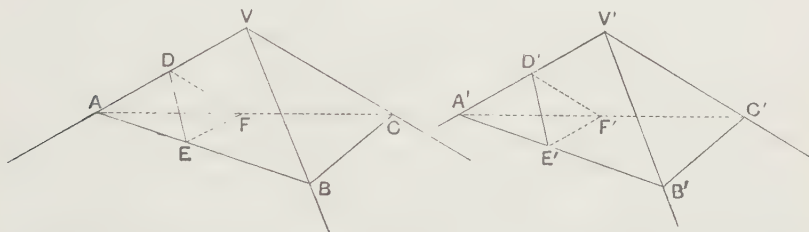
1. *Two trihedral angles are equal if two face angles and the included dihedral angle of the one are respectively equal to two face angles and the included dihedral angle of the other, the equal parts being arranged in the same order.*

2. *Two trihedral angles are equal if two dihedral angles and the included face angle of the one are respectively equal to two dihedral angles and the included face angle of the other, all equal parts being arranged in the same order.*

These propositions are proved immediately by superposition.

3. *Two trihedral angles are equal if the three face angles of the one are respectively equal to the three face angles of the other, and all are arranged in the same order.*

Let $V-ABC$ and $V'-A'B'C'$ be two trihedral angles such that the angle AVB is equal to the angle $A'V'B'$, the angle BVC to the angle $B'V'C'$, and the angle CVA to the angle $C'V'A'$.



We first prove that *corresponding pairs of face angles include equal dihedral angles*.

E.g., the dihedral angle formed by the plane angles CVA , AVB is equal to that formed by the plane angles $C'V'A'$, $A'V'B'$.

Take points A , B , C on VA , VB , VC and points A' , B' , C' on $V'A'$, $V'B'$, $V'C'$, such that VA , VB , VC , $V'A'$, $V'B'$, $V'C'$ are all equal.

Join BC , CA , AB , $B'C'$, $C'A'$, $A'B'$.

Take any point D on AV , and measure $A'D'$ along $A'V'$ equal to AD .

From D draw DE in the plane AVB , and DF in the plane CVA , perpendicular to AV . Then DE , DF will meet AB , AC respectively, the angles VAB , VAC , the base angles of two isosceles triangles, being less than right angles.

Join EF .

Draw the triangle $D'E'F'$ in the same way.

Now, by means of the hypothesis and construction, it appears that the triangles VAB , $V'A'B'$ are equal in all respects.

So are the triangles VAC , $V'A'C'$, and the triangles VBC , $V'B'C'$.

Thus BC , CA , AB are respectively equal to $B'C'$, $C'A'$, $A'B'$, and the triangles ABC , $A'B'C'$ are equal in all respects.

Now, in the triangles ADE , $A'D'E'$,

the angles ADE , DAE are equal to the angles $A'D'E'$, $D'A'E'$ respectively, and AD is equal to $A'D'$.

Therefore the triangles ADE , $A'D'E'$ are equal in all respects.

Similarly the triangles ADF , $A'D'F'$ are equal in all respects.

Thus, in the triangles AEF , $A'E'F'$,

EA , AF are respectively equal to $E'A'$, $A'F'$,

and the angle EAF is equal to the angle $E'A'F'$ (from above);

therefore the triangles AEF , $A'E'F'$ are equal in all respects.

Lastly, in the triangles DEF , $D'E'F'$, the three sides are respectively equal to the three sides;

therefore the triangles are equal in all respects.

Therefore the angles EDF , $E'D'F'$ are equal.

But these angles are the measures of the dihedral angles formed by the planes CVA , AVB and by the planes $C'V'A'$, $A'V'B'$ respectively.

Therefore these dihedral angles are equal.

Similarly for the other two dihedral angles.

Hence the trihedral angles coincide if one is applied to the other;

that is, they are equal.

To understand what is implied by "taken in the same order" we may suppose ourselves to be placed at the vertices, and to take the faces in clockwise direction, or the reverse, for *both* angles.

If the face angles and dihedral angles are *taken in reverse directions*, i.e. in clockwise direction in one and in counterclockwise direction in the other, then, if the other conditions in the above three propositions are fulfilled, the trihedral angles are not equal but *symmetrical*.

If the faces of a trihedral angle be produced beyond the vertex, they form another trihedral angle. It is easily seen that these *vertical trihedral angles* are *symmetrical*.

PROPOSITION 22.

If there be three plane angles of which two, taken together in any manner, are greater than the remaining one, and they are contained by equal straight lines, it is possible to construct a triangle out of the straight lines joining the extremities of the equal straight lines.

Let there be three plane angles ABC , DEF , GHK , of

which two, taken together in any manner, are greater than the remaining one, namely

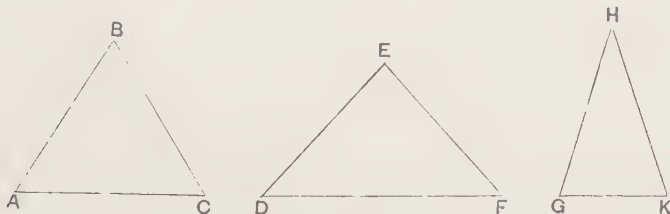
the angles ABC , DEF greater than the angle GHK ,

the angles DEF , GHK greater than the angle ABC ,

and, further, the angles GHK , ABC greater than the angle DEF ;

let the straight lines AB , BC , DE , EF , GH , HK be equal, and let AC , DF , GK be joined;

I say that it is possible to construct a triangle out of straight lines equal to AC , DF , GK , that is, that any two of the straight lines AC , DF , GK are greater than the remaining one.



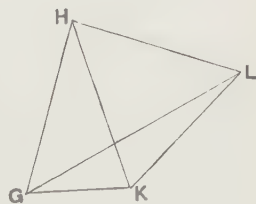
Now, if the angles ABC , DEF , GHK are equal to one another, it is manifest that, AC , DF , GK being equal also, it is possible to construct a triangle out of straight lines equal to AC , DF , GK .

But, if not, let them be unequal,

and on the straight line HK , and at the point H on it, let the angle KHL be constructed equal to the angle ABC ;

let HL be made equal to one of the straight lines AB , BC , DE , EF , GH , HK ,

and let KL , GL be joined.



Now, since the two sides AB , BC are equal to the two sides KH , HL , and the angle at B is equal to the angle KHL , therefore the base AC is equal to the base KL . [1. 4]

And, since the angles ABC , GHK are greater than the angle DEF ,

while the angle ABC is equal to the angle KHL ,
therefore the angle GHL is greater than the angle DEF .

And, since the two sides GH , HL are equal to the two sides DE , EF ,
and the angle GHL is greater than the angle DEF ,
therefore the base GL is greater than the base DF . [I. 24]

But GK , KL are greater than GL .

Therefore GK , KL are much greater than DF .

But KL is equal to AC ;
therefore AC , GK are greater than the remaining straight line DF .

Similarly we can prove that
 AC , DF are greater than GK ,
and further DF , GK are greater than AC .

Therefore it is possible to construct a triangle out of straight lines equal to AC , DF , GK .

Q. E. D.

The Greek text gives an alternative proof, which is relegated by Heiberg to the Appendix. Simson selected the alternative proof in preference to that given above; he objected however to words near the beginning, "If not, let the angles at the points B , E , H be unequal and that at B greater than either of the angles at E , H ," and altered the words so as to take account of the possibility that the angle at B might be equal to one of the other two.

As will be seen, Euclid takes no account of the relative magnitude of the angles except as regards the case when all three are equal. Having proved that *one* base is less than the sum of the two others, he says that "similarly we can prove" the same thing for the other two bases.

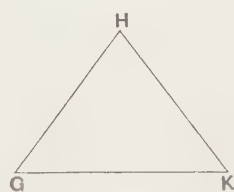
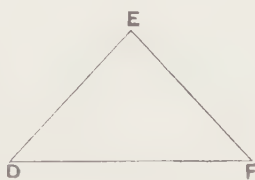
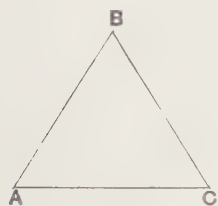
If a distinction is to be made according to the relative magnitude of the three angles, we may say, as in the corresponding place in XI. 21, that, if one of the three angles is either equal to or less than *either* of the other two, the bases subtending those two angles must obviously be together greater than the base subtending the first. Thus it is only necessary to prove, for the case in which one angle is *greater than either of the others*, that the sum of the bases subtending those others is greater than that subtending the first. This is practically the course taken in the interpolated alternative proof.

PROPOSITION 23.

To construct a solid angle out of three plane angles two of which, taken together in any manner, are greater than the remaining one: thus the three angles must be less than four right angles.

Let the angles ABC , DEF , GHK be the three given plane angles, and let two of these, taken together in any manner, be greater than the remaining one, while, further, the three are less than four right angles ;

thus it is required to construct a solid angle out of angles equal to the angles ABC , DEF , GHK .



Let AB , BC , DE , EF , GH , HK be cut off equal to one another,

and let AC , DF , GK be joined ;

it is therefore possible to construct a triangle out of straight lines equal to AC , DF , GK . [XI. 22]

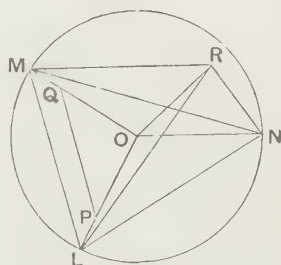
Let LMN be so constructed that AC is equal to LM , DF to MN , and further GK to NL ,

let the circle LMN be described about the triangle LMN ,

let its centre be taken, and let it be O ;

let LO , MO , NO be joined ;

I say that AB is greater than LO .



For, if not, AB is either equal to LO , or less.

First, let it be equal.

Then, since AB is equal to LO ,

while AB is equal to BC , and OL to OM ,

the two sides AB , BC are equal to the two sides LO , OM respectively ;

and, by hypothesis, the base AC is equal to the base LM ;

therefore the angle ABC is equal to the angle LOM . [I. 8]

For the same reason

the angle DEF is also equal to the angle MON ,

and further the angle GHK to the angle NOL ;

therefore the three angles ABC , DEF , GHK are equal to the three angles LOM , MON , NOL .

But the three angles LOM , MON , NOL are equal to four right angles ;

therefore the angles ABC , DEF , GHK are equal to four right angles.

But they are also, by hypothesis, less than four right angles : which is absurd.

Therefore AB is not equal to LO .

I say next that neither is AB less than LO .

For, if possible, let it be so,

and let OP be made equal to AB , and OQ equal to BC , and let PQ be joined.

Then, since AB is equal to BC ,
 OP is also equal to OQ ,
 so that the remainder LP is equal to QM .

Therefore LM is parallel to PQ , [vi. 2]

and LMO is equiangular with PQO ; [I. 29]

therefore, as OL is to LM , so is OP to PQ ; [vi. 4]

and alternately, as LO is to OP , so is LM to PQ . [v. 16]

But LO is greater than OP ;
 therefore LM is also greater than PQ .

But LM was made equal to AC ;
 therefore AC is also greater than PQ .

Since, then, the two sides AB , BC are equal to the two sides PO , OQ ,

and the base AC is greater than the base PQ ,
 therefore the angle ABC is greater than the angle POQ .

[I. 25]

Similarly we can prove that
 the angle DEF is also greater than the angle MON ,
 and the angle GHK greater than the angle NOL .

Therefore the three angles ABC , DEF , GHK are greater than the three angles LOM , MON , NOL .

But, by hypothesis, the angles ABC , DEF , GHK are less than four right angles ;

therefore the angles LOM , MON , NOL are much less than four right angles.

But they are also equal to four right angles :
which is absurd.

Therefore AB is not less than LO .

And it was proved that neither is it equal ;
therefore AB is greater than LO .

Let then OR be set up from the point O at right angles
to the plane of the circle LMN , [xi. 12]
and let the square on OR be equal to that area by which
the square on AB is greater than the square on LO ; [Lemma]
let RL , RM , RN be joined.

Then, since RO is at right angles to the plane of the circle
 LMN ,
therefore RO is also at right angles to each of the straight
lines LO , MO , NO .

And, since LO is equal to OM ,
while OR is common and at right angles,
therefore the base RL is equal to the base RM . [1. 4]

For the same reason
 RN is also equal to each of the straight lines RL , RM ;
therefore the three straight lines RL , RM , RN are equal to
one another.

Next, since by hypothesis the square on OR is equal to
that area by which the square on AB is greater than the
square on LO ,

therefore the square on AB is equal to the squares on LO , OR .

But the square on LR is equal to the squares on LO , OR ,
for the angle LOR is right ; [1. 47]

therefore the square on AB is equal to the square on RL ;
therefore AB is equal to RL .

But each of the straight lines BC , DE , EF , GH , HK is
equal to AB ,

while each of the straight lines RM , RN is equal to RL ;
therefore each of the straight lines AB , BC , DE , EF , GH ,
 HK is equal to each of the straight lines RL , RM , RN .

And, since the two sides LR , RM are equal to the two sides AB , BC ,
and the base LM is by hypothesis equal to the base AC ,
therefore the angle LRM is equal to the angle ABC . [I. 8]

For the same reason
the angle MRN is also equal to the angle DEF ,
and the angle LRN to the angle GHK .

Therefore, out of the three plane angles LRM , MRN , LRN , which are equal to the three given angles ABC , DEF , GHK , the solid angle at R has been constructed, which is contained by the angles LRM , MRN , LRN .

Q. E. F.

LEMMA.

But how it is possible to take the square on OR equal to that area by which the square on AB is greater than the square on LO , we can show as follows.

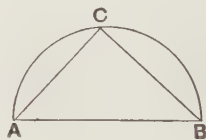
Let the straight lines AB , LO be set out,

and let AB be the greater;

let the semicircle ABC be described on AB ,

and into the semicircle ABC let AC be fitted equal to the straight line LO , not being greater than the diameter AB ; [IV. 1]

let CB be joined



Since then the angle ACB is an angle in the semicircle ACB ,

therefore the angle ACB is right.

[III. 31]

Therefore the square on AB is equal to the squares on AC , CB .

[I. 47]

Hence the square on AB is greater than the square on AC by the square on CB .

But AC is equal to LO .

Therefore the square on AB is greater than the square on LO by the square on CB .

If then we cut off OR equal to BC , the square on AB will be greater than the square on LO by the square on OR .

Q. E. F.

The whole difficulty in this proposition is the proof of a fact which makes the construction *possible*, viz. the fact that, if LMN be a triangle with sides

respectively equal to the bases of the isosceles triangles which have the given angles as vertical angles and the equal sides all of the same length, then one of these equal sides, as AB , is greater than the radius LO of the circle circumscribing the triangle LMN .

Assuming that AB is greater than LO , we have only to draw from O a perpendicular OR to the plane of the triangle LMN , to make OR of such a length that the sum of the squares on LO , OR is equal to the square on AB , and to join RL , RM , RN . (The manner of finding OR such that the square on it is equal to the difference between the squares on AB and LO is shown in the Lemma at the end of the text of the proposition. We have already had the same construction in the Lemma after x. 13.)

Then clearly RL , RM , RN are equal to AB and to one another [1. 4 and 1. 47].

Therefore the triangles LRM , MNR , NRL have their three sides respectively equal to those of the triangles ABC , DEF , GHK respectively.

Hence their vertical angles are equal to the three given angles respectively; and the required solid angle is constructed.

We return now to the proposition to be proved as a preliminary to the construction, viz. that, in the figures, AB is greater than LO .

It will be observed that Euclid, as his manner is, proves it for one case only, that, namely, in which C , the centre of the circle circumscribing the triangle LMN , falls *within* the triangle, leaving the other cases for the reader to prove. As usual, however, the two other cases are found in the Greek text, after the formal conclusion of the proposition, as above, ending with the words $\delta\pi\epsilon\rho\ \epsilon\delta\epsilon\iota\ \pi\omicron\upsilon\eta\sigma\alpha\iota$. This position for the proofs itself suggests that they are not Euclid's but are interpolated; and this is rendered certain by the fact that words distinguishing three cases at the point where the centre O of the circumscribing circle is found, "It [the centre] will then be either within the triangle LMN or on one of its sides or without. First let it be within," are found in the mss. B and V only and are manifestly interpolated. Nevertheless the additional two cases must have been inserted very early, as they are found in all the best mss.

In order to give a clear view of the proof of all three cases as given in the text, we will reproduce all three (Euclid's as well as the others) with abbreviations to make them catch the eye better.

In all three cases the proof is by *reductio ad absurdum*, and it is proved first that AB cannot be equal to LO , and secondly that AB cannot be less than LO .

Case I.

(1) Suppose, if possible, that $AB = LO$.

Then AB , BC are respectively equal to LO , OM ;
and $AC = LM$ (by construction).

Therefore $\angle ABC = \angle LOM$.

Similarly $\angle DEF = \angle MON$,

$\angle GHK = \angle NOL$.

Adding, we have

$$\begin{aligned}\angle ABC + \angle DEF + \angle GHK &= \angle LOM + \angle MON + \angle NOL \\ &= \text{four right angles :}\end{aligned}$$

which contradicts the hypothesis.

Therefore $AB \neq LO$.

(2) Suppose that $AB < LO$.

Make OP, OQ (measured along OL, OM) each equal to AB .

Thus, OL, OM being equal also, it follows that

PQ is $\parallel$ to LM .

Hence

$$LM : PQ = LO : OP;$$

and, since $LO > OP$,

LM , i.e. AC , $> PQ$.

Thus, in $\triangle POQ, ABC$, two sides are equal to two sides, and base $AC >$ base PQ ;

therefore

$$\angle ABC > \angle POQ, \quad \text{i.e. } \angle LOM.$$

Similarly

$$\angle DEF > \angle MON,$$

$$\angle GHK > \angle NOL,$$

and it follows by addition that

$$\angle ABC + \angle DEF + \angle GHK > (\text{four right angles}):$$

which again contradicts the hypothesis.

Case II.

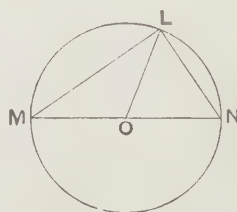
(1) Suppose, if possible, that $AB = LO$.

$$\begin{aligned} \text{Then } (AB + BC), \text{ or } (DE + EF) &= MO + OL \\ &= MN \\ &= DF; \end{aligned}$$

which contradicts the hypothesis.

(2) The supposition that $AB < LO$ is even more impossible; for in this case it would result that

$$DE + EF < DF.$$



Case III.

(1) Suppose, if possible, that $AB = LO$.

Then, in the triangles ABC, LOM , two sides AB, BC are respectively equal to two sides LO, OM , and the bases AC, LM are equal;

therefore $\angle ABC = \angle LOM$.

Similarly $\angle GHK = \angle NOL$.

Therefore, by addition,

$$\begin{aligned} \angle MON &= \angle ABC + \angle GHK \\ &> \angle DEF \quad (\text{by hypothesis}). \end{aligned}$$

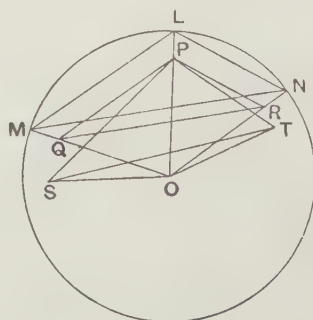
But, in the triangles DEF, MON , which are equal in all respects,

$$\angle MON = \angle DEF.$$

But it was proved that $\angle MON > \angle DEF$: which is impossible.

(2) Suppose, if possible, that $AB < LO$.

Along OL, OM measure OP, OQ each equal to AB .



Then LM, PQ are parallel, and

$$LM : PQ = LO : OP,$$

whence, since $LO > OP$,

$$LM, \text{ or } AC, > PQ.$$

Thus, in the triangles ABC, POQ ,

$$\angle ABC > \angle POQ, \text{ i.e. } \angle LOM.$$

Similarly, by taking OR along ON equal to AB , we prove that

$$\angle GHK > \angle LON.$$

Now, at O , make $\angle POS$ equal to $\angle ABC$, and $\angle POT$ equal to $\angle GHK$.

Make OS, OT each equal to OP , and join ST, SP, TP .

Then, in the equal triangles ABC, POS ,

$$AC = PS,$$

so that

$$LM = PS.$$

Similarly

$$LN = PT.$$

Therefore in the triangles MLN, SPT , since $\angle MLN > \angle SPT$ [this is assumed, but should have been explained],

$$MN > ST,$$

or

$$DF > ST.$$

Lastly, in $\triangle s DEF, SOT$, which have two sides equal to two sides, since $DF > ST$,

$$\angle DEF > \angle SOT$$

$$> \angle ABC + \angle GHK \text{ (by construction):}$$

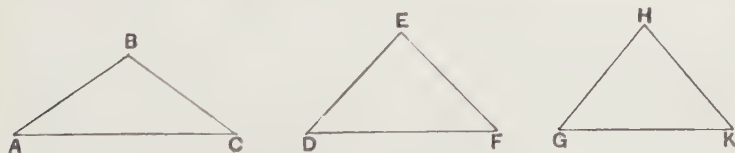
which contradicts the hypothesis.

Simson gives rather different proofs for all three cases; but the essence of them can be put, I think, a little more shortly than in his text, as well as more clearly.

Case I. (O within $\triangle LMN$.)

(1) Let AB be, if possible, equal to LO .

Then the $\triangle s ABC, DEF, GHK$ must be identically equal to the $\triangle s LOM, MON, NOL$ respectively.

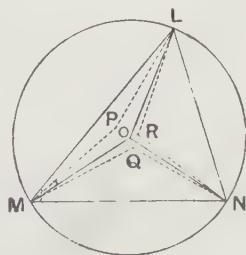


Therefore the vertical angles at O in the latter triangles are equal respectively to the angles at B, E, H .

The latter are therefore together equal to four right angles:

which is impossible.

(2) If AB be less than LO , construct on the bases LM, MN, NL triangles with vertices P, Q, R and identically equal to the $\triangle s ABC, DEF, GHK$ respectively.



Then P, Q, R will fall within the respective angles at O , since $PL = PM$ and $\angle LO$, and similarly in the other cases.

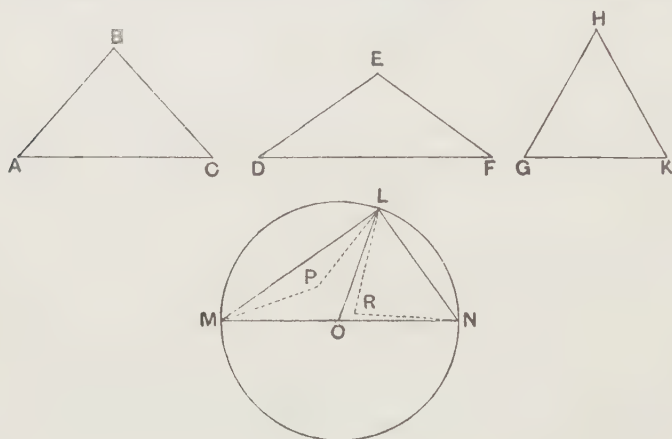
Thus [I. 21] the angles at P, Q, R are respectively greater than the angles at O in which they lie.

Therefore the sum of the angles at P, Q, R , i.e. the sum of the angles at B, E, H , is greater than four right angles:

which again contradicts the hypothesis.

Case II. (O lying on MN .)

In this case, whether (1) $AB = LO$, or (2) $AB < LO$, a triangle cannot be formed with MN as base and each of the other sides equal to AB . In other words, the triangle DEF either reduces to a straight line or is impossible.



Case III. (O lying outside the $\triangle LMN$.)

(1) Suppose, if possible, that $AB = LO$.

Then the triangles LOM, MON, NOL are identically equal to the triangles ABC, DEF, GHK .

Since

$$\angle LOM + \angle LON = \angle MON,$$

$$\angle ABC + \angle GHK = \angle DEF;$$

which contradicts the hypothesis.

(2) Suppose that $AB < OL$.

Draw, as before, on LM, MN, NL as bases triangles with vertices P, Q, R and identically equal to the $\triangle s ABC, DEF, GHK$.

Next, at N on the straight line NR , make $\angle RNS$ equal to the angle PLM , cut off NS equal to LM and join RS, LS .

Then $\triangle NRS$ is identically equal to $\triangle LPM$ or $\triangle ABC$.

Now $(\angle LNR + \angle RNS) < (\angle NLO + \angle OLM)$,

that is,

$$\angle LNS < \angle NLM.$$

Thus, in $\triangle s LNS, NLM$, two sides are equal to two sides, and the included angle in the former is less than the included angle in the other.

Therefore

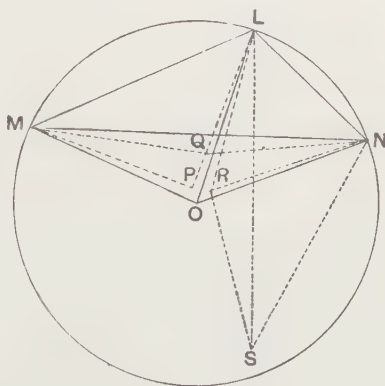
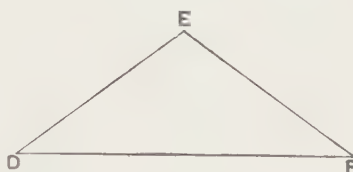
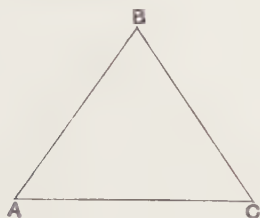
$$LS < MN.$$

Hence, in the triangles MQN , LRS , two sides are equal to two sides, and $MN > LS$.

Therefore

$$\begin{aligned}\angle MQN &> \angle LRS \\ &> (\angle LRN + \angle SRN) \\ &> (\angle LRN + \angle LPM). \\ \angle DEF &> (\angle GHK + \angle ABC): \end{aligned}$$

That is,
which is impossible.



PROPOSITION 24.

If a solid be contained by parallel planes, the opposite planes in it are equal and parallelogrammic.

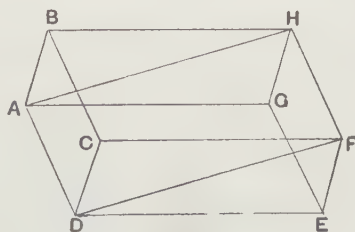
For let the solid $CDHG$ be contained by the parallel planes AC , GF , AH , DF , BF , AE ;

I say that the opposite planes in it are equal and parallelogrammic.

For, since the two parallel planes BG , CE are cut by the plane AC ,

their common sections are parallel.

[xi. 16]



Therefore AB is parallel to DC .

Again, since the two parallel planes BF , AE are cut by the plane AC ,

their common sections are parallel.

[XI. 16]

Therefore BC is parallel to AD .

But AB was also proved parallel to DC ;
therefore AC is a parallelogram.

Similarly we can prove that each of the planes DF , FG , GB , BF , AE is a parallelogram.

Let AH , DF be joined.

Then, since AB is parallel to DC , and BH to CF ,
the two straight lines AB , BH which meet one another are
parallel to the two straight lines DC , CF which meet one
another, not in the same plane;

therefore they will contain equal angles;

[XI. 10]

therefore the angle ABH is equal to the angle DCF .

And, since the two sides AB , BH are equal to the two
sides DC , CF ,

[I. 34]

and the angle ABH is equal to the angle DCF ,

therefore the base AH is equal to the base DF ,

and the triangle ABH is equal to the triangle DCF . [I. 4]

And the parallelogram BG is double of the triangle ABH ,
and the parallelogram CE double of the triangle DCF ; [I. 34]
therefore the parallelogram BG is equal to the parallelo-
gram CE .

Similarly we can prove that
 AC is also equal to GF ,
and AE to BF .

Therefore etc.

Q. E. D.

As Heiberg says, this proposition is carelessly enunciated. Euclid means a solid contained by *six* planes and not more, the planes are parallel *two and two*, and the opposite faces are equal in the sense of *identically* equal, or, as Simson puts it, *equal and similar*. The *similarity* is necessary in order to enable the equality of the parallelepipeds in the next proposition to be inferred from the 10th definition of Book XI. Hence a better enunciation would be:

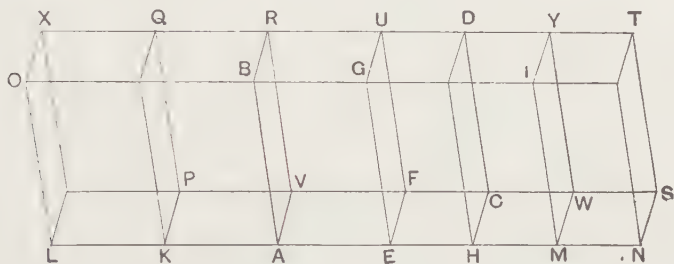
If a solid be contained by six planes parallel two and two, the opposite faces respectively are equal and similar parallelograms.

The proof is simple and requires no elucidation.

PROPOSITION 25.

If a parallelepipedal solid be cut by a plane which is parallel to the opposite planes, then, as the base is to the base, so will the solid be to the solid.

For let the parallelepipedal solid $ABCD$ be cut by the plane FG which is parallel to the opposite planes RA , DH ; I say that, as the base $AEFV$ is to the base $EHCF$, so is the solid $ABFU$ to the solid $EGCD$.



For let AH be produced in each direction, let any number of straight lines whatever, AK , KL , be made equal to AE , and any number whatever, HM , MN , equal to EH ; and let the parallelograms LP , KV , HW , MS and the solids LQ , KR , DM , MT be completed.

Then, since the straight lines LK , KA , AE are equal to one another, the parallelograms LP , KV , AF are also equal to one another, KO , KB , AG are equal to one another, and further LX , KQ , AR are equal to one another, for they are opposite. [xi. 24]

For the same reason the parallelograms EC , HW , MS are also equal to one another, HG , HI , IN are equal to one another, and further DH , MY , NT are equal to one another.

Therefore in the solids LQ , KR , AU three planes are equal to three planes.

But the three planes are equal to the three opposite ; therefore the three solids LQ , KR , AU are equal to one another.

For the same reason the three solids ED , DM , MT are also equal to one another.

Therefore, whatever multiple the base LF is of the base AF , the same multiple also is the solid LU of the solid AU .

For the same reason, whatever multiple the base NF is of the base FH , the same multiple also is the solid NU of the solid HU .

And, if the base LF is equal to the base NF , the solid LU is also equal to the solid NU ;
if the base LF exceeds the base NF , the solid LU also exceeds the solid NU ;
and, if one falls short, the other falls short.

Therefore, there being four magnitudes, the two bases AF , FH , and the two solids AU , UH , equimultiples have been taken of the base AF and the solid AU , namely the base LF and the solid LU , and equimultiples of the base FH and the solid HU , namely the base NF and the solid NU , and it has been proved that, if the base LF exceeds the base FN , the solid LU also exceeds the solid NU , if the bases are equal, the solids are equal, and if the base falls short, the solid falls short.

Therefore, as the base AF is to the base FH , so is the solid AU to the solid UH .

[v. Def. 5]

Q. E. D.

It is to be observed that, as the word *parallelogrammic* was used in Book I. without any definition of its meaning, so *παλληλεπίπεδος*, *parallelepipedal*, is here used without explanation. While it means simply "with parallel planes," i.e. "faces," the term is appropriated to the particular solid which has six plane faces parallel two and two. The proper translation of *στερεὸν παλληλεπίπεδον* is *parallelepipedal solid*, not *solid parallelepiped*, as it is usually translated. Still less is the solid a *parallelopiped*, as the word is not uncommonly written.

The opposite faces in each set of parallelepipedal solids in this proposition are not only equal but equal *and similar*. Euclid infers that the solids in each set are equal from Def. 10 ; but, as we have seen in the note on Defs. 9, 10,

though it is true, where no solid angle in the figures is contained by more than three plane angles, that two solid figures are equal and similar which are contained by the same number of equal and similar faces, similarly arranged, the fact should have been proved. To do this, we have only to prove the proposition, given above in the note on XI. 21, that *two trihedral angles are equal if the three face angles of the one are respectively equal to the three face angles in the other, and all are arranged in the same order*, and then to prove equality by applying one figure to the other as is done by Simson in his proposition C.

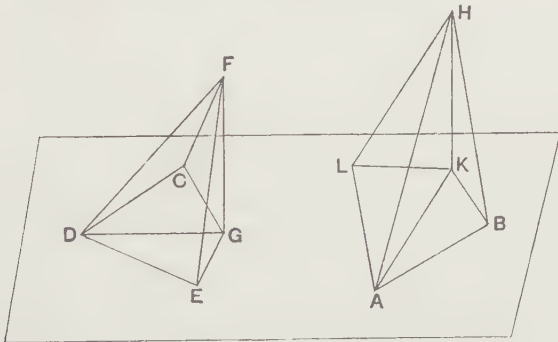
Application will also, of course, establish what is assumed by Euclid of the solids formed by the multiples of the original solids, namely that, if $LF \geq NF$, the solid $LU \geq$ the solid NU .

PROPOSITION 26.

On a given straight line, and at a given point on it, to construct a solid angle equal to a given solid angle.

Let AB be the given straight line, A the given point on it, and the angle at D , contained by the angles EDC , EDF , FDC , the given solid angle ;

thus it is required to construct on the straight line AB , and at the point A on it, a solid angle equal to the solid angle at D .



For let a point F be taken at random on DF ,
 let FG be drawn from F perpendicular to the plane through
 ED , DC , and let it meet the plane at G , [XI. 11]
 let DG be joined,
 let there be constructed on the straight line AB and at the
 point A on it the angle BAL equal to the angle EDC , and
 the angle BAK equal to the angle EDG , [I. 23]
 let AK be made equal to DG ,

let KH be set up from the point K at right angles to the plane through BA , AL , [XI. 12]

let KH be made equal to GF ,

and let HA be joined ;

I say that the solid angle at A , contained by the angles BAL , BAH , HAL is equal to the solid angle at D contained by the angles EDC , EDF , FDC .

For let AB , DE be cut off equal to one another,
and let HB , KB , FE , GE be joined.

Then, since FG is at right angles to the plane of reference, it will also make right angles with all the straight lines which meet it and are in the plane of reference ; [XI. Def. 3]
therefore each of the angles FGD , FGE is right.

For the same reason
each of the angles HKA , HKB is also right.

And, since the two sides KA , AB are equal to the two sides GD , DE respectively,
and they contain equal angles,
therefore the base KB is equal to the base GE . [I. 4]

But KH is also equal to GF ,
and they contain right angles ;
therefore HB is also equal to FE . [I. 4]

Again, since the two sides AK , KH are equal to the two sides DG , GF ,
and they contain right angles,
therefore the base AH is equal to the base FD . [I. 4]

But AB is also equal to DE ;
therefore the two sides HA , AB are equal to the two sides DF , DE .

And the base HB is equal to the base FE ;
therefore the angle BAH is equal to the angle EDF . [I. 8]

For the same reason
the angle HAL is also equal to the angle FDC .

And the angle BAL is also equal to the angle EDC .

Therefore on the straight line AB , and at the point A on it, a solid angle has been constructed equal to the given solid angle at D .

Q. E. F.

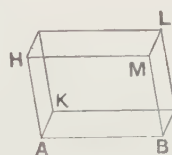
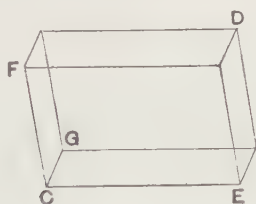
This proposition again assumes the equality of two trihedral angles which have the three plane angles of the one respectively equal to the three plane angles of the other taken in the same order.

PROPOSITION 27.

On a given straight line to describe a parallelepipedal solid similar and similarly situated to a given parallelepipedal solid.

Let AB be the given straight line and CD the given parallelepipedal solid;

thus it is required to describe on the given straight line AB a parallelepipedal solid similar and similarly situated to the given parallelepipedal solid CD .



For on the straight line AB and at the point A on it let the solid angle, contained by the angles BAH , HAK , KAB , be constructed equal to the solid angle at C , so that the angle BAH is equal to the angle ECF , the angle BAK equal to the angle ECG , and the angle KAH to the angle GCF ;

and let it be contrived that,

as EC is to CG , so is BA to AK ,

and, as GC is to CF , so is KA to AH .

[VI. 12]

Therefore also, *ex aequali*,

as EC is to CF , so is BA to AH .

[v. 22]

Let the parallelogram HB and the solid AL be completed.

Now since, as EC is to CG , so is BA to AK , and the sides about the equal angles ECG , BAK are thus proportional,

therefore the parallelogram GE is similar to the parallelogram KB .

For the same reason
the parallelogram KH is also similar to the parallelogram GF ,
and further FE to HB ;
therefore three parallelograms of the solid CD are similar to
three parallelograms of the solid AL .

But the former three are both equal and similar to the
three opposite parallelograms,
and the latter three are both equal and similar to the three
opposite parallelograms;
therefore the whole solid CD is similar to the whole solid AL .

[xi. Def. 9]

Therefore on the given straight line AB there has been
described AL similar and similarly situated to the given
parallelepipedal solid CD .

Q. E. F.

PROPOSITION 28.

*If a parallelepipedal solid be cut by a plane through the
diagonals of the opposite planes, the solid will be bisected by the
plane.*

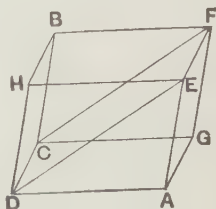
For let the parallelepipedal solid AB be cut by the plane
 $CDEF$ through the diagonals CF , DE of
opposite planes;

I say that the solid AB will be bisected by
the plane $CDEF$.

For, since the triangle CGF is equal
to the triangle CFB , [I. 34]
and ADE to DEH ,

while the parallelogram CA is also equal
to the parallelogram EB , for they are opposite,
and GE to CH ,

therefore the prism contained by the two triangles CGF ,
 ADE and the three parallelograms GE , AC , CE is also equal
to the prism contained by the two triangles CFB , DEH and
the three parallelograms CH , BE , CE ;



for they are contained by planes equal both in multitude and in magnitude.

[xi. Def. 10]

Hence the whole solid AB is bisected by the plane $CDEF$.

Q. E. D.

Simson properly observes that it ought to be proved that the diagonals of two opposite faces *are* in one plane, before we speak of drawing a plane through them. Clavius supplied the proof, which is of course simple enough.

Since EF , CD are both parallel to AG or BH , they are parallel to one another.

Consequently a plane can be drawn through CD , EF and the diagonals DE , CF are in that plane [xi. 7]. Moreover CD , EF are equal as well as parallel; so that CF , DE are also equal and parallel.

Simson does not, however, seem to have noticed a more serious difficulty. The two prisms are shown by Euclid to be contained by equal faces—the faces are in fact equal and similar—and Euclid then infers at once that the prisms are *equal*. But they are not equal in the only sense in which we have, at present, a right to speak of solids being equal, namely in the sense that they can be *applied*, the one to the other. They cannot be so applied because the faces, though equal respectively, are not *similarly arranged*; consequently the prisms are *symmetrical*, and it ought to be proved that they are, though not equal *and similar*, equal in content, or *equivalent*, as Legendre has it.

Legendre addressed himself to proving that the two prisms are equivalent, and his method has been adopted, though his name is not mentioned, by Schultze and Sevenoak and by Holgate. Certain preliminary propositions are necessary.

1. *The sections of a prism made by parallel planes cutting all the lateral edges are equal polygons.*

Suppose a prism MN cut by parallel planes which make sections $ABCDE$, $A'B'C'D'E'$.

Now AB , BC , CD , ... are respectively parallel to $A'B'$, $B'C'$, $C'D'$, ...

[xi. 16]

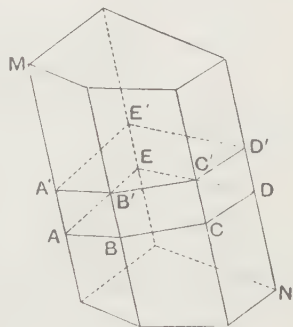
Therefore the angles ABC , BCD , ... are equal to the angles $A'B'C'$, $B'C'D'$, ... respectively.

[xi. 10]

Also AB , BC , CD , ... are respectively equal to $A'B'$, $B'C'$, $C'D'$, ...

[1. 34]

Thus the polygons $ABCDE$, $A'B'C'D'E'$ are equilateral and equiangular to one another.



2. *Two prisms are equal when they have a solid angle in each contained by three faces equal each to each and similarly arranged.*

Let the faces $ABCDE$, AG , AL be equal and similarly placed to the faces $A'B'C'D'E'$, $A'G'$, $A'L'$.

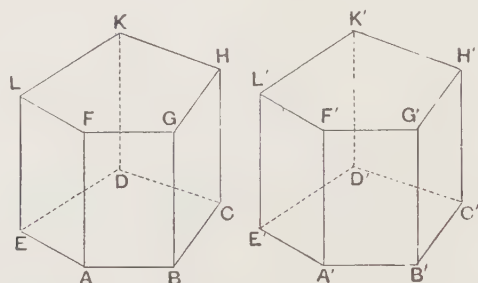
Since the three plane angles at A , A' are equal respectively and are similarly placed, the trihedral angle at A is equal to the trihedral angle at A' .

[(3) in note to xi. 21]

Place the trihedral angle at A on that at A' .

Then the face $ABCDE$ coincides with the face $A'B'C'D'E'$, the face AG with the face $A'G'$, and the face AL with the face $A'L'$.

The point C falls on C' and D on D' .



Since the lateral edges of a prism are parallel, CH will fall on $C'H'$, and DK on $D'K'$.

And the points F, G, L coincide respectively with F', G', L' , so that the planes $GK, G'K'$ coincide.

Hence H, K coincide with H', K' respectively.

Thus the prisms coincide throughout and are equal.

In the same way we can prove that two *truncated* prisms with three faces forming a solid angle related to one another as in the above proposition are identically equal.

In particular,

Cor. *Two right prisms having equal bases and equal heights are equal.*

3. *An oblique prism is equivalent to a right prism whose base is a right section of the oblique prism and whose height is equal to a lateral edge of the oblique prism.*

Suppose GL to be a right section of the oblique prism AD' , and let GL' be a right prism on GL as base and with height equal to a lateral edge of AD' .

Now the lateral edges of GL' are equal to the lateral edges of AD' .

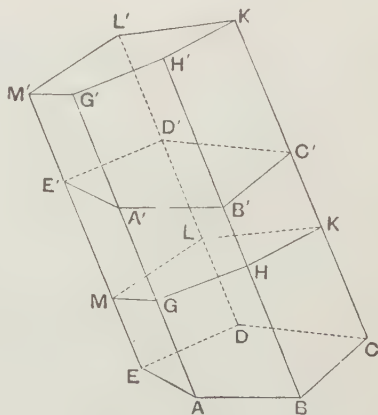
Therefore $AG = A'G', BH = B'H', CK = C'K'$, etc.

Thus the faces AH, BK, CL are equal respectively to the faces $A'H', B'K', C'L'$.

Therefore [by the proposition above]

(truncated prism AL) = (truncated prism $A'L'$).

Subtracting each from the whole solid AL' , we see that the prisms AD', GL' are equivalent.



Now suppose the parallelepiped of Euclid's proposition to be cut by the plane through AG , DF .

Let $KLMN$ be a right section of the parallelepiped cutting the edges AD , BC , GF , HE .

Then $KLMN$ is a parallelogram; and, if the diagonal KM be drawn,

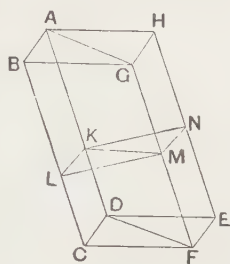
$$\triangle KLM = \triangle MNK.$$

Now the prism of which the $\triangle s$ ABG , DCF are the bases is equal to the right prism on $\triangle KLM$ as base and of height AD .

Similarly the prism of which the $\triangle s$ AGH , DFE are the bases is equal to the right prism on $\triangle MNK$ as base and with height AD .

And the right prisms on $\triangle s$ KLM , MNK as bases and of equal height AD are equal.

Consequently the two prisms into which the parallelepiped is divided are *equivalent*.



[(3) above]

[(2), Cor. above]

PROPOSITION 29.

Parallelepipedal solids which are on the same base and of the same height, and in which the extremities of the sides which stand up are on the same straight lines, are equal to one another.

Let CM , CN be parallelepipedal solids on the same base AB and of the same height,

and let the extremities of their sides which stand up, namely AG , AF , LM , LN , CD , CE , BH , BK , be on the same straight lines FN , DK ;

I say that the solid CM is equal to the solid CN .

For, since each of the figures CH , CK is a parallelogram, CB

is equal to each of the straight lines DH , EK ,

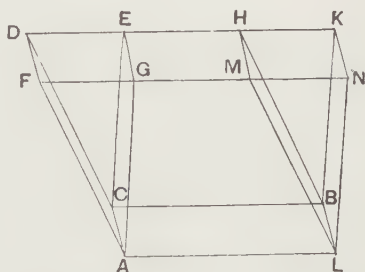
hence DH is also equal to EK .

Let EH be subtracted from each ;

therefore the remainder DE is equal to the remainder HK .

Hence the triangle DCE is also equal to the triangle HBK ,

and the parallelogram DG to the parallelogram HN .



[I. 34]

[I. 8, 4]

[I. 36]

For the same reason
the triangle AFG is also equal to the triangle MLN .

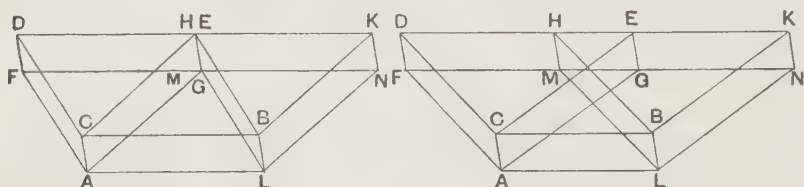
But the parallelogram CF is equal to the parallelogram BM , and CG to BN , for they are opposite ;
therefore the prism contained by the two triangles AFG , DCE and the three parallelograms AD , DG , CG is equal to the prism contained by the two triangles MLN , HBK and the three parallelograms BM , HN , BN .

Let there be added to each the solid of which the parallelogram AB is the base and $GEHM$ its opposite ;
therefore the whole parallelepipedal solid CM is equal to the whole parallelepipedal solid CN .

Therefore etc.

Q. E. D.

As usual, Euclid takes one case only and leaves the reader to prove for himself the two other possible cases shown in the subjoined figures. Euclid's proof holds with a very slight change in each case. With the first figure, the



only difference is that the prism of which the $\triangle s$ GAL , ECB are the bases takes the place of "the solid of which the parallelogram AB is the base and $GEHM$ its opposite"; while with the second figure we have to subtract the prisms which are proved equal successively from the solid of which the parallelogram AB is the base and $FDKN$ its opposite.

Simson, as usual, suspects mutilation by "some unskilful editor," but gives a curious reason why the case in which the two parallelograms opposite to AB have a side common ought not to have been omitted, namely that this case "is immediately deduced from the preceding 28th Prop which seems for this purpose to have been premised to the 29th." But, apart from the fact that Euclid's Prop. 28 does *not* prove the theorem which it enunciates (as we have seen), that theorem is not in the least necessary for the proof of this case of Prop. 29, as Euclid's proof applies to it perfectly well.

PROPOSITION 30.

Parallelepipedal solids which are on the same base and of the same height, and in which the extremities of the sides which stand up are not on the same straight lines, are equal to one another.

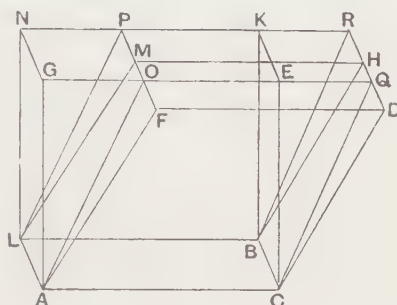
Let CM , CN be parallelepipedal solids on the same base AB and of the same height, and let the extremities of their sides which stand up, namely AF , AG , LM , LN , CD , CE , BH , BK , not be on the same straight lines ;

I say that the solid CM is equal to the solid CN .

For let NK , DH be produced and meet one another at R ,

and further let FM , GE be produced to P , Q ;

let AO , LP , CQ , BR be joined.



Then the solid CM , of which the parallelogram $ACBL$ is the base, and $FDHM$ its opposite, is equal to the solid CP , of which the parallelogram $ACBL$ is the base, and $OQRP$ its opposite ;

for they are on the same base $ACBL$ and of the same height, and the extremities of their sides which stand up, namely AF , AO , LM , LP , CD , CQ , BH , BR , are on the same straight lines FP , DR . [xi. 29]

But the solid CP , of which the parallelogram $ACBL$ is the base, and $OQRP$ its opposite, is equal to the solid CN , of which the parallelogram $ACBL$ is the base and $GEKN$ its opposite ;

for they are again on the same base $ACBL$ and of the same height, and the extremities of their sides which stand up, namely AG , AO , CE , CQ , LN , LP , BK , BR , are on the same straight lines GQ , NR .

Hence the solid CM is also equal to the solid CN .

Therefore etc.

Q. E. D.

This proposition completes the proof of the theorem that

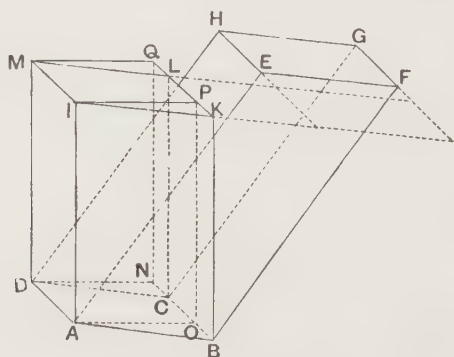
Two parallelepipeds on the same base and of the same height are equivalent.

Legendre deduced the useful theorem that

Every parallelepiped can be changed into an equivalent rectangular parallelepiped having the same height and an equivalent base.

For suppose we have a parallelepiped on the base $AFCD$ with $EFGH$ for the opposite face.

Draw AI, BK, CL, DM perpendicular to the plane through $EFGH$ and all equal to the height of the parallelepiped AG . Then, on joining IK, KL, LM, MI , we have a parallelepiped equivalent to the original one and having its lateral faces AK, BL, CM, DI rectangles.

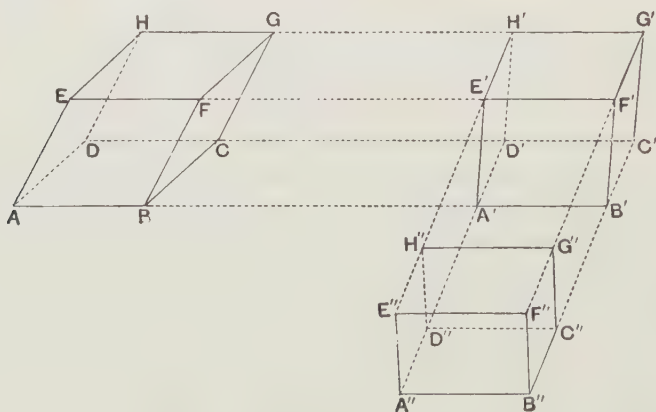


If $ABCD$ is not a rectangle, draw AO, DN in the plane AC perpendicular to BC , and IP, MQ in the plane IL perpendicular to KL .

Joining OP, NQ , we have a *rectangular* parallelepiped on $AOND$ as base which is equivalent to the parallelepiped with $ABCD$ as base and $IKLM$ as opposite face, since we may regard these parallelepipeds as being on the same base $ADMI$ and of the same height (AO).

That is, a rectangular parallelepiped has been constructed which is equivalent to the given parallelepiped and has (1) the same height, (2) an equivalent base.

The American text-books which I have quoted adopt a somewhat different construction shown in the subjoined figure.



The edges AB, DC, EF, HG of the original parallelepiped are produced and cut at right angles by two parallel planes at a distance apart $A'B'$ equal to AB .

Thus a parallelepiped is formed in which all the faces are rectangles except $A'H', B'G'$.

Next produce $D'A'$, $C'B'$, $G'F'$, $H'E'$ and cut them perpendicularly by two parallel planes at a distance apart $B''C''$ equal to $B'C'$.

The points of section determine a *rectangular* parallelepiped.

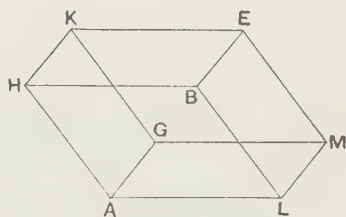
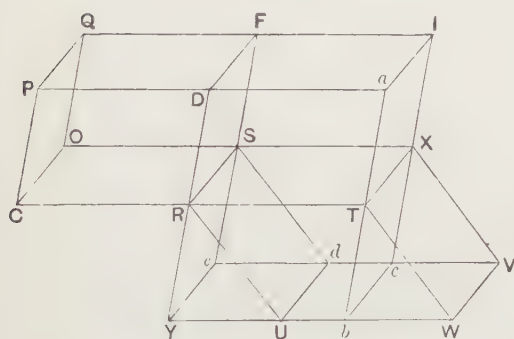
The equivalence of the three parallelepipeds is proved, not by Eucl. xi. 29, 30, but by the proposition about a right section of a prism given above in the note to xi. 28 (3 in that note).

PROPOSITION 31.

Parallelepipedal solids which are on equal bases and of the same height are equal to one another.

Let the parallelepipedal solids AE , CF , of the same height, be on equal bases AB , CD .

I say that the solid AE is equal to the solid CF .



First, let the sides which stand up, HK , BE , AG , LM , PQ , DF , CO , RS , be at right angles to the bases AB , CD ; let the straight line RT be produced in a straight line with CR ;

on the straight line RT , and at the point R on it, let the angle TRU be constructed equal to the angle ALB , [i. 23] let RT be made equal to AL , and RU equal to LB , and let the base RW and the solid XU be completed.

Now, since the two sides TR , RU are equal to the two sides AL , LB ,

and they contain equal angles,

therefore the parallelogram RW is equal and similar to the parallelogram HL .

Since again AL is equal to RT , and LM to RS , and they contain right angles,

therefore the parallelogram RX is equal and similar to the parallelogram AM .

For the same reason

LE is also equal and similar to SU ;

therefore three parallelograms of the solid AE are equal and similar to three parallelograms of the solid XU .

But the former three are equal and similar to the three opposite, and the latter three to the three opposite; [xi. 24]
therefore the whole parallelepipedal solid AE is equal to the whole parallelepipedal solid XU . [xi. Def. 10]

Let DR , WU be drawn through and meet one another at Y ,

let aTb be drawn through T parallel to DY ,

let PD be produced to a ,

and let the solids YX , RI be completed.

Then the solid XY , of which the parallelogram RX is the base and Yc its opposite, is equal to the solid XU of which the parallelogram RX is the base and UV its opposite, for they are on the same base RX and of the same height, and the extremities of their sides which stand up, namely RY , RU , Tb , TW , Se , Sd , Xc , XV , are on the same straight lines YW , eV . [xi. 29]

But the solid XU is equal to AE ;
therefore the solid XY is also equal to the solid AE .

And, since the parallelogram $RUWT$ is equal to the parallelogram YT ,
for they are on the same base RT and in the same parallels RT , YW , [I. 35]
while $RUWT$ is equal to CD , since it is also equal to AB ,
therefore the parallelogram YT is also equal to CD .

But DT is another parallelogram;
therefore, as the base CD is to DT , so is YT to DT . [v. 7]

And, since the parallelepipedal solid CI has been cut by the plane RF which is parallel to opposite planes,
as the base CD is to the base DT , so is the solid CF to the solid RI . [xi. 25]

For the same reason,

since the parallelepipedal solid YI has been cut by the plane RX which is parallel to opposite planes,

as the base YT is to the base TD , so is the solid YX to the solid RI . [xi. 25]

But, as the base CD is to DT , so is YT to DT ;

therefore also, as the solid CF is to the solid RI , so is the solid YX to RI . [v. 11]

Therefore each of the solids CF , YX has to RI the same ratio ;

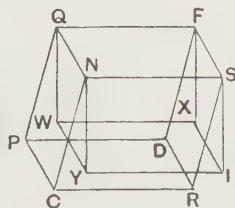
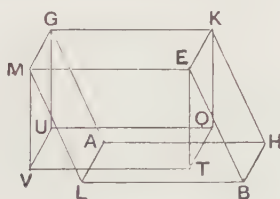
therefore the solid CF is equal to the solid YX . [v. 9]

But YX was proved equal to AE ;

therefore AE is also equal to CF .

Next, let the sides standing up, AG , HK , BE , LM , CN , PQ , DF , RS , not be at right angles to the bases AB , CD ;

I say again that the solid AE is equal to the solid CF .



For from the points K , E , G , M , Q , F , N , S let KO , ET , GU , MV , QW , FX , NY , SI be drawn perpendicular to the plane of reference, and let them meet the plane at the points O , T , U , V , W , X , Y , I ,

and let OT , OU , UV , TV , WX , WY , YI , IX be joined.

Then the solid KV is equal to the solid QI ,

for they are on the equal bases KM , QS and of the same height, and their sides which stand up are at right angles to their bases. [First part of this Prop.]

But the solid KV is equal to the solid AE ,

and QI to CF ;

for they are on the same base and of the same height, while the extremities of their sides which stand up are not on the same straight lines. [xi. 30]

Therefore the solid AE is also equal to the solid CF .
Therefore etc.

Q. E. D.

It is interesting to observe that, in the figure of this proposition, the bases are represented as lying "in the plane of the paper," as it were, and the third dimension as "standing up" from that plane. The figure is that of the manuscript P slightly corrected as regards the solid AE .

Nothing could well be more ingenious than the proof of this proposition, which recalls the brilliant proposition I. 44 and the proofs of VI. 14 and 23.

As the proof occupies considerable space in the text, it will no doubt be well to give a summary.

I. First, suppose that the edges terminating at the angular points of the bases are *perpendicular* to the bases.

AB , CD being the bases, Euclid constructs a solid identically equal to AE (he might simply have *moved* AE itself), placing it so that RS is the edge corresponding to HK ($RS = HK$ because the heights are equal), and the face RX corresponding to HE is in the plane of CS .

The faces CD , RW are in one plane because both are perpendicular to RS . Thus DR , WU meet, if produced, in Y say.

Complete the parallelograms YT , DT and the solids YX , FT .

Then

$$(\text{solid } YX) = (\text{solid } UX),$$

because they are on the same base ST and of the same height. [XI. 29]

Also, CI , YI being parallelepipeds cut by planes RF , RX parallel to pairs of opposite faces respectively,

$$(\text{solid } CF) : (\text{solid } RI) = \square CD : \square DT, \quad [\text{XI. 25}]$$

and

$$(\text{solid } YX) : (\text{solid } RI) = \square YT : \square DT.$$

But [I. 35]

$$\square YT = \square UT$$

$$= \square AB$$

$$= \square CD, \text{ by hypothesis.}$$

Therefore

$$(\text{solid } CF) = (\text{solid } YX)$$

$$= (\text{solid } UX)$$

$$= (\text{solid } AE).$$

II. If the edges terminating at the base are *not* perpendicular to it, turn each solid into an equivalent one on the same base with edges perpendicular to it (by drawing four perpendiculars from the angular points of the base to the plane of the opposite face). (XI. 29, 30 prove the equivalence.)

Then the equivalent solids are equal, by Part I.; so that the original solids are also equal.

Simson observes that Euclid has made no mention of the case in which the bases of the two solids are *equiangular*, and he prefixes this case to Part I. in the text. This is surely unnecessary, as Part I. covers it well enough: the only difference in the figure is that UW would coincide with Yb and dV with e .

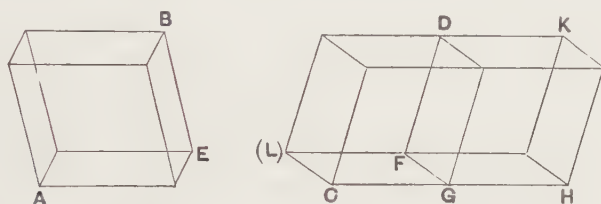
Simson further remarks that in the demonstration of Part II. it is not proved that the new solids constructed in the manner described *are* parallelepipeds. The proof is, however, so simple that it scarcely needed insertion

into the text. He is correct in his remark that the words "while the extremities of their sides which stand up are not on the same straight lines" just before the end of the proposition would be better absent, since they *may* be "on the same straight lines."

PROPOSITION 32.

Parallelepipedal solids which are of the same height are to one another as their bases.

Let AB , CD be parallelepipedal solids of the same height; I say that the parallelepipedal solids AB , CD are to one another as their bases, that is, that, as the base AE is to the base CF , so is the solid AB to the solid CD .



For let FH equal to AE be applied to FG , [I. 45]
and on FH as base, and with the same height as that of CD ,
let the parallelepipedal solid GK be completed.

Then the solid AB is equal to the solid GK ;
for they are on equal bases AE , FH and of the same height. [XI. 31]

And, since the parallelepipedal solid CK is cut by the plane DG which is parallel to opposite planes,
therefore, as the base CF is to the base FH , so is the solid CD to the solid DH . [XI. 25]

But the base FH is equal to the base AE ,
and the solid GK to the solid AB ;
therefore also, as the base AE is to the base CF , so is the solid AB to the solid CD .

Therefore etc.

Q. E. D.

As Clavius observed, Euclid should have said, in applying the parallelogram FH to FG , that it should be applied "in the angle FGH equal to the angle LCG ." Simson is however, I think, hypercritical when he states as regards the completion of the solid GK that it ought to be said, "complete

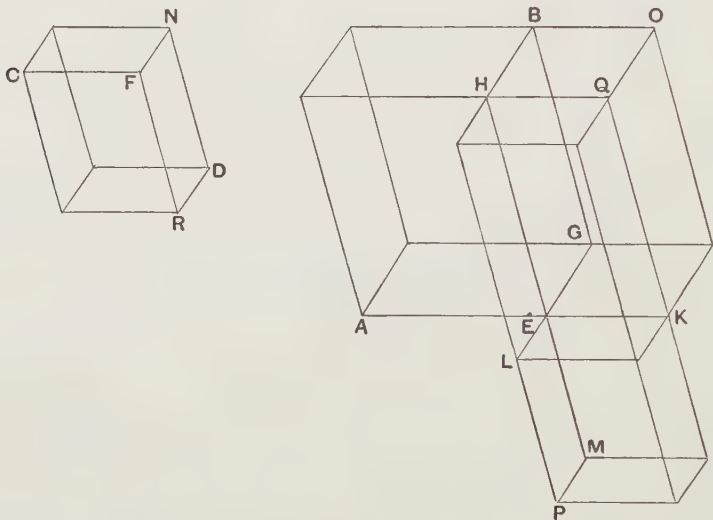
the solid of which the base is FH , and one of its insisting straight lines is FD ." Surely, when we have two faces DG, FH meeting in an edge, to say "complete the solid" is quite sufficient, though the words "on FH as base" might perhaps as well be left out. The same "completion" of a parallelepipedal solid occurs in XI. 31 and 33.

PROPOSITION 33.

Similar parallelepipedal solids are to one another in the triplicate ratio of their corresponding sides.

Let AB, CD be similar parallelepipedal solids,
and let AE be the side corresponding to CF ;

I say that the solid AB has to the solid CD the ratio triplicate of that which AE has to CF .



For let EK, EL, EM be produced in a straight line with AE, GE, HE ,

let EK be made equal to CF , EL equal to FN , and further EM equal to FR ,

and let the parallelogram KL and the solid KP be completed.

Now, since the two sides KE, EL are equal to the two sides CF, FN ,

while the angle KEL is also equal to the angle CFN , inasmuch as the angle AEG is also equal to the angle CFN because of the similarity of the solids AB, CD ,

therefore the parallelogram KL is equal < and similar > to the parallelogram CN .

For the same reason
the parallelogram KM is also equal and similar to CR ,
and further EP to DF ;
therefore three parallelograms of the solid KP are equal and similar to three parallelograms of the solid CD .

But the former three parallelograms are equal and similar to their opposites, and the latter three to their opposites; [xi. 24]
therefore the whole solid KP is equal and similar to the whole solid CD .
[xi. Def. 10]

Let the parallelogram GK be completed,
and on the parallelograms GK , KL as bases, and with the same height as that of AB , let the solids EO , LQ be completed.

Then since; owing to the similarity of the solids AB , CD ,
as AE is to CF , so is EG to FN , and EH to FR ,
while CF is equal to EK , FN to EL , and FR to EM ,
therefore, as AE is to EK , so is GE to EL , and HE to EM .

But, as AE is to EK , so is AG to the parallelogram GK ,
as GE is to EL , so is GK to KL ,
and, as HE is to EM , so is QE to KM ; [vi. 1]
therefore also, as the parallelogram AG is to GK , so is GK to KL , and QE to KM .

But, as AG is to GK , so is the solid AB to the solid EO ,
as GK is to KL , so is the solid OE to the solid QL ,
and, as QE is to KM , so is the solid QL to the solid KP ;
[xi. 32]

therefore also, as the solid AB is to EO , so is EO to QL , and QL to KP .

But, if four magnitudes be continuously proportional, the first has to the fourth the ratio triplicate of that which it has to the second; [v. Def. 10]
therefore the solid AB has to KP the ratio triplicate of that which AB has to EO .

But, as AB is to EO , so is the parallelogram AG to GK ,
and the straight line AE to EK [vi. 1];

hence the solid AB has also to KP the ratio triplicate of that which AE has to EK .

But the solid KP is equal to the solid CD ,
and the straight line EK to CF ;
therefore the solid AB has also to the solid CD the ratio triplicate of that which the corresponding side of it, AE , has to the corresponding side CF .

Therefore etc.

Q. E. D.

PORISM. From this it is manifest that, if four straight lines be < continuously > proportional, as the first is to the fourth, so will a parallelepipedal solid on the first be to the similar and similarly described parallelepipedal solid on the second, inasmuch as the first has to the fourth the ratio triplicate of that which it has to the second.

The proof may be summarised as follows.

The three edges AE , GE , HE of the parallelepiped AB which meet at E , the vertex corresponding to R in the other parallelepiped, are produced, and lengths EK , EL , EM are marked off equal respectively to the edges CF , FN , FR of CD .

The parallelograms and solids are then completed as shown in the figure.

Euclid first shows that the solid CD and the new solid PK are equal and similar according to the criterion in XI. Def. 10, viz. that they are contained by the same number of equal and similar planes. (They are arranged in the same order, and it would be easy to prove equality by proving the equality of a pair of solid angles and then applying one solid to the other.)

We have now, by hypothesis,

$$AE : CF = EG : FN = EH : FR;$$

that is,

$$AE : EK = EG : EL = EH : EM.$$

But

$$AE : EK = \square AG : \square GK,$$

[VI. 1]

$$EG : EL = \square GK : \square KL,$$

$$EH : EM = \square HK : \square KM.$$

Again, by XI. 25 or 32,

$$\square AG : \square GK = (\text{solid } AB) : (\text{solid } EO),$$

$$\square GK : \square KL = (\text{solid } EO) : (\text{solid } QL),$$

$$\square HK : \square KM = (\text{solid } QL) : (\text{solid } KP).$$

Therefore

$(\text{solid } AB) : (\text{solid } EO) = (\text{solid } EO) : (\text{solid } QL) = (\text{solid } QL) : (\text{solid } KP)$,
or the solid AB is to the solid KP (that is, CD) in the ratio triplicate of that which the solid AB has to the solid EO , i.e. the ratio triplicate of that which AE has to EK (or CF).

Heiberg doubts whether the Porism appended to this proposition is genuine.

Simson adds, as Prop. D, a useful theorem which we should have expected to find here, on the analogy of VI. 23 following VI. 19, 20, viz. that *Solid parallelepipeds contained by parallelograms equiangular to one another, each to each, that is, of which the solid angles are equal, each to each, have to one another the ratio compounded of the ratios of their sides.*

The proof follows the method of the proposition XI. 33, and we can use the same figure. In order to obtain one ratio between lines to represent the ratio compounded of the ratios of the sides, after the manner of VI. 23, we take any straight line a , and then determine three other straight lines b, c, d , such that

$$AE : CF = a : b,$$

$$EG : FN = b : c,$$

$$EH : FR = c : d,$$

whence $a : d$ represents the ratio compounded of the ratios of the sides.

We obtain, in the same manner as above,

$$(\text{solid } AB) : (\text{solid } EO) = \square AG : \square GK = AE : EK = AE : CF \\ = a : b,$$

$$(\text{solid } EO) : (\text{solid } QL) = \square GK : \square KL = GE : EL = GE : FN \\ = b : c,$$

$$(\text{solid } QL) : (\text{solid } KP) = \square HK : \square KM = EH : EM = EH : FR \\ = c : d,$$

whence, by composition [V. 22],

$$(\text{solid } AB) : (\text{solid } KP) = a : d,$$

or

$$(\text{solid } AB) : (\text{solid } CD) = a : d.$$

PROPOSITION 34.

In equal parallelepipedal solids the bases are reciprocally proportional to the heights; and those parallelepipedal solids in which the bases are reciprocally proportional to the heights are equal.

Let AB, CD be equal parallelepipedal solids ;

I say that in the parallelepipedal solids AB, CD the bases are reciprocally proportional to the heights,

that is, as the base EH is to the base NQ , so is the height of the solid CD to the height of the solid AB .

First, let the sides which stand up, namely $AG, EF, LB, HK, CM, NO, PD, QR$, be at right angles to their bases ;

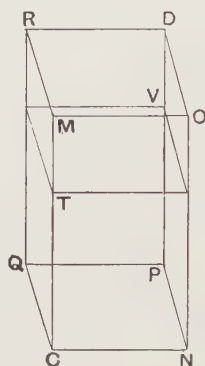
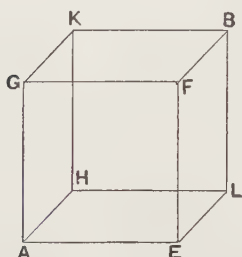
I say that, as the base EH is to the base NQ , so is CM to AG .

If now the base EH is equal to the base NQ , while the solid AB is also equal to the solid CD , CM will also be equal to AG .

For parallelepipedal solids of the same height are to one another as the bases ; [XI. 32]

and, as the base EH is to NQ , so will CM be to AG , and it is manifest that in the parallelepipedal solids AB , CD the bases are reciprocally proportional to the heights.

Next, let the base EH not be equal to the base NQ , but let EH be greater.



Now the solid AB is equal to the solid CD ; therefore CM is also greater than AG .

Let then CT be made equal to AG , and let the parallelepipedal solid VC be completed on NQ as base and with CT as height.

Now, since the solid AB is equal to the solid CD , and CV is outside them, while equals have to the same the same ratio, [v. 7] therefore, as the solid AB is to the solid CV , so is the solid CD to the solid CV .

But, as the solid AB is to the solid CV , so is the base EH to the base NQ , for the solids AB , CV are of equal height ; [XI. 32] and, as the solid CD is to the solid CV , so is the base MQ to the base TQ [XI. 25] and CM to CT [VI. 1] ; therefore also, as the base EH is to the base NQ , so is MC to CT .

But CT is equal to AG ; therefore also, as the base EH is to the base NQ , so is MC to AG .

Therefore in the parallelepipedal solids AB , CD the bases are reciprocally proportional to the heights.

Again, in the parallelepipedal solids AB , CD let the bases be reciprocally proportional to the heights, that is, as the base EH is to the base NQ , so let the height of the solid CD be to the height of the solid AB ;

I say that the solid AB is equal to the solid CD .

Let the sides which stand up be again at right angles to the bases.

Now, if the base EH is equal to the base NQ , and, as the base EH is to the base NQ , so is the height of the solid CD to the height of the solid AB , therefore the height of the solid CD is also equal to the height of the solid AB .

But parallelepipedal solids on equal bases and of the same height are equal to one another ; [xi. 31]

therefore the solid AB is equal to the solid CD .

Next, let the base EH not be equal to the base NQ , but let EH be greater ; therefore the height of the solid CD is also greater than the height of the solid AB , that is, CM is greater than AG .

Let CT be again made equal to AG , and let the solid CV be similarly completed.

Since, as the base EH is to the base NQ , so is MC to AG ,

while AG is equal to CT ,

therefore, as the base EH is to the base NQ , so is CM to CT .

But, as the base EH is to the base NQ , so is the solid AB to the solid CV ,

for the solids AB , CV are of equal height ; [xi. 32]

and, as CM is to CT , so is the base MQ to the base QT [vi. 1] and the solid CD to the solid CV . [xi. 25]

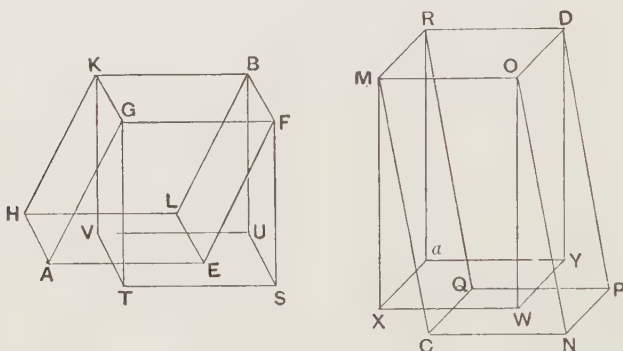
Therefore also, as the solid AB is to the solid CV , so is the solid CD to the solid CV ;

therefore each of the solids AB , CD has to CV the same ratio.

Therefore the solid AB is equal to the solid CD . [v. 9]

Now let the sides which stand up, FE , BL , GA , HK , ON , DP , MC , RQ , not be at right angles to their bases ; let perpendiculars be drawn from the points F , G , B , K , O , M , D , R to the planes through EH , NQ , and let them meet the planes at S , T , U , V , W , X , Y , a , and let the solids FV , Oa be completed ;

I say that, in this case too, if the solids AB , CD are equal, the bases are reciprocally proportional to the heights, that is, as the base EH is to the base NQ , so is the height of the solid CD to the height of the solid AB .



Since the solid AB is equal to the solid CD ,
 while AB is equal to BT ,
 for they are on the same base FK and of the same height ; [XI. 29, 30]
 and the solid CD is equal to DX ,
 for they are again on the same base RO and of the same height ; [id.]
 therefore the solid BT is also equal to the solid DX .

Therefore, as the base FK is to the base OR , so is the height of the solid DX to the height of the solid BT .

[Part 1.]

But the base FK is equal to the base EH ,
 and the base OR to the base NQ ;
 therefore, as the base EH is to the base NQ , so is the height of the solid DX to the height of the solid BT .

But the solids DX , BT and the solids DC , BA have the same heights respectively ;
therefore, as the base EH is to the base NQ , so is the height of the solid DC to the height of the solid AB .

Therefore in the parallelepipedal solids AB , CD the bases are reciprocally proportional to the heights.

Again, in the parallelepipedal solids AB , CD let the bases be reciprocally proportional to the heights,
that is, as the base EH is to the base NQ , so let the height of the solid CD be to the height of the solid AB ;
I say that the solid AB is equal to the solid CD .

For, with the same construction,
since, as the base EH is to the base NQ , so is the height of the solid CD to the height of the solid AB ,
while the base EH is equal to the base FK ,
and NQ to OR ,
therefore, as the base FK is to the base OR , so is the height of the solid CD to the height of the solid AB .

But the solids AB , CD and BT , DX have the same heights respectively ;
therefore, as the base FK is to the base OR , so is the height of the solid DX to the height of the solid BT .

Therefore in the parallelepipedal solids BT , DX the bases are reciprocally proportional to the heights ;
therefore the solid BT is equal to the solid DX . [Part 1.]

But BT is equal to BA ,
for they are on the same base FK and of the same height ;
and the solid DX is equal to the solid DC . [xi. 29, 30] [id.]

Therefore the solid AB is also equal to the solid CD .

Q. E. D.

In this proposition Euclid makes two assumptions which require notice, (1) that, if two parallelepipeds are equal, and have equal bases, their heights are equal, and (2) that, if the bases of two equal parallelepipeds are unequal, that which has the lesser base has the greater height. In justification of the former statement Euclid says, according to what Heiberg holds to be the genuine reading, "for parallelepipedal solids of the same height are to one another as their bases" [xi. 32]. This apparently struck some very early editor as not being sufficient, and he added the explanation appearing in Simson's text, "For if, the bases EH , NQ being equal, the heights AG , CM

were not equal, neither would the solid AB be equal to CD . But it is by hypothesis equal. Therefore the height CM is not unequal to the height AG ; therefore it is equal." Then, it being perceived that there ought not to be two explanations, the genuine one was erased from the inferior MSS. While the interpolated explanation does not take us very far, the truth of the statement may be deduced with perhaps greater ease from xi. 31 than from xi. 32 quoted by Euclid. For, assuming one height greater than the other, while the bases are equal, we have only to cut from the higher solid so much as will make its height equal to that of the other. Then this *part* of the higher solid is equal to the whole of the other solid which is by hypothesis equal to the higher solid itself. That is, the whole is equal to its part: which is impossible.

The genuine text contains no explanation of the second assumption that, if the base EH be greater than the base NQ , while the solids are equal, the height CM is greater than the height AG ; for the added words "for, if not, neither again will the solids AB , CD be equal; but they are equal by hypothesis" are no doubt interpolated. In this case the truth of the assumption is easily deduced from xi. 32 by *reductio ad absurdum*. If the height CM were equal to the height AG , the solid AB would be to the solid CD as the base EH is to the base NQ , i.e. as a greater to a less, so that the solids would not be equal, as they are by hypothesis. Again, if the height CM were less than the height AG , we could increase the height of CD till it was equal to that of AB , and it would then appear that AB is greater than the heightened solid and *a fortiori* greater than CD : which contradicts the hypothesis.

Clavius rather ingeniously puts the first assumption the other way, saying that, if the heights are equal in the equal parallelepipeds, the bases must be equal. This follows *directly* from xi. 32, which proves that the parallelepipeds are to one another as their bases; though Clavius deduces it indirectly from xi. 31. The advantage of Clavius' alternative is that it makes the second assumption unnecessary. He merely says, if the *heights* be not equal, let CM be the greater, and then proceeds with Euclid's construction.

It is also to be observed that, when Euclid comes to the corresponding proposition for cones and cylinders [xii. 15], he begins by supposing the *heights* equal, inferring by xii. 11 (corresponding to xi. 32) that, the solids being equal, the bases are also equal, and then proceeds to the case where the heights are unequal without making any preliminary inference about the bases. The analogy then of xii. 15, and the fact that he quotes xi. 32 here (which directly proves that, if the solids are equal, and also their heights, their bases are also equal), make Clavius' form the more convenient to adopt.

The two assumptions being proved as above, the proposition can be put shortly as follows.

I. Suppose the edges terminating at the corners of the base to be *perpendicular* to it.

Then (a), if the base EH be equal to the base NQ , the parallelepipeds being also equal, the heights must be equal (converse of xi. 31), so that the bases are reciprocally proportional to the heights, the ratio of the bases and the ratio of the heights being both ratios of equality.

(b) If the base EH be greater than the base NQ , and consequently (by deduction from xi. 32) the height CM greater than the height AG , cut off CT from CM equal to AG , and draw the plane TV through T parallel to the base NQ , making the parallelepiped CV , with CT ($= AG$) for its height.

Then, since the solids AB , CD are equal,

$$(\text{solid } AB) : (\text{solid } CV) = (\text{solid } CD) : (\text{solid } CV). \quad [\text{v. 7}]$$

But $(\text{solid } AB) : (\text{solid } CV) = \square HE : \square NQ,$ [xi. 32]
 and $(\text{solid } CD) : (\text{solid } CV) = \square MQ : \square TQ$ [xi. 25]
 $= CM : CT.$ [vi. 1]
 Therefore $\square HE : \square NQ = CM : CT$
 $= CM : AG.$

Conversely (a), if the bases EH, NQ be equal and reciprocally proportional to the heights, the heights must be equal.

Consequently $(\text{solid } AB) = (\text{solid } CD).$ [xi. 31]

(b) If the bases EH, NQ be unequal, if, e.g. $\square EH > \square NQ,$
 then, since $\square EH : \square NQ = CM : AG,$
 $CM > AG.$

Make the same construction as before.

Then $\square EH : \square NQ = (\text{solid } AB) : (\text{solid } CV),$ [xi. 32]
 and $CM : AG = CM : CT$
 $= \square MQ : \square TQ$ [vi. 1]
 $= (\text{solid } CD) : (\text{solid } CV).$ [xi. 25]

Therefore

$$(\text{solid } AB) : (\text{solid } CV) = (\text{solid } CD) : (\text{solid } CV),$$

whence $(\text{solid } AB) = \text{solid } CD.$ [v. 9]

II. Suppose that the edges terminating at the corners of the bases are *not* perpendicular to it.

Drop perpendiculars on the bases from the corners of the faces opposite to the bases.

We thus have two parallelepipeds equal to AB, CD respectively, since they are on the same bases FK, RO and of the same height respectively.

[xi. 29, 30]

If then (1) the solid AB is equal to the solid $CD,$

$$(\text{solid } BT) = (\text{solid } DX),$$

and, by the first part of this proposition,

$$\square KF : \square OR = MX : GT,$$

$$\square HE : \square NQ = MX : GT.$$

$$(2) \text{ If } \square HE : \square NQ = MX : GT,$$

$$\text{then } \square KF : \square OR = MX : GT,$$

so that, by the first half of the proposition, the solids BT, DX are equal, and consequently

$$(\text{solid } AB) = (\text{solid } CD).$$

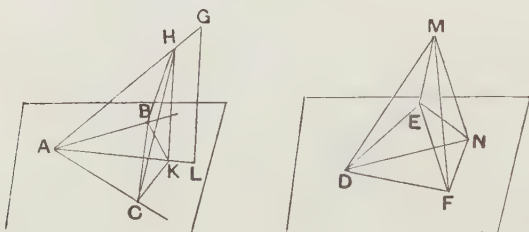
The text of the second part of the proposition four times contains, after the words "of the same height," the words "in which the sides which stand up are not on the same straight lines." As Simson observed, they are inept, as the extremities of the edges may or may not be "on the same straight lines"; cf. the similar words incorrectly inserted at the end of xi. 31.

Words purporting to quote the result of the first part of the proposition are also twice inserted; but they are rejected as unnecessary and as containing an absurd expression—" (solids) in which the heights are at right angles to their bases," as if the heights could be otherwise than perpendicular to the bases.

PROPOSITION 35.

If there be two equal plane angles, and on their vertices there be set up elevated straight lines containing equal angles with the original straight lines respectively, if on the elevated straight lines points be taken at random and perpendiculars be drawn from them to the planes in which the original angles are, and if from the points so arising in the planes straight lines be joined to the vertices of the original angles, they will contain, with the elevated straight lines, equal angles.

Let the angles BAC , EDF be two equal rectilineal angles, and from the points A , D let the elevated straight lines AG , DM be set up containing, with the original straight lines, equal angles respectively, namely, the angle MDE to the angle GAB and the angle MDF to the angle GAC , let points G , M be taken at random on AG , DM , let GL , MN be drawn from the points G , M perpendicular to the planes through BA , AC and ED , DF , and let them meet the planes at L , N , and let LA , ND be joined ;
I say that the angle GAL is equal to the angle MDN .



Let AH be made equal to DM ,
and let HK be drawn through the point H parallel to GL .
But GL is perpendicular to the plane through BA , AC ;
therefore HK is also perpendicular to the plane through BA , AC . [xi. 8]

From the points K , N let KC , NF , KB , NE be drawn perpendicular to the straight lines AC , DF , AB , DE , and let HC , CB , MF , FE be joined.

Since the square on HA is equal to the squares on HK , KA ,

and the squares on KC , CA are equal to the square on KA ,

[I. 47]

therefore the square on HA is also equal to the squares on HK , KC , CA .

But the square on HC is equal to the squares on HK , KC ;

[I. 47]

therefore the square on HA is equal to the squares on HC , CA .

Therefore the angle HCA is right.

[I. 48]

For the same reason

the angle DFM is also right.

Therefore the angle ACH is equal to the angle DFM .

But the angle HAC is also equal to the angle MDF .

Therefore MDF , HAC are two triangles which have two angles equal to two angles respectively, and one side equal to one side, namely, that subtending one of the equal angles, that is, HA equal to MD ;

therefore they will also have the remaining sides equal to the remaining sides respectively.

[I. 26]

Therefore AC is equal to DF .

Similarly we can prove that AB is also equal to DE .

Since then AC is equal to DF , and AB to DE ,

the two sides CA , AB are equal to the two sides FD , DE .

But the angle CAB is also equal to the angle FDE ;

therefore the base BC is equal to the base EF , the triangle to the triangle, and the remaining angles to the remaining angles;

[I. 4]

therefore the angle ACB is equal to the angle DFE .

But the right angle ACK is also equal to the right angle DFN ;

therefore the remaining angle BCK is also equal to the remaining angle EFN .

For the same reason

the angle CBK is also equal to the angle FEN .

Therefore BCK , EFN are two triangles which have two angles equal to two angles respectively, and one side equal to one side, namely, that adjacent to the equal angles, that is, BC equal to EF ;

therefore they will also have the remaining sides equal to the remaining sides. [I. 26]

Therefore CK is equal to FN .

But AC is also equal to DF ;

therefore the two sides AC , CK are equal to the two sides DF , FN ;

and they contain right angles.

Therefore the base AK is equal to the base DN . [I. 4]

And, since AH is equal to DM ,

the square on AH is also equal to the square on DM .

But the squares on AK , KH are equal to the square on AH ,

for the angle AKH is right; [I. 47]

and the squares on DN , NM are equal to the square on DM ,

for the angle DNM is right; [I. 47]

therefore the squares on AK , KH are equal to the squares on DN , NM ;

and of these the square on AK is equal to the square on DN ;

therefore the remaining square on KH is equal to the square on NM ;

therefore HK is equal to MN .

And, since the two sides HA , AK are equal to the two sides MD , DN respectively,

and the base HK was proved equal to the base MN ,

therefore the angle HAK is equal to the angle MDN . [I. 8]

Therefore etc.

PORISM. From this it is manifest that, if there be two equal plane angles, and if there be set up on them elevated straight lines which are equal and contain equal angles with the original straight lines respectively, the perpendiculars drawn from their extremities to the planes in which are the original angles are equal to one another.

Q. E. D.

This proposition is required for the next, where it is necessary to know that, if in two equiangular parallelepipeds equal angles, one in each, be contained by three plane angles respectively, one of which is an angle of the parallelogram forming the *base* in one parallelepiped, while its equal is likewise in the *base* of the other, and the edges in which the two remaining angles forming the solid angles meet are *equal*, the parallelepipeds are of the same height.

Bearing in mind the definition of *the inclination of a straight line to a plane*, we might enunciate the proposition more shortly thus.

If there be two trihedral angles identically equal to one another, corresponding edges in each are equally inclined to the planes through the other two edges respectively.

The proof, which is necessarily somewhat long, may be summarised thus.

It is required to prove that the angles GAL, MDN in the figure are equal, G, M being any points on AG, DM , and GL, MN perpendicular to the planes BAC, EDF respectively.

If AH is made equal to DM , and HK is drawn in the plane GAL parallel to GL ,

HK is also perpendicular to the plane BAC . [xi. 8]

Draw KB, KC perpendicular to AB, AC respectively and NE, NF perpendicular to DE, DF respectively, and complete the figures.

$$\begin{aligned} \text{Now (1)} \quad HA^2 &= HK^2 + KA^2 \\ &= HK^2 + KC^2 + CA^2 \\ &= HC^2 + CA^2 \end{aligned} \quad \left. \vphantom{\begin{aligned} HA^2 &= HK^2 + KA^2 \\ &= HK^2 + KC^2 + CA^2 \\ &= HC^2 + CA^2 \end{aligned}} \right\} \quad [\text{I. 47}]$$

Therefore $\angle HCA = \text{a right angle.}$

Similarly $\angle MFD = \text{a right angle.}$

(2) $\triangle s HAC, MDF$ have therefore two angles equal and one side.

Therefore $\triangle HAC \equiv \triangle MDF$, and $AC = DF$. [I. 26]

(3) Similarly $\triangle HAB \equiv \triangle MDE$, and $AB = DE$.

(4) Hence $\triangle s ABC, DEF$ are equal in all respects, so that $BC = EF$,
and $\angle ABC = \angle DEF$,
 $\angle ACB = \angle DFE$.

(5) Therefore the complements of these angles are equal,

i.e. $\angle KBC = \angle NEF$,

and $\angle KCB = \angle NFE$.

(6) The $\triangle s KBC, NEF$ have two angles equal and one side, and are therefore equal in all respects, so that

$$KB = NE,$$

$$KC = NF.$$

(7) The right-angled triangles KAC, NDF are equal in all respects, since $AC = DF$ [(2) above], $KC = NF$.

Consequently $AK = DN$.

(8) In $\triangle s HAK, MDN$,

$$\begin{aligned} HK^2 + KA^2 &= HA^2 \\ &= MD^2, \text{ by hypothesis,} \\ &= MN^2 + ND^2. \end{aligned}$$

Subtracting the equals KA^2 , ND^2 ,
 we have $HK^2 = MN^2$,
 or $HK = MN$.

(9) $\triangle s HAK$, MDN are now equal in all respects, by 1. 8 and 1. 4, and therefore

$$\angle HAK = \angle MDN.$$

The Porism is merely a statement of the result arrived at in (8).

Legendre uses, practically, the construction and argument of this proposition to prove the theorem given under (3) of the note on xi. 21 above that *In two equal trihedral angles, corresponding pairs of face angles include equal dihedral angles.* This fact is readily deduced from the above proposition.

Since [(1)] HC , KC are both perpendicular to AC , and MF , NF both perpendicular to DF , the angles HCK , MFN are the measures of the *dihedral* angles between the planes HAC , BAC , and MDF , EDF respectively. [xi. Def. 6]

By (6), $KC = NF$,
 and, by (8), $HK = MN$,
 while the angles HKC , MNF , both being right, are equal.

Consequently the $\triangle s HCK$, MFN are equal in all respects, [1. 4]
 so that $\angle HCK = \angle MFN$.

Simson substituted a different proof of (1) in the above summary, as follows.

Since HK is perpendicular to the plane BAC , the plane HBK , passing through HK , is also perpendicular to the plane BAC . [xi. 18]

And AB , being drawn in the plane BAC perpendicular to BK , the common section of the planes HBK , BAC , is perpendicular to the plane HBK [xi. Def. 4], and is therefore perpendicular to every straight line meeting it in that plane [xi. Def. 3].

Hence the angle ABH is a right angle.

I think Euclid's proof much preferable to this with its references to definitions which are more of the nature of theorems.

PROPOSITION 36.

If three straight lines be proportional, the parallelepipedal solid formed out of the three is equal to the parallelepipedal solid on the mean which is equilateral, but equiangular with the aforesaid solid.

Let A , B , C be three straight lines in proportion, so that, as A is to B , so is B to C ;

I say that the solid formed out of A , B , C is equal to the solid on B which is equilateral, but equiangular with the aforesaid solid.

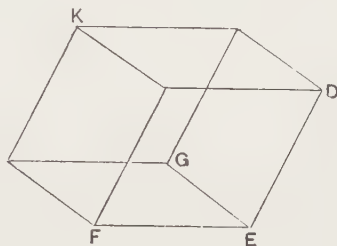
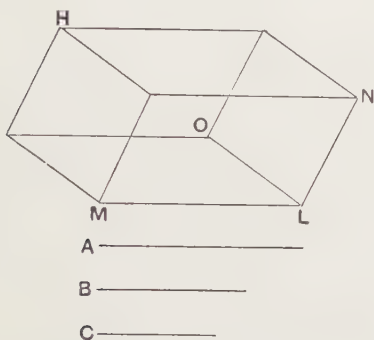
Let there be set out the solid angle at E contained by the angles DEG , GEF , FED ,

let each of the straight lines DE , GE , EF be made equal to B , and let the parallelepipedal solid EK be completed,

let LM be made equal to A ,

and on the straight line LM , and at the point L on it, let there be constructed a solid angle equal to the solid angle at E , namely that contained by NLO , OLM , MLN ;

let LO be made equal to B , and LN equal to C .



Now, since, as A is to B , so is B to C ,
 while A is equal to LM , B to each of the straight lines LO , ED , and C to LN ,
 therefore, as LM is to EF , so is DE to LN .

Thus the sides about the equal angles NLM , DEF are reciprocally proportional;
 therefore the parallelogram MN is equal to the parallelogram DF . [vi. 14]

And, since the angles DEF , NLM are two plane rectilinear angles, and on them the elevated straight lines LO , EG are set up which are equal to one another and contain equal angles with the original straight lines respectively,
 therefore the perpendiculars drawn from the points G , O to the planes through NL , LM and DE , EF are equal to one another; [xi. 35, Por.]

hence the solids LH , EK are of the same height.

But parallelepipedal solids on equal bases and of the same height are equal to one another; [xi. 31]
 therefore the solid HL is equal to the solid EK .

And LH is the solid formed out of A , B , C , and EK the solid on B ;

therefore the parallelepipedal solid formed out of A, B, C is equal to the solid on B which is equilateral, but equiangular with the aforesaid solid.

Q. E. D.

The edges of the parallelepiped HL being respectively equal to A, B, C , and those of the equiangular parallelepiped KE being all equal to B , we regard MN (not containing the edge OL equal to B) as the base of the first parallelepiped, and consequently FD , equiangular to MN , as the base of KE .

Then the solids have the same height.

[XI. 35, Por.]

Hence (solid HL) : (solid KE) = $\square MN$: $\square FD$.

[XI. 32]

But, since A, B, C are in continued proportion,

$$A : B = B : C,$$

or

$$LM : EF = DE : LN.$$

Thus the sides of the equiangular $\square$ s MN, FD are reciprocally proportional, whence

$$\square MN = \square FD,$$

[VI. 14]

and therefore

$$(\text{solid } HL) = (\text{solid } KE).$$

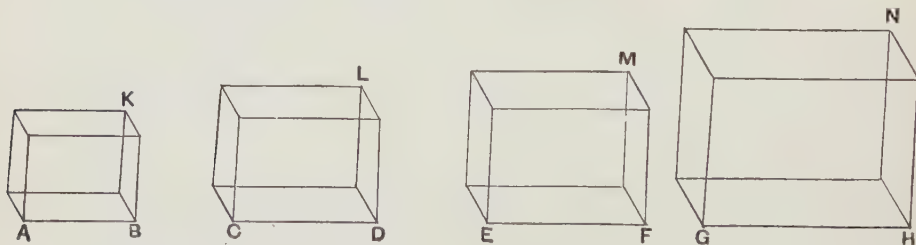
PROPOSITION 37.

If four straight lines be proportional, the parallelepipedal solids on them which are similar and similarly described will also be proportional; and, if the parallelepipedal solids on them which are similar and similarly described be proportional, the straight lines will themselves also be proportional.

Let AB, CD, EF, GH be four straight lines in proportion, so that, as AB is to CD , so is EF to GH ;

and let there be described on AB, CD, EF, GH the similar and similarly situated parallelepipedal solids KA, LC, ME, NG ;

I say that, as KA is to LC , so is ME to NG .



For, since the parallelepipedal solid KA is similar to LC , therefore KA has to LC the ratio triplicate of that which AB has to CD .

[XI. 33]

For the same reason

ME also has to NG the ratio triplicate of that which EF has to GH . [*id.*]

And, as AB is to CD , so is EF to GH .

Therefore also, as AK is to LC , so is ME to NG .

Next, as the solid AK is to the solid LC , so let the solid ME be to the solid NG ;

I say that, as the straight line AB is to CD , so is EF to GH .

For since, again, KA has to LC the ratio triplicate of that which AB has to CD , [xi. 33]

and ME also has to NG the ratio triplicate of that which EF has to GH , [*id.*]

and, as KA is to LC , so is ME to NG ,

therefore also, as AB is to CD , so is EF to GH .

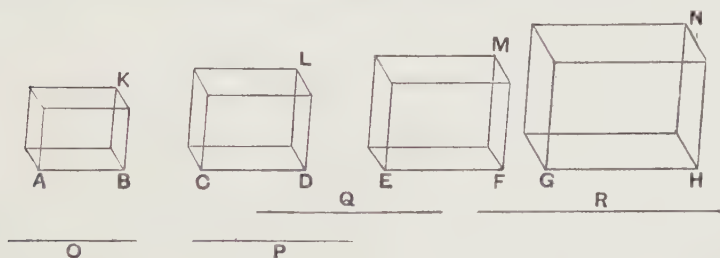
Therefore etc.

Q. E. D.

In this proposition it is assumed that, if two ratios be equal, the ratio triplicate of one is equal to the ratio triplicate of the other and, conversely, that, if ratios which are the triplicate of two other ratios are equal, those other ratios are themselves equal.

To avoid the necessity for these assumptions Simson adopts the alternative proof found in the ms. which Heiberg calls b, and also adopted by Clavius, who, however, gives Euclid's proof as well, attributing it to Theon. The alternative proof proceeds after the manner of vi. 22, thus.

Make AB, CD, O, P continuous proportionals, and also EF, GH, Q, R .



I. Then, since

$$AB : CD = EF : GH,$$

we have, *ex aequali*,

$$AB : P = EF : R. \quad [\text{v. 22}]$$

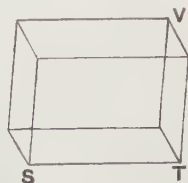
But $(\text{solid } AK) : (\text{solid } CL) = AB : P,$

[xi. 33 and Por.]

and $(\text{solid } EM) : (\text{solid } GN) = EF : R.$

Therefore

$$(\text{solid } AK) : (\text{solid } CL) = (\text{solid } EM) : (\text{solid } GN).$$



II. If the *solids* are proportional, take ST such that

$$AB : CD = EF : ST,$$

and on ST describe the parallelepiped SV similar and similarly situated to either of the parallelepipeds EM , GN .

Then, by the first part,

$$(\text{solid } AK) : (\text{solid } CL) = (\text{solid } EM) : (\text{solid } SV),$$

whence it follows that

$$(\text{solid } GN) = (\text{solid } SV).$$

But these solids are similar and similarly situated ;
therefore their faces are similar and equal ;

[XI. Def. 10]

therefore the corresponding sides GH , ST are equal.

[For this inference cf. note on vi. 22. The equality of GH , ST may readily be proved by application of the two parallelepipeds to one another, since, being similar, they are equiangular.]

Hence

$$AB : CD = EF : GH.$$

The text of the mss. has here a proposition which is as badly placed as it is unnecessary. *If a plane be at right angles to a plane, and from any one of the points in one of the planes a perpendicular be drawn to the other plane, the perpendicular so drawn will fall on the common section of the planes.* It is of the nature of a lemma to XII. 17, where alone the fact is made use of. Heiberg observes that it is omitted in b and that the copyist of P knew other texts which did not contain it. From these facts it is fairly concluded that the proposition was interpolated. The truth of it is of course immediately obvious by *reductio ad absurdum*. Let the plane CAD be perpendicular to the plane AB , and let a perpendicular be drawn to the latter from any point E in the former.

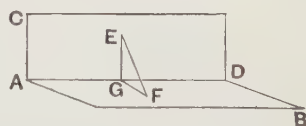
If it does not fall on AD , the common section, let it meet the plane AB in F .

Draw FG in AB perpendicular to AD , and join EG .

Then FG is perpendicular to the plane CAD [XI. Def. 4], and therefore to GE [XI. Def. 3]. Therefore $\angle EGF$ is right.

Also, since EF is perpendicular to AB ,
the angle EFG is right.

That is, the triangle EGF has two right angles :
which is impossible.



PROPOSITION 38.

If the sides of the opposite planes of a cube be bisected, and planes be carried through the points of section, the common section of the planes and the diameter of the cube bisect one another.

For let the sides of the opposite planes CF , AH of the cube AF be bisected at the points K , L , M , N , O , Q , P , R ,

and through the points of section let the planes KN , OR be carried ;

let US be the common section of the planes, and DG the diameter of the cube AF .

I say that UT is equal to TS , and DT to TG .

For let DU , UE , BS , SG be joined.

Then, since DO is parallel to PE ,

the alternate angles DOU , UPE are equal to one another.

[I. 29]

And, since DO is equal to PE , and OU to UP ,

and they contain equal angles,

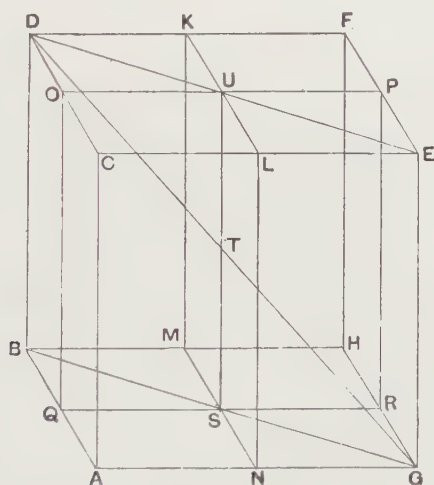
therefore the base DU is equal to the base UE ,

the triangle DOU is equal to the triangle PUE ,

and the remaining angles are equal to the remaining angles ;

[I. 4]

therefore the angle ODU is equal to the angle PUE .



For this reason DUE is a straight line.

[I. 14]

For the same reason, BSG is also a straight line, and BS is equal to SG .

Now, since CA is equal and parallel to DB , while CA is also equal and parallel to EG , therefore DB is also equal and parallel to EG .

[xi. 9]

And the straight lines DE , BG join their extremities ;
therefore DE is parallel to BG . [1. 33]

Therefore the angle EDT is equal to the angle BGT ,
for they are alternate ; [1. 29]
and the angle DTU is equal to the angle GTS . [1. 15]

Therefore DTU , GTS are two triangles which have two
angles equal to two angles, and one side equal to one side,
namely that subtending one of the equal angles, that is, DU
equal to GS ,

for they are the halves of DE , BG ;
therefore they will also have the remaining sides equal to the
remaining sides. [1. 26]

Therefore DT is equal to TG , and UT to TS .

Therefore etc.

Q. E. D.

Euclid enunciates this proposition of a *cube* only, though it is true of any
parallelepiped, no doubt because its truth for a cube is all that was wanted for
the only proposition where it is needed, viz. XIII. 17.

Simson remarks that it should be proved that the straight lines bisecting
the corresponding opposite sides of opposite planes *are* in one plane. This is,
however, clear because e.g. since DK , CL are equal and parallel, KL is equal
and parallel to CD . And, since KL , AB are both parallel to DC , KL is
parallel to AB . And lastly, since KL , MN are both parallel to AB , KL is
parallel to MN and therefore in one plane with it.

The essential thing to be proved is that the plane passing through the
opposite edges DB , EG passes through the straight line US , since, only if
this be the case, can US , DG intersect one another.

To prove this we have only to prove that, if DU , UE and BS , SG be
joined, DUE and BSG are both straight lines.

Now, since DO is parallel to PE ,

$$\angle DOU = \angle EPU.$$

Thus, in the $\triangle DOU$, EUP , two sides DO , OU are equal to two sides
 EP , PU , and the included angles are equal.

Therefore $\triangle DOU \equiv \triangle EUP$,

$$DU = UE,$$

and

$$\angle DUO = \angle EUP,$$

so that DUE is a straight line, bisected at U . Similarly BSG is a straight
line, bisected at S .

Thus the plane through DB , EG (DB , EG being equal and parallel)
contains the straight lines DUE , BSG (which are therefore equal and parallel
also) and also [xi. 7] the straight lines US , DG (which accordingly intersect).

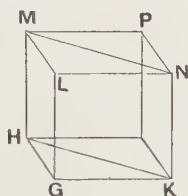
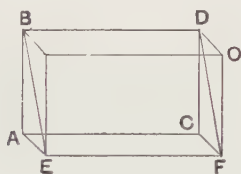
In $\triangle s$ DTU , GTS , the angles UDT , SGT are equal (being alternate),
and the angles UTD , STG are also equal (being vertically opposite), while
 DU (half of DE) is equal to GS (half of BG).

Therefore [I. 26] the triangles DTU , GTS are equal in all respects, so that
 $DT = TG$,
 $UT = TS$.

PROPOSITION 39.

If there be two prisms of equal height, and one have a parallelogram as base and the other a triangle, and if the parallelogram be double of the triangle, the prisms will be equal.

Let $ABCDEF$, $GHKLMN$ be two prisms of equal height,
 let one have the parallelogram AF as base, and the other the triangle GHK ,
 and let the parallelogram AF be double of the triangle GHK ;
 I say that the prism $ABCDEF$ is equal to the prism $GHKLMN$.



For let the solids AO , GP be completed.

Since the parallelogram AF is double of the triangle GHK ,
 while the parallelogram HK is also double of the triangle GHK , [I. 34]
 therefore the parallelogram AF is equal to the parallelogram HK .

But parallelepipedal solids which are on equal bases and of the same height are equal to one another; [XI. 31]
 therefore the solid AO is equal to the solid GP .

And the prism $ABCDEF$ is half of the solid AO ,
 and the prism $GHKLMN$ is half of the solid GP ; [XI. 28]
 therefore the prism $ABCDEF$ is equal to the prism $GHKLMN$.

Therefore etc.

Q. E. D.

This proposition is made use of in XII. 3, 4. The phraseology is interesting because we find one of the *parallelogrammic* faces of one of the triangular prisms called its *base*, and the perpendicular on this plane from that vertex of either *triangular* face which is not in this plane the *height*.

The proof is simple because we have only to complete parallelepipeds which are double the prisms respectively and then use XI. 31. It has to be borne in mind, however, that, if the parallelepipeds are not rectangular, the proof in XI. 28 is not sufficient to establish the fact that the parallelepipeds are double of the prisms, but has to be supplemented as shown in the note on that proposition. XII. 4 does, however, require the theorem in its general form.

BOOK XII.

HISTORICAL NOTE.

The predominant feature of Book XII. is the use of the *method of exhaustion*, which is applied in Propositions 2, 3—5, 10, 11, 12, and (in a slightly different form) in Propositions 16—18. We conclude therefore that for the content of this Book Euclid was greatly indebted to Eudoxus, to whom the discovery of the method of exhaustion is attributed. The evidence for this attribution comes mainly from Archimedes. (1) In the preface to *On the Sphere and Cylinder* I., after stating the main results obtained by himself regarding the surface of a sphere or a segment thereof, and the volume and surface of a right cylinder with height equal to its diameter as compared with those of a sphere with the same diameter, Archimedes adds: "Having now discovered that the properties mentioned are true of these figures, I cannot feel any hesitation in setting them side by side both with my former investigations and with those of the theorems of Eudoxus on solids which are held to be most irrefragably established, namely that any pyramid is one third part of the prism which has the same base with the pyramid and equal height [i.e. Eucl. XII. 7], and that any cone is one third part of the cylinder which has the same base with the cone and equal height [i.e. Eucl. XII. 10]. For, though these properties also were naturally inherent in the figures all along, yet they were in fact unknown to all the many able geometers who lived before Eudoxus and had not been observed by any one." (2) In the preface to the treatise known as the *Quadrature of the Parabola* Archimedes states the "lemma" assumed by him and known as the "Axiom of Archimedes" (see note on x. 1 above) and proceeds: "Earlier geometers (οἱ πρότερον γεωμέτραι) have also used this lemma; for it is by the use of this same lemma that they have shown that circles are to one another in the duplicate ratio of their diameters [Eucl. XII. 2], and that spheres are to one another in the triplicate ratio of their diameters [Eucl. XII. 18], and further that every pyramid is one third part of the prism which has the same base with the pyramid and equal height [Eucl. XII. 7]; also, that every cone is one third part of the cylinder which has the same base with the cone and equal height [Eucl. XII. 10] they proved by assuming a certain lemma similar to that aforesaid." Thus in the first passage two theorems of Eucl. XII. are definitely attributed to Eudoxus; and, when Archimedes says, in the second passage, that "earlier geometers" proved these two theorems by means of the lemma known as the "Axiom of Archimedes" and of a lemma similar to it respectively, we can hardly suppose him to be alluding to

any other proof than that given by Eudoxus. As a matter of fact, the lemma used by Euclid to prove both propositions (XII. 3—5 and 7, and XII. 10) is the theorem of Eucl. x. 1. As regards the connexion between the two “lemmas” see note on x. 1.

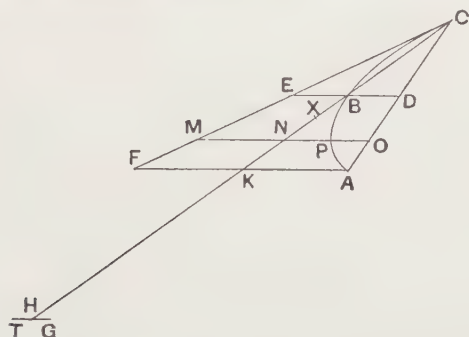
We are not, however, to suppose that none of the *results* obtained by the method of exhaustion had been discovered before the time of Eudoxus (fl. about 368—5 B.C.). Two at least are of earlier date, those of Eucl. XII. 2 and XII. 7.

(a) Simplicius (*Comment. in Aristot. Phys.* p. 61, ed. Diels) quotes Eudemus as saying, in his *History of Geometry*, that Hippocrates of Chios (fl. say 430 B.C.) first laid it down (ἔθετο) that similar segments of circles are in the ratio of the squares on their bases and that he proved this (ἐδείκνυνεν) by proving (ἐκ τοῦ δείξαι) that the squares on the diameters have the same ratio as the (whole) circles. We know nothing of the method by which Hippocrates proved this proposition; but, having regard to the evidence from Archimedes quoted above, it is not permissible to suppose that the method was the fully developed *method of exhaustion* as we know it.

(b) As regards the two theorems about the volume of a pyramid and of a cone respectively, which Eudoxus was the first to prove, we now have authentic evidence in the short treatise by Archimedes discovered by Heiberg in a MS. at Constantinople in 1906 and published in *Hermes* the following year (see now *Archimedis opera omnia*, ed. Heiberg, 2. ed., Vol. II., 1913, pp. 425—507; T. L. Heath, *The Method of Archimedes*, Cambridge, 1912). The said treatise, complete in all essentials, bears the title Ἀρχιμήδους περὶ τῶν μηχανικῶν θεωρημάτων πρὸς Ἐρατοσθένην ἔφοδος. This “Method” (or “Plan of attack”), addressed to Eratosthenes, is none other than the ἐφόδιον on which, according to Suidas, Theodosius wrote a commentary, and which is several times cited by Heron in his *Metrica*; its discovery adds a new and important chapter to the history of the integral calculus. In the preface to this work Archimedes alludes to the theorems which he first discovered by means of mechanical considerations, but proved afterwards by geometry, because the investigation by means of mechanics did not constitute a rigid proof; he observes, however, that the mechanical method is of great use for the *discovery* of theorems, and it is much easier to provide the rigid proof when the fact to be proved has once been discovered than it would be if nothing were known to begin with. He goes on: “Hence too, in the case of those theorems the proof of which was first discovered by Eudoxus, namely those relating to the cone and the pyramid, that the cone is one third part of the cylinder, and the pyramid one third part of the prism, having the same base and equal height, no small part of the credit will naturally be assigned to Democritus, who was the first to make the statement (of the fact) regarding the said figure [i.e. property], though without proving it.” Hence the *discovery* of the two theorems must now be attributed to Democritus (fl. towards the end of 5th cent. B.C.). The words “without proving it” (χωρὶς ἀποδείξεως) do not mean that Democritus gave no sort of proof, but only that he did not give a proof on the rigorous lines required later; for the same words are used by Archimedes of his own investigations by means of mechanics, which, however, do constitute a reasoned argument. The character of Archimedes’ mechanical arguments combined with a passage of Plutarch about a particular question in infinitesimals said to have been raised by Democritus may perhaps give a clue to the line of Democritus’ argument as regards the pyramid. The essential

feature of Archimedes' mechanical arguments in this tract is that he regards an area as the sum of an infinite number of *straight lines* parallel to one another and terminated by the boundary or boundaries of the closed figure the area of which is to be found, and a volume as the sum of an infinite number of *plane sections* parallel to one another: which is of course the same thing as taking (as we do in the integral calculus) the sum of an infinite number of strips of breadth dx (say), when dx becomes indefinitely small, or the sum of an infinite number of parallel laminae of depth dz (say), when dz becomes indefinitely small. To give only one instance, we may take the case of the area of a segment of a parabola cut off by a chord.

Let CBA be the parabolic segment, CE the tangent at C meeting the



diameter EBD through the middle point of the chord CA in E , so that

$$EB = BD.$$

Draw AF parallel to ED meeting CE produced in F . Produce CB to H so that $CK = KH$, where K is the point in which CH meets AF ; and suppose CH to be a lever.

Let any diameter $MNPO$ be drawn meeting the curve in P and CF , CK , CA in M , N , O respectively.

Archimedes then observes that

$$CA \cdot AO = MO : OP$$

("for this is proved in a lemma"),

whence

$$HK : KN = MO : OP,$$

so that, if a straight line TG equal to PO be placed with its middle point at H , the straight line MO with centre of gravity at N , and the straight line TG with centre of gravity at H , will balance about K .

Taking all other parts of diameters like PO intercepted between the curve and CA , and placing equal straight lines with their centres of gravity at H , these straight lines collected at H will balance (about K) all the lines like MO parallel to FA intercepted within the triangle CFA in the positions in which they severally lie in the figure.

Hence Archimedes infers that an area equal to that of the parabolic segment hung at H will balance (about K) the triangle CFA hung at its centre of gravity, the point X (a point on CK such that $CK = 3XK$), and therefore that

$$\begin{aligned} (\text{area of triangle } CFA) : (\text{area of segment}) &= HK : KX \\ &= 3 : 1, \end{aligned}$$

from which it follows that

$$\text{area of parabolic segment} = \frac{4}{3} \triangle ABC.$$

The same sort of argument is used for solids, *plane sections* taking the place of *straight lines*.

Archimedes is careful to state once more that this method of argument does not constitute a *proof*. Thus, at the end of the above proposition about the parabolic segment, he adds: "This property is of course not proved by what has just been said; but it has furnished a sort of *indication* (ἐμφασίν τινα) that the conclusion is true."

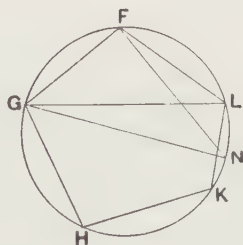
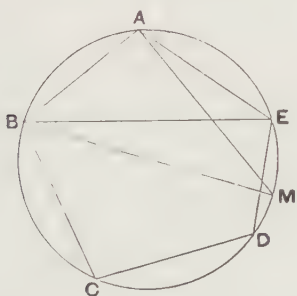
Let us now turn to the passage of Plutarch (*De Comm. Not. adv. Stoicos* xxxix. 3) about Democritus above referred to. Plutarch speaks of Democritus as having raised the question in natural philosophy (φυσικῶς): "if a cone were cut by a plane parallel to the base [by which is clearly meant a plane indefinitely near to the base], what must we think of the surfaces of the sections, that they are equal or unequal? For, if they are unequal, they will make the cone irregular, as having many indentations, like steps, and unevennesses; but, if they are equal, the sections will be equal, and the cone will appear to have the property of the cylinder and to be made up of equal, not unequal circles, which is very absurd." The phrase "*made up* of equal...circles" (ἐξ ἴσων συγκείμενος...κύκλων) shows that Democritus already had the idea of a solid being the sum of an infinite number of parallel planes, or indefinitely thin laminae, indefinitely near together: a most important anticipation of the same thought which led to such fruitful results in Archimedes. If then one may hazard a conjecture as to Democritus' argument with regard to a pyramid, it seems probable that he would notice that, if two pyramids of the same height and equal triangular bases are respectively cut by planes parallel to the base and dividing the heights in the same ratio, the corresponding sections of the two pyramids are equal, whence he would infer that the pyramids are equal as being the sum of the same infinite number of equal plane sections or indefinitely thin laminae. (This would be a particular anticipation of Cavalieri's proposition that the areal or solid contents of two figures are equal if two sections of them taken at the same height, whatever the height may be, always give equal straight lines or equal surfaces respectively.) And Democritus would of course see that the three pyramids into which a prism on the same base and of equal height with the original pyramid is divided (as in Eucl. xii. 7) satisfy this test of equality, so that the pyramid would be one third part of the prism. The extension to a pyramid with a polygonal base would be easy. And Democritus may have stated the proposition for the cone (of course without an absolute proof) as a natural inference from the result of increasing indefinitely the number of sides in a regular polygon forming the base of a pyramid.

BOOK XII. PROPOSITIONS.

PROPOSITION I.

Similar polygons inscribed in circles are to one another as the squares on the diameters.

Let ABC , FGH be circles,
let $ABCDE$, $FGHKL$ be similar polygons inscribed in them,
and let BM , GN be diameters of the circles ;
I say that, as the square on BM is to the square on GN , so
is the polygon $ABCDE$ to the polygon $FGHKL$.



For let BE , AM , GL , FN be joined.

Now, since the polygon $ABCDE$ is similar to the polygon $FGHKL$,

the angle BAE is equal to the angle GFL ,

and, as BA is to AE , so is GF to FL .

[vi. Def. 1]

Thus BAE , GFL are two triangles which have one angle equal to one angle, namely the angle BAE to the angle GFL , and the sides about the equal angles proportional ;

therefore the triangle ABE is equiangular with the triangle FLG .

[vi. 6]

Therefore the angle AEB is equal to the angle FLG .

But the angle AEB is equal to the angle AMB ,
 for they stand on the same circumference ; [III. 27]
 and the angle FLG to the angle FNG ;
 therefore the angle AMB is also equal to the angle FNG .

But the right angle BAM is also equal to the right angle GFN ; [III. 31]
 therefore the remaining angle is equal to the remaining angle.
 [I. 32]

Therefore the triangle ABM is equiangular with the triangle FGN .

Therefore, proportionally, as BM is to GN , so is BA to GF . [VI. 4]

But the ratio of the square on BM to the square on GN is duplicate of the ratio of BM to GN ,
 and the ratio of the polygon $ABCDE$ to the polygon $FGHKL$ is duplicate of the ratio of BA to GF ; [VI. 20]
 therefore also, as the square on BM is to the square on GN , so is the polygon $ABCDE$ to the polygon $FGHKL$.

Therefore etc.

Q. E. D.

As, from this point onward, the text of each proposition usually occupies considerable space, I shall generally give in the notes a summary of the argument, to enable it to be followed more easily.

Here we have to prove that a pair of corresponding sides are in the ratio of the corresponding diameters.

Since $\angle s$ BAE , GFL are equal, and the sides about those angles proportional,

$\triangle s$ ABE , FGL are equiangular,

so that

$$\angle AEB = \angle FLG$$

Hence their equals in the same segments, $\angle s$ AMB , FNG , are equal.

And the right angles BAM , GFN are equal.

Therefore $\triangle s$ ABM , FGN are equiangular, so that

$$BM : GN = BA : GF.$$

The duplicates of these ratios are therefore equal,
 whence (polygon $ABCDE$) : (polygon $FGHKL$)

$$= \text{duplicate ratio of } BA \text{ to } GF$$

$$= \text{duplicate ratio of } BM \text{ to } GN$$

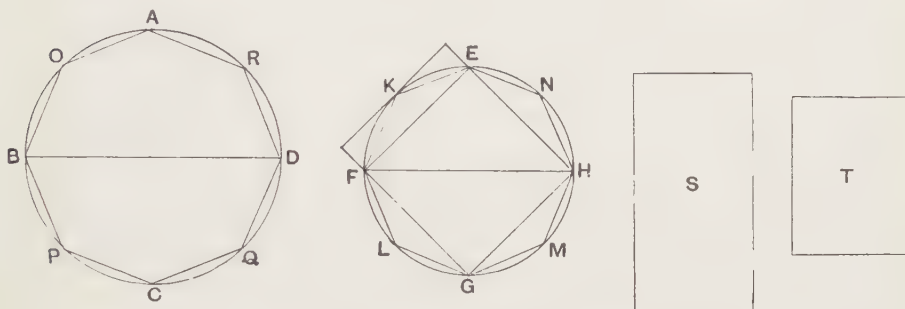
$$= BM^2 : GN^2.$$

PROPOSITION 2.

Circles are to one another as the squares on the diameters.

Let $ABCD$, $EFGH$ be circles, and BD , FH their diameters;

I say that, as the circle $ABCD$ is to the circle $EFGH$, so is the square on BD to the square on FH .



For, if the square on BD is not to the square on FH as the circle $ABCD$ is to the circle $EFGH$, then, as the square on BD is to the square on FH , so will the circle $ABCD$ be either to some less area than the circle $EFGH$, or to a greater.

First, let it be in that ratio to a less area S .

Let the square $EFGH$ be inscribed in the circle $EFGH$; then the inscribed square is greater than the half of the circle $EFGH$, inasmuch as, if through the points E , F , G , H we draw tangents to the circle, the square $EFGH$ is half the square circumscribed about the circle, and the circle is less than the circumscribed square; hence the inscribed square $EFGH$ is greater than the half of the circle $EFGH$.

Let the circumferences EF , FG , GH , HE be bisected at the points K , L , M , N , and let EK , KF , FL , LG , GM , MH , HN , NE be joined; therefore each of the triangles EKF , FLG , GMH , HNE is also greater than the half of the segment of the circle about it, inasmuch as, if through the points K , L , M , N we draw tangents to the circle and complete the parallelograms on the straight lines EF , FG , GH , HE , each of the triangles EKF ,

FLG, *GMH*, *HNE* will be half of the parallelogram about it,

while the segment about it is less than the parallelogram ; hence each of the triangles *EKF*, *FLG*, *GMH*, *HNE* is greater than the half of the segment of the circle about it.

Thus, by bisecting the remaining circumferences and joining straight lines, and by doing this continually, we shall leave some segments of the circle which will be less than the excess by which the circle *EFGH* exceeds the area *S*.

For it was proved in the first theorem of the tenth book that, if two unequal magnitudes be set out, and if from the greater there be subtracted a magnitude greater than the half, and from that which is left a greater than the half, and if this be done continually, there will be left some magnitude which will be less than the lesser magnitude set out.

Let segments be left such as described, and let the segments of the circle *EFGH* on *EK*, *KF*, *FL*, *LG*, *GM*, *MH*, *HN*, *NE* be less than the excess by which the circle *EFGH* exceeds the area *S*.

Therefore the remainder, the polygon *EKFLGMHN*, is greater than the area *S*.

Let there be inscribed, also, in the circle *ABCD* the polygon *AOBPCQDR* similar to the polygon *EKFLGMHN* ; therefore, as the square on *BD* is to the square on *FH*, so is the polygon *AOBPCQDR* to the polygon *EKFLGMHN*.

[XII. 1]

But, as the square on *BD* is to the square on *FH*, so also is the circle *ABCD* to the area *S* ;

therefore also, as the circle *ABCD* is to the area *S*, so is the polygon *AOBPCQDR* to the polygon *EKFLGMHN* ;

[v. 11]

therefore, alternately, as the circle *ABCD* is to the polygon inscribed in it, so is the area *S* to the polygon *EKFLGMHN*.

[v. 16]

But the circle *ABCD* is greater than the polygon inscribed in it ;

therefore the area *S* is also greater than the polygon *EKFLGMHN*.

But it is also less :
which is impossible.

Therefore, as the square on BD is to the square on FH , so is not the circle $ABCD$ to any area less than the circle $EFGH$.

Similarly we can prove that neither is the circle $EFGH$ to any area less than the circle $ABCD$ as the square on FH is to the square on BD .

I say next that neither is the circle $ABCD$ to any area greater than the circle $EFGH$ as the square on BD is to the square on FH .

For, if possible, let it be in that ratio to a greater area S .

Therefore, inversely, as the square on FH is to the square on DB , so is the area S to the circle $ABCD$.

But, as the area S is to the circle $ABCD$, so is the circle $EFGH$ to some area less than the circle $ABCD$;

therefore also, as the square on FH is to the square on BD , so is the circle $EFGH$ to some area less than the circle $ABCD$;

[V. 11]

which was proved impossible.

Therefore, as the square on BD is to the square on FH , so is not the circle $ABCD$ to any area greater than the circle $EFGH$.

And it was proved that neither is it in that ratio to any area less than the circle $EFGH$;

therefore, as the square on BD is to the square on FH , so is the circle $ABCD$ to the circle $EFGH$.

Therefore etc.

Q. E. D.

LEMMA.

I say that, the area S being greater than the circle $EFGH$, as the area S is to the circle $ABCD$, so is the circle $EFGH$ to some area less than the circle $ABCD$.

For let it be contrived that, as the area S is to the circle $ABCD$, so is the circle $EFGH$ to the area T .

I say that the area T is less than the circle $ABCD$.

For since, as the area S is to the circle $ABCD$, so is the circle $EFGH$ to the area T ,

therefore, alternately, as the area S is to the circle $EFGH$, so is the circle $ABCD$ to the area T . [v. 16]

But the area S is greater than the circle $EFGH$; .
therefore the circle $ABCD$ is also greater than the area T .

Hence, as the area S is to the circle $ABCD$, so is the circle $EFGH$ to some area less than the circle $ABCD$.

Q. E. D.

Though this theorem is said to have been proved by Hippocrates, we may with tolerable certainty attribute the proof of it given by Euclid to Eudoxus, to whom XII. 7 Por. and XII. 10 (which Euclid proves in exactly the same manner) are specifically attributed by Archimedes. As regards the lemma used herein (Eucl. x. 1) and the somewhat different lemma by means of which Archimedes says that the theorems of XII. 2, XII. 7 Por. and XII. 18 were proved, see my note on x. 1 above.

The first essential in this proposition is to prove that we can *exhaust* a circle, in the sense of x. 1, by successively inscribing in it regular polygons, each of which has twice as many sides as the preceding one. We take first an inscribed square, then bisect the arcs subtended by the sides and so form an equilateral polygon of eight sides, then do the same with the latter, forming a polygon of 16 sides, and so on. And we have to prove that what is left over when any one of these polygons is taken away from the circle is *more than half exhausted* when the next polygon is made and subtracted from the circle.

Euclid proves that the inscribed square is greater than half the circle and that the regular octagon when subtracted takes away more than half of what was left by the square. He then infers that the same thing will happen whenever the number of sides is doubled.

This can be seen generally by taking *any* arc of a circle cut off by a chord AB . Bisect the arc in C . Draw a tangent to the circle at C , and let AD , BE be drawn perpendicular to the tangent. Join AC , CB .

Then DE is parallel to AB , since

$$\begin{aligned} \angle ECB &= \angle CAB, \text{ in alternate segment, [III. 32]} \\ &= \angle CBA. \end{aligned} \quad [\text{III. 29, 1. 5}]$$

Thus $ABED$ is a $\square$;

and it is greater than the segment ACB .

Therefore its half, the $\triangle ACB$, is greater than half the segment.

Thus, by x. 1, Euclid's construction of successive regular polygons in a circle, if continued far enough, will at length leave segments which are together less than any given area.

Now let X , X' be the areas of the circles, d , d' their diameters, respectively.

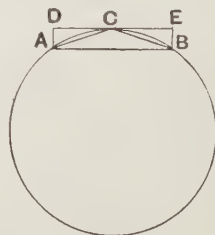
Then, if

$$\begin{aligned} X : X' &\neq d^2 : d'^2, \\ d^2 : d'^2 &= X : S, \end{aligned}$$

where S is some area either greater or less than X' .

I. Suppose $S < X'$.

Continue the construction of polygons in X' until we arrive at one which



leaves over segments together less than the excess of X' over S , i.e. a polygon such that

$$X' > (\text{polygon in } X') > S.$$

Inscribe in the circle X a polygon similar to that in X' .

$$\begin{aligned} \text{Then} \quad (\text{polygon in } X) : (\text{polygon in } X') &= d^2 : d'^2 & [\text{xii. 1}] \\ &= X : S, \text{ by hypothesis;} \end{aligned}$$

and, alternately,

$$(\text{polygon in } X) : X = (\text{polygon in } X') : S.$$

$$\begin{aligned} \text{But} \quad (\text{polygon in } X) &< X; \\ \text{therefore} \quad (\text{polygon in } X') &< S. \end{aligned}$$

$$\begin{aligned} \text{But, by construction.} \quad (\text{polygon in } X') &> S; \\ \text{which is impossible.} \end{aligned}$$

Hence S cannot be *less* than X' as supposed.

II. Suppose $S > X'$.

$$\begin{aligned} \text{Since} \quad d^2 : d'^2 &= X : S, \\ \text{we have, inversely,} \quad d'^2 : d^2 &= S : X. \end{aligned}$$

$$\begin{aligned} \text{Suppose that} \quad S : X &= X' : T, \\ \text{whence, since } S > X', \quad X &> T. & [\text{v. 14}] \end{aligned}$$

$$\begin{aligned} \text{Consequently} \quad d'^2 : d^2 &= X' : T, \\ \text{where } T &< X. \end{aligned}$$

This can be proved impossible in exactly the same way as shown in Part I.

Hence S cannot be *greater* than X' as supposed.

Since then S is neither greater nor less than X' ,

$$S = X',$$

$$\text{and therefore} \quad d^2 : d'^2 = X : X'.$$

With reference to the assumption that there is some space S such that

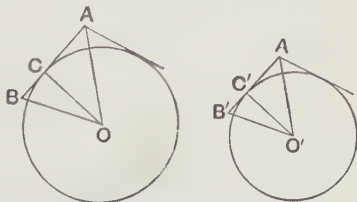
$$d^2 : d'^2 = X : S,$$

i.e. that there is a fourth proportional to the areas d^2 , d'^2 , X , Simson observes that it is sufficient, in this and the like cases, that a thing made use of in the reasoning can possibly *exist*, though it cannot be exhibited by a geometrical construction. As regards the assumption see note on v. 18 above.

There is grave reason for suspecting the genuineness of the Lemma at the end of the proposition; though, if it be rejected, it will be necessary to delete the words "as was before proved" in corresponding places in xii. 5, 18.

It will be observed that Euclid proves the impossibility in the second case by reducing it to the first. If it is desired to prove the second case independently, we must *circumscribe* successive polygons to the circles instead of inscribing them, in the way shown by Archimedes in his first proposition on the *Measurement of a circle*. Of course we require, as a preliminary, the proposition corresponding to xii. 1, that *Similar polygons circumscribed about circles are to one another as the squares on the diameters*.

Let AB , $A'B'$ be corresponding sides of the two similar polygons. Then $\angle s$ OAB , $O'A'B'$ are equal, since AO , $A'O'$ bisect equal angles.



Similarly $\angle ABO = \angle A'B'O'$.

Therefore $\triangle AOB, A'O'B'$ are similar, so that their areas are in the duplicate ratio of AB to $A'B'$.

The radii $OC, O'C'$ drawn to the points of contact are perpendicular to $AB, A'B'$, and it follows that

$$AB : A'B' = CO : C'O'.$$

Thus the polygons are to one another in the duplicate ratio of the radii, and therefore of the diameters.

Now suppose a square $ABCD$ described about a circle.

Make an octagon described about the circle by drawing tangents at the points E etc., where OA etc. meet the circle.

Then shall the tangent at E cut off more than half of the area between AK, AH and the arc HEK .

For the angle AEG is right, and is therefore $> \angle EAG$.

Therefore

$$\begin{aligned} AG &> EG \\ &> GK. \end{aligned}$$

Therefore

$$\triangle AGE > \triangle EGK.$$

Similarly

$$\triangle AFE > \triangle EFH.$$

Hence

$$\triangle AFG > \frac{1}{2} (\text{re-entrant quadrilateral } AHEK),$$

and *a fortiori*, $\triangle AFG > \frac{1}{2} (\text{area between } AH, AK \text{ and the arc}).$

Thus the octagon takes from the square more than half the space between the square and the circle.

Similarly, if a figure of 16 equal sides be circumscribed by cutting off symmetrically the corners of the octagon, it will take away more than half of the space between the octagon and circle.

Suppose now, with the original notation, that

$$d^2 : d'^2 = X : S,$$

where S is greater than X' .

Continue the construction of circumscribed polygons about X' until the total area between the polygon and the circle is less than the difference between S and X' , i.e. till

$$S > (\text{polygon about } X') > X'.$$

Circumscribe a similar polygon about X .

$$\text{Then } (\text{polygon about } X) : (\text{polygon about } X') = d^2 : d'^2$$

$$= X : S, \text{ by hypothesis,}$$

and, alternately,

$$(\text{polygon about } X) : X = (\text{polygon about } X') : S.$$

But

$$(\text{polygon about } X) > X.$$

Therefore

$$(\text{polygon about } X') > S.$$

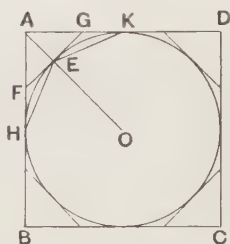
But

$$S > (\text{polygon about } X') :$$

[above]

which is impossible.

Hence S cannot be *greater* than X' .



Legendre proves this proposition by a method equally rigorous but not, I think, possessing any advantages over Euclid's. It depends on a lemma corresponding to Eucl. xii. 16, but with another part added to it.

Two concentric circles being given, we can always inscribe in the greater a regular polygon such that its sides do not meet the circumference of the lesser, and we can also circumscribe about the lesser a regular polygon such that its sides do not meet the circumference of the greater.

Let CA , CB be the radii of the circles.

I. At A on the inner circle draw the tangent DE meeting the outer circle in D , E .

Inscribe in the outer circle any of the regular polygons which we can inscribe, e.g. a square.

Bisect the arc subtended by a side, bisect the half, bisect that again, and so on, until we arrive at an arc less than the arc DBE .

Let this arc be MN , and suppose it so placed that B is its middle point.

Then the chord MN is clearly more distant from the centre C than DE is; and the regular polygon, of which MN is a side, does not anywhere meet the circumference of the inner circle.

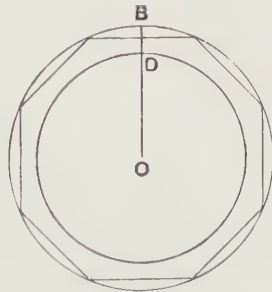
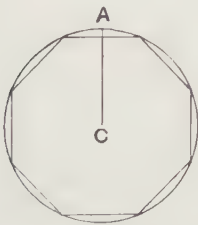
II. Join CM , CN , meeting DE in P , Q .

Then PQ will be the side of a polygon circumscribed about the inner circle and similar to the polygon inscribed in the outer; and the circumscribed polygon of which PQ is a side will not anywhere meet the outer circle.

Legendre now proves xii. 2 after the following manner.

For brevity, let us denote the area of the circle with radius CA by (circ. CA).

Then it is required to prove that, if OB be the radius of a second circle,

$$(\text{circ. } CA) : (\text{circ. } OB) = CA^2 : OB^2.$$


Suppose, if possible, that this relation is not true. Then CA^2 will be to OB^2 as (circ. CA) is to an area greater or less than (circ. OB).

I. Suppose, first, that

$$CA^2 : OB^2 = (\text{circ. } CA) : (\text{circ. } OD),$$

where OD is less than OB .

Inscribe in the circle with radius OB a regular polygon such that its sides do not anywhere meet the circumference of the circle with centre OD ;

[Lemma]

and inscribe a similar polygon in the other circle.

The areas of the polygons will then be in the duplicate ratio of CA to OB ,
or [XII. 1]

$$\begin{aligned} (\text{polygon in circ. } CA) : (\text{polygon in circ. } OB) \\ &= CA^2 : OB^2 \\ &= (\text{circ. } CA) : (\text{circ. } OD), \text{ by hypothesis.} \end{aligned}$$

But this is impossible, because the polygon in (circ. CA) is *less* than (circ. CA), but the polygon in (circ. OB) is *greater* than (circ. OD).

Therefore CA^2 cannot be to OB^2 as (circ. CA) is to a *less* circle than (circ. OB).

II. Suppose, if possible, that

$$CA^2 : OB^2 = (\text{circ. } CA) : (\text{some circle} > \text{circ. } OB).$$

Then inversely

$$OB^2 : CA^2 = (\text{circ. } OB) : (\text{some circle} < \text{circ. } CA),$$

and this is proved impossible exactly as in Part I.

$$\text{Therefore } CA^2 : OB^2 = (\text{circ. } CA) : (\text{circ. } OB).$$

PROPOSITION 3.

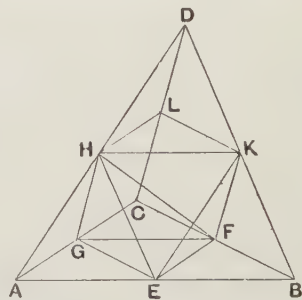
Any pyramid which has a triangular base is divided into two pyramids equal and similar to one another, similar to the whole and having triangular bases, and into two equal prisms; and the two prisms are greater than the half of the whole pyramid.

Let there be a pyramid of which the triangle ABC is the base and the point D the vertex;

I say that the pyramid $ABCD$ is divided into two pyramids equal to one another, having triangular bases and similar to the whole pyramid, and into two equal prisms; and the two prisms are greater than the half of the whole pyramid.

For let AB, BC, CA, AD, DB, DC be bisected at the points E, F, G, H, K, L , and let $HE, EG, GH, HK, KL, LH, KF, FG$ be joined.

Since AE is equal to EB , and AH to DH ,
therefore EH is parallel to DB .



[VI. 2]

For the same reason

HK is also parallel to AB .

Therefore $HEBK$ is a parallelogram ;

therefore HK is equal to EB .

[i. 34]

But EB is equal to EA ;

therefore AE is also equal to HK .

But AH is also equal to HD ;

therefore the two sides EA, AH are equal to the two sides KH, HD respectively ,

and the angle EAH is equal to the angle KHD ;

therefore the base EH is equal to the base KD .

[i. 4]

Therefore the triangle AEH is equal and similar to the triangle HKD .

For the same reason

the triangle AHG is also equal and similar to the triangle HLD .

Now, since two straight lines EH, HG meeting one another are parallel to two straight lines KD, DL meeting one another, and are not in the same plane, they will contain equal angles.

[xi. 10]

Therefore the angle EHG is equal to the angle KDL .

And, since the two straight lines EH, HG are equal to the two KD, DL respectively,

and the angle EHG is equal to the angle KDL ,

therefore the base EG is equal to the base KL ;

[i. 4]

therefore the triangle EHG is equal and similar to the triangle KDL .

For the same reason

the triangle AEG is also equal and similar to the triangle HKL .

Therefore the pyramid of which the triangle AEG is the base and the point H the vertex is equal and similar to the pyramid of which the triangle HKL is the base and the point D the vertex.

[xi. Def. 10]

And, since HK has been drawn parallel to AB , one of the sides of the triangle ADB ,

the triangle ADB is equiangular to the triangle DHK , [I. 29]
and they have their sides proportional;
therefore the triangle ADB is similar to the triangle DHK .
[VI. Def. 1]

For the same reason

the triangle DBC is also similar to the triangle DKL , and
the triangle ADC to the triangle DLH .

Now, since the two straight lines BA , AC meeting one
another are parallel to the two straight lines KH , HL meeting
one another, not in the same plane, they will contain equal
angles. [XI. 10]

Therefore the angle BAC is equal to the angle KHL .

And, as BA is to AC , so is KH to HL ;

therefore the triangle ABC is similar to the triangle HKL .

Therefore also the pyramid of which the triangle ABC is
the base and the point D the vertex is similar to the pyramid
of which the triangle HKL is the base and the point D the
vertex.

But the pyramid of which the triangle HKL is the base
and the point D the vertex was proved similar to the pyramid
of which the triangle AEG is the base and the point H the
vertex.

Therefore each of the pyramids $AEGH$, $HKLD$ is
similar to the whole pyramid $ABCD$.

Next, since BF is equal to FC ,

the parallelogram $EBFG$ is double of the triangle GFC .

And since, if there be two prisms of equal height, and one
have a parallelogram as base, and the other a triangle, and if
the parallelogram be double of the triangle, the prisms are
equal, [XI. 39]

therefore the prism contained by the two triangles BKF ,
 EHG , and the three parallelograms $EBFG$, $EBKH$, $HKFG$
is equal to the prism contained by the two triangles GFC ,
 HKL and the three parallelograms $KFCL$, $LCGH$, $HKFG$.

And it is manifest that each of the prisms, namely that in
which the parallelogram $EBFG$ is the base and the straight
line HK is its opposite, and that in which the triangle GFC is
the base and the triangle HKL its opposite, is greater than
each of the pyramids of which the triangles AEG , HKL are
the bases and the points H , D the vertices,

inasmuch as, if we join the straight lines EF , EK , the prism in which the parallelogram $EBFG$ is the base and the straight line HK its opposite is greater than the pyramid of which the triangle EBF is the base and the point K the vertex.

But the pyramid of which the triangle EBF is the base and the point K the vertex is equal to the pyramid of which the triangle AEG is the base and the point H the vertex ; for they are contained by equal and similar planes.

Hence also the prism in which the parallelogram $EBFG$ is the base and the straight line HK its opposite is greater than the pyramid of which the triangle AEG is the base and the point H the vertex.

But the prism in which the parallelogram $EBFG$ is the base and the straight line HK its opposite is equal to the prism in which the triangle GFC is the base and the triangle HKL its opposite,

and the pyramid of which the triangle AEG is the base and the point H the vertex is equal to the pyramid of which the triangle HKL is the base and the point D the vertex.

Therefore the said two prisms are greater than the said two pyramids of which the triangles AEG , HKL are the bases and the points H , D the vertices.

Therefore the whole pyramid, of which the triangle ABC is the base and the point D the vertex, has been divided into two pyramids equal to one another and into two equal prisms, and the two prisms are greater than the half of the whole pyramid.

Q. E. D.

We will denote a pyramid with vertex D and base ABC by $D(ABC)$ or $D\text{-}ABC$ and the triangular prism with triangles GCF , HLK for bases by (GCF, HLK) .

The following are the steps of the proof.

I. To prove pyramid $H(AEG)$ equal and similar to pyramid $D(HKL)$.
Since sides of $\triangle DAB$ are bisected at H , E , K ,

$$HE \parallel DB, \text{ and } HK \parallel AB.$$

Hence

$$HK = EB = EA,$$

$$HE = KB = DK.$$

Therefore (1) $\triangle s HAE$, DHK are equal and similar.

Similarly (2) $\triangle s HAG$, DHL are equal and similar.

Again, LH , HK are respectively $\parallel$ to GA , AE in a different plane ;

therefore

$$\angle GAE = \angle LHK.$$

And LH , HK are respectively equal to GA , AE .

Therefore (3) $\triangle s GAE$, LHK are equal and similar.

Similarly (4) $\triangle s HGE$, DLK are equal and similar.

Therefore [xi. Def. 10] the pyramids $H(AEG)$ and $D(HKL)$ are equal and similar.

II. To prove the pyramid $D(HKL)$ similar to the pyramid $D(ABC)$.

(1) The $\triangle s DHK$, DAB are equiangular and therefore similar.

Similarly (2) $\triangle s DLH$, DCA are similar, as also (3) the $\triangle s DLK$, DCB .

Again, BA , AC are respectively parallel to KH , HL in a different plane ;
therefore $\angle BAC = \angle KHL$.

And $BA : AC = KH : HL$.

Therefore (4) $\triangle s BAC$, KHL are similar.

Consequently the pyramid $D(ABC)$ is similar to the pyramid $D(HKL)$, and therefore also to the pyramid $H(AEG)$.

III. To prove prism (GCF, HLK) equal to prism (HGE, KFB) .

The prisms may be regarded as having the same *height* (the distance between the planes HKL , ABC) and having for bases (1) the $\triangle CGF$ and (2) the $\square EBFH$, which is the double of the $\triangle CGF$.

Therefore, by xi. 39, the prisms are equal.

IV. To prove the prisms greater than the small pyramids.

Prism (HGE, KFB) is clearly greater than pyramid $K(EFB)$ and therefore greater than pyramid $H(AEG)$.

Therefore each of the prisms is greater than each of the small pyramids ; and the sum of the two prisms is greater than the sum of the two small pyramids, which, with the two prisms, make up the whole pyramid.

PROPOSITION 4.

If there be two pyramids of the same height which have triangular bases, and each of them be divided into two pyramids equal to one another and similar to the whole, and into two equal prisms, then, as the base of the one pyramid is to the base of the other pyramid, so will all the prisms in the one pyramid be to all the prisms, being equal in multitude, in the other pyramid.

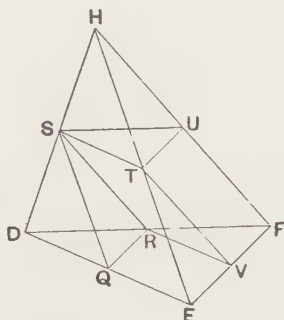
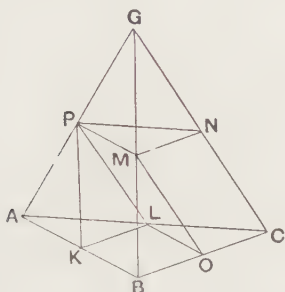
Let there be two pyramids of the same height which have the triangular bases ABC , DEF , and vertices the points G , H ,

and let each of them be divided into two pyramids equal to one another and similar to the whole and into two equal prisms ;

[xii. 3]

I say that, as the base ABC is to the base DEF , so are all the prisms in the pyramid $ABCG$ to all the prisms, being equal in multitude, in the pyramid $DEFH$,

For, since BO is equal to OC , and AL to LC ,
therefore LO is parallel to AB ,
and the triangle ABC is similar to the triangle LOC .



For the same reason
the triangle DEF is also similar to the triangle RVF .

And, since BC is double of CO , and EF of FV , therefore, as BC is to CO , so is EF to FV .

And on BC , CO are described the similar and similarly situated rectilinear figures ABC , LOC ,
and on EF , FV the similar and similarly situated figures DEF , RVF ;

therefore, as the triangle ABC is to the triangle LOC , so is the triangle DEF to the triangle RVF ; [VI. 22]

therefore, alternately, as the triangle ABC is to the triangle DEF , so is the triangle LOC to the triangle RVF . [v. 16]

But, as the triangle LOC is to the triangle RVF , so is the prism in which the triangle LOC is the base and PMN its opposite to the prism in which the triangle RVF is the base and STU its opposite ; [Lemma following]

therefore also, as the triangle ABC is to the triangle DEF , so is the prism in which the triangle LOC is the base and PMN its opposite to the prism in which the triangle RVF is the base and STU its opposite.

But, as the said prisms are to one another, so is the prism in which the parallelogram $KBOL$ is the base and the straight line PM its opposite to the prism in which the parallelogram $OEVR$ is the base and the straight line ST its opposite.

[XI. 39; cf. XII. 3]

Therefore also the two prisms, that in which the parallelogram $KBOL$ is the base and PM its opposite, and that in which the triangle LOC is the base and PMN its opposite, are to the prisms in which $QEV R$ is the base and the straight line ST its opposite and in which the triangle RVF is the base and STU its opposite in the same ratio [v. 12]

Therefore also, as the base ABC is to the base DEF , so are the said two prisms to the said two prisms.

And similarly, if the pyramids $PMNG$, $STUH$ be divided into two prisms and two pyramids, as the base PMN is to the base STU , so will the two prisms in the pyramid $PMNG$ be to the two prisms in the pyramid $STUH$.

But, as the base PMN is to the base STU , so is the base ABC to the base DEF ; for the triangles PMN , STU are equal to the triangles LOC , RVF respectively.

Therefore also, as the base ABC is to the base DEF , so are the four prisms to the four prisms.

And similarly also, if we divide the remaining pyramids into two pyramids and into two prisms, then, as the base ABC is to the base DEF , so will all the prisms in the pyramid $ABCG$ be to all the prisms, being equal in multitude, in the pyramid $DEFH$.

Q. E. D.

LEMMA.

But that, as the triangle LOC is to the triangle RVF , so is the prism in which the triangle LOC is the base and PMN its opposite to the prism in which the triangle RVF is the base and STU its opposite, we must prove as follows.

For in the same figure let perpendiculars be conceived drawn from G , H to the planes ABC , DEF ; these are of course equal because, by hypothesis, the pyramids are of equal height.

Now, since the two straight lines GC and the perpendicular from G are cut by the parallel planes ABC , PMN ; they will be cut in the same ratios. [xi. 17]

And GC is bisected by the plane PMN at N ;
therefore the perpendicular from G to the plane ABC will
also be bisected by the plane PMN .

For the same reason

the perpendicular from H to the plane DEF will also be
bisected by the plane STU .

And the perpendiculars from G, H to the planes ABC, DEF are equal;

therefore the perpendiculars from the triangles PMN, STU
to the planes ABC, DEF are also equal.

Therefore the prisms in which the triangles LOC, RVF
are bases, and PMN, STU their opposites, are of equal
height.

Hence also the parallelepipedal solids described from the
said prisms are of equal height and are to one another as their
bases; [xi. 32]

therefore their halves, namely the said prisms, are to one another
as the base LOC is to the base RVF .

Q. E. D.

We can incorporate the lemma at the end of the proposition and summarise the proof thus.

Since LO is parallel to AB ,

$\triangle s\ ABC, LOC$ are similar.

In like manner $\triangle s\ DEF, RVF$ are similar.

And, since $BC : CO = EF : FV$,

$\triangle ABC : \triangle LOC = \triangle DEF : \triangle RVF$, [vi. 22]

and, alternately,

$\triangle ABC : \triangle DEF = \triangle LOC : \triangle RVF$.

Now the prisms (LOC, PMN) and (RVF, STU) are equal in height:

for the perpendiculars from G, H on the bases ABC, DEF are divided by
the planes PMN, STU (parallel to the bases) in the same proportion as GC, HF
are divided by those planes [xi. 17], i.e. they are bisected;

hence the heights of the prisms, being half the equal heights of the pyramids,
are equal.

And the prisms are the halves respectively of parallelepipeds of the same
height on parallelogrammic bases double of the $\triangle s\ LOC, RVF$ respectively;
[xi. 28 and note]

hence they are in the same ratio as those parallelepipeds, and therefore as
their bases [xi. 32].

Therefore

$$\begin{aligned} (\text{prism } LOC, PMN) : (\text{prism } RVF, STU) &= \triangle LOC . \triangle RVF \\ &= \triangle ABC : \triangle DEF. \end{aligned}$$

And since the other prisms in the pyramids are equal to these prisms respectively,

$$(\text{sum of prisms in } GABC) : (\text{sum of prisms in } HDEF) = \triangle ABC : \triangle DEF.$$

Similarly, if the pyramids $GPMN$, $HSTU$ be divided in like manner, and also the pyramids $PAKL$, $SDQR$, we shall have e.g.

$$\begin{aligned} (\text{sum of prisms in } GPMN) : (\text{sum of prisms in } HSTU) &= \triangle PMN : \triangle STU \\ &= \triangle ABC : \triangle DEF, \end{aligned}$$

and similarly for the second pair of pyramids.

The process may be continued indefinitely, and we shall always have

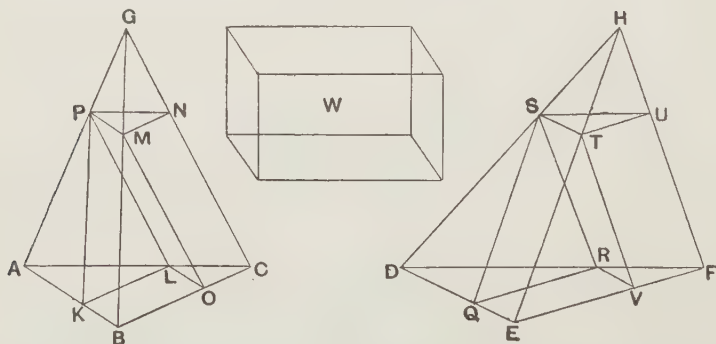
$$(\text{sum of prisms in } GABC) : (\text{sum of prisms in } HDEF) = \triangle ABC : \triangle DEF.$$

PROPOSITION 5.

Pyramids which are of the same height and have triangular bases are to one another as the bases.

Let there be pyramids of the same height, of which the triangles ABC , DEF are the bases and the points G , H the vertices;

I say that, as the base ABC is to the base DEF , so is the pyramid $ABCG$ to the pyramid $DEFH$.



For, if the pyramid $ABCG$ is not to the pyramid $DEFH$ as the base ABC is to the base DEF ,

then, as the base ABC is to the base DEF , so will the pyramid $ABCG$ be either to some solid less than the pyramid $DEFH$ or to a greater.

Let it, first, be in that ratio to a less solid W , and let the pyramid $DEFH$ be divided into two pyramids equal to one another and similar to the whole and into two equal prisms; then the two prisms are greater than the half of the whole pyramid.

[XII. 3]

Again, let the pyramids arising from the division be similarly divided,

and let this be done continually until there are left over from the pyramid $DEFH$ some pyramids which are less than the excess by which the pyramid $DEFH$ exceeds the solid W .

[x. 1]

Let such be left, and let them be, for the sake of argument, $DQRS$, $STUH$;

therefore the remainders, the prisms in the pyramid $DEFH$, are greater than the solid W .

Let the pyramid $ABCG$ also be divided similarly, and a similar number of times, with the pyramid $DEFH$;

therefore, as the base ABC is to the base DEF , so are the prisms in the pyramid $ABCG$ to the prisms in the pyramid $DEFH$.

[xii. 4]

But, as the base ABC is to the base DEF , so also is the pyramid $ABCG$ to the solid W ;

therefore also, as the pyramid $ABCG$ is to the solid W , so are the prisms in the pyramid $ABCG$ to the prisms in the pyramid $DEFH$;

[v. 11]

therefore, alternately, as the pyramid $ABCG$ is to the prisms in it, so is the solid W to the prisms in the pyramid $DEFH$.

[v. 16]

But the pyramid $ABCG$ is greater than the prisms in it; therefore the solid W is also greater than the prisms in the pyramid $DEFH$.

But it is also less :
which is impossible.

Therefore the prism $ABCG$ is not to any solid less than the pyramid $DEFH$ as the base ABC is to the base DEF .

Similarly it can be proved that neither is the pyramid $DEFH$ to any solid less than the pyramid $ABCG$ as the base DEF is to the base ABC .

I say next that neither is the pyramid $ABCG$ to any solid greater than the pyramid $DEFH$ as the base ABC is to the base DEF .

For, if possible, let it be in that ratio to a greater solid W ; therefore, inversely, as the base DEF is to the base ABC , so is the solid W to the pyramid $ABCG$.

But, as the solid W is to the solid $ABCG$, so is the pyramid $DEFH$ to some solid less than the pyramid $ABCG$, as was before proved ; [XII. 2, Lemma]

therefore also, as the base DEF is to the base ABC , so is the pyramid $DEFH$ to some solid less than the pyramid $ABCG$: [v. 11]

which was proved absurd.

Therefore the pyramid $ABCG$ is not to any solid greater than the pyramid $DEFH$ as the base ABC is to the base DEF .

But it was proved that neither is it in that ratio to a less solid.

Therefore, as the base ABC is to the base DEF , so is the pyramid $ABCG$ to the pyramid $DEFH$.

Q. E. D.

In the two preceding propositions it has been shown how we can divide a pyramid with a triangular base into (1) two equal prisms which are together greater than half the pyramid and (2) two equal pyramids similar to the original one, and that, if this process be continued with the two pyramids, then with the four resulting pyramids, and so on, and if, further, another pyramid of the same height as the original one be similarly divided, the subdivision being made the same number of times, the sum of all the prisms in one pyramid is to the sum of all the prisms in the other as the base of the first is to the base of the second.

We can now prove in the manner of XII. 2 that the volumes of the pyramids themselves are as the bases.

Let us call the pyramids P, P' and their respective bases B, B' .

If $P : P' \neq B : B'$,
suppose that $B : B' = P : W$.

I. Let W be $< P'$.

Divide P' into two prisms and two pyramids, subdivide the latter similarly, and so on, until the sum of the *pyramids* remaining is less than the difference between P' and W [x. 1], so that

$$P' > (\text{prisms in } P') > W.$$

Then divide P similarly, the same number of times.

Now $(\text{prisms in } P) : (\text{prisms in } P') = B : B'$ [XII. 4]
 $= P : W$, by hypothesis,

and, alternately,

$$(\text{prisms in } P) : P = (\text{prisms in } P') : W.$$

But $(\text{prisms in } P) < P$;

therefore $(\text{prisms in } P') < W$.

But, by construction, $(\text{prisms in } P') > W$.

Hence W cannot be less than P' .

II. Suppose, if possible, that $W > P'$.

Then, inversely, $B' : B = W : P$.

$$= P' : V,$$

where V is some solid less than P .

[Cf. xii. 2, Lemma, and note.]

But this can be proved impossible exactly as in Part I.

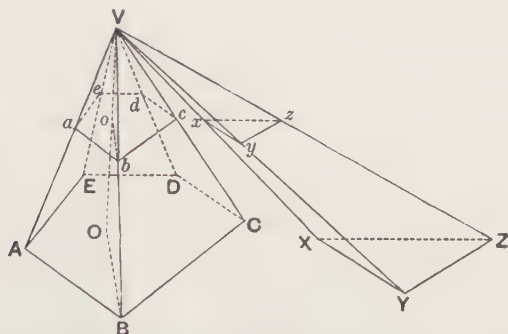
Therefore W is neither less nor greater than P' ,

so that

$$B : B' = P : P'.$$

Legendre, followed by the American editors already mentioned, and by others, approaches the subject by a different route, proving the following propositions.

1. If a pyramid be cut by a plane parallel to the base, (a) the lateral edges and the height will be cut in the same proportion, (b) the section by the plane will be a polygon similar to the base.



(a) Since a lateral face VAB of the pyramid $V(ABCDE)$ is cut by two parallel planes in AB , ab ,

$$AB \parallel ab;$$

Similarly $BC \parallel bc$, and so on.

Therefore $VA : Va = VB : Vb = VC : Vc = \dots$

And, if VO the height be cut in O, o ,

$BO \parallel bo$; and each of the above ratios is equal to $VO : Vo$.

(b) Since $BA \parallel ba$, and $BC \parallel bc$,

$$\angle ABC = \angle abc.$$

[xi. 10]

Similarly for all the other angles of the polygons, which are therefore equiangular.

Also, by similar triangles,

$$VA : Va = AB : ab,$$

and so on.

Therefore, by the ratios above,

$$AB : ab = BC : bc = \dots$$

Therefore the polygons are similar.

2. If two pyramids of the same height be cut by planes which are at the same perpendicular distance from the vertices, the sections are as the respective bases.

For, if we place the pyramids so that the vertices coincide and the bases are in one plane, the planes of the sections will coincide.

If, e.g., the base of the second pyramid be XYZ and the section xyz , we shall have, by the argument of the last proposition,

$$VX : Vx = VY : Vy = VZ : Vz = VO : Vo = VA : Va = \dots,$$

and XYZ , xyz will be similar.

$$\begin{aligned} \text{Now } (\text{polygon } ABCDE) : (\text{polygon } abcde) &= AB^2 : ab^2 \\ &= VA^2 : Va^2, \end{aligned}$$

and

$$\begin{aligned} \triangle XYZ : \triangle xyz &= XY^2 : xy^2 \\ &= VX^2 : Vx^2 \\ &= VA^2 : Va^2. \end{aligned}$$

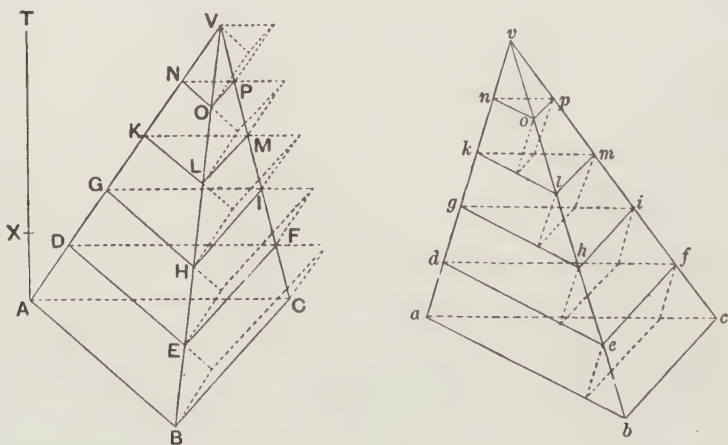
Therefore

$$(\text{polygon } ABCDE) : (\text{polygon } abcde) = \triangle XYZ : \triangle xyz.$$

As a particular case, *if the bases of the two pyramids are equivalent, the sections are also equivalent.*

3. *Two triangular pyramids which have equivalent bases and equal heights are equivalent.*

Let $VABC$, $vabc$ be pyramids with equivalent bases ABC , abc , which for convenience we will suppose placed in one plane, and let TA be the common height.



Then, if the pyramids are not equivalent, one must be greater than the other.

Let $VABC$ be the greater; and let AX be the height of a prism on ABC as base which is equal in volume to the difference of the pyramids.

Divide the height AT into equal parts such that each is less than AX , and let each part be equal to z .

Through the points of division draw planes parallel to the bases cutting both pyramids in the sections DEF , GHI , ... and def , ghi , ...

The sections DEF , def will then be equivalent; so will the sections GHI , ghi , and so on. [(2) above]

On the triangles ABC , DEF , GHI , ... as bases draw *exterior* prisms having for edges the parts AD , DG , GK , ... of the edge AV ;

and on the triangles def , ghi , ... as bases draw *interior* prisms having for edges the parts ad , dg , ... of av .

All the partial prisms will then have the same height z .

Now the sum of the exterior prisms of the pyramid $VABC$ is *greater* than that pyramid ;

and the sum of the interior prisms in the pyramid $vabc$ is *less* than that pyramid.

Consequently the difference between the sum of the first set of prisms and the sum of the second set of prisms is greater than the difference between the two pyramids.

Again, if we start from the bases ABC , abc , the *second* exterior prism $DEFG$ is equivalent to the *first* interior prism $defa$, since their bases are equivalent and they have the same height z . [xi. 28 and note ; xi. 32]

Similarly the third exterior prism is equivalent to the second interior prism, and so on, until we arrive at the last of each.

Therefore the prism $ABCD$, the *first* exterior prism, is the difference between the sums of the exterior and interior prisms respectively.

Therefore the difference between the two pyramids is *less* than the prism $ABCD$, which should therefore be greater than the prism with base ABC and height AX .

But the prism $ABCD$ is, by hypothesis, less than the latter prism : which is impossible.

Consequently the pyramid $VABC$ cannot be greater than the pyramid $vabc$.

Similarly it may be proved that $vabc$ cannot be greater than $VABC$.

Therefore the pyramids are *equivalent*.

Legendre next establishes a proposition corresponding to Eucl. XII. 7, viz.

4. *Any triangular pyramid is one third of the triangular prism on the same base and of the same height,*
and from this he deduces that

COR. *The volume of a triangular pyramid is equal to a third of the product of its base by its height.*

He has previously proved that the volume of a triangular prism is equal to the product of its base and height, since (1) the prism is half of a parallelepiped of the same height and with a parallelogram for base which is double of the base of the prism, and (2) this parallelepiped can be transformed into an equivalent *rectangular* parallelepiped with the same height and an equivalent base.

The theorem (4) is then extended to *any* pyramid in the proposition

5. *Any pyramid has for its measure the third part of the product of its base and its height,* from which follow

COR. I. *Any pyramid is the third part of the prism on the same base and of the same height.*

COR. II. *Two pyramids of the same height are to one another as their bases, and two pyramids on the same base are to one another as their heights.*

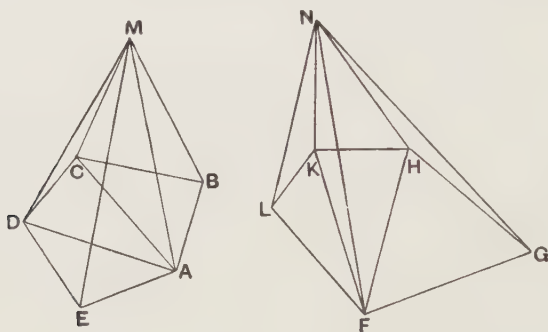
The first part of the second corollary corresponds to the present proposition as extended by the next, XII. 6,

PROPOSITION 6.

Pyramids which are of the same height and have polygonal bases are to one another as the bases.

Let there be pyramids of the same height of which the polygons $ABCDE$, $FGHKL$ are the bases and the points M , N the vertices;

I say that, as the base $ABCDE$ is to the base $FGHKL$, so is the pyramid $ABCDEM$ to the pyramid $FGHKLN$.



For let AC , AD , FH , FK be joined.

Since then $ABCM$, $ACDM$ are two pyramids which have triangular bases and equal height,

they are to one another as the bases ; [XII. 5]

therefore, as the base ABC is to the base ACD , so is the pyramid $ABCM$ to the pyramid $ACDM$.

And, *componendo*, as the base $ABCD$ is to the base ACD , so is the pyramid $ABCDM$ to the pyramid $ACDM$. [v. 18]

But also, as the base ACD is to the base ADE , so is the pyramid $ACDM$ to the pyramid $ADEM$. [XII. 5]

Therefore, *ex aequali*, as the base $ABCD$ is to the base ADE , so is the pyramid $ABCDM$ to the pyramid $ADEM$. [v. 22]

And, again *componendo*, as the base $ABCDE$ is to the base ADE , so is the pyramid $ABCDEM$ to the pyramid $ADEM$. [v. 18]

Similarly also it can be proved that, as the base $FGHKL$ is to the base FGH , so is the pyramid $FGHKLN$ to the pyramid $FGHN$.

And, since $ADEM$, $FGHN$ are two pyramids which have triangular bases and equal height,

therefore, as the base ADE is to the base FGH , so is the pyramid $ADEM$ to the pyramid $FGHN$. [xii. 5]

But, as the base ADE is to the base $ABCDE$, so was the pyramid $ADEM$ to the pyramid $ABCDEM$.

Therefore also, *ex aequali*, as the base $ABCDE$ is to the base FGH , so is the pyramid $ABCDEM$ to the pyramid $FGHN$. [v. 22]

But further, as the base FGH is to the base $FGHKL$, so also was the pyramid $FGHN$ to the pyramid $FGHKLN$.

Therefore also, *ex aequali*, as the base $ABCDE$ is to the base $FGHKL$, so is the pyramid $ABCDEM$ to the pyramid $FGHKLN$. [v. 22]

Q. E. D.

It will be seen that, in order to obtain the proportion

(base $ABCDE$) : $\triangle ADE$ = (pyramid $MABCDE$) : (pyramid $MADE$),

Euclid employs v. 18 (*componendo*) twice over, with an *ex aequali* step [v. 22] intervening.

We might arrive at it more concisely by using v. 24 extended to any number of antecedents.

Thus

$\triangle ABC$: $\triangle ADE$ = (pyramid $MABC$) : (pyramid $MADE$),

$\triangle ACD$: $\triangle ADE$ = (pyramid $MACD$) : (pyramid $MADE$),

and lastly

$\triangle ADE$: $\triangle ADE$ = (pyramid $MADE$) : (pyramid $MADE$).

Therefore, adding the antecedents [v. 24], we have

(polygon $ABCDE$) : $\triangle ADE$ = (pyramid $MABCDE$) : (pyramid $MADE$).

Again, since the pyramids $MADE$, $NFGH$ are of the same height,

$\triangle ADE$: $\triangle FGH$ = (pyramid $MADE$) : (pyramid $NFGH$).

Lastly, using the same argument for the pyramid $NFGHKL$ as for $MABCDE$, and inverting, we have

$\triangle FGH$: (polygon $FGHKL$) = (pyramid $NFGH$) : (pyramid $NFGHKL$).

Thus from the three proportions, *ex aequali*,

(polygon $ABCDE$) : (polygon $FGHKL$)
= (pyramid $MABCDE$) : (pyramid $NFGHKL$).

PROPOSITION 7.

Any prism which has a triangular base is divided into three pyramids equal to one another which have triangular bases.

Let there be a prism in which the triangle ABC is the base and DEF its opposite;

I say that the prism $ABCDEF$ is divided into three pyramids equal to one another, which have triangular bases.

For let BD , EC , CD be joined.

Since $ABED$ is a parallelogram, and BD is its diameter, therefore the triangle ABD is equal to the triangle EBD ; [I. 34]

therefore also the pyramid of which the triangle ABD is the base and the point C the vertex is equal to the pyramid of which the triangle DEB is the base and the point C the vertex. [XII. 5]

But the pyramid of which the triangle DEB is the base and the point C the vertex is the same with the pyramid of which the triangle EBC is the base and the point D the vertex;

for they are contained by the same planes.

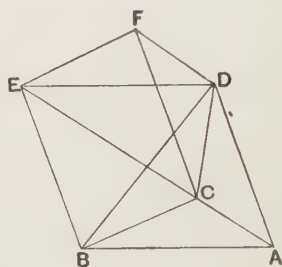
Therefore the pyramid of which the triangle ABD is the base and the point C the vertex is also equal to the pyramid of which the triangle EBC is the base and the point D the vertex.

Again, since $FCBE$ is a parallelogram, and CE is its diameter,

the triangle CEF is equal to the triangle CBE . [I. 34]

Therefore also the pyramid of which the triangle BCE is the base and the point D the vertex is equal to the pyramid of which the triangle ECF is the base and the point D the vertex. [XII. 5]

But the pyramid of which the triangle BCE is the base and the point D the vertex was proved equal to the pyramid of which the triangle ABD is the base and the point C the vertex;



therefore also the pyramid of which the triangle CEF is the base and the point D the vertex is equal to the pyramid of which the triangle ABD is the base and the point C the vertex ;

therefore the prism $ABCDEF$ has been divided into three pyramids equal to one another which have triangular bases.

And, since the pyramid of which the triangle ABD is the base and the point C the vertex is the same with the pyramid of which the triangle CAB is the base and the point D the vertex,

for they are contained by the same planes,

while the pyramid of which the triangle ABD is the base and the point C the vertex was proved to be a third of the prism in which the triangle ABC is the base and DEF its opposite, therefore also the pyramid of which the triangle ABC is the base and the point D the vertex is a third of the prism which has the same base, the triangle ABC , and DEF as its opposite.

PORISM. From this it is manifest that any pyramid is a third part of the prism which has the same base with it and equal height.

Q. E. D.

If we denote by $C-ABD$ a pyramid with vertex C and base ABD , Euclid's argument is easily followed thus.

The $\square ABED$ being bisected by BD ,

$$\begin{aligned} (\text{pyramid } C-ABD) &= (\text{pyramid } C-DEB) & [\text{XII. 5}] \\ &\equiv (\text{pyramid } D-EBC). \end{aligned}$$

And, the $\square EBCF$ being bisected by EC ,

$$(\text{pyramid } D-EBC) = (\text{pyramid } D-ECF).$$

Thus $(\text{pyramid } C-ABD) = (\text{pyramid } D-EBC) = (\text{pyramid } D-ECF)$, and these three pyramids make up the whole prism, so that each is one-third of the prism.

$$\begin{aligned} \text{And, since } (\text{pyramid } C-ABD) &\equiv (\text{pyramid } D-ABC), \\ (\text{pyramid } D-ABC) &= \frac{1}{3} (\text{prism } ABC, DEF). \end{aligned}$$

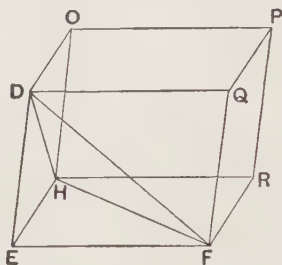
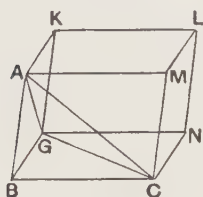
PROPOSITION 8.

Similar pyramids which have triangular bases are in the triplicate ratio of their corresponding sides.

Let there be similar and similarly situated pyramids of

which the triangles ABC , DEF , are the bases and the points G , H the vertices ;

I say that the pyramid $ABCG$ has to the pyramid $DEFH$ the ratio triplicate of that which BC has to EF .



For let the parallelepipedal solids $BGML$, $EHQP$ be completed.

Now, since the pyramid $ABCG$ is similar to the pyramid $DEFH$,

therefore the angle ABC is equal to the angle DEF ,

the angle GBC to the angle HEF ,

and the angle ABG to the angle DEH ;

and, as AB is to DE , so is BC to EF , and BG to EH .

And since, as AB is to DE , so is BC to EF , and the sides are proportional about equal angles, therefore the parallelogram BM is similar to the parallelogram EQ .

For the same reason

BN is also similar to ER , and BK to EO ;

therefore the three parallelograms MB , BK , BN are similar to the three EQ , EO , ER .

But the three parallelograms MB , BK , BN are equal and similar to their three opposites,

and the three EQ , EO , ER are equal and similar to their three opposites. [xi. 24]

Therefore the solids $BGML$, $EHQP$ are contained by similar planes equal in multitude.

Therefore the solid $BGML$ is similar to the solid $EHQP$.

But similar parallelepipedal solids are in the triplicate ratio of their corresponding sides. [xi. 33]

Therefore the solid $BGML$ has to the solid $EHQP$ the ratio triplicate of that which the corresponding side BC has to the corresponding side EF .

But, as the solid $BGML$ is to the solid $EHQP$, so is the pyramid $ABCG$ to the pyramid $DEFH$,

inasmuch as the pyramid is a sixth part of the solid, because the prism which is half of the parallelepipedal solid [XI. 28] is also triple of the pyramid. [XII. 7]

Therefore the pyramid $ABCG$ also has to the pyramid $DEFH$ the ratio triplicate of that which BC has to EF .

Q. E. D.

PORISM. From this it is manifest that similar pyramids which have polygonal bases are also to one another in the triplicate ratio of their corresponding sides.

For, if they are divided into the pyramids contained in them which have triangular bases, by virtue of the fact that the similar polygons forming their bases are also divided into similar triangles equal in multitude and corresponding to the wholes [VI. 20],

then, as the one pyramid which has a triangular base in the one complete pyramid is to the one pyramid which has a triangular base in the other complete pyramid, so also will all the pyramids which have triangular bases contained in the one pyramid be to all the pyramids which have triangular bases contained in the other pyramid [v. 12], that is, the pyramid itself which has a polygonal base to the pyramid which has a polygonal base.

But the pyramid which has a triangular base is to the pyramid which has a triangular base in the triplicate ratio of the corresponding sides ;

therefore also the pyramid which has a polygonal base has to the pyramid which has a similar base the ratio triplicate of that which the side has to the side.

It is at once proved that, the pyramids being similar, the parallelepipeds constructed as shown in the figure are also similar.

Consequently, as these latter are in the triplicate ratio of their corresponding sides [XI. 33], so are the pyramids which are their sixth parts respectively (being one third of the respective *prisms* on the same bases, i.e. of the halves of the respective parallelepipeds, XI. 28).

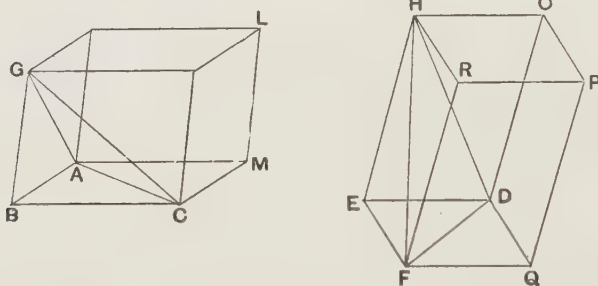
As the Porism is not used where Euclid might have been expected to use it (see note on XII. 12, p. 416), there is some reason to doubt its genuineness. P only has it in the margin, though in the first hand.

PROPOSITION 9.

In equal pyramids which have triangular bases the bases are reciprocally proportional to the heights; and those pyramids in which the bases are reciprocally proportional to the heights are equal.

For let there be equal pyramids which have the triangular bases ABC , DEF and vertices the points G , H ;

I say that in the pyramids $ABCG$, $DEFH$ the bases are reciprocally proportional to the heights, that is, as the base ABC is to the base DEF , so is the height of the pyramid $DEFH$ to the height of the pyramid $ABCG$.



For let the parallelepipedal solids $BGML$, $EHQP$ be completed.

Now, since the pyramid $ABCG$ is equal to the pyramid $DEFH$,

and the solid $BGML$ is six times the pyramid $ABCG$,
and the solid $EHQP$ six times the pyramid $DEFH$,
therefore the solid $BGML$ is equal to the solid $EHQP$.

But in equal parallelepipedal solids the bases are reciprocally proportional to the heights ;

[XI. 34]

therefore, as the base BM is to the base EQ , so is the height of the solid $EHQP$ to the height of the solid $BGML$.

But, as the base BM is to EQ , so is the triangle ABC to the triangle DEF .

[I. 34]

Therefore also, as the triangle ABC is to the triangle DEF , so is the height of the solid $EHQP$ to the height of the solid $BGML$.

[V. 11]

But the height of the solid $EHQP$ is the same with the height of the pyramid $DEFH$,
 and the height of the solid $BGML$ is the same with the height of the pyramid $ABCG$,
 therefore, as the base ABC is to the base DEF , so is the height of the pyramid $DEFH$ to the height of the pyramid $ABCG$.

Therefore in the pyramids $ABCG$, $DEFH$ the bases are reciprocally proportional to the heights.

Next, in the pyramids $ABCG$, $DEFH$ let the bases be reciprocally proportional to the heights;
 that is, as the base ABC is to the base DEF , so let the height of the pyramid $DEFH$ be to the height of the pyramid $ABCG$;

I say that the pyramid $ABCG$ is equal to the pyramid $DEFH$.

For, with the same construction,
 since, as the base ABC is to the base DEF , so is the height of the pyramid $DEFH$ to the height of the pyramid $ABCG$,
 while, as the base ABC is to the base DEF , so is the parallelogram BM to the parallelogram EQ ,
 therefore also, as the parallelogram BM is to the parallelogram EQ , so is the height of the pyramid $DEFH$ to the height of the pyramid $ABCG$. [v. 11]

But the height of the pyramid $DEFH$ is the same with the height of the parallelepiped $EHQP$,
 and the height of the pyramid $ABCG$ is the same with the height of the parallelepiped $BGML$;
 therefore, as the base BM is to the base EQ , so is the height of the parallelepiped $EHQP$ to the height of the parallelepiped $BGML$.

But those parallelepipedal solids in which the bases are reciprocally proportional to the heights are equal; [xi. 34]
 therefore the parallelepipedal solid $BGML$ is equal to the parallelepipedal solid $EHQP$.

And the pyramid $ABCG$ is a sixth part of $BGML$, and the pyramid $DEFH$ a sixth part of the parallelepiped $EHQP$;

therefore the pyramid $ABCG$ is equal to the pyramid $DEFH$.

Therefore etc.

Q. E. D.

The volumes of the pyramids are respectively one sixth part of the volumes of the parallelepipeds described, as in the figure, on double the bases and with the same heights as the pyramids.

I. Thus the parallelepipeds are equal if the pyramids are equal.

And, the parallelepipeds being equal, their bases are reciprocally proportional to their heights; [XI. 34]

hence the bases of the equal pyramids (which are the halves of the bases of the parallelepipeds) are proportional to their heights.

II. If the bases of the pyramids are reciprocally proportional to their heights, so are the bases of the parallelepipeds to their heights (since the bases of the parallelepipeds are double of the bases of the pyramids respectively).

Consequently the parallelepipeds are equal.

[XI. 34]

Therefore their sixth parts, the pyramids, are also equal.

PROPOSITION 10.

Any cone is a third part of the cylinder which has the same base with it and equal height.

For let a cone have the same base, namely the circle $ABCD$, with a cylinder and equal height;

I say that the cone is a third part of the cylinder, that is, that the cylinder is triple of the cone.

For if the cylinder is not triple of the cone, the cylinder will be either greater than triple or less than triple of the cone.

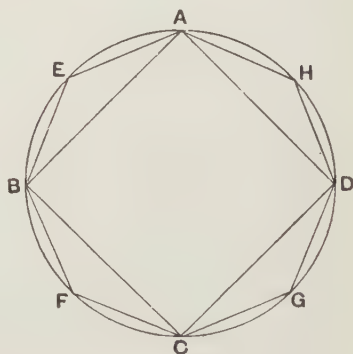
First let it be greater than triple,

and let the square $ABCD$ be inscribed in the circle $ABCD$;

then the square $ABCD$ is greater than the half of the circle $ABCD$. [iv. 6]

From the square $ABCD$ let there be set up a prism of equal height with the cylinder.

Then the prism so set up is greater than the half of the cylinder,



inasmuch as, if we also circumscribe a square about the circle $ABCD$ [iv. 7], the square inscribed in the circle $ABCD$ is half of that circumscribed about it,

and the solids set up from them are parallelepipedal prisms of equal height,

while parallelepipedal solids which are of the same height are to one another as their bases; [xi. 32]

therefore also the prism set up on the square $ABCD$ is half of the prism set up from the square circumscribed about the circle $ABCD$; [cf. xi. 28, or xii. 6 and 7, Por.]

and the cylinder is less than the prism set up from the square circumscribed about the circle $ABCD$;

therefore the prism set up from the square $ABCD$ and of equal height with the cylinder is greater than the half of the cylinder.

Let the circumferences AB , BC , CD , DA be bisected at the points E , F , G , H ,

and let AE , EB , BF , FC , CG , GD , DH , HA be joined;

then each of the triangles AEB , BFC , CGD , DHA is greater than the half of that segment of the circle $ABCD$ which is about it, as we proved before. [xii. 2]

On each of the triangles AEB , BFC , CGD , DHA let prisms be set up of equal height with the cylinder;

then each of the prisms so set up is greater than the half part of that segment of the cylinder which is about it,

inasmuch as, if we draw through the points E , F , G , H parallels to AB , BC , CD , DA , complete the parallelograms on AB , BC , CD , DA , and set up from them parallelepipedal solids of equal height with the cylinder, the prisms on the triangles AEB , BFC , CGD , DHA are halves of the several solids set up;

and the segments of the cylinder are less than the parallelepipedal solids set up;

hence also the prisms on the triangles AEB , BFC , CGD , DHA are greater than the half of the segments of the cylinder about them.

Thus, bisecting the circumferences that are left, joining

straight lines, setting up on each of the triangles prisms of equal height with the cylinder,

and doing this continually,

we shall leave some segments of the cylinder which will be less than the excess by which the cylinder exceeds the triple of the cone. [X. 1]

Let such segments be left, and let them be $AE, EB, BF, FC, CG, GD, DH, HA$;

therefore the remainder, the prism of which the polygon $AEBFCGDH$ is the base and the height is the same as that of the cylinder, is greater than triple of the cone.

But the prism of which the polygon $AEBFCGDH$ is the base and the height the same as that of the cylinder is triple of the pyramid of which the polygon $AEBFCGDH$ is the base and the vertex is the same as that of the cone; [XII. 7, Por.]

therefore also the pyramid of which the polygon $AEBFCGDH$ is the base and the vertex is the same as that of the cone is greater than the cone which has the circle $ABCD$ as base.

But it is also less, for it is enclosed by it:
which is impossible.

Therefore the cylinder is not greater than triple of the cone.

I say next that neither is the cylinder less than triple of the cone,

For, if possible, let the cylinder be less than triple of the cone,

therefore, inversely, the cone is greater than a third part of the cylinder.

Let the square $ABCD$ be inscribed in the circle $ABCD$; therefore the square $ABCD$ is greater than the half of the circle $ABCD$.

Now let there be set up from the square $ABCD$ a pyramid having the same vertex with the cone;

therefore the pyramid so set up is greater than the half part of the cone,

seeing that, as we proved before, if we circumscribe a square

about the circle, the square $ABCD$ will be half of the square circumscribed about the circle,

and if we set up from the squares parallelepipedal solids of equal height with the cone, which are also called prisms, the solid set up from the square $ABCD$ will be half of that set up from the square circumscribed about the circle;

for they are to one another as their bases. [XI. 32]

Hence also the thirds of them are in that ratio;

therefore also the pyramid of which the square $ABCD$ is the base is half of the pyramid set up from the square circumscribed about the circle.

And the pyramid set up from the square about the circle is greater than the cone,
for it encloses it.

Therefore the pyramid of which the square $ABCD$ is the base and the vertex is the same with that of the cone is greater than the half of the cone.

Let the circumferences AB, BC, CD, DA be bisected at the points E, F, G, H ,

and let $AE, EB, BF, FC, CG, GD, DH, HA$ be joined;

therefore also each of the triangles AEB, BFC, CGD, DHA is greater than the half part of that segment of the circle $ABCD$ which is about it.

Now, on each of the triangles AEB, BFC, CGD, DHA let pyramids be set up which have the same vertex as the cone;

therefore also each of the pyramids so set up is, in the same manner, greater than the half part of that segment of the cone which is about it.

Thus, by bisecting the circumferences that are left, joining straight lines, setting up on each of the triangles a pyramid which has the same vertex as the cone,

and doing this continually,

we shall leave some segments of the cone which will be less than the excess by which the cone exceeds the third part of the cylinder. [X. 1]

Let such be left, and let them be the segments on $AE, EB, BF, FC, CG, GD, DH, HA$;

therefore the remainder, the pyramid of which the polygon $AEBFCGDH$ is the base and the vertex the same with that of the cone, is greater than a third part of the cylinder.

But the pyramid of which the polygon $AEBFCGDH$ is the base and the vertex the same with that of the cone is a third part of the prism of which the polygon $AEBFCGDH$ is the base and the height is the same with that of the cylinder;

therefore the prism of which the polygon $AEBFCGDH$ is the base and the height is the same with that of the cylinder is greater than the cylinder of which the circle $ABCD$ is the base.

But it is also less, for it is enclosed by it :
which is impossible.

Therefore the cylinder is not less than triple of the cone.

But it was proved that neither is it greater than triple ;
therefore the cylinder is triple of the cone ;
hence the cone is a third part of the cylinder.

Therefore etc.

Q. E. D.

We observe the use in this proposition of the term "parallelepipedal prism," which recalls Heron's "parallelogrammic" or "parallel-sided prism."

The course of the proof is exactly the same as in XII. 2, except that an arithmetical fraction takes the place of a ratio which, being incommensurable, could only be expressed as a ratio. Consequently we do not need *proportions* in this proposition, as we did in XII. 2, and shall again in XII. 11, etc.

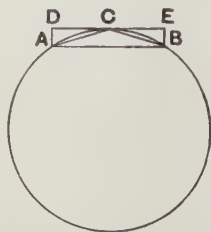
Euclid *exhausts* the cylinder and cone respectively by setting up prisms and pyramids of the same height on the successive regular polygons inscribed in the circle which is the common base, viz. the square, the regular polygon of 8 sides, that of 16 sides, etc.

If AB be the side of one polygon, we obtain two sides of the next by bisecting the arc ACB and joining AC , CB . Draw the tangent DE at C and complete the parallelogram $ABED$.

Now suppose a prism erected on the polygon of which AB is a side, and of the same height as that of the cylinder.

To obtain the prism of the same height on the next polygon we add all the triangular prisms of the same height on the bases ACB and the rest.

Now the prism on ACB is half the prism of the same height on the $\square ABED$ as base.



[cf. xi. 28]

And the prism on $\square ABED$ includes, and is greater than, the portion of the cylinder standing on the segment ACB of the circle.

The same thing is true in regard to the other sides of the polygon of which AB is one side.

Thus the process begins with a prism on the square inscribed in the circle, which is more than half the cylinder, the next prism (with eight lateral faces) takes away more than half the remainder, and so on ;

hence [x. 1], if we proceed far enough, we shall ultimately arrive at a prism leaving over portions of the cylinder together less than any assigned volume.

The construction of pyramids on the successive polygons exhausts the cone in exactly the same way.

Now, if the cone is not equal to one-third of the cylinder, it must be either greater or less.

I. Suppose, if possible, that, V, O being their volumes respectively,
 $O > 3V$.

Construct successive inscribed polygons in the bases and prisms on them until we arrive at a prism P leaving over portions of the cylinder together less than $(O - 3V)$, i.e. such that

$$O > P > 3V.$$

But P is triple of the pyramid on the same base and of the same height ; and this pyramid is included by, and is therefore less than, V ;

therefore $P < 3V$.

But, by construction, $P > 3V$:

which is impossible.

Therefore $O \nless 3V$.

II. Suppose, if possible, that $O < 3V$,

Therefore $V > \frac{1}{3}O$.

Construct successive pyramids in the cone in the manner described until we arrive at a pyramid Π leaving over portions of the cone together less than $(V - \frac{1}{3}O)$, i.e. such that

$$V > \Pi > \frac{1}{3}O.$$

Now Π is one-third of the prism on the same base and of the same height ; and this prism is included by, and is therefore less than, the cylinder ;

therefore $\Pi < \frac{1}{3}O$.

But, by construction, $\Pi > \frac{1}{3}O$:

which is impossible.

Therefore O is neither greater nor less than $3V$, so that

$$O = 3V.$$

It will be observed that here, as in XII. 2, Euclid always *exhausts* the solid by (as it were) building up to it from inside. Hence the solid to be exhausted must, with him, be supposed *greater* than the solid to which it is to be proved equal ; and this is the reason why, in the second part, the initial supposition is turned round.

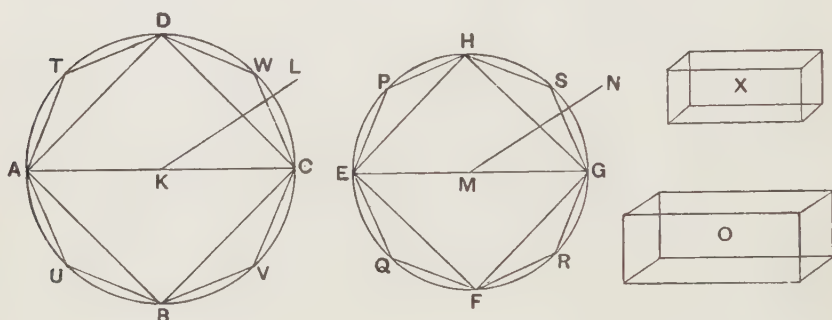
In this case too Euclid might have approximated to the cone and cylinder by *circumscribing* successive pyramids and prisms in the way shown, after Archimedes, in the note on XII. 2.

PROPOSITION 11.

Cones and cylinders which are of the same height are to one another as their bases.

Let there be cones and cylinders of the same height, let the circles $ABCD$, $EFGH$ be their bases, KL , MN their axes and AC , EG the diameters of their bases;

I say that, as the circle $ABCD$ is to the circle $EFGH$, so is the cone AL to the cone EN .



For, if not, then, as the circle $ABCD$ is to the circle $EFGH$, so will the cone AL be either to some solid less than the cone EN or to a greater.

First, let it be in that ratio to a less solid O , and let the solid X be equal to that by which the solid O is less than the cone EN ;

therefore the cone EN is equal to the solids O , X .

Let the square $EFGH$ be inscribed in the circle $EFGH$; therefore the square is greater than the half of the circle.

Let there be set up from the square $EFGH$ a pyramid of equal height with the cone;

therefore the pyramid so set up is greater than the half of the cone,

inasmuch as, if we circumscribe a square about the circle, and set up from it a pyramid of equal height with the cone, the inscribed pyramid is half of the circumscribed pyramid,

for they are to one another as their bases, [XII. 6]

while the cone is less than the circumscribed pyramid.

Let the circumferences EF , FG , GH , HE be bisected at the points P , Q , R , S ,
and let HP , PE , EQ , QF , FR , RG , GS , SH be joined.

Therefore each of the triangles HPE , EQF , FRG , GSH is greater than the half of that segment of the circle which is about it.

On each of the triangles HPE , EQF , FRG , GSH let there be set up a pyramid of equal height with the cone ;
therefore, also, each of the pyramids so set up is greater than the half of that segment of the cone which is about it.

Thus, bisecting the circumferences which are left, joining straight lines, setting up on each of the triangles pyramids of equal height with the cone,
and doing this continually,

we shall leave some segments of the cone which will be less than the solid X . [x. 1]

Let such be left, and let them be the segments on HP , PE , EQ , QF , FR , RG , GS , SH ;

therefore the remainder, the pyramid of which the polygon $HPEQFRGS$ is the base and the height the same with that of the cone, is greater than the solid O .

Let there also be inscribed in the circle $ABCD$ the polygon $DTAUBVCW$ similar and similarly situated to the polygon $HPEQFRGS$,
and on it let a pyramid be set up of equal height with the cone AL .

Since then, as the square on AC is to the square on EG , so is the polygon $DTAUBVCW$ to the polygon $HPEQFRGS$, [xii. 1]

while, as the square on AC is to the square on EG , so is the circle $ABCD$ to the circle $EFGH$, [xii. 2]

therefore also, as the circle $ABCD$ is to the circle $EFGH$, so is the polygon $DTAUBVCW$ to the polygon $HPEQFRGS$.

But, as the circle $ABCD$ is to the circle $EFGH$, so is the cone AL to the solid O ,

and, as the polygon $DTAUBVCW$ is to the polygon $HPEQFRGS$, so is the pyramid of which the polygon $DTAUBVCW$ is the base and the point L the vertex to the pyramid of which the polygon $HPEQFRGS$ is the base and the point N the vertex. [xii. 6]

Therefore also, as the cone AL is to the solid O , so is the pyramid of which the polygon $DTAUBVCW$ is the base and the point L the vertex to the pyramid of which the polygon $HPEQFRGS$ is the base and the point N the vertex ; [v. 11] therefore, alternately, as the cone AL is to the pyramid in it, so is the solid O to the pyramid in the cone EN . [v. 16]

But the cone AL is greater than the pyramid in it ; therefore the solid O is also greater than the pyramid in the cone EN .

But it is also less :
which is absurd.

Therefore the cone AL is not to any solid less than the cone EN as the circle $ABCD$ is to the circle $EFGH$.

Similarly we can prove that neither is the cone EN to any solid less than the cone AL as the circle $EFGH$ is to the circle $ABCD$.

I say next that neither is the cone AL to any solid greater than the cone EN as the circle $ABCD$ is to the circle $EFGH$.

For, if possible, let it be in that ratio to a greater solid O ; therefore, inversely, as the circle $EFGH$ is to the circle $ABCD$, so is the solid O to the cone AL .

But, as the solid O is to the cone AL , so is the cone EN to some solid less than the cone AL ; therefore also, as the circle $EFGH$ is to the circle $ABCD$, so is the cone EN to some solid less than the cone AL :
which was proved impossible.

Therefore the cone AL is not to any solid greater than the cone EN as the circle $ABCD$ is to the circle $EFGH$.

But it was proved that neither is it in this ratio to a less solid ;
therefore, as the circle $ABCD$ is to the circle $EFGH$, so is the cone AL to the cone EN .

But, as the cone is to the cone, so is the cylinder to the cylinder,
for each is triple of each ;

Therefore also, as the circle $ABCD$ is to the circle $EFGH$, so are the cylinders on them which are of equal height.

Therefore etc.

Q. E. D.

We need not again repeat the preliminary construction of successive pyramids and prisms exhausting the cones and cylinders.

Let Z, Z' be the volumes of the two cones, β, β' their respective bases.

If $\beta : \beta' \neq Z : Z'$,

then must $\beta : \beta' = Z : O$,

where O is either less or greater than Z' .

I. Suppose, if possible, that O is *less* than Z' .

Inscribe in Z' a pyramid (Π') leaving over portions of it together less than $(Z' - O)$, i.e. such that

$$Z' > \Pi' > O.$$

Inscribe in Z a pyramid Π on a polygon inscribed in the circular base of Z similar to the polygon which is the base of Π' .

Now, if d, d' be the diameters of the bases,

$$\beta : \beta' = d^2 : d'^2 \quad [\text{XII. 2}]$$

$$= (\text{polygon in } \beta) : (\text{polygon in } \beta') \quad [\text{XII. 1}]$$

$$= \Pi : \Pi'. \quad [\text{XII. 6}]$$

Therefore $Z : O = \Pi : \Pi'$,

and, alternately, $Z : \Pi = O : \Pi'$.

But $Z > \Pi$, since it includes it ;

therefore $O > \Pi'$.

But, by construction, $O < \Pi'$:

which is impossible.

Therefore $O \nless Z$.

II. Suppose, if possible, that

$$\beta : \beta' = Z : O,$$

where O is *greater* than Z' .

Therefore $\beta : \beta' = O' : Z'$,

where O' is some solid less than Z .

That is, $\beta' : \beta = Z' : O'$,

where $O' < Z$.

This is proved impossible exactly in the same way as the assumption in Part I. was proved impossible.

Therefore Z has not either to a less solid than Z' or to a greater solid than Z' the ratio of β to β' ;

therefore $\beta : \beta' = Z : Z'$.

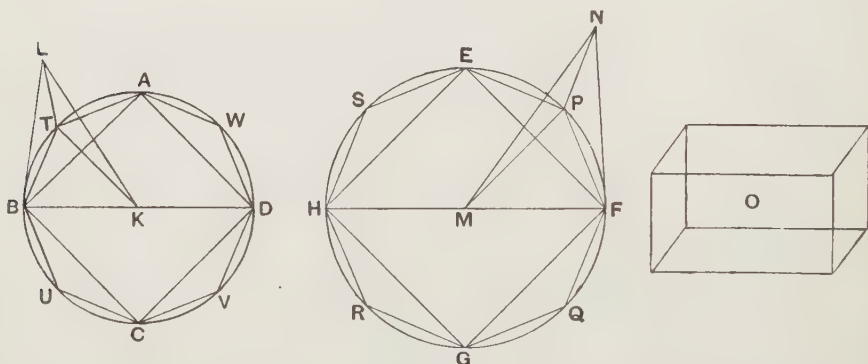
The same is true of the cylinders which are equal to $3Z, 3Z'$ respectively.

PROPOSITION 12.

Similar cones and cylinders are to one another in the triplicate ratio of the diameters in their bases.

Let there be similar cones and cylinders,
let the circles $ABCD$, $EFGH$ be their bases, BD , FH the diameters of the bases, and KL , MN the axes of the cones and cylinders;

I say that the cone of which the circle $ABCD$ is the base and the point L the vertex has to the cone of which the circle $EFGH$ is the base and the point N the vertex the ratio triplicate of that which BD has to FH .



For, if the cone $ABCDL$ has not to the cone $EFGHN$ the ratio triplicate of that which BD has to FH ,
the cone $ABCDL$ will have that triplicate ratio either to some solid less than the cone $EFGHN$ or to a greater.

First, let it have that triplicate ratio to a less solid O .

Let the square $EFGH$ be inscribed in the circle $EFGH$;
[IV. 6]

therefore the square $EFGH$ is greater than the half of the circle $EFGH$.

Now let there be set up on the square $EFGH$ a pyramid having the same vertex with the cone;

therefore the pyramid so set up is greater than the half part of the cone.

Let the circumferences EF , FG , GH , HE be bisected at the points P , Q , R , S ,

and let EP , PF , FQ , QG , GR , RH , HS , SE be joined.

Therefore each of the triangles EPF , FQG , GRH , HSE is also greater than the half part of that segment of the circle $EFGH$ which is about it.

Now on each of the triangles EPF , FQG , GRH , HSE let a pyramid be set up having the same vertex with the cone; therefore each of the pyramids so set up is also greater than the half part of that segment of the cone which is about it.

Thus, bisecting the circumferences so left, joining straight lines, setting up on each of the triangles pyramids having the same vertex with the cone, and doing this continually,

we shall leave some segments of the cone which will be less than the excess by which the cone $EFGHN$ exceeds the solid O . [x. 1]

Let such be left, and let them be the segments on EP , PF , FQ , QG , GR , RH , HS , SE ;

therefore the remainder, the pyramid of which the polygon $EPFQGRHS$ is the base and the point N the vertex, is greater than the solid O .

Let there be also inscribed in the circle $ABCD$ the polygon $ATBUCVDW$ similar and similarly situated to the polygon $EPFQGRHS$,

and let there be set up on the polygon $ATBUCVDW$ a pyramid having the same vertex with the cone;

of the triangles containing the pyramid of which the polygon $ATBUCVDW$ is the base and the point L the vertex let LBT be one,

and of the triangles containing the pyramid of which the polygon $EPFQGRHS$ is the base and the point N the vertex let NFP be one;

and let KT , MP be joined.

Now, since the cone $ABCDL$ is similar to the cone $EFGHN$,

therefore, as BD is to FH , so is the axis KL to the axis MN .

[xi. Def. 24]

But, as BD is to FH , so is BK to FM ;
therefore also, as BK is to FM , so is KL to MN .

And, alternately, as BK is to KL , so is FM to MN .

[v. 16]

And the sides are proportional about equal angles, namely the angles BKL , FMN ;

therefore the triangle BKL is similar to the triangle FMN .

[vi. 6]

Again, since, as BK is to KT , so is FM to MP ,
and they are about equal angles, namely the angles BKT , FMP ,

inasmuch as, whatever part the angle BKT is of the four right angles at the centre K , the same part also is the angle FMP of the four right angles at the centre M ;

since then the sides are proportional about equal angles,
therefore the triangle BKT is similar to the triangle FMP .

[vi. 6]

Again, since it was proved that, as BK is to KL , so is FM to MN ,

while BK is equal to KT , and FM to PM ,

therefore, as TK is to KL , so is PM to MN ;

and the sides are proportional about equal angles, namely the angles TKL , PMN , for they are right;

therefore the triangle LKT is similar to the triangle NMP .

[vi. 6]

And since, owing to the similarity of the triangles LKB , NMF ,

as LB is to BK , so is NF to FM ,

and, owing to the similarity of the triangles BKT , FMP ,

as KB is to BT , so is MF to FP ,

therefore, *ex aequali*, as LB is to BT , so is NF to FP . [v. 22]

Again since, owing to the similarity of the triangles LTK , NPM ,

as LT is to TK , so is NP to PM ,

and, owing to the similarity of the triangles TKB , PMF ,

as KT is to TB , so is MP to PF ;

therefore, *ex aequali*, as LT is to TB , so is NP to PF . [v. 22]

But it was also proved that, as TB is to BL , so is PF to FN .

Therefore, *ex aequali*, as TL is to LB , so is PN to NF .

[v. 22]

Therefore in the triangles LTB , NPF the sides are proportional;

therefore the triangles LTB , NPF are equiangular; [vi. 5]

hence they are also similar. [vi. Def. 1]

Therefore the pyramid of which the triangle BKT is the base and the point L the vertex is also similar to the pyramid of which the triangle FMP is the base and the point N the vertex,

for they are contained by similar planes equal in multitude.

[xi. Def. 9]

But similar pyramids which have triangular bases are to one another in the triplicate ratio of their corresponding sides.

[xii. 8]

Therefore the pyramid $BKTL$ has to the pyramid $FMPN$ the ratio triplicate of that which BK has to FM .

Similarly, by joining straight lines from A , W , D , V , C , U to K , and from E , S , H , R , G , Q to M , and setting up on each of the triangles pyramids which have the same vertex with the cones,

we can prove that each of the similarly arranged pyramids will also have to each similarly arranged pyramid the ratio triplicate of that which the corresponding side BK has to the corresponding side FM , that is, which BD has to FH .

And, as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents;

[v. 12]

therefore also, as the pyramid $BKTL$ is to the pyramid $FMPN$, so is the whole pyramid of which the polygon $ATBUCVDW$ is the base and the point L the vertex to the whole pyramid of which the polygon $EPFQGRHS$ is the base and the point N the vertex;

hence also the pyramid of which $ATBUCVDW$ is the base and the point L the vertex has to the pyramid of which the polygon $EPFQGRHS$ is the base and the point N the vertex the ratio triplicate of that which BD has to FH .

But, by hypothesis, the cone of which the circle $ABCD$

is the base and the point L the vertex has also to the solid O the ratio triplicate of that which BD has to FH ;

therefore, as the cone of which the circle $ABCD$ is the base and the point L the vertex is to the solid O , so is the pyramid of which the polygon $ATBUCVDW$ is the base and L the vertex to the pyramid of which the polygon $EPFQGRHS$ is the base and the point N the vertex;

therefore, alternately, as the cone of which the circle $ABCD$ is the base and L the vertex is to the pyramid contained in it of which the polygon $ATBUCVDW$ is the base and L the vertex, so is the solid O to the pyramid of which the polygon $EPFQGRHS$ is the base and N the vertex. [v. 16]

But the said cone is greater than the pyramid in it;
for it encloses it.

Therefore the solid O is also greater than the pyramid of which the polygon $EPFQGRHS$ is the base and N the vertex.

But it is also less :
which is impossible.

Therefore the cone of which the circle $ABCD$ is the base and L the vertex has not to any solid less than the cone of which the circle $EFGH$ is the base and the point N the vertex the ratio triplicate of that which BD has to FH :

Similarly we can prove that neither has the cone $EFGHN$ to any solid less than the cone $ABCDL$ the ratio triplicate of that which FH has to BD .

I say next that neither has the cone $ABCDL$ to any solid greater than the cone $EFGHN$ the ratio triplicate of that which BD has to FH .

For, if possible, let it have that ratio to a greater solid O .

Therefore, inversely, the solid O has to the cone $ABCDL$ the ratio triplicate of that which FH has to BD .

But, as the solid O is to the cone $ABCDL$, so is the cone $EFGHN$ to some solid less than the cone $ABCDL$.

Therefore the cone $EFGHN$ also has to some solid less than the cone $ABCDL$ the ratio triplicate of that which FH has to BD :

which was proved impossible.

Therefore the cone $ABCDL$ has not to any solid greater than the cone $EFGHN$ the ratio triplicate of that which BD has to FH .

But it was proved that neither has it this ratio to a less solid than the cone $EFGHN$.

Therefore the cone $ABCDL$ has to the cone $EFGHN$ the ratio triplicate of that which BD has to FH .

But, as the cone is to the cone, so is the cylinder to the cylinder,

for the cylinder which is on the same base as the cone and of equal height with it is triple of the cone; [XII. 10]

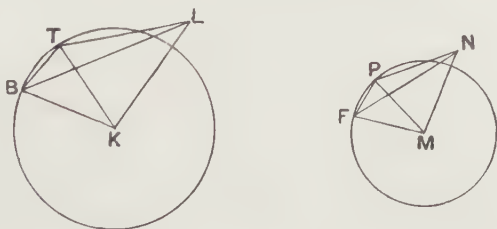
therefore the cylinder also has to the cylinder the ratio triplicate of that which BD has to FH .

Therefore etc.

Q. E. D.

The method of proof is precisely that of the previous proposition. The only addition is caused by the necessity of proving that, if similar equilateral polygons be inscribed in the bases of two similar cones, and pyramids be erected on them with the same vertices as those of the cones, the pyramids (are similar and) are to one another in the triplicate ratio of corresponding edges.

Let KL , MN be the axes of the cones, L , N the vertices, and let BT , FP be sides of similar polygons inscribed in the bases. Join BK , TK , BL , TL , PM , FM , PN , FN .



Now BKL , FMN are right-angled triangles, and, since the cones are similar,

$$BK : KL = FM : MN. \quad [\text{XI. Def. 24}]$$

Therefore (1) $\triangle s BKL, FMN$ are similar. [VI. 6]

Similarly (2) $\triangle s TKL, PMN$ are similar.

Next, in $\triangle s BKT, FMP$, the angles BKT, FMP are equal, since each is the same fraction of four right angles; and the sides about the equal angles are proportional;

therefore (3) $\triangle s BKT, FMP$ are similar.

Again, since from the similar $\triangle s BKL, FMN$, and the similar $\triangle s BKT, FMP$ respectively,

$$LB : BK = NF : FM,$$

$$BK : BT = MF : FP,$$

ex aequali,

$$LB : BT = NF : FP.$$

Similarly

$$LT : TB = NP : PF.$$

Inverting the latter ratio and compounding it with the preceding one, we have, *ex aequali*,

$$LB : LT = NF : NP.$$

Thus in $\triangle s LTB, NPF$ the sides are proportional in pairs ;
therefore (4) $\triangle s LTB, NPF$ are similar.

Thus the partial pyramids $L-BKT, N-FMP$ are similar.

In exactly the same way it is proved that all the other partial pyramids are similar.

Now

(pyramid $L-BKT$) : (pyramid $N-FMP$) = ratio triplicate of $(BK : FM)$.

The other partial pyramids are to one another in the same triplicate ratio.

The sum of the antecedents is therefore to the sum of the consequents in the same triplicate ratio,

i.e. (pyramid $L-ATBU\dots$) : (pyramid $N-EPFQ\dots$)
= ratio triplicate of ratio $(BK : FM)$
= ratio triplicate of ratio $(BD : FH)$.

[The fact that Euclid makes this transition from the partial pyramids to the whole pyramids in the body of this proposition seems to me to suggest grave doubts as to the genuineness of the Porism to XII. 8, which contains a similar but rather more general extension from the case of triangular pyramids to pyramids with polygonal bases. Were that Porism genuine, Euclid would have been more likely to refer to it than to repeat here the same arguments which it contains.]

Now we are in a position to apply the method of exhaustion.

If X, X' be the volumes of the cones, d, d' the diameters of their bases, and if

$$(\text{ratio triplicate of } d : d') \neq X : X',$$

then must

$$(\text{ratio triplicate of } d : d') = X : O,$$

where O is either less or greater than X' .

I. Suppose that O is *less* than X' .

Construct in the way described a pyramid (Π') in X' leaving over portions of X' together less than $(X' - O)$, so that $X' > \Pi' > O$,

and construct in X a pyramid (Π) , with the same vertex as X has, on a polygon inscribed in its base similar to the base of Π' .

Then, by what has just been proved,

$$\Pi : \Pi' = (\text{ratio triplicate of } d : d')$$

$$= X : O, \text{ by hypothesis,}$$

and, alternately,

$$\Pi : X = \Pi' : O.$$

But X includes, and is therefore greater than, Π ;

therefore

$$O > \Pi'.$$

But, by construction,

$$O < \Pi' :$$

which is impossible.

Therefore

O cannot be less than X' .

II. Suppose, if possible, that

(ratio triplicate of $d : d'$) = $X : O$,

where O is *greater* than X' ;

then (ratio triplicate of $d : d'$) = $Z : X'$,

or, inversely, (ratio triplicate of $d' : d$) = $X' : Z$,

where Z is *some solid less* than X .

This is proved impossible by the exact method of Part I.

Hence O cannot be either greater or less than X' ,

and $X : X' =$ (ratio triplicate of ratio $d : d'$).

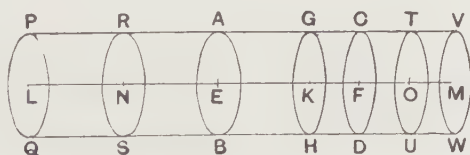
PROPOSITION 13.

If a cylinder be cut by a plane which is parallel to its opposite planes, then, as the cylinder is to the cylinder, so will the axis be to the axis.

For let the cylinder AD be cut by the plane GH which is parallel to the opposite planes AB, CD ,

and let the plane GH meet the axis at the point K ;

I say that, as the cylinder BG is to the cylinder GD , so is the axis EK to the axis KF .



For let the axis EF be produced in both directions to the points L, M ,

and let there be set out any number whatever of axes EN, NL equal to the axis EK ,

and any number whatever FO, OM equal to FK ;

and let the cylinder PW on the axis LM be conceived of which the circles PQ, VW are the bases.

Let planes be carried through the points N, O parallel to AB, CD and to the bases of the cylinder PW ,

and let them produce the circles RS, TU about the centres N, O .

Then, since the axes LN, NE, EK are equal to one another,

therefore the cylinders QR , RB , BG are to one another as their bases. [XII. 11]

But the bases are equal;
therefore the cylinders QR , RB , BG are also equal to one another.

Since then the axes LN , NE , EK are equal to one another,
and the cylinders QR , RB , BG are also equal to one another,
and the multitude of the former is equal to the multitude of the latter,
therefore, whatever multiple the axis KL is of the axis EK ,
the same multiple also will the cylinder QG be of the cylinder GB .

For the same reason, whatever multiple the axis MK is of the axis KF , the same multiple also is the cylinder WG of the cylinder GD .

And, if the axis KL is equal to the axis KM , the cylinder QG will also be equal to the cylinder GW ,
if the axis is greater than the axis, the cylinder will also be greater than the cylinder,
and if less, less.

Thus, there being four magnitudes, the axes EK , KF and the cylinders BG , GD ,
there have been taken equimultiples of the axis EK and of the cylinder BG , namely the axis LK and the cylinder QG ,
and equimultiples of the axis KF and of the cylinder GD , namely the axis KM and the cylinder GW ;
and it has been proved that,
if the axis KL is in excess of the axis KM , the cylinder QG is also in excess of the cylinder GW ,
if equal, equal,
and if less, less.

Therefore, as the axis EK is to the axis KF , so is the cylinder BG to the cylinder GD . [v. Def. 5]

Q. E. D.

It is not necessary to reproduce the proof, as it follows exactly the method of VI. 1 and XI. 25.

The fact that cylinders described about axes of equal length and having

equal bases are equal is inferred from XII. 11 to the effect that cylinders of equal height are to one another as their bases.

That, of two cylinders with unequal axes but equal bases, the greater is that which has the longer axis is of course obvious either by application or by cutting off from the cylinder with the longer axis a cylinder with an axis of the same length as that of the other given cylinder.

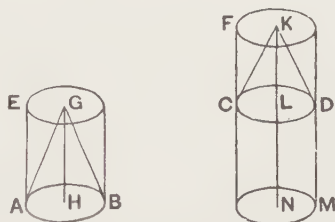
PROPOSITION 14.

Cones and cylinders which are on equal bases are to one another as their heights.

For let EB , FD be cylinders on equal bases, the circles AB , CD :

I say that, as the cylinder EB is to the cylinder FD , so is the axis GH to the axis KL .

For let the axis KL be produced to the point N ,
let LN be made equal to the axis GH ,



and let the cylinder CM be conceived about LN as axis.

Since then the cylinders EB , CM are of the same height, they are to one another as their bases. [XII. 11]

But the bases are equal to one another :
therefore the cylinders EB , CM are also equal.

And, since the cylinder FM has been cut by the plane CD which is parallel to its opposite planes, therefore, as the cylinder CM is to the cylinder FD , so is the axis LN to the axis KL . [XII. 13]

But the cylinder CM is equal to the cylinder EB ,
and the axis LN to the axis GH ;
therefore, as the cylinder EB is to the cylinder FD , so is the axis GH to the axis KL .

But, as the cylinder EB is to the cylinder FD , so is the cone ABG to the cone CDK . [XII. 10]

Therefore also, as the axis GH is to the axis KL , so is the cone ABG to the cone CDK and the cylinder EB to the cylinder FD .

Q. E. D.

No separate proposition corresponding to this is necessary in the case of parallelepipeds, for XI. 25 really contains the property corresponding to that in this proposition as well as the property corresponding to that in XII. 13.

PROPOSITION 15.

In equal cones and cylinders the bases are reciprocally proportional to the heights; and those cones and cylinders in which the bases are reciprocally proportional to the heights are equal.

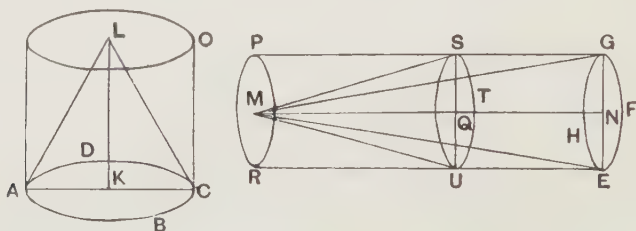
Let there be equal cones and cylinders of which the circles $ABCD$, $EFGH$ are the bases;

let AC , EG be the diameters of the bases,

and KL , MN the axes, which are also the heights of the cones or cylinders;

let the cylinders AO , EP be completed.

I say that in the cylinders AO , EP the bases are reciprocally proportional to the heights, that is, as the base $ABCD$ is to the base $EFGH$, so is the height MN to the height KL .



For the height LK is either equal to the height MN or not equal.

First, let it be equal.

Now the cylinder AO is also equal to the cylinder EP .

But cones and cylinders which are of the same height are to one another as their bases; [XII. 11]

therefore the base $ABCD$ is also equal to the base $EFGH$.

Hence also, reciprocally, as the base $ABCD$ is to the base $EFGH$, so is the height MN to the height KL .

Next, let the height LK not be equal to MN , but let MN be greater; from the height MN let QN be cut off equal to KL , through the point Q let the cylinder EP be cut by the plane TUS parallel to the planes of the circles $EFGH$, RP ,

and let the cylinder ES be conceived erected from the circle $EFGH$ as base and with height NQ .

Now, since the cylinder AO is equal to the cylinder EP , therefore, as the cylinder AO is to the cylinder ES , so is the cylinder EP to the cylinder ES . [v. 7]

But, as the cylinder AO is to the cylinder ES , so is the base $ABCD$ to the base $EFGH$,

for the cylinders AO , ES are of the same height ; [xii. 11]

and, as the cylinder EP is to the cylinder ES , so is the height MN to the height QN ,

for the cylinder EP has been cut by a plane which is parallel to its opposite planes. [xii. 13]

Therefore also, as the base $ABCD$ is to the base $EFGH$, so is the height MN to the height QN . [v. 11]

But the height QN is equal to the height KL ;

therefore, as the base $ABCD$ is to the base $EFGH$, so is the height MN to the height KL .

Therefore in the cylinders AO , EP the bases are reciprocally proportional to the heights.

Next, in the cylinders AO , EP let the bases be reciprocally proportional to the heights,

that is, as the base $ABCD$ is to the base $EFGH$, so let the height MN be to the height KL ;

I say that the cylinder AO is equal to the cylinder EP .

For, with the same construction,

since, as the base $ABCD$ is to the base $EFGH$, so is the height MN to the height KL ,

while the height KL is equal to the height QN ,

therefore, as the base $ABCD$ is to the base $EFGH$, so is the height MN to the height QN

But, as the base $ABCD$ is to the base $EFGH$, so is the cylinder AO to the cylinder ES ,

for they are of the same height ; [xii. 11]

and, as the height MN is to QN , so is the cylinder EP to the cylinder ES ; [xii. 13]

therefore, as the cylinder AO is to the cylinder ES , so is the cylinder EP to the cylinder ES . [v. 11]

Circumscribe about the circle with radius CD a regular polygon $GHI\dots$ such that its sides do not anywhere meet the circle with radius CA . [See note on XII. 2, p. 393 above, for Legendre's lemma relating to this construction.]

Imagine a prism erected on the polygon as base and with height h .

Then $(\text{volume of prism}) = (\text{polygon } GHI\dots) \times h$.

[Legendre has previously proved this proposition, first for a parallelepiped (by transforming it into a rectangular one), then for a triangular prism (half of a parallelepiped of the same height), and lastly for a prism with a polygonal base.]

But $(\text{polygon } GHI\dots) < (\text{surf. } CA)$.

Therefore $(\text{volume of prism}) < (\text{surf. } CA) \times h$
 $< (\text{cylinder on circle of rad. } CD),$
 by hypothesis.

But the prism is *greater* than the latter cylinder, since it includes it :
 which is impossible.

II. In order not to multiply figures let us, in this second case, suppose that CD is the radius of the base of the given cylinder, and that $(\text{surf. } CD) \times h$ is the measure of a cylinder greater than it, e.g. a cylinder on the circle with radius CA as base and of height h .

Then, with the same construction,

$(\text{volume of prism}) = (\text{polygon } GHI\dots) \times h$.

And $(\text{polygon } GHI\dots) > (\text{surf. } CD)$.

Therefore $(\text{volume of prism}) > (\text{surf. } CD) \times h$
 $> (\text{cylinder on surf. } CA),$ by hypothesis.

But the volume of the prism is also *less* than that cylinder, being included by it :
 which is impossible.

Therefore $(\text{volume of cylinder}) = (\text{its base}) \times (\text{its height})$.

It follows as a corollary that

Cylinders of the same height are to one another as their bases [XII. 13], and
cylinders on the same base are to one another as their heights [XII. 14].

Also

Similar cylinders are as the cubes of their heights, or as the cubes of the diameters of their bases [Eucl. XII. 12].

For the bases are as the squares on their diameters; and, since the cylinders are similar, the diameters of the bases are as their heights.

Therefore the bases are as the squares on the heights, and the bases multiplied by the heights, or the cylinders themselves, are as the cubes of the heights.

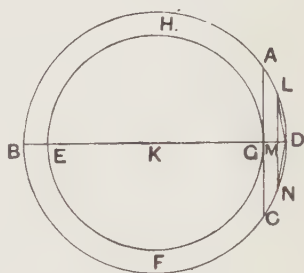
I need not reproduce Legendre's proofs of the corresponding propositions for the cone.

PROPOSITION 16.

Given two circles about the same centre, to inscribe in the greater circle an equilateral polygon with an even number of sides which does not touch the lesser circle.

Let $ABCD$, $EFGH$ be the two given circles about the same centre K ;

thus it is required to inscribe in the greater circle $ABCD$ an equilateral polygon with an even number of sides which does not touch the circle $EFGH$.



For let the straight line BKD be drawn through the centre K ,

and from the point G let GA be drawn at right angles to the straight line BD and carried through to C ;

therefore AC touches the circle $EFGH$.

[III. 16, Por.]

Then, bisecting the circumference BAD , bisecting the half of it, and doing this continually, we shall leave a circumference less than AD .

[X. 1]

Let such be left, and let it be LD ;

from L let LM be drawn perpendicular to BD and carried through to N ,

and let LD , DN be joined;

therefore LD is equal to DN .

[III. 3, I. 4]

Now, since LN is parallel to AC ,

and AC touches the circle $EFGH$,

therefore LN does not touch the circle $EFGH$;

therefore LD , DN are far from touching the circle $EFGH$.

If then we fit into the circle $ABCD$ straight lines equal to the straight line LD and placed continuously, there will be inscribed in the circle $ABCD$ an equilateral polygon with an even number of sides which does not touch the lesser circle $EFGH$.

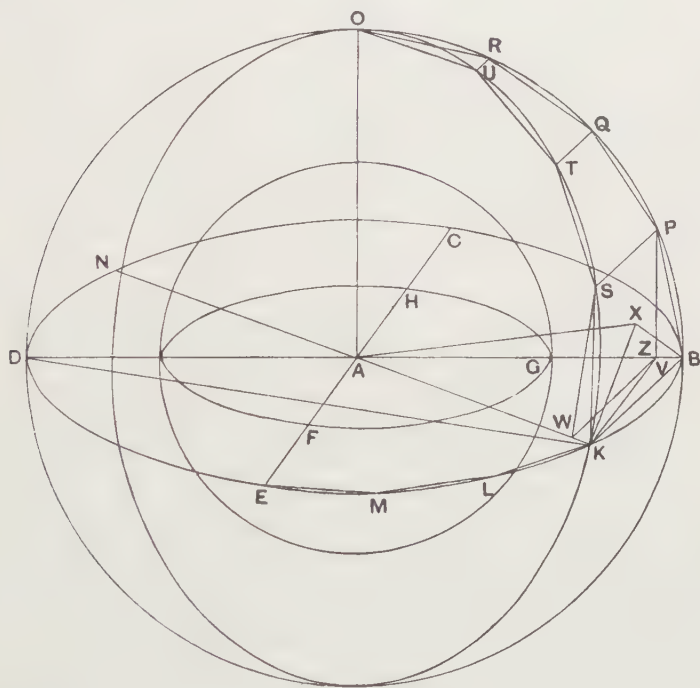
Q. E. F.

It must be carefully observed that the polygon inscribed in the outer circle in this proposition is such that not only do its own sides not touch the inner circle, but also the chords, as LN , joining angular points next but one to each other do not touch the inner circle either. In other words, the polygon is the second in order, not the first, which satisfies the condition of the enunciation. This is important, because such a polygon is wanted in the next proposition; hence in that proposition the *exact* construction here given must be followed.

PROPOSITION 17.

Given two spheres about the same centre, to inscribe in the greater sphere a polyhedral solid which does not touch the lesser sphere at its surface.

Let two spheres be conceived about the same centre A ; thus it is required to inscribe in the greater sphere a polyhedral solid which does not touch the lesser sphere at its surface.



Let the spheres be cut by any plane through the centre ; then the sections will be circles, inasmuch as the sphere was produced by the diameter remaining fixed and the semicircle being carried round it ;

[XI. Def. 14]

hence, in whatever position we conceive the semicircle to be, the plane carried through it will produce a circle on the circumference of the sphere.

And it is manifest that this circle is the greatest possible,

inasmuch as the diameter of the sphere, which is of course the diameter both of the semicircle and of the circle, is greater than all the straight lines drawn across in the circle or the sphere.

Let then $BCDE$ be the circle in the greater sphere, and FGH the circle in the lesser sphere ;
let two diameters in them, BD , CE , be drawn at right angles to one another ;
then, given the two circles $BCDE$, FGH about the same centre, let there be inscribed in the greater circle $BCDE$ an equilateral polygon with an even number of sides which does not touch the lesser circle FGH ,
let BK , KL , LM , ME be its sides in the quadrant BE .
let KA be joined and carried through to N ,
let AO be set up from the point A at right angles to the plane of the circle $BCDE$, and let it meet the surface of the sphere at O ,
and through AO and each of the straight lines BD , KN let planes be carried ;
they will then make greatest circles on the surface of the sphere, for the reason stated.

Let them make such,
and in them let BOD , KON be the semicircles on BD , KN .

Now, since OA is at right angles to the plane of the circle $BCDE$,
therefore all the planes through OA are also at right angles to the plane of the circle $BCDE$; [xi. 18]
hence the semicircles BOD , KON are also at right angles to the plane of the circle $BCDE$.

And, since the semicircles BED , BOD , KON are equal, for they are on the equal diameters BD , KN ,
therefore the quadrants BE , BO , KO are also equal to one another.

Therefore there are as many straight lines in the quadrants BO , KO equal to the straight lines BK , KL , LM , ME as there are sides of the polygon in the quadrant BE .

Let them be inscribed, and let them be BP , PQ , QR , RO and KS , ST , TU , UO ,
let SP , TQ , UR be joined,

and from P, S let perpendiculars be drawn to the plane of the circle $BCDE$; [XI. 11]

these will fall on BD, KN , the common sections of the planes, inasmuch as the planes of BOD, KON are also at right angles to the plane of the circle $BCDE$. [cf. XI. Def. 4]

Let them so fall, and let them be PV, SW , and let WV be joined.

Now since, in the equal semicircles BOD, KON , equal straight lines BP, KS have been cut off, and the perpendiculars PV, SW have been drawn, therefore PV is equal to SW , and BV to KW . [III. 27, I. 26]

But the whole BA is also equal to the whole KA ; therefore the remainder VA is also equal to the remainder WA ; therefore, as BV is to VA , so is KW to WA ; therefore WV is parallel to KB . [VI. 2]

And, since each of the straight lines PV, SW is at right angles to the plane of the circle $BCDE$, therefore PV is parallel to SW . [XI. 6]

But it was also proved equal to it; therefore WV, SP are also equal and parallel. [I. 33]

And, since WV is parallel to SP , while WV is parallel to KB , therefore SP is also parallel to KB . [XI. 9]

And BP, KS join their extremities; therefore the quadrilateral $KBPS$ is in one plane, inasmuch as, if two straight lines be parallel, and points be taken at random on each of them, the straight line joining the points is in the same plane with the parallels. [XI. 7]

For the same reason each of the quadrilaterals $SPQT, TQRU$ is also in one plane.

But the triangle URO is also in one plane. [XI. 2]

If then we conceive straight lines joined from the points P, S, Q, T, R, U to A , there will be constructed a certain polyhedral solid figure between the circumferences BO, KO , consisting of pyramids of which the quadrilaterals $KBPS, SPQT, TQRU$ and the triangle URO are the bases and the point A the vertex.

And, if we make the same construction in the case of each of the sides KL , LM , ME as in the case of BK , and further in the case of the remaining three quadrants,

there will be constructed a certain polyhedral figure inscribed in the sphere and contained by pyramids, of which the said quadrilaterals and the triangle URO , and the others corresponding to them, are the bases and the point A the vertex.

I say that the said polyhedron will not touch the lesser sphere at the surface on which the circle FGH is.

Let AX be drawn from the point A perpendicular to the plane of the quadrilateral $KBPS$, and let it meet the plane at the point X ; [XI. 11]

let XB , XK be joined.

Then, since AX is at right angles to the plane of the quadrilateral $KBPS$,

therefore it is also at right angles to all the straight lines which meet it and are in the plane of the quadrilateral.

[XI. Def. 3]

Therefore AX is at right angles to each of the straight lines BX , XK .

And, since AB is equal to AK ,

the square on AB is also equal to the square on AK .

And the squares on AX , XB are equal to the square on AB ,

for the angle at X is right; [I. 47]

and the squares on AX , XK are equal to the square on AK .

[*id.*]

Therefore the squares on AX , XB are equal to the squares on AX , XK .

Let the square on AX be subtracted from each;

therefore the remainder, the square on BX , is equal to the remainder, the square on XK ;

therefore BX is equal to XK .

Similarly we can prove that the straight lines joined from X to P , S are equal to each of the straight lines BX , XK .

Therefore the circle described with centre X and distance one of the straight lines XB , XX will pass through P , S also, and $KBPS$ will be a quadrilateral in a circle.

Now, since KB is greater than WV , while WV is equal to SP , therefore KB is greater than SP .

But KB is equal to each of the straight lines KS , BP ; therefore each of the straight lines KS , BP is greater than SP .

And, since $KBPS$ is a quadrilateral in a circle, and KB , BP , KS are equal, and PS less, and BX is the radius of the circle, therefore the square on KB is greater than double of the square on BX .

Let KZ be drawn from K perpendicular to BV .

Then, since BD is less than double of DZ , and, as BD is to DZ , so is the rectangle DB , BZ to the rectangle DZ , ZB , if a square be described upon BZ and the parallelogram on ZD be completed, then the rectangle DB , BZ is also less than double of the rectangle DZ , ZB .

And, if KD be joined, the rectangle DB , BZ is equal to the square on BK , and the rectangle DZ , ZB equal to the square on KZ ; [III. 31, VI. 8 and Por.] therefore the square on KB is less than double of the square on KZ .

But the square on KB is greater than double of the square on BX ; therefore the square on KZ is greater than the square on BX .

And, since BA is equal to KA , the square on BA is equal to the square on KA .

And the squares on BX , XA are equal to the square on BA , and the squares on KZ , ZA equal to the square on KA ;

[I. 47]

therefore the squares on BX , XA are equal to the squares on KZ , ZA ,

and of these the square on KZ is greater than the square on BX ;

therefore the remainder, the square on ZA , is less than the square on XA .

Therefore AX is greater than AZ ;

therefore AX is much greater than AG .

And AX is the perpendicular on one base of the polyhedron,

and AG on the surface of the lesser sphere;

hence the polyhedron will not touch the lesser sphere on its surface.

Therefore, given two spheres about the same centre, a polyhedral solid has been inscribed in the greater sphere which does not touch the lesser sphere at its surface.

Q. E. F.

PORISM. But if in another sphere also a polyhedral solid be inscribed similar to the solid in the sphere $BCDE$,

the polyhedral solid in the sphere $BCDE$ has to the polyhedral solid in the other sphere the ratio triplicate of that which the diameter of the sphere $BCDE$ has to the diameter of the other sphere.

For, the solids being divided into their pyramids similar in multitude and arrangement, the pyramids will be similar.

But similar pyramids are to one another in the triplicate ratio of their corresponding sides;

[XII. 8, Por.]

therefore the pyramid of which the quadrilateral $KBPS$ is the base, and the point A the vertex, has to the similarly arranged pyramid in the other sphere the ratio triplicate of that which the corresponding side has to the corresponding side, that is, of that which the radius AB of the sphere about A as centre has to the radius of the other sphere.

Similarly also each pyramid of those in the sphere about A as centre has to each similarly arranged pyramid of those in the other sphere the ratio triplicate of that which AB has to the radius of the other sphere.

And, as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents;

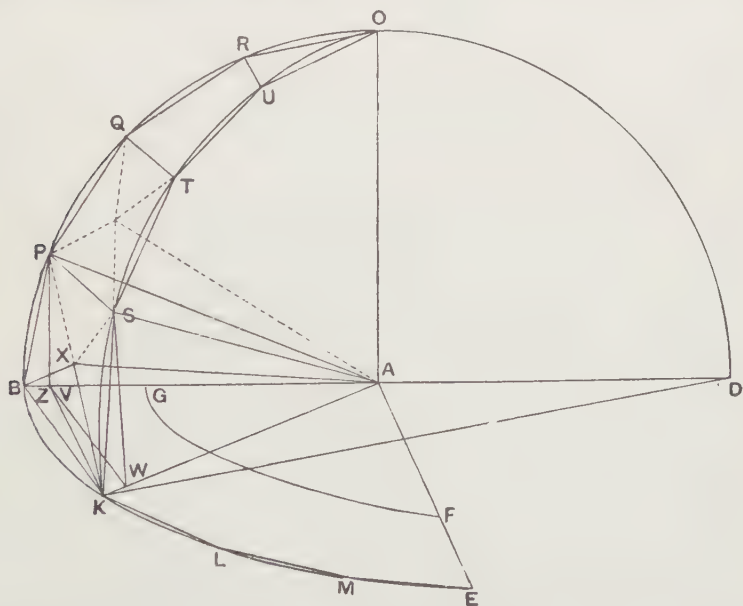
[V. 12]

hence the whole polyhedral solid in the sphere about A as centre has to the whole polyhedral solid in the other sphere the ratio triplicate of that which AB has to the radius of the other sphere, that is, of that which the diameter BD has to the diameter of the other sphere.

Q. E. D.

This proposition is of great length and therefore requires summarising in order to make it easier to grasp. Moreover there are some assumptions in it which require to be proved, and some omissions to be supplied. The figure also is one of some complexity, and, in addition, the text and the figure treat two points Z and V , which are really one and the same, as different.

The first thing needed is to know that all sections of a sphere by planes through the centre are circles and equal to one another (great circles or "greatest circles" as Euclid calls them, more appropriately). Euclid uses his definition of a sphere as the figure described by a semicircle revolving about its diameter. This of course establishes that all planes through the particular diameter make equal circular sections; but it is also assumed that the same sphere is generated by *any other* semicircle of the same size and with its centre at the same point.



The construction and argument of the proposition may be shortly given as follows.

A plane through the centre of two concentric spheres cuts them in great circles of which BE , GF are quadrants.

A regular polygon with an even number of sides is inscribed (*exactly* as in Prop. 16) to the outer circle such that its sides do not touch the inner circle. BK , KL , LM , ME are the sides in the quadrant BE .

AO is drawn at right angles to the plane ABE , and through AO are drawn planes passing through B, K, L, M, E , etc., cutting the sphere in great circles.

OB, OK are quadrants of two of these great circles.

As these quadrants are equal to the quadrant BE , they will be divisible into arcs equal in number and magnitude to the arcs BK, KL, LM, ME .

Dividing the other quadrants of these circles, and also all the quadrants of the other circles through OA , in this way we shall have in all the circles a polygon equal to that in the circle of which BE is a quadrant.

BP, PQ, QR, RO and KS, ST, TU, UO are the sides of these polygons in the quadrants BO, KO .

Joining PS, QT, RU , and making the same construction all round the circles through AO , we have a certain polyhedron inscribed in the outer sphere.

Draw PV perpendicular to AB and therefore (since the planes OAB, BAE are at right angles) perpendicular to the plane BAE ; [xi. Def. 4] draw SW perpendicular to AK and therefore (for a like reason) perpendicular to the plane BAE .

Draw KZ perpendicular to BA . (Since $BK = BP$, and $DB \cdot BV = BP^2$, $DB \cdot BZ = BK^2$, it follows that $BV = BZ$, and Z, V coincide.)

Now, since $\angle s PAV, SAW$, being angles subtended at the centre by equal arcs of equal circles, are equal,

and since $\angle s PVA, SWA$ are right,

while $AS = AP$,

$\triangle s PAV, SAW$ are equal in all respects, [I. 26]

and $AV = AW$.

Consequently $AB : AV = AK : AW$;

and VW, BK are parallel.

But PV, SW are parallel (being both perpendicular to one plane) and equal (by the equal $\triangle s PAV, SAW$),

therefore VW, PS are equal and parallel.

Therefore BK (being parallel to VW) is parallel to PS .

Consequently (1) $BPSK$ is a quadrilateral in one plane.

Similarly the other quadrilaterals $PQTS, QRUT$ are in one plane; and the triangle ORU is in one plane.

In order now to prove that the plane $BPSK$ does not anywhere touch the inner sphere we have to prove that the shortest distance from A to the plane is greater than AL , which by the construction in XII. 16 is greater than AG .

Draw AX perpendicular to the plane $BPSK$.

Then $AX^2 + XB^2 = AX^2 + XK^2 = AX^2 + XS^2 = AX^2 + XP^2 = AB^2$,

whence $XB = XK = XS = XP$,

or (2) the quadrilateral $BPSK$ is inscribable in a circle with X as centre and radius XB .

Now $BK > VW$
 $> PS$;

therefore in the quadrilateral $BPSK$ three sides BK, BP, KS are equal, but PS is less.

Consequently the angles about X are three equal angles and one smaller angle;

therefore any one of the equal angles is greater than a right angle, i.e. $\angle BXX$ is obtuse.

Therefore (3) $BK^2 > 2BX^2$. [ii. 12]

Next, consider the semicircle BKD with KZ drawn perpendicular to BD .

We have $BD < 2DZ$,

so that $DB \cdot BZ < 2DZ \cdot ZB$,

or $BK^2 < 2KZ^2$;

therefore, *a fortiori*, [by (3) above]

(4) $BX^2 < KZ^2$.

Now $AK^2 = AB^2$;

therefore $AZ^2 + ZK^2 = AX^2 + XB^2$.

And $BX^2 < KZ^2$;

therefore $AX^2 > AZ^2$,

or (5) $AX > AZ$.

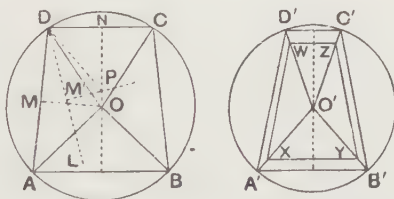
But, by the construction in xii. 16, $AZ > AG$; therefore, *a fortiori*, $AX > AG$.

And, since the perpendicular AX is the shortest distance from A to the plane $BPSK$,

(6) the plane $BPSK$ does not anywhere meet the inner sphere.

Euclid omits to prove that, *a fortiori*, the other quadrilaterals $PQTS$, $QRUT$, and the triangle ROU , do not anywhere meet the inner sphere.

For this purpose it is only necessary to show that the radii of the circles circumscribing $BPSK$, $PQTS$, $QRUT$ and ROU are in descending order of magnitude.



We have therefore to prove that, if $ABCD$, $A'B'C'D'$ are two quadrilaterals inscribable in circles, and

$$AD = BC = A'D' = B'C',$$

while AB is not greater than AD , $A'B' = CD$, and $AB > CD > C'D'$,

then the radius OA of the circle circumscribing the first quadrilateral is greater than the radius $O'A'$ of the circle circumscribing the second.

Clavius, and Simson after him, prove this by *reductio ad absurdum*.

(1) If $OA = O'A'$,

it follows that $\angle s AOD$, BOC , $A'O'D'$, $B'O'C'$ are all equal.

Also $\angle AOB > \angle A'O'B'$,

$$\angle COD > \angle C'O'D',$$

whence the four angles about O are together greater than the four angles about O' , i.e. greater than four right angles;

which is impossible.

(2) If $O'A' > OA$,

cut off from $O'A'$, $O'B'$, $O'C'$, $O'D'$ lengths equal to OA , and draw the inner quadrilateral as shown in the figure ($XYZW$).

Then

$$AB > A'B' > XY,$$

$$CD > C'D' > ZW,$$

$$AD = A'D' > WX,$$

$$BC = B'C' > YZ.$$

Consequently the same absurdity as in (1) follows *a fortiori*.

Therefore, since OA is neither equal to nor less than $O'A'$,

$$OA > O'A'.$$

The fact is also sufficiently clear if we draw MO , NO bisecting DA , DC perpendicularly and therefore meeting in O , the centre of the circumscribed circle, and then suppose the side DA with the perpendicular MO to turn inwards about D as centre. Then the intersection of MO and NO , as P , will gradually move towards N .

Simson gives his proof as "Lemma 11." immediately before XII. 17. He adds to the Porism some words explaining how we may construct a similar polyhedron in another sphere and how we may prove that the polyhedra are similar.

The Porism is of course of the essence of the matter because it is the porism which as much as the construction is wanted in the next proposition. It would therefore not have been amiss to include the Porism in the enunciation of XII. 17 so as to call attention to it.

PROPOSITION 18.

Spheres are to one another in the triplicate ratio of their respective diameters.

Let the spheres ABC , DEF be conceived,
and let BC , EF be their diameters ;

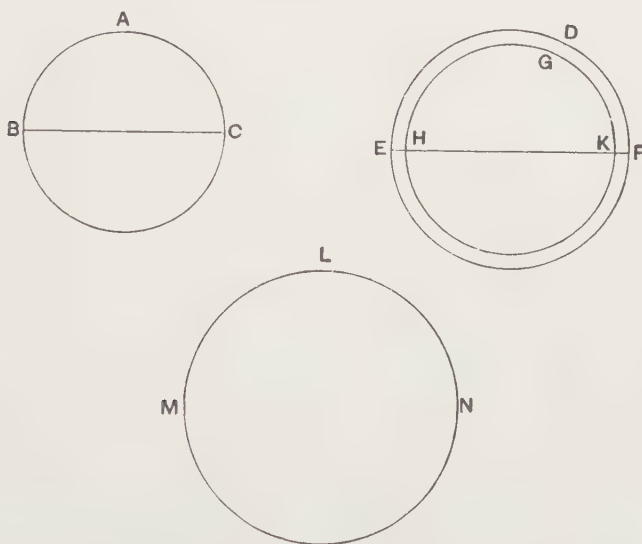
I say that the sphere ABC has to the sphere DEF the ratio triplicate of that which BC has to EF .

For, if the sphere ABC has not to the sphere DEF the ratio triplicate of that which BC has to EF ,
then the sphere ABC will have either to some less sphere than the sphere DEF , or to a greater, the ratio triplicate of that which BC has to EF .

First, let it have that ratio to a less sphere GHK ,
let DEF be conceived about the same centre with GHK ,
let there be inscribed in the greater sphere DEF a polyhedral solid which does not touch the lesser sphere GHK at its surface,

[XII. 17]

and let there also be inscribed in the sphere ABC a polyhedral solid similar to the polyhedral solid in the sphere DEF ; therefore the polyhedral solid in ABC has to the polyhedral solid in DEF the ratio triplicate of that which BC has to EF .
[XII. 17, Por.]



But the sphere ABC also has to the sphere GHK the ratio triplicate of that which BC has to EF ;

therefore, as the sphere ABC is to the sphere GHK , so is the polyhedral solid in the sphere ABC to the polyhedral solid in the sphere DEF ;

and, alternately, as the sphere ABC is to the polyhedron in it, so is the sphere GHK to the polyhedral solid in the sphere DEF . [v. 16]

But the sphere ABC is greater than the polyhedron in it; therefore the sphere GHK is also greater than the polyhedron in the sphere DEF .

But it is also less,
for it is enclosed by it.

Therefore the sphere ABC has not to a less sphere than the sphere DEF the ratio triplicate of that which the diameter BC has to EF .

Similarly we can prove that neither has the sphere DEF to a less sphere than the sphere ABC the ratio triplicate of that which EF has to BC .

I say next that neither has the sphere ABC to any greater sphere than the sphere DEF the ratio triplicate of that which BC has to EF .

For, if possible, let it have that ratio to a greater, LMN ; therefore, inversely, the sphere LMN has to the sphere ABC the ratio triplicate of that which the diameter EF has to the diameter BC .

But, inasmuch as LMN is greater than DEF , therefore, as the sphere LMN is to the sphere ABC , so is the sphere DEF to some less sphere than the sphere ABC , as was before proved. [XII. 2, Lemma]

Therefore the sphere DEF also has to some less sphere than the sphere ABC the ratio triplicate of that which EF has to BC :

which was proved impossible.

Therefore the sphere ABC has not to any sphere greater than the sphere DEF the ratio triplicate of that which BC has to EF .

But it was proved that neither has it that ratio to a less sphere.

Therefore the sphere ABC has to the sphere DEF the ratio triplicate of that which BC has to EF .

Q. E. D.

It is the method of this proposition which Legendre adopted for his proof of XII. 2 (see note on that proposition).

The argument can be put very shortly. We will suppose S, S' to be the volumes of the spheres, and d, d' to be their diameters; and we will for brevity express the triplicate ratio of d to d' by $d^3 : d'^3$.

If $d^3 : d'^3 \neq S : S'$,
then $d^3 : d'^3 = S : T$,

where T is the volume of some sphere either greater or less than S' .

I. Suppose, if possible, that $T < S'$.

Let T be supposed concentric with S' .

As in XII. 17, inscribe a polyhedron in S' such that its faces do not anywhere touch T ;

and inscribe in S a polyhedron similar to that in S' .

Then

$$S : T = d^3 : d'^3$$

$$= (\text{polyhedron in } S) : (\text{polyhedron in } S') ;$$

or, alternately,

$$S : (\text{polyhedron in } S) = T : (\text{polyhedron in } S').$$

And

$$S > (\text{polyhedron in } S) ;$$

therefore

$$T > (\text{polyhedron in } S').$$

But, by construction, $T < (\text{polyhedron in } S') :$
which is impossible.

Therefore

$$T \nless S'.$$

II. Suppose, if possible, that $T > S'.$

Now

$$d^3 : d'^3 = S : T$$

$$= X : S',$$

where X is the volume of some sphere less than $S,$

[XII. 2, Lemma]

or, inversely,

$$d'^3 : d^3 = S' : X,$$

where $X < S.$

This is proved impossible exactly as in Part I.

Therefore

$$T \nless S'.$$

Hence $T,$ not being greater or less than $S',$ is equal to it, and

$$d^3 : d'^3 = S : S'.$$

BOOK XIII.

HISTORICAL NOTE.

I have already given, in the note to IV. 10, the evidence upon which the construction of the five regular solids is attributed to the Pythagoreans. Some of them, the cube, the tetrahedron (which is nothing but a pyramid), and the octahedron (which is only a double pyramid with a square base), cannot but have been known to the Egyptians. And it appears that dodecahedra have been found, of bronze or other material, which may belong to periods earlier than Pythagoras' time by some centuries (for references see Cantor's *Geschichte der Mathematik* I, pp. 175—6).

It is true that the author of the scholium No. 1 to Eucl. XIII. says that the Book is about "the five so-called Platonic figures, which however do not belong to Plato, three of the aforesaid five figures being due to the Pythagoreans, namely the cube, the pyramid and the dodecahedron, while the octahedron and the icosahedron are due to Theaetetus." This statement (taken probably from Geminus) may perhaps rest on the fact that Theaetetus was the first to write at any length about the two last-mentioned solids. We are told indeed by Suidas (s. v. Θεαιήτης) that Theaetetus "first wrote on the 'five solids' as they are called." This no doubt means that Theaetetus was the first to write a complete and systematic treatise on all the regular solids; it does not exclude the possibility that Hippasus or others had already written on the dodecahedron. The fact that Theaetetus wrote upon the regular solids agrees very well with the evidence which we possess of his contributions to the theory of irrationals, the connexion between which and the investigation of the regular solids is seen in Euclid's Book XIII.

Theaetetus flourished about 380 B.C., and his work on the regular solids was soon followed by another, that of Aristaëus, an elder contemporary of Euclid, who also wrote an important book on *Solid Loci*, i.e. on conics treated as loci. This Aristaëus (known as "the elder") wrote in the period about 320 B.C. We hear of his *Comparison of the five regular solids* from Hypsicles (2nd cent. B.C.), the writer of the short book commonly included in the editions of the *Elements* as Book XIV. Hypsicles gives in this Book some six propositions supplementing Eucl. XIII.; and he introduces the second of the propositions (Heiberg's Euclid, Vol. v. p. 6) as follows:

"The same circle circumscribes both the pentagon of the dodecahedron and the triangle of the icosahedron when both are inscribed in the same sphere. This is proved by Aristaëus in the book entitled *Comparison of the five figures*."

Hypsicles proceeds (pp. 7 sqq.) to give a proof of this theorem. Allman pointed out (*Greek Geometry from Thales to Euclid*, 1889, pp. 201—2) that this proof depends on eight theorems, six of which appear in Euclid's Book XIII. (in Propositions 8, 10, 12, 15, 16 with Por., 17); two other propositions not mentioned by Allman are also used, namely XIII. 4 and 9. This seems, as Allman says, to confirm the inference of Bretschneider (p. 171) that, as Aristaeus' work was the newest and latest in which, before Euclid's time, this subject was treated, we have in Eucl. XIII. at least a partial recapitulation of the contents of the treatise of Aristaeus.

After Euclid, Apollonius wrote on the comparison of the dodecahedron and the icosahedron inscribed in one and the same sphere. This we also learn from Hypsicles, who says in the next words following those about Aristaeus above quoted: "But it is proved by Apollonius in the second edition of his *Comparison of the dodecahedron with the icosahedron* that, as the surface of the dodecahedron is to the surface of the icosahedron [inscribed in the same sphere], so is the dodecahedron itself [i.e. its volume] to the icosahedron, because the perpendicular is the same from the centre of the sphere to the pentagon of the dodecahedron and to the triangle of the icosahedron."

BOOK XIII. PROPOSITIONS.

PROPOSITION I.

If a straight line be cut in extreme and mean ratio, the square on the greater segment added to the half of the whole is five times the square on the half.

For let the straight line AB be cut in extreme and mean ratio at the point C , and let AC be the greater segment; let the straight line AD be produced in a straight line with CA , and let AD be made half of AB ; I say that the square on CD is five times the square on AD .

For let the squares AE , DF be described on AB , DC , and let the figure in DF be drawn ; let FC be carried through to G .

Now, since AB has been cut in extreme and mean ratio at C , therefore the rectangle AB, BC is equal to the square on AC .

[vi. Def. 3, vi. 17]

And CE is the rectangle AB, BC , and FH the square on AC ;

therefore CE is equal to FH .

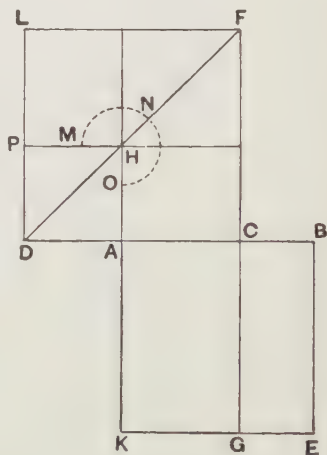
And, since BA is double of AD ,
while BA is equal to KA , and AD to AH ,
therefore KA is also double of AH .

But, as KA is to AH , so is CK to CH ; therefore CK is double of CH .

[VI. I]

But LH , HC are also double of CH .

Therefore KC is equal to LH, HC .



But CE was also proved equal to HF ;
therefore the whole square AE is equal to the gnomon MNO .

And, since BA is double of AD ,
the square on BA is quadruple of the square on AD ,
that is, AE is quadruple of DH .

But AE is equal to the gnomon MNO ;
therefore the gnomon MNO is also quadruple of AP ;
therefore the whole DF is five times AP .

And DF is the square on DC , and AP the square on DA ;
therefore the square on CD is five times the square on DA .

Therefore etc.

Q. E. D.

The first five propositions are in the nature of lemmas, which are required for later propositions but are not in themselves of much importance.

It will be observed that, while the method of the propositions is that of Book II., being strictly geometrical and not algebraical, none of the results of that Book are made use of (except indeed in the Lemma to XIII. 2, which is probably not genuine). It would therefore appear as though these propositions were taken from an earlier treatise without being revised or rewritten in the light of Book II. It will be remembered that, according to Proclus (p. 67, 6), Eudoxus "greatly added to the number of the theorems which originated with Plato regarding *the section*" (i.e. presumably the "*golden section*"); and it is therefore probable that the five theorems are due to Eudoxus.

That, if AB is divided at C in extreme and mean ratio, the rectangle AB, BC is equal to the square on AC is inferred from VI. 17.

AD is made equal to half AB , and we have to prove that

$$(\text{sq. on } CD) = 5 (\text{sq. on } AD).$$

The figure shows at once that

$$\square CH = \square HL,$$

$$\text{so that } \square CH + \square HL = 2 (\square CH) \\ = \square AG.$$

$$\text{Also } \text{sq. } HF = (\text{sq. on } AC) \\ = \text{rect. } AB, BC \\ = CE.$$

By addition,

$$(\text{gnomon } MNO) = \text{sq. on } AB \\ = 4 (\text{sq. on } AD);$$

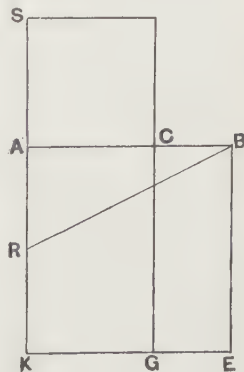
whence, adding the sq. on AD to each, we have

$$(\text{sq. on } CD) = 5 (\text{sq. on } AD).$$

The result here, and in the next propositions, is really seen more readily by means of the figure of II. 11.

In this figure $SR = AC + \frac{1}{2}AB$, by construction;
and we have therefore to prove that

$$(\text{sq. on } SR) = 5 (\text{sq. on } AR).$$



This is obvious, for

$$\begin{aligned} (\text{sq. on } SR) &= (\text{sq. on } RB) \\ &= \text{sum of sqs. on } AB, AR \\ &= 5 (\text{sq. on } AR). \end{aligned}$$

The MSS. contain a curious addition to XIII. 1—5 in the shape of analyses and syntheses for each proposition prefaced by the heading :

“What is analysis and what is synthesis.

“Analysis is the assumption of that which is sought as if it were admitted <and the arrival> by means of its consequences at something admitted to be true.

“Synthesis is an assumption of that which is admitted <and the arrival> by means of its consequences at something admitted to be true.”

There must apparently be some corruption in the text ; it does not, in the case of synthesis, give what is wanted. B and V have, instead of “something admitted to be true,” the words “the end or attainment of what is sought.”

The whole of this addition is evidently interpolated. To begin with, the analyses and syntheses of the five propositions are placed all together in four MSS. ; in P, q they come after an alternative proof of XIII. 5 (which alternative proof P gives after XIII. 6, while q gives it instead of XIII. 6), in B (which has not the alternative proof of XIII. 5) after XIII. 6, and in b (in which XIII. 6 is wanting, and the alternative proof of XIII. 5 is in the margin, in the first hand) after XIII. 5, while V has the analyses of 1—3 in the text after XIII. 6 and those of 4—5 in the same place in the margin, by the second hand. Further, the addition is altogether alien from the plan and manner of the *Elements*. The interpolation took place before Theon's time, and the probability is that it was originally in the margin, whence it crept into the text of P after XIII. 5. Heiberg (after Bretschneider) suggested in his edition (Vol. v. p. lxxxiv.) that it might be a relic of analytical investigations by Theaetetus or Eudoxus, and he cited the remark of Pappus (v. p. 410) at the beginning of his “comparisons of the five [regular solid] figures which have an equal surface,” to the effect that he will not use “the so-called analytical investigation by means of which some of the ancients effected their demonstrations.” More recently (*Paralipomena zu Euklid in Hermes* xxxviii., 1903) Heiberg conjectures that the author is Heron, on the ground that the sort of analysis and synthesis recalls Heron's remarks on analysis and synthesis in his commentary on the beginning of Book II. (quoted by an-Nairizī, ed. Curtze, p. 89) and his quasi-algebraical alternative proofs of propositions in that Book.

To show the character of the interpolated matter I need only give the analysis and synthesis of one proposition. In the case of XIII. 1 it is in substance as follows. The figure is a mere straight line.

Let AB be divided in extreme and mean ratio at C , AC being the greater segment ; and let

$$AD = \frac{1}{2} AB.$$

I say that

(Analysis.)

“For, since

and $(\text{sq. on } CD) = (\text{sq. on } CA) + (\text{sq. on } AD) + 2 (\text{rect. } CA, AD),$

therefore $(\text{sq. on } CA) + 2 (\text{rect. } CA, AD) = 4 (\text{sq. on } AD).$

But

and

$$\text{rect. } BA \cdot AC = 2 (\text{rect. } CA \cdot AD),$$

$$(\text{sq. on } CA) = (\text{rect. } AB, BC).$$



Therefore

$$(\text{rect. } BA, AC) + (\text{rect. } AB, BC) = 4 (\text{sq. on } AD),$$

$$\text{or} \quad (\text{sq. on } AB) = 4 (\text{sq. on } AD);$$

$$\text{and this is true, since} \quad AD = \frac{1}{2} AB.$$

(Synthesis.)

$$\text{Since} \quad (\text{sq. on } AB) = 4 (\text{sq. on } AD),$$

$$\text{and} \quad (\text{sq. on } AB) = (\text{rect. } BA, AC) + (\text{rect. } AB, BC),$$

$$\text{therefore} \quad 4 (\text{sq. on } AD) = 2 (\text{rect. } DA, AC) + \text{sq. on } AC.$$

Adding to each the square on AD , we have

$$(\text{sq. on } CD) = 5 (\text{sq. on } AD).$$

PROPOSITION 2.

If the square on a straight line be five times the square on a segment of it, then, when the double of the said segment is cut in extreme and mean ratio, the greater segment is the remaining part of the original straight line.

For let the square on the straight line AB be five times the square on the segment AC

of it,

and let CD be double of AC ;

I say that, when CD is cut in extreme and mean ratio, the greater segment is CB .

Let the squares AF , CG be described on AB , CD respectively, let the figure in AF be drawn, and let BE be drawn through.

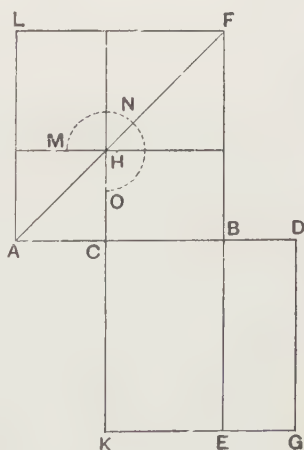
Now, since the square on BA is five times the square on AC , AF is five times AH .

Therefore the gnomon MNO is quadruple of AH .

And, since DC is double of CA , therefore the square on DC is quadruple of the square on CA , that is, CG is quadruple of AH .

But the gnomon MNO was also proved quadruple of AH ; therefore the gnomon MNO is equal to CG .

And, since DC is double of CA , while DC is equal to CK , and AC to CH , therefore KB is also double of BH .



But LH , HB are also double of HB ;
therefore KB is equal to LH , HB .

But the whole gnomon MNO was also proved equal to the whole CG ;
therefore the remainder HF is equal to BG .

And BG is the rectangle CD , DB ,
for CD is equal to DG ;
and HF is the square on CB ;
therefore the rectangle CD , DB is equal to the square on CB .

Therefore, as DC is to CB , so is CB to BD .

But DC is greater than CB ;
therefore CB is also greater than BD .

Therefore, when the straight line CD is cut in extreme and mean ratio, CB is the greater segment.

Therefore etc.

Q. E. D.

LEMMA.

That the double of AC is greater than BC is to be proved thus.

If not, let BC be, if possible, double of CA .

Therefore the square on BC is quadruple of the square on CA ;
therefore the squares on BC , CA are five times the square on CA .

But, by hypothesis, the square on BA is also five times the square on CA ;

therefore the square on BA is equal to the squares on BC , CA :
which is impossible. [II. 4]

Therefore CB is not double of AC .

Similarly we can prove that neither is a straight line less than CB double of CA ;
for the absurdity is much greater.

Therefore the double of AC is greater than CB .

Q. E. D.

This proposition is the converse of Prop. 1. We have to prove that, if AB be so divided at C that

$$(\text{sq. on } AB) = 5 (\text{sq. on } AC),$$

and if $CD = 2AC$,

then

$$(\text{rect. } CD, DB) = (\text{sq. on } CB).$$

Subtract from each side the sq. on AC ;

then $(\text{gnomon } MNO) = 4 (\text{sq. on } AC)$
 $= (\text{sq. on } CD).$

Now, as in the last proposition,

$$\square CE = 2 (\square BH) \\ = \square BH + \square HL.$$

Subtracting these equals from the equals, the square on CD and the gnomon MNO respectively, we have

$$\square BG = (\text{square } HF),$$

i.e. $(\text{rect. } CD, DB) = (\text{sq. on } CB).$

Here again the proposition can readily be proved by means of a figure similar to that of II. 11.

Draw CA through C at right angles to CB and of length equal to CA in the original figure; make CD double of CA ; produce AC to R so that $CR = CB$.

Complete the squares on CB and CD , and join AD .

Now we are given the fact that

$$(\text{sq. on } AR) = 5 (\text{sq. on } CA).$$

But

$$5 (\text{sq. on } AC) = (\text{sq. on } AC) + (\text{sq. on } CD) \\ = (\text{sq. on } AD).$$

Therefore

$$(\text{sq. on } AR) = (\text{sq. on } AD), \\ AR = AD.$$

or

Now

$$(\text{rect. } KR, RC) + (\text{sq. on } AC) = (\text{sq. on } AR) \\ = (\text{sq. on } AD) \\ = (\text{sq. on } AC) + (\text{sq. on } CD).$$

Therefore $(\text{rect. } KR, RC) = (\text{sq. on } CD).$

That is, $(\text{rectangle } RE) = (\text{square } CG).$

Subtract the common part CE ,

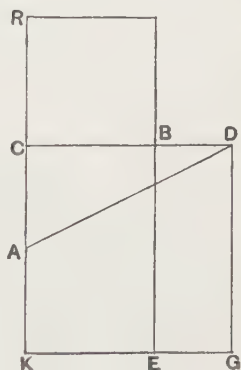
and

$$(\text{rect. } BG) = (\text{sq. } RB),$$

or

$$\text{rect. } CD, DB = (\text{sq. on } CB).$$

Heiberg, with reason, doubts the genuineness of the Lemma following this proposition.



PROPOSITION 3.

If a straight line be cut in extreme and mean ratio, the square on the lesser segment added to the half of the greater segment is five times the square on the half of the greater segment.

In this case we have

$$\begin{aligned}
 (\text{sq. on } BD) &= (\text{sq. } FG) + (\text{rect. } CG) + (\text{rect. } CN) \\
 &= (\text{sq. } FG) + (\text{rect. } FE) + (\text{rect. } CN) \\
 &= (\text{sq. } FG) + (\text{rect. } CE) \\
 &= (\text{sq. } FG) + (\text{rect. } AB, BC) \\
 &= (\text{sq. } FG) + (\text{sq. on } AC), \text{ by hypothesis,} \\
 &= 5 (\text{sq. on } DC).
 \end{aligned}$$

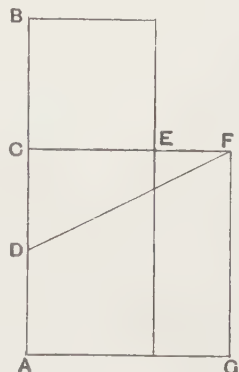
The theorem is still more obvious if the figure of II. 11 be used. Let CF be divided in extreme and mean ratio at E , by the method of II. 11.

Then, since

$$\begin{aligned}
 (\text{rect. } AB, BC) + (\text{sq. on } CD) &= \text{sq. on } BD \\
 &= \text{sqs. on } CD, CF, \\
 (\text{rect. } AB, BC) &= (\text{sq. on } CF) \\
 &= (\text{sq. on } CA),
 \end{aligned}$$

and AB is divided at C in extreme and mean ratio.

$$\begin{aligned}
 \text{And } (\text{sq. on } BD) &= (\text{sq. on } DF) \\
 &= 5 (\text{sq. on } CD).
 \end{aligned}$$



PROPOSITION 4.

If a straight line be cut in extreme and mean ratio, the square on the whole and the square on the lesser segment together are triple of the square on the greater segment.

Let AB be a straight line,

let it be cut in extreme and mean ratio at C ,
and let AC be the greater segment;

I say that the squares on AB , BC are
triple of the square on CA .

For let the square $ADEB$ be described on AB ,

and let the figure be drawn.

Since then AB has been cut in extreme
and mean ratio at C ,

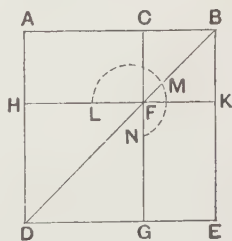
and AC is the greater segment,

therefore the rectangle AB , BC is equal to the square on AC .

[vi. Def. 3, vi. 17]

And AK is the rectangle AB , BC , and HG the square
on AC ;

therefore AK is equal to HG .



And, since AF is equal to FE ,
 let CK be added to each ;
 therefore the whole AK is equal to the whole CE ;
 therefore AK, CE are double of AK .

But AK, CE are the gnomon LMN and the square CK ;
 therefore the gnomon LMN and the square CK are double
 of AK .

But, further, AK was also proved equal to HG ;
 therefore the gnomon LMN and the squares CK, HG are
 triple of the square HG .

And the gnomon LMN and the squares CK, HG are
 the whole square AE and CK , which are the squares on
 AB, BC ,
 while HG is the square on AC .

Therefore the squares on AB, BC are triple of the square
 on AC .

Q. E. D.

Here, as in the preceding propositions, the results are proved *de novo* by
 the method of Book II., without reference to that Book. Otherwise the proof
 might have been shorter.

For, by II. 7,

$$\begin{aligned} (\text{sq. on } AB) + (\text{sq. on } BC) &= 2 (\text{rect. } AB, BC) + (\text{sq. on } AC) \\ &= 3 (\text{sq. on } AC). \end{aligned}$$

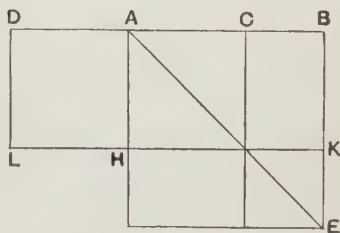
PROPOSITION 5.

*If a straight line be cut in extreme and mean ratio, and
 there be added to it a straight line equal to the greater segment,
 the whole straight line has been cut in extreme and mean ratio,
 and the original straight line is the greater segment.*

For let the straight line AB be cut in extreme and mean
 ratio at the point C ,

let AC be the greater segment,
 and let AD be equal to AC .

I say that the straight line
 DB has been cut in extreme and
 mean ratio at A , and the original
 straight line AB is the greater
 segment.



For let the square AE be described on AB ,
 and let the figure be drawn.

Since AB has been cut in extreme and mean ratio at C ,
therefore the rectangle AB, BC is equal to the square on AC .
[vi. Def. 3, vi. 17]

And CE is the rectangle AB, BC , and CH the square
on AC ;

therefore CE is equal to HC .

But HE is equal to CE ,

and DH is equal to HC ;

therefore DH is also equal to HE .

Therefore the whole DK is equal to the whole AE .

And DK is the rectangle BD, DA ,

for AD is equal to DL ;

and AE is the square on AB ;

therefore the rectangle BD, DA is equal to the square
on AB .

Therefore, as DB is to BA , so is BA to AD . [vi. 17]

And DB is greater than BA ;

therefore BA is also greater than AD . [v. 14]

Therefore DB has been cut in extreme and mean ratio at
 A , and AB is the greater segment.

Q. E. D.

We have

$$\begin{aligned} (\text{sq. } DH) &= (\text{sq. } HC) \\ &= (\text{rect. } CE), \text{ by hypothesis,} \\ &= (\text{rect. } HE). \end{aligned}$$

Add to each side the rectangle AK , and

$$\begin{aligned} (\text{rect. } DK) &= (\text{sq. } AE), \\ \text{or } (\text{rect. } BD, DA) &= (\text{sq. on } AB). \end{aligned}$$

The result is of course obvious from II. 11.

There is an alternative proof given in P after XIII. 6, which depends on
Book v.

By hypothesis, $BA : AC = AC : CB$,

or, inversely, $AC : AB = CB : AC$.

Componendo, $(AB + AC) : AB = AB : AC$,

or $DB : BA = BA : AD$.

PROPOSITION 6.

*If a rational straight line be cut in extreme and mean ratio,
each of the segments is the irrational straight line called
apotome.*

Let AB be a rational straight line,
let it be cut in extreme and mean
ratio at C ,



and let AC be the greater segment ;

I say that each of the straight lines AC , CB is the irrational
straight line called apotome.

For let BA be produced, and let AD be made half of BA .

Since then the straight line AB has been cut in extreme
and mean ratio,

and to the greater segment AC is added AD which is half
of AB ,

therefore the square on CD is five times the square on DA .

[XIII. 1]

Therefore the square on CD has to the square on DA the
ratio which a number has to a number ;

therefore the square on CD is commensurable with the square
on DA .

[x. 6]

But the square on DA is rational,

for DA is rational, being half of AB which is rational ;

therefore the square on CD is also rational ;

[x. Def. 4]

therefore CD is also rational.

And, since the square on CD has not to the square on
 DA the ratio which a square number has to a square number,
therefore CD is incommensurable in length with DA ;

[x. 9]

therefore CD , DA are rational straight lines commensurable
in square only ;

therefore AC is an apotome.

[x. 73]

Again, since AB has been cut in extreme and mean ratio,
and AC is the greater segment,

therefore the rectangle AB , BC is equal to the square on AC .

[vi. Def. 3, vi. 17]

Therefore the square on the apotome AC , if applied to
the rational straight line AB , produces BC as breadth.

But the square on an apotome, if applied to a rational
straight line, produces as breadth a first apotome ;

[x. 97]

therefore CB is a first apotome.

And CA was also proved to be an apotome.
Therefore etc.

Q. E. D.

It seems certain that this proposition is an interpolation. P has it, but the copyist (or rather the copyist of its archetype) says that "this theorem is not found in most copies of the new recension, but is found in those of the old." In the first place, there is a scholium to XIII. 17 in P itself which proves the same thing as XIII. 6, and which would therefore have been useless if XIII. 6 had preceded. Hence, when the scholium was written, this proposition had not yet been interpolated. Secondly, P has it before the alternative proof of XIII. 5; this proof is considered, on general grounds, to be interpolated, and it would appear that it must have been a *later* interpolation (XIII. 6) which divorced it from the proposition to which it belonged. Thirdly, there is cause for suspicion in the proposition itself, for, while the enunciation states that each segment of the straight line is an *apotome*, the proposition adds that the lesser segment is a *first* apotome. The scholium in P referred to has not this blot. What is actually wanted in XIII. 17 is the fact that the *greater* segment is an apotome. It is probable that Euclid *assumed* this fact as evident enough from XIII. 1 without further proof, and that he neither wrote XIII. 6 nor the quotation of its enunciation in XIII. 17.

PROPOSITION 7.

If three angles of an equilateral pentagon, taken either in order or not in order, be equal, the pentagon will be equiangular.

For in the equilateral pentagon $ABCDE$ let, first, three angles taken in order, those at A, B, C , be equal to one another;

I say that the pentagon $ABCDE$ is equiangular.

For let AC, BE, FD be joined.

Now, since the two sides CB, BA are equal to the two sides BA, AE respectively,

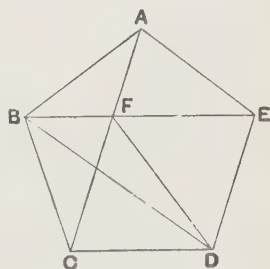
and the angle CBA is equal to the angle BAE ,

therefore the base AC is equal to the base BE ,
the triangle ABC is equal to the triangle ABE ,

and the remaining angles will be equal to the remaining angles,
namely those which the equal sides subtend, [1. 4]

that is, the angle BCA to the angle BEA and the angle ABE to the angle CAB ;

hence the side AF is also equal to the side BF . [1. 6]



But the whole AC was also proved equal to the whole BE ;
therefore the remainder FC is also equal to the remainder FE .

But CD is also equal to DE .

Therefore the two sides FC , CD are equal to the two sides FE , ED ;

and the base FD is common to them;

therefore the angle FCD is equal to the angle FED . [I. 8]

But the angle BCA was also proved equal to the angle AEB ;

therefore the whole angle BCD is also equal to the whole angle AED .

But, by hypothesis, the angle BCD is equal to the angles at A , B ;

therefore the angle AED is also equal to the angles at A , B .

Similarly we can prove that the angle CDE is also equal to the angles at A , B , C ;

therefore the pentagon $ABCDE$ is equiangular.

Next, let the given equal angles not be angles taken in order, but let the angles at the points A , C , D be equal;

I say that in this case too the pentagon $ABCDE$ is equiangular.

For let BD be joined.

Then, since the two sides BA , AE are equal to the two sides BC , CD ,

and they contain equal angles,

therefore the base BE is equal to the base BD ,

the triangle ABE is equal to the triangle BCD ,

and the remaining angles will be equal to the remaining angles, namely those which the equal sides subtend; [I. 4]

therefore the angle AEB is equal to the angle CDB .

But the angle BED is also equal to the angle BDE ,

since the side BE is also equal to the side BD . [I. 5]

Therefore the whole angle AED is equal to the whole angle CDE .

But the angle CDE is, by hypothesis, equal to the angles at A , C ;

therefore the angle AED is also equal to the angles at A , C .

For the same reason

the angle ABC is also equal to the angles at A, C, D .

Therefore the pentagon $ABCDE$ is equiangular.

Q. E. D.

This proposition is required in XIII. 17.

The steps of the proof may be shown thus.

I. Suppose that the angles at A, B, C are all equal.

Then the isosceles triangles BAE, ABC are equal in all respects ;

thus $BE = AC, \angle BCA = \angle BEA, \angle CAB = \angle EBA$.

By the last equality, $FA = FB,$

so that, since $BE = AC, \quad FC = FE.$

The $\triangle s FED, FCD$ are now equal in all respects,

[I. 8, 4]

and $\angle FCD = \angle FED.$

But $\angle ACB = \angle AEB$, from above,

whence, by addition, $\angle BCD = \angle AED.$

Similarly it may be proved that $\angle CDE$ is also equal to any one of the angles at A, B, C .

II. Suppose the angles at A, C, D to be equal.

Then the isosceles triangles ABE, CBD are equal in all respects, and hence $BE = BD$ (so that $\angle BDE = \angle BED$),

and $\angle CDB = \angle AEB.$

By addition of the equal angles,

$$\angle CDE = \angle DEA.$$

Similarly it may be proved that $\angle ABC$ is also equal to each of the angles at A, C, D .

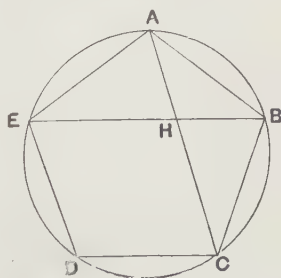
PROPOSITION 8.

If in an equilateral and equiangular pentagon straight lines subtend two angles taken in order, they cut one another in extreme and mean ratio, and their greater segments are equal to the side of the pentagon.

For in the equilateral and equiangular pentagon $ABCDE$ let the straight lines AC, BE , cutting one another at the point H , subtend two angles taken in order, the angles at A, B ;

I say that each of them has been cut in extreme and mean ratio at the point H , and their greater segments are equal to the side of the pentagon.

For let the circle $ABCDE$ be circumscribed about the pentagon $ABCDE$.



[IV. 14]

Then, since the two straight lines EA , AB are equal to the two AB , BC ,
and they contain equal angles,
therefore the base BE is equal to the base AC ,
the triangle ABE is equal to the triangle ABC ,
and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend. [I. 4]

Therefore the angle BAC is equal to the angle ABE ;
therefore the angle AHE is double of the angle BAH . [I. 32]

But the angle EAC is also double of the angle BAC ,
inasmuch as the circumference EDC is also double of the circumference CB ;
[III. 28, VI. 33]

therefore the angle HAE is equal to the angle AHE ;
hence the straight line HE is also equal to EA , that is, to AB .
[I. 6]

And, since the straight line BA is equal to AE ,
the angle ABE is also equal to the angle AEB . [I. 5]

But the angle ABE was proved equal to the angle BAH ;
therefore the angle BEA is also equal to the angle BAH .

And the angle ABE is common to the two triangles ABE and ABH ;

therefore the remaining angle BAE is equal to the remaining angle AHB ;
[I. 32]

therefore the triangle ABE is equiangular with the triangle ABH ;

therefore, proportionally, as EB is to BA , so is AB to BH .
[VI. 4]

But BA is equal to EH ;
therefore, as BE is to EH , so is EH to HB .

And BE is greater than EH ;
therefore EH is also greater than HB . [V. 14]

Therefore BE has been cut in extreme and mean ratio at H , and the greater segment HE is equal to the side of the pentagon.

Similarly we can prove that AC has also been cut in extreme and mean ratio at H , and its greater segment CH is equal to the side of the pentagon.

In order to prove this theorem we have to show (1) that the $\triangle s$ AEB , HAB are similar, and (2) that $EH = EA (= AB)$.

To prove (2) we have

$\triangle s$ AEB , BAC equal in all respects,

whence

$$EB = AC,$$

and

$$\angle BAC = \angle ABE.$$

Therefore

$$\angle AHE = 2 \angle BAC$$

$$= \angle EAC,$$

so that

$$EH = EA$$

$$= AB.$$

To prove (1) we have, in the $\triangle s$ AEB , HAB ,

$$\angle BAH = \angle EBA$$

$$= \angle AEB,$$

and

$$\angle ABE \text{ is common,}$$

therefore the third $\angle s$ AHB , EAB are equal,

and

$$\triangle s$$
 AEB , HAB are similar.

Now, since these triangles are similar,

$$EB : BA = BA : BH,$$

or

$$(\text{rect. } EB, BH) = (\text{sq. on } BA)$$

$$= (\text{sq. on } EH),$$

so that EB is divided in extreme and mean ratio at H .

Similarly its equal, CA , is divided in extreme and mean ratio at H .

PROPOSITION 9.

If the side of the hexagon and that of the decagon inscribed in the same circle be added together, the whole straight line has been cut in extreme and mean ratio, and its greater segment is the side of the hexagon.

Let ABC be a circle ;

of the figures inscribed in the circle ABC let BC be the side of a decagon, CD that of a hexagon,

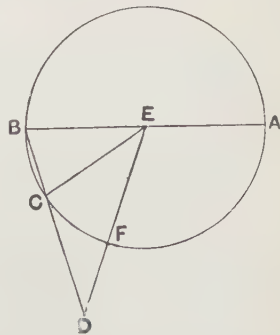
and let them be in a straight line ;

I say that the whole straight line BD has been cut in extreme and mean ratio, and CD is its greater segment.

For let the centre of the circle, the point E , be taken,

let EB , EC , ED be joined,

and let BE be carried through to A .



Since BC is the side of an equilateral decagon,

therefore the circumference ACB is five times the circumference BC ;

therefore the circumference AC is quadruple of CB .

But, as the circumference AC is to CB , so is the angle AEC to the angle CEB ; [vi. 33]

therefore the angle AEC is quadruple of the angle CEB .

And, since the angle EBC is equal to the angle ECB , [i. 5]
therefore the angle AEC is double of the angle ECB . [i. 32]

And, since the straight line EC is equal to CD ,
for each of them is equal to the side of the hexagon inscribed in the circle ABC , [iv. 15, Por.]

the angle CED is also equal to the angle CDE ; [i. 5]

therefore the angle ECB is double of the angle EDC . [i. 32]

But the angle AEC was proved double of the angle ECB ;
therefore the angle AEC is quadruple of the angle EDC .

But the angle AEC was also proved quadruple of the angle BEC ;

therefore the angle EDC is equal to the angle BEC .

But the angle EBD is common to the two triangles BEC and BED ;

therefore the remaining angle BED is also equal to the remaining angle ECB ; [i. 32]

therefore the triangle EBD is equiangular with the triangle EBC .

Therefore, proportionally, as DB is to BE , so is EB to BC . [vi. 4]

But EB is equal to CD .

Therefore, as BD is to DC , so is DC to CB .

And BD is greater than DC ;

therefore DC is also greater than CB .

Therefore the straight line BD has been cut in extreme and mean ratio, and DC is its greater segment.

Q. E. D.

BC is the side of a regular decagon inscribed in the circle; CD is the side of the inscribed regular hexagon, and is therefore equal to the radius BE or EC .

Therefore, in order to prove our theorem, we have only to show that *the triangles EBC, DBE are similar.*

Since BC is the side of a regular decagon;

$$(\text{arc } BCA) = 5 (\text{arc } BC),$$

so that

$$(\text{arc } CFA) = 4 (\text{arc } BC),$$

whence

$$\angle CEA = 4 \angle BEC.$$

But

$$\angle CEA = 2 \angle ECB.$$

Therefore

$$\angle ECB = 2 \angle BEC \dots\dots\dots(1).$$

But, since $CD = CE$,

$$\angle CDE = \angle CED,$$

so that

$$\angle ECB = 2 \angle CDE.$$

It follows from (1) that $\angle BEC = \angle CDE$.

Now, in the $\triangle$ s EBC , DBE ,

$$\angle BEC = \angle BDE,$$

and

$$\angle EBC \text{ is common,}$$

so that

$$\angle ECB = \angle DEB,$$

and

$$\triangle\text{s } EBC, DBE \text{ are similar.}$$

Hence

$$DB : BE = EB : BC,$$

or

$$\begin{aligned} (\text{rect. } DB, BC) &= (\text{sq. on } EB) \\ &= (\text{sq. on } CD), \end{aligned}$$

and DB is divided at C in extreme and mean ratio.

To find the side of the decagon algebraically in terms of the radius we have, if x be the side required,

$$(r + x)x = r^2,$$

whence

$$x = \frac{r}{2} (\sqrt{5} - 1).$$

PROPOSITION 10.

If an equilateral pentagon be inscribed in a circle, the square on the side of the pentagon is equal to the squares on the side of the hexagon and on that of the decagon inscribed in the same circle.

Let $ABCDE$ be a circle,

and let the equilateral pentagon $ABCDE$ be inscribed in the circle $ABCDE$.

I say that the square on the side of the pentagon $ABCDE$ is equal to the squares on the side of the hexagon and on that of the decagon inscribed in the circle $ABCDE$.

For let the centre of the circle, the point F , be taken,

let AF be joined and carried through to the point G ,

let FB be joined,

let FH be drawn from F perpendicular to AB and be carried through to K ,

let AK , KB be joined,

let FL be again drawn from F perpendicular to AK , and be carried through to M ,

and let KN be joined.

Since the circumference $ABCG$ is equal to the circumference $AEDG$,

and in them ABC is equal to AED ,

therefore the remainder, the circumference CG , is equal to the remainder GD .

But CD belongs to a pentagon ;

therefore CG belongs to a decagon.

And, since FA is equal to FB ,

and FH is perpendicular,

therefore the angle AFK is also equal to the angle KFB .

[I. 5, I. 26]

Hence the circumference AK is also equal to KB ; [III. 26]
therefore the circumference AB is double of the circumference BK ;

therefore the straight line AK is a side of a decagon.

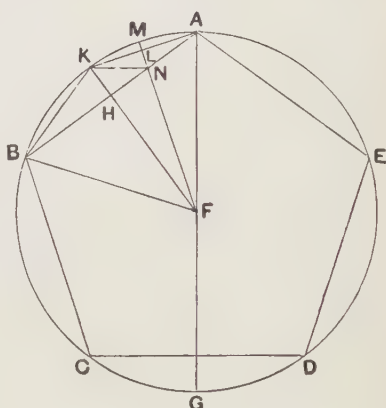
For the same reason
 AK is also double of KM .

Now, since the circumference AB is double of the circumference BK ,

while the circumference CD is equal to the circumference AB ,
therefore the circumference CD is also double of the circumference BK .

But the circumference CD is also double of CG ;
therefore the circumference CG is equal to the circumference BK .

But BK is double of KM , since KA is so also ;
therefore CG is also double of KM .



But, further, the circumference CB is also double of the circumference BK ,

for the circumference CB is equal to BA .

Therefore the whole circumference GB is also double of BM ;

hence the angle GFB is also double of the angle BFM . [vi. 33]

But the angle GFB is also double of the angle FAB ,
for the angle FAB is equal to the angle ABF .

Therefore the angle BFN is also equal to the angle FAB .

But the angle ABF is common to the two triangles ABF and BFN ;

therefore the remaining angle AFB is equal to the remaining angle BNF ; [i. 32]

therefore the triangle ABF is equiangular with the triangle BFN .

Therefore, proportionally, as the straight line AB is to BF ,
so is FB to BN ; [vi. 4]

therefore the rectangle AB, BN is equal to the square on BF . [vi. 17]

Again, since AL is equal to LK ,
while LN is common and at right angles,
therefore the base KN is equal to the base AN ; [i. 4]
therefore the angle LKN is also equal to the angle LAN .

But the angle LAN is equal to the angle KBN ;
therefore the angle LKN is also equal to the angle KBN .

And the angle at A is common to the two triangles AKB and AKN .

Therefore the remaining angle AKB is equal to the remaining angle KNA ; [i. 32]
therefore the triangle KBA is equiangular with the triangle KNA .

Therefore, proportionally, as the straight line BA is to AK , so is KA to AN ; [vi. 4]
therefore the rectangle BA, AN is equal to the square on AK . [vi. 17]

But the rectangle AB, BN was also proved equal to the square on BF ;

therefore the rectangle AB, BN together with the rectangle BA, AN , that is, the square on BA [II. 2], is equal to the square on BF together with the square on AK .

And BA is a side of the pentagon, BF of the hexagon [iv. 15, Por.], and AK of the decagon.

Therefore etc.

Q. E. D.

$ABCDE$ being a regular pentagon inscribed in a circle, and AG the diameter through A , it follows that

$$(\text{arc } CG) = (\text{arc } GD),$$

and CG, GD are sides of an inscribed regular decagon.

FHK being drawn perpendicular to AB , it follows, by I. 26, that $\angle s$ AFK, BFK are equal, and BK, KA are sides of the regular decagon.

Similarly it may be proved that, FLM being perpendicular to AK ,

$$AL = LK,$$

and $(\text{arc } AM) = (\text{arc } MK).$

The main facts to prove are that

(1) the triangles ABF, FBN are similar, and (2) the triangles ABK, AKN are similar.

$$(1) \quad 2(\text{arc } CG) = (\text{arc } CD)$$

$$= (\text{arc } AB)$$

$$= 2(\text{arc } BK),$$

$$\text{or} \quad (\text{arc } CG) = (\text{arc } BK) = (\text{arc } AK) \\ = 2(\text{arc } KM).$$

$$\text{And} \quad (\text{arc } CB) = 2(\text{arc } BK).$$

Therefore, by addition,

$$(\text{arc } BCG) = 2(\text{arc } BKM).$$

$$\text{Therefore} \quad \angle BFG = 2\angle BFN.$$

$$\text{But} \quad \angle BFG = 2\angle FAB,$$

$$\text{so that} \quad \angle FAB = \angle BFN.$$

Hence, in the $\triangle s$ ABF, FBN ,

$$\angle FAB = \angle BFN,$$

and $\angle ABF$ is common;

$$\text{therefore} \quad \angle AFB = \angle BNF,$$

and $\triangle s$ ABF, FBN are similar.

(2) Since $AL = LK$, and the angles at L are right,

$$AN = NK,$$

$$\text{and} \quad \angle NKA = \angle NAK$$

$$= \angle KBA.$$

Hence, in the $\triangle s$ ABK, AKN ,

$$\angle ABK = \angle AKN,$$

and $\angle KAN$ is common,

whence the third angles are equal;

therefore the triangles ABK, AKN are similar.

Now from the similarity of $\triangle s ABF, FBN$ it follows that

$$AB : BF = BF : BN,$$

or $(\text{rect. } AB, BN) = (\text{sq. on } BF).$

And, from the similarity of ABK, AKN ,

$$BA : AK = AK : AN,$$

or $(\text{rect. } BA, AN) = (\text{sq. on } AK).$

Therefore, by addition,

$$(\text{rect. } AB, BN) + (\text{rect. } BA, AN) = (\text{sq. on } BF) + (\text{sq. on } AK),$$

that is, $(\text{sq. on } AB) = (\text{sq. on } BF) + (\text{sq. on } AK).$

* If r be the radius of the circle, we have seen (XIII. 9, note) that

$$AK = \frac{r}{2} (\sqrt{5} - 1).$$

$$\text{Therefore } (\text{side of pentagon})^2 = r^2 + \frac{r^2}{4} (6 - 2\sqrt{5})$$

$$= \frac{r^2}{4} (10 - 2\sqrt{5}),$$

$$\text{so that } (\text{side of pentagon}) = \frac{r}{2} \sqrt{10 - 2\sqrt{5}}.$$

PROPOSITION 11.

If in a circle which has its diameter rational an equilateral pentagon be inscribed, the side of the pentagon is the irrational straight line called minor.

For in the circle $ABCDE$ which has its diameter rational let the equilateral pentagon $ABCDE$ be inscribed;

I say that the side of the pentagon is the irrational straight line called minor.

For let the centre of the circle, the point F , be taken, let AF, FB be joined and carried through to the points, G, H , let AC be joined,

and let FK be made a fourth part of AF .

Now AF is rational;

therefore FK is also rational.

But BF is also rational;

therefore the whole BK is rational.

And, since the circumference ACG is equal to the circumference ADG ,

and in them ABC is equal to AED ,

therefore the remainder CG is equal to the remainder GD .

And DC is double of LC , CM is half of CA , and FK a fourth part of FA ;

therefore, as DC is to CM , so is MF to FK .

Componendo also, as the sum of DC , CM is to CM , so is MK to KF ;

[v. 18]

therefore also, as the square on the sum of DC , CM is to the square on CM , so is the square on MK to the square on KF .

And since, when the straight line subtending two sides of the pentagon, as AC , is cut in extreme and mean ratio, the greater segment is equal to the side of the pentagon, that is, to DC ,

[XIII. 8]

while the square on the greater segment added to the half of the whole is five times the square on the half of the whole,

[XIII. 1]

and CM is half of the whole AC ,

therefore the square on DC , CM taken as one straight line is five times the square on CM .

But it was proved that, as the square on DC , CM taken as one straight line is to the square on CM , so is the square on MK to the square on KF ;

therefore the square on MK is five times the square on KF .

But the square on KF is rational,

for the diameter is rational;

therefore the square on MK is also rational;

therefore MK is rational.

And, since BF is quadruple of FK ,

therefore BK is five times KF ;

therefore the square on BK is twenty-five times the square on KF .

But the square on MK is five times the square on KF ;

therefore the square on BK is five times the square on KM ;

therefore the square on BK has not to the square on KM the ratio which a square number has to a square number;

therefore BK is incommensurable in length with KM . [x. 9]

And each of them is rational.

Therefore BK , KM are rational straight lines commensurable in square only.

But, if from a rational straight line there be subtracted a rational straight line which is commensurable with the whole in square only, the remainder is irrational, namely an apotome; therefore MB is an apotome and MK the annex to it. [x. 73]

I say next that MB is also a fourth apotome.

Let the square on N be equal to that by which the square on BK is greater than the square on KM ;
therefore the square on BK is greater than the square on KM by the square on N .

And, since KF is commensurable with FB ,
componendo also, KB is commensurable with FB . [x. 15]

But BF is commensurable with BH ;
therefore BK is also commensurable with BH . [x. 12]

And, since the square on BK is five times the square on KM ,
therefore the square on BK has to the square on KM the ratio which 5 has to 1.

Therefore, *convertendo*, the square on BK has to the square on N the ratio which 5 has to 4 [v. 19, Por.], and this is not the ratio which a square number has to a square number;
therefore BK is incommensurable with N ; [x. 9]

therefore the square on BK is greater than the square on KM by the square on a straight line incommensurable with BK .

Since then the square on the whole BK is greater than the square on the annex KM by the square on a straight line incommensurable with BK ,

and the whole BK is commensurable with the rational straight line, BH , set out,

therefore MB is a fourth apotome. [x. Def. III. 4]

But the rectangle contained by a rational straight line and a fourth apotome is irrational,

and its square root is irrational, and is called minor. [x. 94]

But the square on AB is equal to the rectangle HB, BM , because, when AH is joined, the triangle ABH is equiangular with the triangle ABM , and, as HB is to BA , so is AB to BM .

Therefore the side AB of the pentagon is the irrational straight line called minor.

Q. E. D.

Here we require certain definitions and propositions of Book x.

First we require the definition of an *apotome* [see x. 73], which is a straight line of the form $(\rho \sim \sqrt{k} \cdot \rho)$, where ρ is a "rational" straight line and k is any integer or numerical fraction, the square root of which is not integral or expressible in integers. The lesser of the straight lines $\rho, \sqrt{k} \cdot \rho$ is the *annex*.

Next we require the definition of the *fourth apotome* [x. Def. III. (after x. 84)], which is a straight line of the form $(x - y)$, where x, y (being both rational and commensurable in square only) are also such that $\sqrt{x^2 - y^2}$ is incommensurable with x , while x is commensurable with a given rational straight line ρ . As shown on x. 88 (note), the *fourth apotome* is of the form

$$\left(k\rho - \sqrt{1 + k}\rho\right).$$

Lastly the *minor* (straight line) is the irrational straight line defined in x. 76. It is of the form $(x - y)$, where x, y are incommensurable in square, and $(x^2 + y^2)$ is 'rational,' while xy is 'medial.' As shown in the note on x. 76, the *minor* irrational straight line is of the form

$$\frac{\rho}{\sqrt{2}} \sqrt{1 + \frac{k}{1+k^2}} - \frac{\rho}{\sqrt{2}} \sqrt{1 - \frac{k}{1+k^2}}.$$

The proposition may be put as follows. $ABCDE$ being a regular pentagon inscribed in a circle, AG, BH the diameters through A, B meeting CD in L and AC in M respectively, FK is made equal to $\frac{1}{4}AF$.

Now, the radius $AF(r)$ being rational, so are FK, BK .

The arcs CG, GD are equal;

hence $\angle$ s at L are right, and $CD = 2CL$

Similarly $\angle$ s at M are right, and $AC = 2CM$.

We have to prove

- (1) that BM is an apotome,
- (2) that BM is a fourth apotome,
- (3) that BA is a minor irrational straight line.

Remembering that, if CA is divided in extreme and mean ratio, the greater segment is equal to the side of the pentagon [xiii. 8], and that accordingly [xiii. 1] $(CD + \frac{1}{2}CA)^2 = 5(\frac{1}{2}CA)^2$, we work towards a proportion containing the ratio $(CD + CM)^2 : CM^2$, thus.

The Δ s ACL, AFM are equiangular and therefore similar.

Therefore

$$LC : CA = MF : FA,$$

and accordingly

$$2LC : CA = MF : \frac{1}{2}FA;$$

thus

$$2LC : \frac{1}{2}CA = MF : \frac{1}{4}FA,$$

or

$$DC : CM = MF : FK;$$

whence, *componendo*, and squaring,

$$(DC + CM)^2 : CM^2 = MK^2 : KF^2.$$

But

$$(DC + CM)^2 = 5CM^2;$$

therefore

$$MK^2 = 5KF^2.$$

[This means that

$$MK^2 = \frac{5}{16} r^2,$$

or

$$MK = \frac{\sqrt{5}}{4} r.]$$

It follows that, KF being rational, MK^2 , and therefore MK , is rational.

(1) To prove that BM is an *apotome* and MK its *annex*.

We have

$$BF = 4FK;$$

therefore

$$BK = 5FK,$$

$$BK^2 = 25FK^2$$

$$= 5MK^2, \text{ from above;}$$

therefore BK^2 has not to MK^2 the ratio of a square number to a square number;

therefore BK , MK are incommensurable in length.

They are therefore rational and commensurable in square only;

accordingly BM is an *apotome*.

$$[BK^2 = 5MK^2 = \frac{25}{16} r^2, \text{ and } BK = \frac{5}{4} r.]$$

$$\text{Consequently } BK - MK = \left(\frac{5}{4} r - \frac{\sqrt{5}}{4} r \right).]$$

(2) To prove that BM is a *fourth apotome*.

First, since KF , FB are commensurable,

BK , BF are commensurable, i.e. BK is commensurable with BH , a given rational straight line.

Secondly, if

$$N^2 = BK^2 - KM^2,$$

since

$$BK^2 : KM^2 = 5 : 1,$$

it follows that

$$BK^2 : N^2 = 5 : 4,$$

whence BK , N are incommensurable.

Therefore BM is a *fourth apotome*.

(3) To prove that BA is a *minor* irrational straight line.

If a fourth apotome form a rectangle with a rational straight line, the side of the square equivalent to the rectangle is *minor* [x. 94].

Now

$$BA^2 = HB \cdot BM,$$

HB is rational, and BM is a fourth apotome;

therefore BA is a *minor* irrational straight line.

$$[BA = r\sqrt{2} \cdot \sqrt{\frac{5}{4} - \frac{\sqrt{5}}{4}} = \frac{r}{2} \sqrt{10 - 2\sqrt{5}}.]$$

If this is separated into the difference between two straight lines, we have

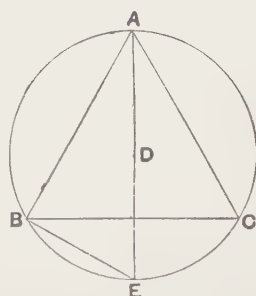
$$BA = \frac{r}{2} \sqrt{5 + 2\sqrt{5}} - \frac{r}{2} \sqrt{5 - 2\sqrt{5}}.]$$

PROPOSITION 12.

If an equilateral triangle be inscribed in a circle, the square on the side of the triangle is triple of the square on the radius of the circle.

Let ABC be a circle,
and let the equilateral triangle ABC be inscribed in it ;
I say that the square on one side of
the triangle ABC is triple of the square
on the radius of the circle.

For let the centre D of the circle
 ABC be taken,
let AD be joined and carried through
to E ,
and let BE be joined.



Then, since the triangle ABC is
equilateral,
therefore the circumference BEC is a third part of the circum-
ference of the circle ABC .

Therefore the circumference BE is a sixth part of the
circumference of the circle ;
therefore the straight line BE belongs to a hexagon ;
therefore it is equal to the radius DE . [iv. 15, Por.]

And, since AE is double of DE ,
the square on AE is quadruple of the square on ED , that is,
of the square on BE .

But the square on AE is equal to the squares on AB , BE ;
[III. 31, I. 47]
therefore the squares on AB , BE are quadruple of the square
on BE .

Therefore, *separando*, the square on AB is triple of the
square on BE .

But BE is equal to DE ;
therefore the square on AB is triple of the square on DE .

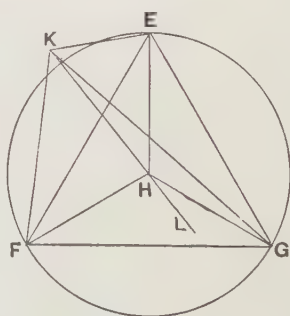
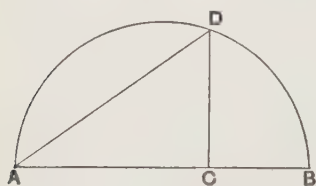
Therefore the square on the side of the triangle is triple
of the square on the radius.

Q. E. D.

PROPOSITION 13.

*To construct a pyramid, to comprehend it in a given sphere,
and to prove that the square on the diameter of the sphere is
one and a half times the square on the side of the pyramid.*

Let the diameter AB of the given sphere be set out,
 and let it be cut at the point C so that AC is double of CB ;
 let the semicircle ADB be described on AB ,
 let CD be drawn from the point C at right angles to AB ,
 and let DA be joined ;
 let the circle EFG which has its radius equal to DC be
 set out,
 let the equilateral triangle EFG be inscribed in the circle EFG ,
 [IV. 2]
 let the centre of the circle, the point H , be taken, [III. 1]
 let EH , HF , HG be joined ;
 from the point H let HK be set up at right angles to the plane
 of the circle EFG , [XI. 12]
 let HK equal to the straight line AC be cut off from HK ,
 and let KE , KF , KG be joined.



Now, since KH is at right angles to the plane of the circle EFG ,

therefore it will also make right angles with all the straight lines which meet it and are in the plane of the circle EFG .

[XI. Def. 3]

But each of the straight lines HE , HF , HG meets it :
 therefore HK is at right angles to each of the straight lines HE , HF , HG .

And, since AC is equal to HK , and CD to HE ,
 and they contain right angles,

therefore the base DA is equal to the base KE .

[I. 4]

For the same reason

each of the straight lines KF , KG is also equal to DA ;
therefore the three straight lines KE , KF , KG are equal to one another.

And, since AC is double of CB ,
therefore AB is triple of BC .

But, as AB is to BC , so is the square on AD to the square on DC , as will be proved afterwards.

Therefore the square on AD is triple of the square on DC .

But the square on FE is also triple of the square on EH ,
[XIII. 12]

and DC is equal to EH ;

therefore DA is also equal to EF .

But DA was proved equal to each of the straight lines KE , KF , KG ;

therefore each of the straight lines EF , FG , GE is also equal to each of the straight lines KE , KF , KG ;

therefore the four triangles EFG , KEF , KFG , KEG are equilateral.

Therefore a pyramid has been constructed out of four equilateral triangles, the triangle EFG being its base and the point K its vertex.

It is next required to comprehend it in the given sphere and to prove that the square on the diameter of the sphere is one and a half times the square on the side of the pyramid.

For let the straight line HL be produced in a straight line with KH ,

and let HL be made equal to CB .

Now, since, as AC is to CD , so is CD to CB , [vi. 8, Por.] while AC is equal to KH , CD to HE , and CB to HL ,

therefore, as KH is to HE , so is EH to HL ;

therefore the rectangle KH , HL is equal to the square on EH .
[vi. 17]

And each of the angles KHE , EHL is right;

therefore the semicircle described on KL will pass through E also.
[cf. vi. 8, III. 31.]

If then, KL remaining fixed, the semicircle be carried round and restored to the same position from which it began to be moved, it will also pass through the points F, G , since, if FL, LG be joined, the angles at F, G similarly become right angles;

and the pyramid will be comprehended in the given sphere.

For KL , the diameter of the sphere, is equal to the diameter AB of the given sphere; inasmuch as KH was made equal to AC , and HL to CB .

I say next that the square on the diameter of the sphere is one and a half times the square on the side of the pyramid

For, since AC is double of CB ,

therefore AB is triple of BC ;

and, *convertendo*, BA is one and a half times AC .

But, as BA is to AC , so is the square on BA to the square on AD .

Therefore the square on BA is also one and a half times the square on AD .

And BA is the diameter of the given sphere, and AD is equal to the side of the pyramid.

Therefore the square on the diameter of the sphere is one and a half times the square on the side of the pyramid.

Q. E. D.

LEMMA.

It is to be proved that, as AB is to BC , so is the square on AD to the square on DC .

For let the figure of the semicircle be set out,

let DB be joined,

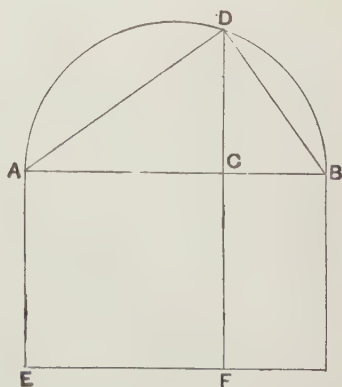
let the square EC be described on AC ,

and let the parallelogram FB be completed.

Since then, because the triangle DAB is equiangular with the triangle DAC ,

as BA is to AD , so is DA to AC ,

[VI. 8, VI. 4]



therefore the rectangle BA, AC is equal to the square on AD .
[VI. 17]

And since, as AB is to BC , so is EB to BF ,
[VI. 1]
and EB is the rectangle BA, AC , for EA is equal to AC ,
and BF is the rectangle AC, CB ,
therefore, as AB is to BC , so is the rectangle BA, AC to the
rectangle AC, CB .

And the rectangle BA, AC is equal to the square on AD ,
and the rectangle AC, CB to the square on DC ,
for the perpendicular DC is a mean proportional between the
segments AC, CB of the base, because the angle ADB is
right.
[VI. 8, Por.]

Therefore, as AB is to BC , so is the square on AD to
the square on DC .

Q. E. D.

The Lemma is with reason suspected. Euclid commonly takes more
difficult theorems for granted in the stereometrical Books. It is also clumsy
in itself, while, from a gloss in the proposition rejected as an interpolation, it
is clear that the interpolator of the gloss had not the Lemma. With the
Lemma should disappear the words "as will be proved afterwards" (p. 469).

In the figure of the proposition, the semicircle really represents half of
a section of the sphere through its centre and one edge of the inscribed
tetrahedron (AD being the length of that edge).

The proof is in three parts, the object of which is to prove

- (1) that $KEFG$ is a tetrahedron with all its edges equal to AD ,
- (2) that it is inscribable in a sphere of diameter equal to AB ,
- (3) that $AB^2 = \frac{3}{2} AD^2$.

To prove (1) we have to show

(a) that $KE = KF = KG = AD$,

(b) that $AD = EF$.

(a) Since $HE = HF = HG = CD$,
 $KH = AC$,

and $\angle s ACD, KHE, KHF, KHG$ are right,

$\triangle s ACD, KHE, KHF, KHG$ are equal in all respects;

therefore $KE = KF = KG = AD$.

(b) Since $AB = 3BC$,

and $AB : BC = AB \cdot AC : AC \cdot CB$
 $= AD^2 : CD^2$,

it follows that $AD^2 = 3CD^2$.

But [XIII. 12] $EF^2 = 3EH^2$;

and $EH = CD$, by construction

Therefore $AD = EF$.

Thus $EFGK$ is a *regular* tetrahedron.

(2) We now observe the usefulness of Euclid's description of a sphere [in XI. Def. 14].

Producing $KH (= AC)$ to L so that $HL = CB$,

we have KL equal to AB ;

thus KL is a diameter of the sphere which should circumscribe our tetrahedron,

and we have only to prove that E, F, G lie on semicircles described on KL as diameter.

E.g. for the point E ,

since $AC : CD = CD : CB$,

while $AC = KH, CD = HE, CB = HL$,

we have $KH : HE = HE : HL$,

or $KH \cdot HL = HE^2$,

whence, the angles KHE, EHL being right,

EKL is a triangle right-angled at E [cf. VI. 8].

Hence E lies on a semicircle on KL as diameter.

Similarly for F, G .

Thus a semicircle on KL as diameter revolving round KL passes successively through E, F, G .

(3) $AB = 3BC$;

therefore $BA = \frac{3}{2}AC$.

And $BA : AC = BA^2 : BA \cdot AC$
 $= BA^2 : AD^2$.

Therefore $BA^2 = \frac{3}{2}AD^2$.

If r be the radius of the circumscribed sphere,

$$(\text{edge of tetrahedron}) = \frac{2\sqrt{2}}{\sqrt{3}} \cdot r = \frac{2}{3}\sqrt{6} \cdot r.$$

It will be observed that, although in these cases Euclid's construction is equivalent to inscribing the particular regular solid in a given sphere, he does not actually construct the solid *in* the sphere but constructs a solid which a sphere *equal* to the given sphere will circumscribe. Pappus, on the other hand, in dealing with the same problems, actually constructs the respective solids in the given spheres. His method is to find circular sections in the given spheres containing a certain number of the angular points of the given solids. His solutions are interesting, although they require a knowledge of some properties of a sphere which are of course not found in the *Elements* but belonged to treatises such as the *Sphaerica* of Theodosius.

Pappus' solution of the problem of Eucl. XIII. 13.

In order to inscribe a regular pyramid or tetrahedron in a given sphere, Pappus (III. pp. 142—144) finds two circular sections equal and parallel to one another, each of which contains one of two opposite edges as its diameter. In this and the other similar problems he proceeds in the orthodox manner by

analysis and synthesis. The following is a reproduction of his solution of this case.

Analysis.

Suppose the problem solved, A, B, C, D being the angular points of the required pyramid.

Through A draw EF parallel to CD ; this will make equal angles with AC, AD ; and, since AB does so too, EF is perpendicular to AB [Pappus has a lemma for this, p. 140, 12—24], and is therefore a tangent to the sphere (for EF is parallel to CD , the base of the triangle ACD , and therefore touches the circle circumscribing it, while it also touches the circular section AB made by the plane passing through AB and EF perpendicular to it).

Similarly GH drawn through D parallel to AB touches the sphere.

And the plane through GH, CD makes a circular section equal and parallel to AB .

Through the centre K of that circular section, and in the plane of the section, draw LM perpendicular to CD and therefore parallel to AB . Join BL, BM .

BM is then perpendicular to AB, LM , and LB is a diameter of the sphere. Join MC .

Then

$$LM^2 = 2MC^2,$$

and

$$BC = AB = LM,$$

so that

$$BC^2 = 2MC^2.$$

And BM , being perpendicular to the plane of the circle LM , is perpendicular to CM ,

whence

$$BC^2 = BM^2 + MC^2,$$

so that

$$BM = MC.$$

But

$$BC = LM;$$

therefore

$$LM^2 = 2BM^2.$$

And, since the angle LMB is right,

$$BL^2 = LM^2 + MB^2 = \frac{3}{2} LM^2.$$

Synthesis.

Draw two parallel circular sections of the sphere with diameter d' , such that

$$d'^2 = \frac{2}{3} d^2,$$

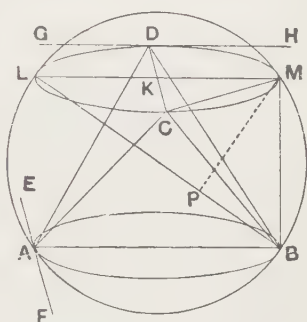
where d is the diameter of the sphere.

[This is easily done by dividing BL , any diameter of the sphere, at P , so that $LP = 2PB$, and then drawing PM at right angles to LB meeting the great circle LMB of the sphere in M . Then $LM^2 : LB^2 = LP : LB = 2 : 3$.]

Draw sections through M, B perpendicular to MB , and in these sections respectively draw the parallel diameters LM, AB .

Lastly, in the section LM draw CD through the centre K perpendicular to LM .

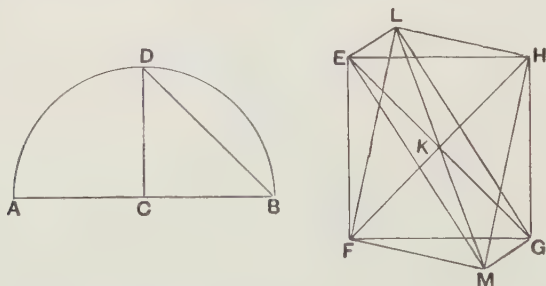
$ABCD$ is then the required regular pyramid or tetrahedron.



PROPOSITION 14.

To construct an octahedron and comprehend it in a sphere, as in the preceding case; and to prove that the square on the diameter of the sphere is double of the square on the side of the octahedron.

Let the diameter AB of the given sphere be set out,
and let it be bisected at C ;
let the semicircle ADB be described on AB ,
let CD be drawn from C at right angles to AB ,
let DB be joined;
let the square $EFGH$, having each of its sides equal to DB ,
be set out,
let HF , EG be joined,
from the point K let the straight line KL be set up at right
angles to the plane of the square $EFGH$ [XI. 12], and let it be
carried through to the other side of the plane, as KM ;
from the straight lines KL , KM let KL , KM be respectively
cut off equal to one of the straight lines EK , FK , GK , HK ,
and let LE , LF , LG , LH , ME , MF , MG , MH be joined.



Then, since KE is equal to KH ,
and the angle EKH is right,
therefore the square on HE is double of the square on EK .

[1. 47]

Again, since LK is equal to KE ,
and the angle LKE is right,
therefore the square on EL is double of the square on EK .

[*id.*]

But the square on HE was also proved double of the square on EK ;

therefore the square on LE is equal to the square on EH ;

therefore LE is equal to EH .

For the same reason

LH is also equal to HE ;

therefore the triangle LEH is equilateral.

Similarly we can prove that each of the remaining triangles of which the sides of the square $EFGH$ are the bases, and the points L, M the vertices, is equilateral;

therefore an octahedron has been constructed which is contained by eight equilateral triangles.

It is next required to comprehend it in the given sphere, and to prove that the square on the diameter of the sphere is double of the square on the side of the octahedron.

For, since the three straight lines LK, KM, KE are equal to one another,

therefore the semicircle described on LM will also pass through E .

And for the same reason,

if, LM remaining fixed, the semicircle be carried round and restored to the same position from which it began to be moved,

it will also pass through the points F, G, H ,

and the octahedron will have been comprehended in a sphere.

I say next that it is also comprehended in the given sphere.

For, since LK is equal to KM ,

while KE is common,

and they contain right angles,

therefore the base LE is equal to the base EM . [I. 4]

And, since the angle LEM is right, for it is in a semicircle, [III. 31]

therefore the square on LM is double of the square on LE . [I. 47]

Again, since AC is equal to CB ,

AB is double of BC .

But, as AB is to BC , so is the square on AB to the square on BD ;

therefore the square on AB is double of the square on BD .

But the square on LM was also proved double of the square on LE .

And the square on DB is equal to the square on LE , for EH was made equal to DB .

Therefore the square on AB is also equal to the square on LM ;

therefore AB is equal to LM .

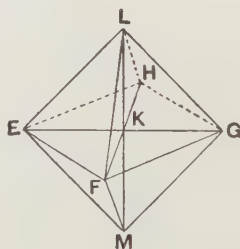
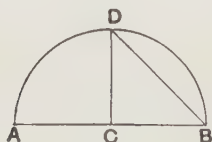
And AB is the diameter of the given sphere ;
therefore LM is equal to the diameter of the given sphere.

Therefore the octahedron has been comprehended in the given sphere, and it has been demonstrated at the same time that the square on the diameter of the sphere is double of the square on the side of the octahedron.

Q. E. D.

I think the accompanying figure will perhaps be clearer than that in Euclid's text.

$EFGH$ being a square with side equal to BD , it follows that KE, KF, KG, KH are all equal to CB .



So are KL, KM , by construction ;
hence LE, LF, LG, LH and ME, MF, MG, MH are all equal to EF or BD .

Thus (1) the figure is made up of eight equilateral triangles and is therefore a regular octahedron.

(2) Since $KE = KL = KM$,
the semicircle on LM in the plane LKE passes through E .

Similarly F, G, H lie on semicircles on LM as diameter.

Thus all the vertices of the tetrahedron lie on the sphere of which LM is a diameter.

(3) $LE = EM = BD$;
therefore $LM^2 = 2EL^2 = 2BD^2$
 $= AB^2$,
or $LM = AB$.

(4)

$$AB^2 = 2BD^2$$

$$= 2EF^2.$$

If r be the radius of the circumscribed sphere,

$$(\text{edge of octahedron}) = \sqrt{2} \cdot r.$$

Pappus' method.

Pappus (III. pp. 148—150) finds the two equal and parallel sections of the sphere which circumscribe two opposite faces of the octahedron thus.

Analysis.

Suppose the octahedron inscribed, $A, B, C; D, E, F$ being the vertices.

Through ABC, DEF describe planes
making the circular sections ABC, DEF .

Since the straight lines DA, DB, DE, DF are equal, the points A, E, F, B lie on a circle of which D is the pole.

Again, since AB, BF, FE, EA are equal, $ABFE$ is a square inscribed in the said circle, and AB, EF are parallel.

Similarly DE is parallel to BC , and DF to AC .

Therefore the circles through D, E, F and A, B, C are parallel; and they are also equal because the equilateral triangles inscribed in them are equal.

Now, ABC, DEF being equal and parallel circular sections, and AB, EF equal and parallel chords "not on the same side of the centres,"

AF is a diameter of the sphere.

[Pappus has a lemma for this, pp. 136—138].

And $AE = EF$, so that $AF^2 = 2FE^2$.

But, if d' be the diameter of the circle DEF ,

$$d'^2 = \frac{4}{3} EF^2.$$

[cf. XIII. 12]

Therefore, if d be the diameter of the sphere,

$$d^2 : d'^2 = 3 : 2.$$

Now d is given, and therefore d' is given; hence the circles DEF, ABC are given.

Synthesis.

Draw two equal and parallel circular sections with diameter d' , such that

$$d^2 = \frac{3}{2} d'^2,$$

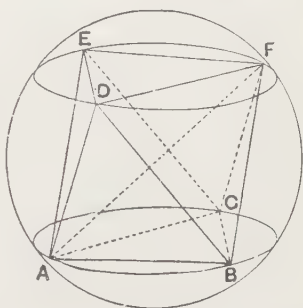
where d is the diameter of the sphere.

Inscribe an equilateral triangle ABC in either circle (ABC).

In the other circle draw EF equal and parallel to AB but on the opposite side of the centre, and complete the inscribed equilateral triangle DEF .

$ABCDEF$ is the octahedron required.

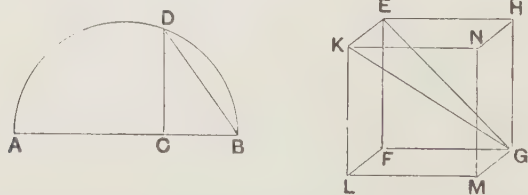
It will be observed that, whereas in the problem of XIII. 13 Euclid first finds the circle circumscribing a face and Pappus first finds an edge, in this problem Euclid finds the edge first and Pappus the circle circumscribing a face.



PROPOSITION 15.

To construct a cube and comprehend it in a sphere, like the pyramid; and to prove that the square on the diameter of the sphere is triple of the square on the side of the cube.

Let the diameter AB of the given sphere be set out, and let it be cut at C so that AC is double of CB ; let the semicircle ADB be described on AB , let CD be drawn from C at right angles to AB , and let DB be joined; let the square $EFGH$ having its side equal to DB be set out, from E, F, G, H let EK, FL, GM, HN be drawn at right angles to the plane of the square $EFGH$, from EK, FL, GM, HN let EK, FL, GM, HN respectively be cut off equal to one of the straight lines EF, FG, GH, HE , and let KL, LM, MN, NK be joined; therefore the cube FN has been constructed which is contained by six equal squares.



It is then required to comprehend it in the given sphere, and to prove that the square on the diameter of the sphere is triple of the square on the side of the cube.

For let KG, EG be joined.

Then, since the angle KEG is right, because KE is also at right angles to the plane EG and of course to the straight line EG also,

[xi. Def. 3]

therefore the semicircle described on KG will also pass through the point E .

Again, since GF is at right angles to each of the straight lines FL, FE ,

GF is also at right angles to the plane FK ;

hence also, if we join FK, GF will be at right angles to FK ;

and for this reason again the semicircle described on GK will also pass through F .

Similarly it will also pass through the remaining angular points of the cube.

If then, KG remaining fixed, the semicircle be carried round and restored to the same position from which it began to be moved,

the cube will be comprehended in a sphere.

I say next that it is also comprehended in the given sphere.

For, since GF is equal to FE ,

and the angle at F is right,

therefore the square on EG is double of the square on EF .

But EF is equal to EK ;

therefore the square on EG is double of the square on EK ;

hence the squares on GE , EK , that is the square on GK [I. 47], is triple of the square on EK .

And, since AB is triple of BC ,

while, as AB is to BC , so is the square on AB to the square on BD .

therefore the square on AB is triple of the square on BD .

But the square on GK was also proved triple of the square on KE .

And KE was made equal to DB ;

therefore KG is also equal to AB .

And AB is the diameter of the given sphere;

therefore KG is also equal to the diameter of the given sphere.

Therefore the cube has been comprehended in the given sphere; and it has been demonstrated at the same time that the square on the diameter of the sphere is triple of the square on the side of the cube.

Q. E. D.

AB is divided so that $AC = 2CB$; CD is drawn at right angles to AB , and BD is joined.

KG is, by construction, a cube of side equal to BD .

To prove (1) that it is inscribable in a sphere.

Since KE is perpendicular to EH , EF ,

KE is perpendicular to EG .

Thus, KEG being a right angle, E lies on a semicircle with diameter KG . The same thing is proved in the same way of the other vertices F, H, L, M, N .

Thus the cube is inscribed in the sphere of which KG is a diameter.

$$\begin{aligned}
 (2) \quad KG^2 &= KE^2 + EG^2 \\
 &= KE^2 + 2EF^2 \\
 &= 3EK^2.
 \end{aligned}$$

Also

$$AB = 3BC,$$

while

$$\begin{aligned}
 AB : BC &= AB^2 : AB \cdot BC \\
 &= AB^2 : BD^2;
 \end{aligned}$$

therefore

$$AB^2 = 3BD^2.$$

But

$$BD = EK;$$

therefore

$$KG = AB.$$

(3)

$$\begin{aligned}
 AB^2 &= 3BD^2 \\
 &= 3KE^2.
 \end{aligned}$$

If r be the radius of the circumscribed sphere,

$$(\text{edge of cube}) = \frac{2}{\sqrt{3}} \cdot r = \frac{2}{3} \sqrt{3} \cdot r.$$

Pappus' solution.

In this case too Pappus (III. pp. 144—148) gives the full analysis and synthesis.

Analysis.

Suppose the problem solved, and let the vertices of the cube be A, B, C, D, E, F, G, H .

Draw planes through A, B, C, D and E, F, G, H respectively; these will produce parallel circular sections, which are also equal since the inscribed squares are equal.

And CE will be a diameter of the sphere.

Join EG .

Now, since $EG^2 = 2EH^2 = 2GC^2$,

and the angle CGE is right,

$$CE^2 = GC^2 + EG^2 = \frac{3}{2}EG^2.$$

But CE^2 is given;

therefore EG^2 is given, so that the circles

$EFGH, ABCD$, and the squares inscribed in them, are given.

Synthesis.

Draw two parallel circular sections with equal diameters d' , such that

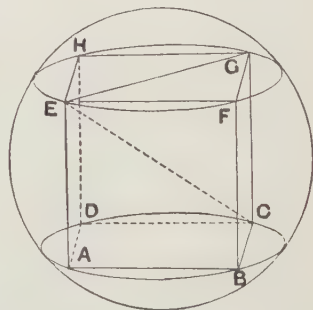
$$d^2 = \frac{3}{2}d'^2,$$

where d is the diameter of the given sphere.

Inscribe a square in one of the circles, as $ABCD$.

In the other circle draw FG equal and parallel to BC , and complete the square on FG inscribed in the circle $EFGH$.

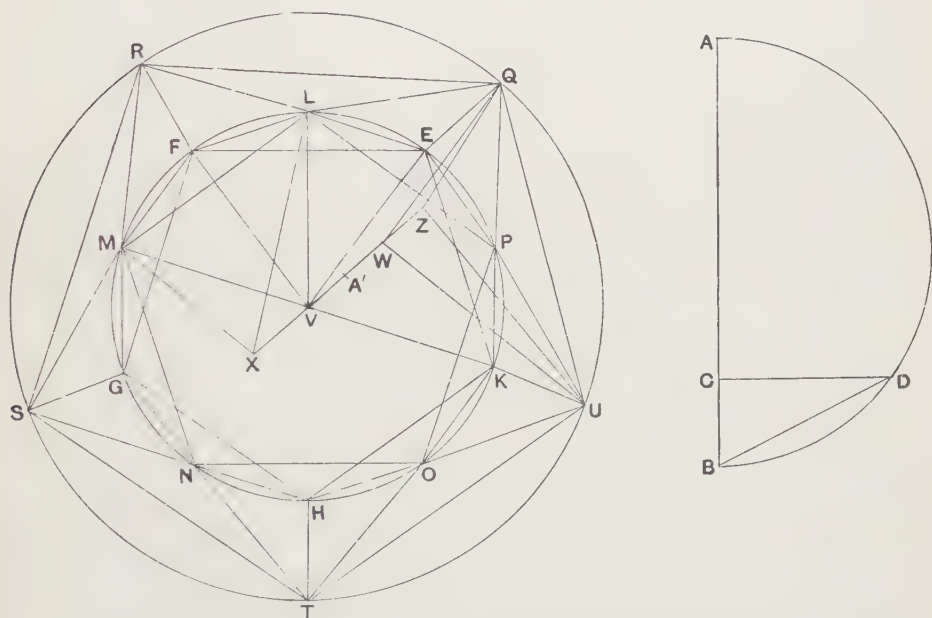
The eight vertices of the required cube are thus determined.



PROPOSITION 16.

To construct an icosahedron and comprehend it in a sphere, like the aforesaid figures; and to prove that the side of the icosahedron is the irrational straight line called minor.

Let the diameter AB of the given sphere be set out, and let it be cut at C so that AC is quadruple of CB , let the semicircle ADB be described on AB , let the straight line CD be drawn from C at right angles to AB , and let DB be joined;



let the circle $EFGHK$ be set out and let its radius be equal to DB ,

let the equilateral and equiangular pentagon $EFGHK$ be inscribed in the circle $EFGHK$,

let the circumferences EF , FG , GH , HK , KE be bisected at the points L , M , N , O , P ,

and let LM , MN , NO , OP , PL , EP be joined.

Therefore the pentagon $LMNOP$ is also equilateral, and the straight line EP belongs to a decagon.

Now from the points E, F, G, H, K let the straight lines EQ, FR, GS, HT, KU be set up at right angles to the plane of the circle, and let them be equal to the radius of the circle $EFGHK$,

let $QR, RS, ST, TU, UQ, QL, LR, RM, MS, SN, NT, TO, OU, UP, PQ$ be joined.

Now, since each of the straight lines EQ, KU is at right angles to the same plane, therefore EQ is parallel to KU . [xi. 6]

But it is also equal to it ;

and the straight lines joining those extremities of equal and parallel straight lines which are in the same direction are equal and parallel. [i. 33]

Therefore QU is equal and parallel to EK .

But EK belongs to an equilateral pentagon ;

therefore QU also belongs to the equilateral pentagon inscribed in the circle $EFGHK$.

For the same reason

each of the straight lines QR, RS, ST, TU also belongs to the equilateral pentagon inscribed in the circle $EFGHK$; therefore the pentagon $QRSTU$ is equilateral.

And, since QE belongs to a hexagon,

and EP to a decagon,

and the angle QEP is right,

therefore QP belongs to a pentagon ;

for the square on the side of the pentagon is equal to the square on the side of the hexagon and the square on the side of the decagon inscribed in the same circle. [xiii. 10]

For the same reason

PU is also a side of a pentagon.

But QU also belongs to a pentagon ;

therefore the triangle QPU is equilateral.

For the same reason

each of the triangles QLR, RMS, SNT, TOU is also equilateral.

And, since each of the straight lines QL , QP was proved to belong to a pentagon,
and LP also belongs to a pentagon,
therefore the triangle QLP is equilateral.

For the same reason
each of the triangles LRM , MSN , NTO , OUP is also equilateral.

Let the centre of the circle $EFGHK$. the point V , be taken;
from V let VZ be set up at right angles to the plane of the circle,
let it be produced in the other direction, as VX ,
let there be cut off VW , the side of a hexagon, and each of the straight lines VX , WZ , being sides of a decagon,
and let QZ , QW , UZ , EV , LV , LX , XM be joined.

Now, since each of the straight lines VW , QE is at right angles to the plane of the circle,
therefore VW is parallel to QE . [xi. 6]

But they are also equal;
therefore EV , QW are also equal and parallel. [i. 33]

But EV belongs to a hexagon;
therefore QW also belongs to a hexagon.

And, since QW belongs to a hexagon,
and WZ to a decagon,
and the angle QWZ is right,
therefore QZ belongs to a pentagon. [xiii. 10]

For the same reason
 UZ also belongs to a pentagon,
inasmuch as, if we join VK , WU , they will be equal and opposite, and VK , being a radius; belongs to a hexagon;
[iv. 15, Por.]
therefore WU also belongs to a hexagon.

But WZ belongs to a decagon,
and the angle UWZ is right;
therefore UZ belongs to a pentagon. [xiii. 10]

But QU also belongs to a pentagon;
therefore the triangle QUZ is equilateral.

For the same reason
each of the remaining triangles of which the straight lines QR , RS , ST , TU are the bases, and the point Z the vertex, is also equilateral.

Again, since VL belongs to a hexagon,
and VX to a decagon,
and the angle LVX is right,
therefore LX belongs to a pentagon. [XIII. 10]

For the same reason,
if we join MV , which belongs to a hexagon,
 MX is also inferred to belong to a pentagon.

But LM also belongs to a pentagon ;
therefore the triangle LMX is equilateral.

Similarly it can be proved that each of the remaining triangles of which MN , NO , OP , PL are the bases, and the point X the vertex, is also equilateral.

Therefore an icosahedron has been constructed which is contained by twenty equilateral triangles.

It is next required to comprehend it in the given sphere, and to prove that the side of the icosahedron is the irrational straight line called minor.

For, since VW belongs to a hexagon,
and WZ to a decagon,
therefore VZ has been cut in extreme and mean ratio at W ,
and VW is its greater segment ; [XIII. 9]
therefore, as ZV is to VW , so is VW to WZ .

But VW is equal to VE , and WZ to VX ;
therefore, as ZV is to VE , so is EV to VX .

And the angles ZVE , EVX are right ;
therefore, if we join the straight line EZ , the angle XEZ will be right because of the similarity of the triangles XEZ , VEZ .

For the same reason,
since, as ZV is to VW , so is VW to WZ ,
and ZV is equal to XW , and VW to WQ ,
therefore, as XW is to WQ , so is QW to WZ .

And for this reason again,
 if we join QX , the angle at Q will be right ; [vi. 8]
 therefore the semicircle described on XZ will also pass
 through Q . [iii. 31]

And if, XZ remaining fixed, the semicircle be carried
 round and restored to the same position from which it began
 to be moved, it will also pass through Q and the remaining
 angular points of the icosahedron,
 and the icosahedron will have been comprehended in a
 sphere.

I say next that it is also comprehended in the given sphere.
 For let VW be bisected at A' .

Then, since the straight line VZ has been cut in extreme
 and mean ratio at W ,

and ZW is its lesser segment,

therefore the square on ZW added to the half of the greater
 segment, that is WA' , is five times the square on the half
 of the greater segment ; [xiii. 3]

therefore the square on ZA' is five times the square on
 $A'W$.

And ZX is double of ZA' , and VW double of $A'W$;
 therefore the square on ZX is five times the square on
 WV .

And, since AC is quadruple of CB ,
 therefore AB is five times BC .

But, as AB is to BC , so is the square on AB to the square
 on BD ; [vi. 8, v. Def. 9]

therefore the square on AB is five times the square on BD .

But the square on ZX was also proved to be five times
 the square on VW .

And DB is equal to VW ,

for each of them is equal to the radius of the circle $EFGHK$;
 therefore AB is also equal to XZ .

And AB is the diameter of the given sphere ;
 therefore XZ is also equal to the diameter of the given sphere.

Therefore the icosahedron has been comprehended in the
 given sphere.

I say next that the side of the icosahedron is the irrational straight line called minor.

For, since the diameter of the sphere is rational, and the square on it is five times the square on the radius of the circle $EFGHK$,

therefore the radius of the circle $EFGHK$ is also rational ; hence its diameter is also rational.

But, if an equilateral pentagon be inscribed in a circle which has its diameter rational, the side of the pentagon is the irrational straight line called minor. [XIII. 11]

And the side of the pentagon $EFGHK$ is the side of the icosahedron.

Therefore the side of the icosahedron is the irrational straight line called minor.

PORISM. From this it is manifest that the square on the diameter of the sphere is five times the square on the radius of the circle from which the icosahedron has been described, and that the diameter of the sphere is composed of the side of the hexagon and two of the sides of the decagon inscribed in the same circle.

Q. E. D.

Euclid's method is

- (1) to find the pentagons in the two parallel circular sections of the sphere, the sides of which form ten (five in each circle) of the edges of the icosahedron,
- (2) to find the two points which are the poles of the two circular sections,
- (3) to prove that the triangles formed by joining the angular points of the pentagons which are nearest to one another two and two are equilateral,
- (4) to prove that the triangles of which the poles are the vertices and the sides of the pentagons the bases are also equilateral,
- (5) that all the angular points other than the poles lie on a sphere the diameter of which is the straight line joining the poles,
- (6) that this sphere is of the same size as the given sphere,
- (7) that, if the diameter of the sphere is rational, the edge of the icosahedron is the *minor* irrational straight line.

I have drawn another figure which will perhaps show the pentagons, and the position of the poles with regard to them, more clearly than does Euclid's figure.

- (1) If AB is the diameter of the given sphere, divide AB at C so that

$$AC = 4CB;$$

draw CD at right angles to AB meeting the semicircle on AB in D .

Join BD .

BD is the radius of the circular sections containing the pentagons.

[If r is the radius of the sphere,

since

$$AB : BC = AB^2 : AB \cdot BC \\ = AB^2 : BD^2,$$

while

$$AB = 5BC,$$

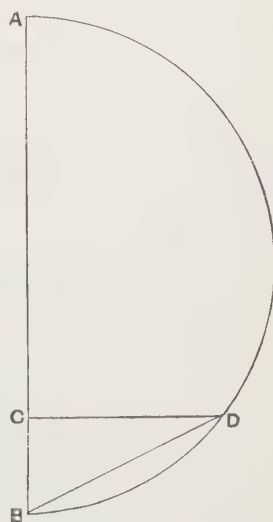
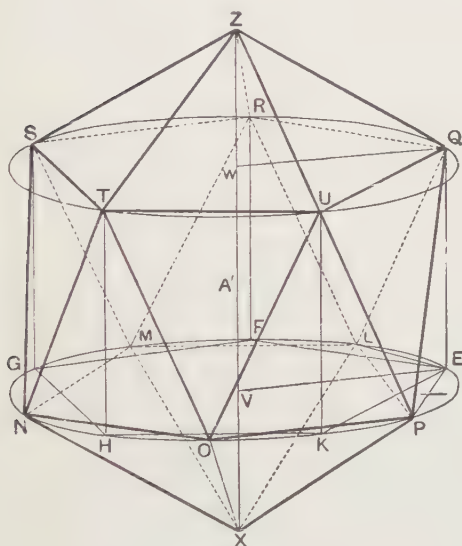
it follows that

$$AB^2 = 5BD^2,$$

or

$$(\text{radius of section})^2 = \frac{4}{5}r^2.$$

Thus [XIII. 10, note] $(\text{side of pentagon})^2 = \frac{r^2}{5} (10 - 2\sqrt{5}).$



Inscribe the regular pentagon $EFGHK$ in the circle $EFGHK$ of radius equal to BD .

Bisect the arcs EF , FG , ..., so forming a decagon in the circle.

Joining successive points of bisection, we obtain another regular pentagon $LMNOP$.

$LMNOP$ is one of the pentagons containing five edges of the icosahedron.

The other circle and inscribed pentagon are obtained by drawing perpendiculars from E , F , G , H , K to the plane of the circle, as EQ , FR , GS , HT , KU , and making each of these perpendiculars equal to the radius of the circle, or, as Euclid says, the side of the regular hexagon in it.

$QRSTU$ is the second pentagon (of course equal to the first) containing five edges of the icosahedron.

Joining each angular point of one of the two pentagons to the two nearest angular points in the other pentagon, we complete ten triangles each of which has for one side a side of one or other of the two pentagons.

V , W are the centres of the two circles, and VW is of course perpendicular to the planes of both.

(2) Produce VW in both directions, making VX and WZ both equal to a side of the regular decagon in the circle (as EL).

Joining X , Z to the angular points of the corresponding pentagons, we

have five more triangles formed with the sides of each pentagon as bases, ten more triangles in all.

Now we come to the proof.

(3) Taking two adjacent perpendiculars, EQ , KU , to the plane of the circle $EFGHK$, we see that they are parallel as well as equal; therefore QU , EK are equal and parallel.

Similarly for QR , EF etc.

Thus the pentagons have their sides equal.

To prove that the triangles QPL etc., are equilateral, we have, e.g.

$$\begin{aligned} QL^2 &= LE^2 + EQ^2 \\ &= (\text{side of decagon})^2 + (\text{side of hexagon})^2 \\ &= (\text{side of pentagon})^2, \end{aligned} \quad [\text{XIII. 10}]$$

$$\begin{aligned} \text{i.e. } QL &= (\text{side of pentagon in circle}) \\ &= LP. \end{aligned}$$

Similarly $QP = LP$,

and $\triangle QPL$ is equilateral.

So for the other triangles between the two pentagons.

(4) Since VW , EQ are equal and parallel,

VE , WQ are equal and parallel.

Thus WQ is equal to the side of a regular hexagon in the circles.

Now the angle ZWQ is right;

$$\begin{aligned} \text{therefore } ZQ^2 &= ZW^2 + WQ^2 \\ &= (\text{side of decagon})^2 + (\text{side of hexagon})^2 \\ &= (\text{side of pentagon})^2. \end{aligned} \quad [\text{XIII. 10}]$$

Thus ZQ , ZR , ZS , ZT , ZU are all equal to QR , RS etc.; and the triangles with Z as vertex and bases QR , RS etc. are equilateral.

Similarly for the triangles with X as vertex and LM , MN etc. as bases.

Hence the figure is an icosahedron, being contained by twenty equal equilateral triangles.

(5) To prove that all the vertices of the icosahedron lie on the sphere which has XZ for diameter.

VW being equal to the side of a regular hexagon, and WZ to the side of a regular decagon inscribed in the same circle,

VZ is divided at W in extreme and mean ratio. [XIII. 9]

$$\text{Therefore } ZV : VW = VW : WZ,$$

$$\text{or, since } VW = VE, \quad WZ = VX,$$

$$ZV : VE = VE : VX.$$

Thus E lies on the semicircle on ZX as diameter. [VI. 8]

Similarly for all the other vertices of the icosahedron.

Hence the sphere with diameter XZ circumscribes it.

(6) To prove $XZ = AB$.

Since VZ is divided in extreme and mean ratio at W , and VW is bisected at A' ,

$$A'Z^2 = 5A'W^2. \quad [\text{XIII. 3}]$$

Taking the doubles of $A'Z$, $A'W$, we have

$$XZ^2 = 5VW^2$$

$$= 5BD^2$$

$$= AB^2.$$

[see under (1) above]

That is, $XZ = AB$.

[If r is the radius of the sphere,

$$VW = BD = \frac{2}{\sqrt{5}} r,$$

$VX =$ (side of decagon in circle of radius BD)

$$= \frac{BD}{2} (\sqrt{5} - 1) \quad [\text{XIII. 9, note}]$$

$$= \frac{r}{\sqrt{5}} (\sqrt{5} - 1).$$

Consequently

$$XZ = VW + 2 VX$$

$$= \frac{2}{\sqrt{5}} r + \frac{2}{\sqrt{5}} r (\sqrt{5} - 1) \\ = 2r.]$$

(6) The radius of the circle $EFGHK$ is equal to $\frac{2}{\sqrt{5}} r$, and is therefore "rational" in Euclid's sense.

Hence the side of the inscribed pentagon is the irrational straight line called *minor*. [XIII. 11]

[The side of this pentagon is the edge of the icosahedron, and its value is (note on XIII. 10)

$$\frac{BD}{2} \sqrt{10 - 2\sqrt{5}} \\ = \frac{r}{\sqrt{5}} \sqrt{10 - 2\sqrt{5}} \\ = \frac{r}{5} \sqrt{10(5 - \sqrt{5})}.]$$

Pappus' solution.

This solution (Pappus, III. pp. 150—6) differs considerably from that of Euclid. Whereas Euclid uses *two* circular sections of the sphere (those circumscribing the pentagons of his construction), Pappus finds *four* parallel circular sections each passing through *three* of the vertices of the icosahedron; two of the circles are small circles circumscribing two opposite triangular faces respectively, and the other two circles are between these two circles, parallel to them and equal to one another.

Analysis.

Suppose the problem solved, the vertices of the icosahedron being $A, B, C; D, E, F; G, H, K; L, M, N$.

Since the straight lines BA, BC, BF, BG, BE drawn from B to the surface of the sphere are equal,

A, C, F, G, E are in one plane.

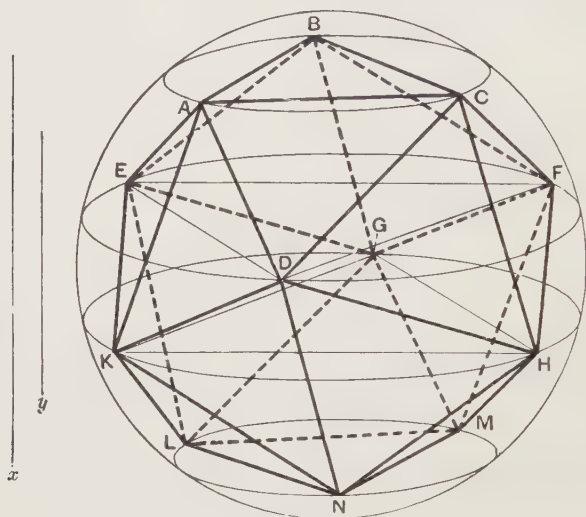
And AC, CF, FG, GE, EA are equal;

therefore $ACFGE$ is an equilateral and equiangular pentagon.

So are the figures $KEBCD, DHFBA, AKLG\beta, AKNHC$, and $CHMGB$.

Join EF, KH .

Now AC will be parallel to EF (in the pentagon $ACFGE$) and to KH (in the pentagon $AKNHC$), so that EF, KH are also parallel; and further KH is parallel to LM (in the pentagon $LKDHM$).



Similarly BC, ED, GH, LN are all parallel; and likewise BA, FD, GK, MN are all parallel.

Since BC is equal and parallel to LN , and BA to MN , the circles ABC, LMN are equal and parallel.

Similarly the circles DEF, KGH are equal and parallel; for the triangles inscribed in them are equal (since each of the sides in both is the chord subtending an angle of equal pentagons), and their sides are parallel respectively.

Now in the equal and parallel circles DEF, KGH the chords EF, KH are equal and parallel, and on opposite sides of the centres; therefore FK is a diameter of the sphere [Pappus' lemma, pp. 136—8], and the angle FEK is right [Pappus' lemma, p. 138, 20—26].

[The diameter FK is not actually drawn in the figure.]

In the pentagon $GEACF$, if EF be divided in extreme and mean ratio, the greater segment is equal to AC . [Eucl. XIII. 8]

Therefore $EF : AC = (\text{side of hexagon}) : (\text{side of decagon in same circle})$. [XIII. 9]

$$\text{And} \quad EF^2 + AC^2 = EF^2 + EK^2 = d^2,$$

where d is the diameter of the sphere.

Thus FK, EF, AC are as the sides of the pentagon, hexagon and decagon respectively inscribed in the same circle. [XIII. 10]

But FK , the diameter of the sphere, is given;

therefore EF, AC are given respectively;

thus the radii of the circles EFD, ACB are given (if r, r' are their radii, $r^2 = \frac{1}{3}EF^2, r'^2 = \frac{1}{3}AC^2$).

Hence the circles are given ;

and so are the circles KHG , LMN which are equal and parallel to them respectively.

Synthesis.

If d be the diameter of the sphere, set out two straight lines x, y , such that d, x, y are in the ratio of the sides of the pentagon, hexagon and decagon respectively inscribed in one and the same circle.

Draw (1) two equal and parallel circular sections in the sphere, with radii equal to r , where $r^2 = \frac{1}{3}x^2$, as DEF , KGH ,

and (2) two equal and parallel circular sections as ABC , LMN , with radius r' such that $r'^2 = \frac{1}{3}y^2$.

In the circles (1) draw EF , KH as sides of inscribed equilateral triangles, parallel to one another, and on opposite sides of the centres ;

and in the circles (2) draw AC , LM as sides of inscribed equilateral triangles parallel to one another and to EF , KH , and so that AC , EF are on opposite sides of the centres, and likewise KH , LM .

Complete the figure.

The correctness of the construction is proved as in the analysis.

It follows also (says Pappus) that

$$(\text{diam. of sphere})^2 = 3 (\text{side of pentagon in } DEF)^2.$$

For, by construction, $KF : FE = p : h$,

where p, h are the sides of the pentagon and hexagon inscribed in the same circle DEF .

And $FE : h$ = the ratio of the side of an equilateral triangle to that of a hexagon inscribed in the same circle ;

that is, $FE : h = \sqrt{3} : 1$,

whence $KF : p = \sqrt{3} : 1$,

or $KF^2 = 3p^2$.

Another construction.

Mr H. M. Taylor has a neat construction for an icosahedron of edge a .

Let l be the length of the diagonal of a regular pentagon with side equal to a .

Then (figure of XIII. 8), by Ptolemy's theorem,

$$l^2 = la + a^2.$$

Construct a cube with edge equal to l .

Let O be the centre of the cube.

From O draw OL , OM , ON perpendicular to three adjacent faces, and in these draw PP' , QQ' , RR' parallel to AB , AD , AE respectively.

Make LP , LP' , MQ , MQ' , NR , NR' all equal to $\frac{1}{2}a$.

Let p, p', q, q', r, r' be the reflexes of P, P', Q, Q', R, R' respectively.

Then will $P, P', Q, Q', R, R', p, p', q, q', r, r'$ be the vertices of a regular icosahedron.

The projections of PQ on AB , AD , AE are equal to $\frac{1}{2}(l - a)$, $\frac{1}{2}a$, $\frac{1}{2}l$ respectively.

Therefore

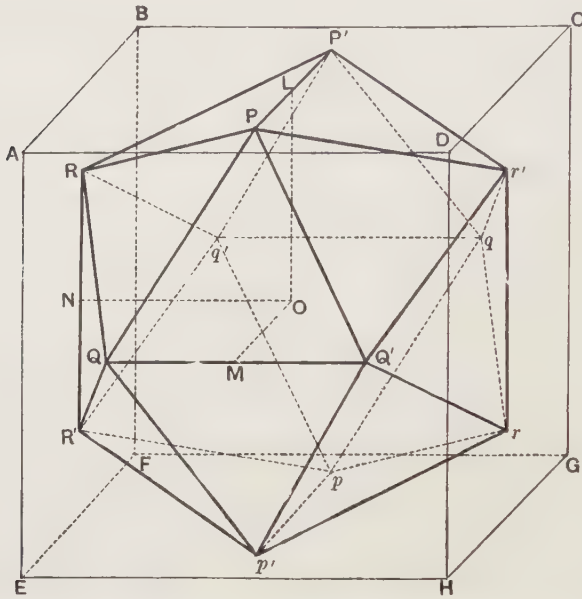
$$\begin{aligned} PQ^2 &= \frac{1}{4}(l - a)^2 + \frac{1}{4}a^2 + \frac{1}{4}l^2 \\ &= \frac{1}{2}(l^2 - al + a^2) \\ &= a^2. \end{aligned}$$

Therefore

$$PQ = a.$$

Similarly it may be proved that every other edge is equal to a .
All the angular points lie on a sphere with radius OP , and

$$OP^2 = \frac{1}{4} (a^2 + l^2).$$



Each solid pentahedral angle is composed of five equal plane angles, each of which is the angle of an equilateral triangle.

Therefore the icosahedron is regular.

$$[a^2 = 4OP^2 - l^2.$$

And, from the equation $l^2 = la + a^2$, we derive

$$l = \frac{a}{2} (\sqrt{5} + 1).$$

Therefore, if r be the radius of the sphere,

$$a^2 \left\{ 1 + \frac{(\sqrt{5} + 1)^2}{4} \right\} = 4r^2,$$

whence

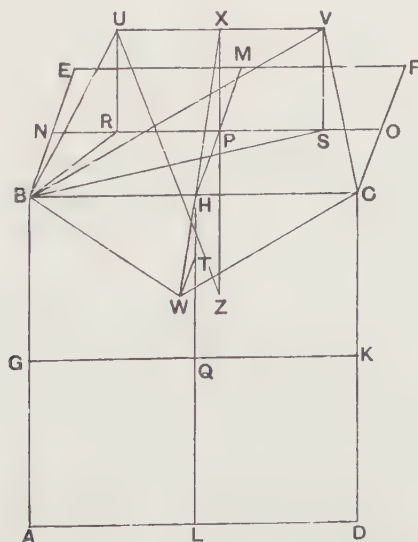
$$\begin{aligned} a &= 4r / \sqrt{10 + 2\sqrt{5}} \\ &= 4r \sqrt{10 - 2\sqrt{5}} / \sqrt{80} \\ &= \frac{r}{\sqrt{5}} \sqrt{10 - 2\sqrt{5}} \\ &= \frac{r}{5} \sqrt{10(5 - \sqrt{5})}, \end{aligned}$$

as above.]

PROPOSITION 17.

To construct a dodecahedron and comprehend it in a sphere, like the aforesaid figures, and to prove that the side of the dodecahedron is the irrational straight line called apotome.

Let $ABCD$, $CBEF$, two planes of the aforesaid cube be set out, at right angles to one another, be set out,
 let the sides AB , BC , CD , DA , EF , EB , FC be bisected at G , H , K , L , M , N , O respectively,
 let GK , HL , MH , NO be joined,
 let the straight lines NP , PO , HQ be cut in extreme and mean ratio at the points R , S , T respectively,
 and let RP , PS , TQ be their greater segments ;
 from the points R , S , T let RU , SV , TW be set up at right angles to the planes of the cube towards the outside of the cube,
 let them be made equal to RP , PS , TQ ,
 and let UB , BW , WC , CV , VU be joined.



I say that the pentagon $UBWCV$ is equilateral, and in one plane, and is further equiangular.
 For let RB , SB , VB be joined.

Then, since the straight line NP has been cut in extreme and mean ratio at R ,
and RP is the greater segment,
therefore the squares on PN , NR are triple of the square on RP . [XIII. 4]

But PN is equal to NB , and PR to RU ;
therefore the squares on BN , NR are triple of the square on RU .

But the square on BR is equal to the squares on BN , NR ;
[I. 47]
therefore the square on BR is triple of the square on RU ;
hence the squares on BR , RU are quadruple of the square on RU .

But the square on BU is equal to the squares on BR , RU ;
therefore the square on BU is quadruple of the square on RU ;
therefore BU is double of RU .

But VU is also double of UR ,
inasmuch as SR is also double of PR , that is, of RU ;
therefore BU is equal to UV .

Similarly it can be proved that each of the straight lines BW , WC , CV is also equal to each of the straight lines BU , UV .

Therefore the pentagon $BUVCW$ is equilateral.

I say next that it is also in one plane.

For let PX be drawn from P parallel to each of the straight lines RU , SV and towards the outside of the cube, and let XH , HW be joined;

I say that XHW is a straight line.

For, since HQ has been cut in extreme and mean ratio at T , and QT is its greater segment,
therefore, as HQ is to QT , so is QT to TH .

But HQ is equal to HP , and QT to each of the straight lines TW , PX ;
therefore, as HP is to PX , so is WT to TH .

And HP is parallel to TW ,
for each of them is at right angles to the plane BD ; [XI. 6]
and TH is parallel to PX ,
for each of them is at right angles to the plane BF . [*id.*]

But if two triangles, as XPH , HTW , which have two sides proportional to two sides be placed together at one angle so that their corresponding sides are also parallel, the remaining straight lines will be in a straight line; [vi. 32] therefore XH is in a straight line with HW .

But every straight line is in one plane; [xi. 1] therefore the pentagon $UBWCV$ is in one plane.

I say next that it is also equiangular.

For, since the straight line NP has been cut in extreme and mean ratio at R , and PR is the greater segment, while PR is equal to PS , therefore NS has also been cut in extreme and mean ratio at P ,

and NP is the greater segment; [xiii. 5]

therefore the squares on NS , SP are triple of the square on NP . [xiii. 4]

But NP is equal to NB , and PS to SV ; therefore the squares on NS , SV are triple of the square on NB ;

hence the squares on VS , SN , NB are quadruple of the square on NB .

But the square on SB is equal to the squares on SN , NB ; therefore the squares on BS , SV , that is, the square on BV —for the angle $VS B$ is right—is quadruple of the square on NB ;

therefore VB is double of BN .

But BC is also double of BN ; therefore BV is equal to BC .

And, since the two sides BU , UV are equal to the two sides BW , WC ,

and the base BV is equal to the base BC , therefore the angle BUV is equal to the angle BWC . [i. 8]

Similarly we can prove that the angle UVC is also equal to the angle BWC ;

therefore the three angles BWC , BUV , UVC are equal to one another.

But if in an equilateral pentagon three angles are equal to one another, the pentagon will be equiangular, [XIII. 7]
therefore the pentagon $BUVCW$ is equiangular.

And it was also proved equilateral;
therefore the pentagon $BUVCW$ is equilateral and equiangular, and it is on one side BC of the cube.

Therefore, if we make the same construction in the case of each of the twelve sides of the cube,
a solid figure will have been constructed which is contained by twelve equilateral and equiangular pentagons, and which is called a dodecahedron.

It is then required to comprehend it in the given sphere, and to prove that the side of the dodecahedron is the irrational straight line called apotome.

For let XP be produced, and let the produced straight line be XZ ;

therefore PZ meets the diameter of the cube, and they bisect one another,

for this has been proved in the last theorem but one of the eleventh book. [XI. 38]

Let them cut at Z ;

therefore Z is the centre of the sphere which comprehends the cube,

and ZP is half of the side of the cube.

Let UZ be joined.

Now, since the straight line NS has been cut in extreme and mean ratio at P ,

and NP is its greater segment,

therefore the squares on NS , SP are triple of the square on NP . [XIII. 4]

But NS is equal to XZ ,

inasmuch as NP is also equal to PZ , and XP to PS .

But further PS is also equal to XU ,

since it is also equal to RP ;

therefore the squares on ZX , XU are triple of the square on NP .

But the square on UZ is equal to the squares on ZX , XU ;
therefore the square on UZ is triple of the square on NP .

But the square on the radius of the sphere which comprehends the cube is also triple of the square on the half of the side of the cube,

for it has previously been shown how to construct a cube and comprehend it in a sphere, and to prove that the square on the diameter of the sphere is triple of the square on the side of the cube.

[XIII. 15]

But, if whole is so related to whole, so is half to half also; and NP is half of the side of the cube; therefore UZ is equal to the radius of the sphere which comprehends the cube.

And Z is the centre of the sphere which comprehends the cube;

therefore the point U is on the surface of the sphere.

Similarly we can prove that each of the remaining angles of the dodecahedron is also on the surface of the sphere; therefore the dodecahedron has been comprehended in the given sphere.

I say next that the side of the dodecahedron is the irrational straight line called apotome.

For since, when NP has been cut in extreme and mean ratio, RP is the greater segment,

and, when PO has been cut in extreme and mean ratio, PS is the greater segment,

therefore, when the whole NO is cut in extreme and mean ratio, RS is the greater segment.

[Thus, since, as NP is to PR , so is PR to RN ,

the same is true of the doubles also,

for parts have the same ratio as their equimultiples; [v. 15]

therefore as NO is to RS , so is RS to the sum of NR , SO .

But NO is greater than RS ;

therefore RS is also greater than the sum of NR , SO ;

therefore NO has been cut in extreme and mean ratio,

and RS is its greater segment.]

But RS is equal to UV ;

therefore, when NO is cut in extreme and mean ratio, UV is the greater segment.

And, since the diameter of the sphere is rational, and the square on it is triple of the square on the side of the cube,

therefore NO , being a side of the cube, is rational.

[But if a rational line be cut in extreme and mean ratio, each of the segments is an irrational apotome.]

Therefore UV , being a side of the dodecahedron, is an irrational apotome. [XIII. 6]

PORISM. From this it is manifest that, when the side of the cube is cut in extreme and mean ratio, the greater segment is the side of the dodecahedron.

Q. E. D.

In this proposition we find Euclid using two propositions which precede but are used nowhere else, notably VI. 32, which some authors, in consequence of their having overlooked its use here, have been hard put to it to explain.

Euclid's construction in this case is really identical with that given by Mr H. M. Taylor, and also referred to by Henrici and Treutlein under "crystal-formation."

Euclid starts from the cube inscribed in a sphere, as in XIII. 15, and then finds the side of the regular pentagon in which the side of the cube is a diagonal.

Mr Taylor takes l to be the diagonal of a regular pentagon of side a , so that, by Ptolemy's theorem,

$$l^2 = al + a^2,$$

constructs a cube of which l is the edge, and gets the side of the pentagon by drawing ZX from Z , the centre of the cube, perpendicular to the face BF and equal to $\frac{1}{2}(l + a)$, then drawing UV through X parallel to BC , and making UX , XV both equal to $\frac{1}{2}a$.

Euclid finds UV thus.

Draw NO , MH bisecting pairs of opposite sides in the square BF and meeting in P .

Draw GK , HL bisecting pairs of opposite sides in the square BD and meeting in Q .

Divide PN , PO , QH respectively in extreme and mean ratio at R , S , T (PR , PS , QT being the greater segments); draw RU , SV , TW outwards perpendicular to the respective faces of the cube, and all equal in length to PR , PS , TQ .

Join BU , UV , VC , CW , WB .

Then $BUVCW$ is one of the pentagonal faces of the dodecahedron; and the others can be constructed in the same way.

Euclid now proves

- (1) that the pentagon $BUVCW$ is equilateral,
- (2) that it is in one plane,
- (3) that it is equiangular,

(4) that the vertex U is on the sphere which circumscribes the cube, and hence

(5) that all the other vertices lie on the same sphere, and (6) that the side of the dodecahedron is an *apotome*.

(1) To prove that the pentagon $BUVCW$ is *equilateral*.

We have

$$BU^2 = BR^2 + RU^2$$

$$= (BN^2 + NR^2) + RP^2$$

$$= (PN^2 + NR^2) + RP^2$$

$$= 3RP^2 + RP^2$$

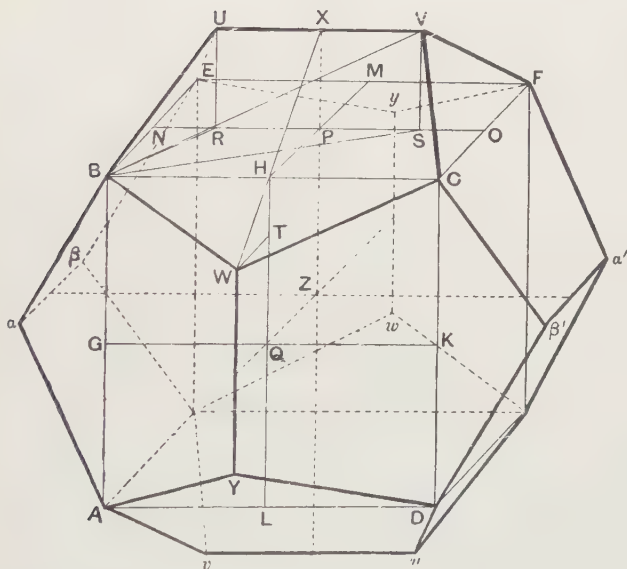
$$= 4RP^2$$

$$= UV^2.$$

[XIII. 4]

Therefore

$$BU = UV.$$



Similarly it may be proved that BW , WC , CV are all equal to UV or BU .

[Mr Taylor proceeds in this way. With his notation, the projections of BU on BA , BC , BE are respectively $\frac{1}{2}a$, $\frac{1}{2}(l-a)$, $\frac{1}{2}l$.

Therefore

$$\begin{aligned} BU^2 &= \frac{1}{4}a^2 + \frac{1}{4}(l-a)^2 + \frac{1}{4}l^2 \\ &= \frac{1}{2}(l^2 - al + a^2) \\ &= a^2. \end{aligned}$$

Similarly for BW , WC etc.]

(2) To prove that the pentagon $BUVCW$ is in one plane.

Draw PX parallel to RU or SV meeting UV in X .

Join XH , HW .

Then we have to prove that XH, HW are *in one straight line*.

Now HP, WT , being both perpendicular to the face BD , are parallel.

For the same reason XP, HT are parallel.

Also, since QH is divided at T in extreme and mean ratio,

$$QH : QT = QT : TH.$$

And

$$QH = HP, QT = WT = PX.$$

Therefore

$$HP : PX = WT : TH.$$

Consequently the triangles HPX, WTH satisfy the conditions of VI. 32 ; hence XHW is a straight line.

[Mr Taylor proves this as follows :

The projections of WH, WX on BE are $\frac{1}{2}a$ and $\frac{1}{2}(a+l)$,
and the projections of WH, WX on BA are $\frac{1}{2}(l-a)$ and $\frac{1}{2}l$;
and

$$a : (a+l) = (l-a) : l,$$

since

$$al = l^2 - a^2.$$

Therefore WHX is a straight line.]

(3) To prove that the pentagon $BUVCW$ is *equiangular*.

We have

$$\begin{aligned} BV^2 &= BS^2 + SV^2 \\ &= (BN^2 + NS^2) + SP^2 \\ &= PN^2 + (NS^2 + SP^2) \\ &= PN^2 + 3PN^2, \end{aligned}$$

since NS is divided in extreme and mean ratio at P [XIII. 5], so that

$$NS^2 + SP^2 = 3PN^2. \quad [\text{XIII. 4}]$$

Consequently

$$\begin{aligned} BV^2 &= 4PN^2 \\ &= BC^2, \end{aligned}$$

or

$$BV = BC.$$

The $\triangle s$ UBV, WBC are therefore equal in all respects,

and

$$\angle BUV = \angle BWC.$$

Similarly

$$\angle CVU = \angle BWC.$$

Therefore the pentagon is *equiangular*.

[XIII. 7]

(4) To prove that the sphere which circumscribes the cube also circumscribes the dodecahedron we have only to prove that, if Z be the centre of the sphere, $ZU = ZB$, for example.

Now, by XI. 38, XP produced meets the diagonal of the cube, and the portion of XP produced which is within the cube and the diagonal bisect one another.

And

$$\begin{aligned} ZU^2 &= ZX^2 + XU^2 \\ &= NS^2 + PS^2 \\ &= 3PN^2, \end{aligned}$$

as before.

Also (cf. XIII. 15)

$$\begin{aligned} ZB^2 &= ZP^2 + PB^2 \\ &= ZP^2 + PN^2 + NB^2 \\ &= 3PN^2. \end{aligned}$$

Hence

$$ZU = ZB.$$

(5) Similarly for ZV, ZW etc.

(6) Since PN is divided in extreme and mean ratio at R ,

$$NP : PR = PR : RN.$$

Doubling the terms, we have

$$NO : RS = RS : (NR + SO),$$

so that, if NO is divided in extreme and mean ratio, the greater segment is equal to RS .

Now, since the diameter of the sphere is rational,

and $(\text{diam. of sphere})^2 = 3 (\text{edge of cube})^2$,

the edge of the cube (i.e. NO) is rational.

Consequently RS is an *apotome*.

[This is proved in the spurious XIII. 6 above; Euclid assumes it, and the words purporting to quote the theorem are probably interpolated, like XIII. 6 itself.]

As a matter of fact, with Mr Taylor's notation,

$$l^2 = la + a^2,$$

and

$$a = \frac{\sqrt{5} - 1}{2} l.$$

Since, if r is the radius of the circumscribing sphere, $r = \sqrt{3} \cdot \frac{l}{2}$,

$$a = \frac{r}{\sqrt{3}} (\sqrt{5} - 1) = \frac{r}{3} (\sqrt{15} - \sqrt{3}).$$

Pappus' solution.

Here too Pappus (III. pp. 156—162) finds four circular sections of the sphere all parallel to one another and all passing through five of the vertices of the dodecahedron.

Analysis.

Suppose (he says) the problem solved, and let the vertices of the dodecahedron be $A, B, C, D, E; F, G, H, K, L; M, N, O, P, Q; R, S, T, U, V$.

Then, as before, FD is parallel to FL , and AE to FG ; therefore the planes $ABCDE, FGHL$ are parallel.

But, since PA is parallel to BH , and BH to OC , PA is parallel to OC ; and they are equal; therefore PO, AC are parallel, so that ST, ED are also parallel.

Similarly RS, DC are parallel, and likewise the pairs $(TU, EA), (UV, AB), (VR, BC)$.

Therefore the planes $ABCDE, RSTUV$ are parallel; and the circles $ABCDE, RSTUV$ are equal, since the inscribed pentagons are equal.

Similarly the circles $FGHL, MNOPQ$ are equal, since the pentagons inscribed in them are equal.

Now CL, OU are parallel because each is parallel to KN ; therefore L, C, O, U are in one plane.

And LC, CO, OU, UL are all equal, since they subtend angles of equal pentagons.

Also L, C, O, U are on a plane section, i.e. a circle;

therefore $LCOU$ is a square.

Therefore $OL^2 = 2LC^2 = 2LF^2$

(for LC, LF subtend angles of equal pentagons).

Thus the two circles are given, and so accordingly are the equal and parallel circular sections.

Synthesis.

Set out two straight lines x, y such that d, x, y are in the ratio of the sides of a regular pentagon, hexagon and decagon respectively inscribed in one and the same circle.

Find two circular sections of the sphere with radii r, r' , where

$$r^2 = \frac{1}{3}x^2, \quad r'^2 = \frac{1}{3}y^2.$$

Let these be the circles $FGHKL, ABCDE$ respectively, and draw the equal and parallel circles on the other side of the centre, namely $MNOPQ, RSTUV$.

In the first two circles inscribe regular pentagons with their sides respectively parallel, ED being parallel to FL .

Draw equal and parallel chords (on the other sides of the centres) in the other two circles, namely ST equal and parallel to ED , and PO equal and parallel to FL ; and complete the regular pentagons on ST, PO inscribed in the circles.

Thus all the vertices of the dodecahedron are determined.

The proof of the correctness of the construction is clear from the analysis.

Pappus adds that the construction shows that the circles containing five vertices of the dodecahedron are the same respectively as those containing three vertices of the icosahedron, and that the same circle circumscribes the triangle of the icosahedron and the pentagonal face of the dodecahedron in the same sphere.

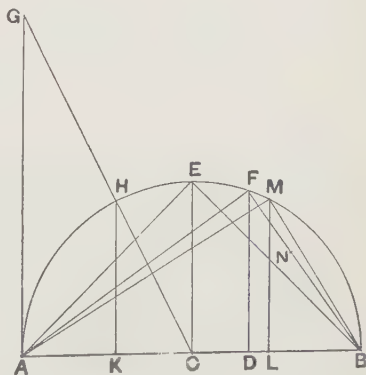
PROPOSITION 18.

To set out the sides of the five figures and to compare them with one another.

Let AB , the diameter of the given sphere, be set out, and let it be cut at C so that AC is equal to CB , and at D so that AD is double of DB ; let the semicircle AEB be described on AB , from C, D let CE, DF be drawn at right angles to AB , and let AF, FB, EB be joined.

Then, since AD is double of DB , therefore AB is triple of BD .

Convertendo, therefore, BA is one and a half times AD .



But, as BA is to AD , so is the square on BA to the square on AF , [v. Def. 9, vi. 8]

for the triangle AFB is equiangular with the triangle AFD ; therefore the square on BA is one and a half times the square on AF .

But the square on the diameter of the sphere is also one and a half times the square on the side of the pyramid.

[XIII. 13]

And AB is the diameter of the sphere;
therefore AF is equal to the side of the pyramid.

Again, since AD is double of DB ,
therefore AB is triple of BD .

But, as AB is to BD , so is the square on AB to the square on BF ; [vi. 8, v. Def. 9]

therefore the square on AB is triple of the square on BF .

But the square on the diameter of the sphere is also triple of the square on the side of the cube. [XIII. 15]

And AB is the diameter of the sphere;
therefore BF is the side of the cube.

And, since AC is equal to CB ,
therefore AB is double of BC .

But, as AB is to BC , so is the square on AB to the square on BE ;

therefore the square on AB is double of the square on BE .

But the square on the diameter of the sphere is also double of the square on the side of the octahedron. [XIII. 14]

And AB is the diameter of the given sphere;
therefore BE is the side of the octahedron.

Next, let AG be drawn from the point A at right angles to the straight line AB ,

let AG be made equal to AB ,

let GC be joined,

and from H let HK be drawn perpendicular to AB .

Then, since GA is double of AC ,

for GA is equal to AB ,

and, as GA is to AC , so is HK to KC ,

therefore HK is also double of KC .

Therefore the square on HK is quadruple of the square on KC ;

therefore the squares on HK , KC , that is, the square on HC , is five times the square on KC .

But HC is equal to CB ;

therefore the square on BC is five times the square on CK .

And, since AB is double of CB ,

and, in them, AD is double of DB ,

therefore the remainder BD is double of the remainder DC .

Therefore BC is triple of CD ;

therefore the square on BC is nine times the square on CD .

But the square on BC is five times the square on CK ;

therefore the square on CK is greater than the square on CD ;

therefore CK is greater than CD .

Let CL be made equal to CK ,

from L let LM be drawn at right angles to AB ,

and let MB be joined.

Now, since the square on BC is five times the square on CK ,

and AB is double of BC , and KL double of CK ,

therefore the square on AB is five times the square on KL .

But the square on the diameter of the sphere is also five times the square on the radius of the circle from which the icosahedron has been described. [XIII. 16, Por.]

And AB is the diameter of the sphere;

therefore KL is the radius of the circle from which the icosahedron has been described;

therefore KL is a side of the hexagon in the said circle.

[IV. 15, Por.]

And, since the diameter of the sphere is made up of the side of the hexagon and two of the sides of the decagon inscribed in the same circle, [XIII. 16, Por.]

and AB is the diameter of the sphere,

while KL is a side of the hexagon,

and AK is equal to LB ,

therefore each of the straight lines AK , LB is a side of the decagon inscribed in the circle from which the icosahedron has been described.

And, since LB belongs to a decagon, and ML to a hexagon,

for ML is equal to KL , since it is also equal to HK , being the same distance from the centre, and each of the straight lines HK , KL is double of KC ,

therefore MB belongs to a pentagon. [XIII. 10]

But the side of the pentagon is the side of the icosahedron; [XIII. 16]

therefore MB belongs to the icosahedron.

Now, since FB is a side of the cube,
let it be cut in extreme and mean ratio at N ,
and let NB be the greater segment;

therefore NB is a side of the dodecahedron. [XIII. 17, Por.]

And, since the square on the diameter of the sphere was proved to be one and a half times the square on the side AF of the pyramid, double of the square on the side BE of the octahedron and triple of the side FB of the cube,

therefore, of parts of which the square on the diameter of the sphere contains six, the square on the side of the pyramid contains four, the square on the side of the octahedron three, and the square on the side of the cube two.

Therefore the square on the side of the pyramid is four-thirds of the square on the side of the octahedron, and double of the square on the side of the cube;

and the square on the side of the octahedron is one and a half times the square on the side of the cube.

The said sides, therefore, of the three figures, I mean the pyramid, the octahedron and the cube, are to one another in rational ratios.

But the remaining two, I mean the side of the icosahedron and the side of the dodecahedron, are not in rational ratios either to one another or to the aforesaid sides; for they are irrational, the one being minor [XIII. 16] and the other an apotome [XIII. 17].

That the side MB of the icosahedron is greater than the side NB of the dodecahedron we can prove thus.

For, since the triangle FDB is equiangular with the triangle FAB , [VI. 8]
proportionally, as DB is to BF , so is BF to BA . [VI. 4]

And, since the three straight lines are proportional,
as the first is to the third, so is the square on the first to the
square on the second ;

[v. Def. 9, vi. 20, Por.]

therefore, as DB is to BA , so is the square on DB to the
square on BF ;

therefore, inversely, as AB is to BD , so is the square on FB
to the square on BD .

But AB is triple of BD ;

therefore the square on FB is triple of the square on BD .

But the square on AD is also quadruple of the square
on DB ,

for AD is double of DB ;

therefore the square on AD is greater than the square on FB ;

therefore AD is greater than FB ;

therefore AL is by far greater than FB .

And, when AL is cut in extreme and mean ratio,

KL is the greater segment,

inasmuch as LK belongs to a hexagon, and KA to a decagon ;

[xiii. 9]

and, when FB is cut in extreme and mean ratio, NB is the
greater segment ;

therefore KL is greater than NB .

But KL is equal to LM ;

therefore LM is greater than NB .

Therefore MB , which is a side of the icosahedron, is by
far greater than NB which is a side of the dodecahedron.

Q. E. D

I say next that *no other figure, besides the said five figures,
can be constructed which is contained by equilateral and equi-
angular figures equal to one another.*

For a solid angle cannot be constructed with two triangles,
or indeed planes.

With three triangles the angle of the pyramid is constructed,
with four the angle of the octahedron, and with five the angle
of the icosahedron ;

but a solid angle cannot be formed by six equilateral and equi-
angular triangles placed together at one point,

for, the angle of the equilateral triangle being two-thirds of a right angle, the six will be equal to four right angles : which is impossible, for any solid angle is contained by angles less than four right angles. [XI. 21]

For the same reason, neither can a solid angle be constructed by more than six plane angles.

By three squares the angle of the cube is contained, but by four it is impossible for a solid angle to be contained, for they will again be four right angles.

By three equilateral and equiangular pentagons the angle of the dodecahedron is contained ;

but by four such it is impossible for any solid angle to be contained,

for, the angle of the equilateral pentagon being a right angle and a fifth, the four angles will be greater than four right angles :

which is impossible.

Neither again will a solid angle be contained by other polygonal figures by reason of the same absurdity.

Therefore etc.

Q. E. D.

LEMMA.

But that *the angle of the equilateral and equiangular pentagon is a right angle and a fifth* we must prove thus.

Let $ABCDE$ be an equilateral and equiangular pentagon,

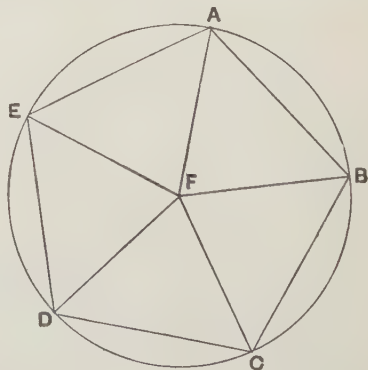
let the circle $ABCDE$ be circumscribed about it,

let its centre F be taken,

and let FA, FB, FC, FD, FE be joined.

Therefore they bisect the angles of the pentagon at A, B, C, D, E .

And, since the angles at F are equal to four right angles and are equal,



therefore one of them, as the angle AFB , is one right angle less a fifth;

therefore the remaining angles FAB , ABF consist of one right angle and a fifth.

But the angle FAB is equal to the angle FBC ;
therefore the whole angle ABC of the pentagon consists of one right angle and a fifth.

Q. E. D.

We have seen in the preceding notes that, if r be the radius of the sphere circumscribing the five solid figures,

$$(\text{edge of tetrahedron}) = \frac{2}{3}\sqrt{6} \cdot r,$$

$$(\text{edge of octahedron}) = \sqrt{2} \cdot r,$$

$$(\text{edge of cube}) = \frac{2}{3}\sqrt{3} \cdot r,$$

$$(\text{edge of icosahedron}) = \frac{r}{5}\sqrt{10(5-\sqrt{5})},$$

$$(\text{edge of dodecahedron}) = \frac{r}{3}(\sqrt{15} - \sqrt{3}).$$

Euclid here exhibits the edges of all the five regular solids in one figure.

(1) Make AD equal to $2DB$.

Thus

$$BA = \frac{3}{2}AD,$$

and

$$BA : AD = BA^2 : AF^2;$$

therefore

$$BA^2 = \frac{3}{2}AF^2.$$

Thus

$$AF = \sqrt{\frac{2}{3}} \cdot 2r = \frac{2}{3}\sqrt{6} \cdot r = (\text{edge of tetrahedron}).$$

(2)

$$AB^2 : BF^2 = AB : BD$$

$$= 3 : 1.$$

Therefore

$$BF^2 = \frac{1}{3}AB^2,$$

or

$$BF = \frac{2}{\sqrt{3}} \cdot r = \frac{2}{3}\sqrt{3} \cdot r = (\text{edge of cube}).$$

(3)

$$AB^2 = 2BE^2.$$

Therefore

$$BE = \sqrt{2} \cdot r = (\text{edge of octahedron}).$$

(4) Draw AG perpendicular and equal to AB . Join GC , meeting the semicircle in H , and draw HK perpendicular to AB .

Then

$$GA = 2AC;$$

therefore, by similar triangles, $HK = 2KC$.

Hence

$$HK^2 = 4KC^2,$$

and therefore

$$5KC^2 = HK^2 + KC^2$$

$$= HC^2$$

$$= CB^2.$$

Again, since $AB = 2CB$, and $AD = 2DB$,

by subtraction,

$$BD = 2DC,$$

or

$$BC = 3DC.$$

Therefore

$$\begin{aligned} 9DC^2 &= BC^2 \\ &= 5KC^2. \end{aligned}$$

Hence $KC > CD$.

Make CL equal to KC , draw LM at right angles to AB , and join AM , MB .

Since

$$\begin{aligned} CB^2 &= 5KC^2, \\ AB^2 &= 5KL^2. \end{aligned}$$

It follows that KL ($= \sqrt{\frac{4}{5}} \cdot r$) is the radius of, or the side of the regular hexagon in, the circle containing the pentagonal sections of the icosahedron.

[XIII. 16]

And, since

$$2r = (\text{side of hexagon}) + 2 (\text{side of decagon in same circle})$$

[XIII. 16, Por.]

$$AK = LB = (\text{side of decagon in the said circle}).$$

But

$$LM = HK = KL = (\text{side of hexagon in circle}).$$

$$\begin{aligned} \text{Therefore } LM^2 + LB^2 (= BM^2) &= (\text{side of pentagon in circle})^2 & [\text{XIII. 10}] \\ &= (\text{edge of icosahedron})^2, \end{aligned}$$

and

$$BM = (\text{edge of icosahedron}).$$

[More shortly,

$$HK = 2KC,$$

whence

$$HK^2 = 4KC^2,$$

and

$$5KC^2 = HC^2 = r^2.$$

Also

$$AK = r - CK = r \left(1 - \frac{1}{\sqrt{5}} \right).$$

Thus

$$\begin{aligned} BM^2 &= HK^2 + AK^2 \\ &= \frac{4}{5} r^2 + r^2 \left(1 - \frac{1}{\sqrt{5}} \right)^2 \\ &= r^2 \left(\frac{10}{5} - \frac{2}{\sqrt{5}} \right) \\ &= \frac{r^2}{5} (10 - 2\sqrt{5}), \end{aligned}$$

and

$$BM = \frac{r}{5} \sqrt{10(5 - \sqrt{5})} = (\text{edge of icosahedron}).]$$

(5) Cut BF (the edge of the cube) in extreme and mean ratio at N .

Then, if BN be the greater segment,

$$BN = (\text{edge of dodecahedron}).$$

[XIII. 17]

[Solving, we obtain

$$\begin{aligned} BN &= \frac{\sqrt{5-1}}{2} \cdot BF \\ &= \frac{\sqrt{5-1}}{2} \cdot \frac{2}{\sqrt{3}} \cdot r \\ &= \frac{r}{3} (\sqrt{15} - \sqrt{3}) \\ &= (\text{edge of dodecahedron}).] \end{aligned}$$

(6) If t , o , c are the edges of the tetrahedron, octahedron and cube respectively,

$$4t^2 = \frac{3}{2}t^2 = 2o^2 = 3c^2.$$

If each of these equals is put equal to X ,

$$4t^2 = X,$$

$$t^2 = \frac{2}{3} \cdot X,$$

$$o^2 = \frac{1}{2} \cdot X,$$

$$c^2 = \frac{1}{3} \cdot X,$$

whence

$$4t^2 : t^2 : o^2 : c^2 = 6 : 4 : 3 : 2,$$

and the ratios between $2t$, t , o , c are all *rational* (in Euclid's sense).

The ratios between these and the edges of the icosahedron and the dodecahedron are *irrational*.

(7) To prove that

(edge of icosahedron) > (edge of dodecahedron),

i.e. that

$$MB > NB.$$

By similar $\triangle s$ FDB , AFB ,

$$DB : BF = BF : BA,$$

or

$$DB : BA = DB^2 : BF^2.$$

But

$$3DB = BA;$$

therefore

$$BF^2 = 3DB^2.$$

By hypothesis,

$$AD^2 = 4DB^2;$$

therefore

$$AD > BF,$$

and, *a fortiori*,

$$AL > BF.$$

Now LK is the side of a hexagon, and AK the side of a decagon in the same circle;

therefore, when AL is divided in extreme and mean ratio, KL is the greater segment.

And, when BF is divided in extreme and mean ratio, BN is the greater segment.

Therefore, since

$$AL > BF,$$

$$KL > BN,$$

or

$$LM > BN.$$

And therefore, *a fortiori*,

$$MB > BN.$$

BOOK VII.

DEFINITIONS.

1. An **unit** is that by virtue of which each of the things ~~that~~ exist is called one.

2. A **number** is a multitude composed of units.

3. A number is **a part** of a number, the less of the greater, when it measures the greater ;

4. **but parts** when it does not measure it.

5. The greater number is a **multiple** of the less when it is measured by the less.

6. An **even number** is that which is divisible into two equal parts.

7. An **odd number** is that which is not divisible into two equal parts, or that which differs by an unit from an even number.

8. An **even-times even number** is that which is measured by an even number according to an even number.

9. An **even-times odd number** is that which is measured by an even number according to an odd number.

10. An **odd-times odd number** is that which is measured by an odd number according to an odd number.

11. A **prime number** is that which is measured by an unit alone.

12. Numbers **prime to one another** are those which are measured by an unit alone as a common measure.

13. A **composite number** is that which is measured by some number.

14. Numbers **composite to one another** are those which are measured by some number as a common measure.

15. A number is said to **multiply** a number when that which is multiplied is added to itself as many times as there are units in the other, and thus some number is produced.

16. And, when two numbers having multiplied one another make some number, the number so produced is called **plane**, and its **sides** are the numbers which have multiplied one another.

17. And, when three numbers having multiplied one another make some number, the number so produced is **solid**, and its **sides** are the numbers which have multiplied one another.

18. A **square number** is equal multiplied by equal, or a number which is contained by two equal numbers.

19. And a **cube** is equal multiplied by equal and again by equal, or a number which is contained by three equal numbers.

20. Numbers are **proportional** when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth.

21. **Similar plane** and **solid** numbers are those which have their sides proportional.

22. A **perfect number** is that which is equal to its own parts.

DEFINITION I.

Μονάς ἐστίν, καθ' ἣν ἕκαστον τῶν ὄντων ἐν λέγεται.

Iamblichus (fl. *circa* 300 A.D.) tells us (*Comm. on Nicomachus*, ed. Pistelli, p. 11, 5) that the Euclidean definition of an *unit* or a *monad* was the definition given by "more recent" writers (οἱ νεώτεροι), and that it lacked the words "even though it be collective" (κὰν συστηματικὸν ᾗ). He also gives (*ibid.* p. 11) a number of other definitions. (1) According to "some of the Pythagoreans," "an unit is the boundary between number and parts" (μονάς ἐστὶν ἀριθμοῦ καὶ μορίων μεθόριον), "because from it, as from a seed and eternal root, ratios increase reciprocally on either side," i.e. on one side we have multiple ratios continually increasing and on the other (if the unit be subdivided) submultiple ratios with denominators continually increasing. (2) A somewhat similar definition is that of Thymaridas, an ancient Pythagorean, who defined a monad as "limiting quantity" (περαίνουσα ποσότης), the beginning and the end of a thing being equally an extremity (πέρας). Perhaps the words together with their explanation may best be expressed by "limit of fewness." Theon of Smyrna (p. 18, 6, ed. Hiller) adds the explanation that the monad is "that which, when the multitude is diminished by way of continued subtraction, is deprived of all number and takes an abiding position (μονήν) and rest." If, after arriving at an unit in this way, we proceed to divide the unit itself into parts, we straightway have multitude again. (3) Some, according to Iamblichus (p. 11, 16), defined it as the "form of forms" (εἰδῶν εἶδος) because it potentially comprehends all forms of number, e.g. it is a polygonal number of any number of sides from three upwards, a solid number in all forms, and so on. (We are forcibly reminded of the latest theories of number as a "Gattung" of "Mengen" or as a "class of classes.") (4) Again an unit, says Iamblichus, is the first, or smallest, in the category of *how many* (ποσόν), the common part or beginning of *how many*. Aristotle defines it as "the indivisible in the (category of) quantity," τὸ κατὰ τὸ ποσὸν ἀδιαίρετον (*Metaph.* 1089 b 35), ποσόν including in Aristotle continuous as well as discrete quantity; hence it is distinguished from a point by the fact that it has not position: "Of the indivisible in the category of, and *quâ*, quantity, that which is every way (indivisible) and destitute of position is called an *unit*, and that which is every way indivisible and has position is a *point*" (*Metaph.* 1016 b 25). (5) In accordance with the last distinction, Aristotle calls the unit "a point without position," στιγμή ἄθετος (*Metaph.* 1084 b 26). (6) Lastly, Iamblichus says that the school of Chrysippus defined it in a confused manner (συγκεχυμένως) as "multitude *one* (πλήθος ἓν)," whereas it is alone contrasted with multitude. On a comparison of these definitions, it would seem that Euclid intended his to be a more *popular* one than those of his predecessors, δημώδης, as Nicomachus called Euclid's definition of an *even number*.

The etymological signification of the word *μονάς* is supposed by Theon of Smyrna (p. 19, 7—13) to be either (1) that it remains unaltered if it be multiplied by itself any number of times, or (2) that it is separated and *isolated* (μεμονώσθαι) from the rest of the multitude of numbers. Nicomachus also observes (1. 8, 2) that, while any number is half the sum (1) of the adjacent numbers on each side, (2) of numbers equidistant on each side, the unit is *most solitary* (μονωτάτη) in that it has not a number on each side but only on one side, and it is half of the latter alone, i.e. of 2.

DEFINITION 2.

Ἄριθμός δὲ τὸ ἐκ μονάδων συγκείμενον πλήθος.

The definition of a *number* is again only one out of many that are on record. Nicomachus (i. 7, 1) combines several into one, saying that it is "a defined multitude (πλήθος ὠρισμένον), or a collection of units (μονάδων σύστημα), or a flow of quantity made up of units" (ποσότητος χύμα ἐκ μονάδων συγκείμενον). Theon, in words almost identical with those attributed by Stobaeus (*Eclogae*, i. 1, 8) to Moderatus, a Pythagorean, says (p. 18, 3—5): "A number is a collection of units, or a progression (προποδισμός) of multitude beginning from an unit and a retrogression (ἀναποδισμός) ceasing at an unit." According to Iamblichus (p. 10) the description "collection of units" (μονάδων σύστημα) was applied to the *how many*, i.e. to number, by Thales, following the Egyptian view (κατὰ τὸ Αἰγυπτιακὸν ἀρέσκον), while it was Eudoxus the Pythagorean who said that a number was "a defined multitude" (πλήθος ὠρισμένον). Aristotle has a number of definitions which come to the same thing: "limited multitude" (πλήθος τὸ πεπερασμένον, *Metaph.* 1020 a 13), "multitude" (or "combination") "of units" or "multitude of indivisibles" (*ibid.* 1053 a 30, 1039 a 12, 1085 b 22), "several ones" (ἕνα πλείω, *Phys.* iii. 7, 207 b 7), "multitude measurable by one" (*Metaph.* 1057 a 3) and "multitude measured and multitude of measures," the "measure" being unity, τὸ ἓν (*ibid.* 1088 a 5).

DEFINITION 3.

Μέρος ἐστὶν ἀριθμὸς ἀριθμοῦ ὃ ἐλάσσων τοῦ μείζονος, ὅταν καταμετρήῃ τὸν μείζονα.

By a *part* Euclid means a submultiple, as he does in v. Def. 1, with which definition this one is identical except for the substitution of *number* (ἀριθμός) for *magnitude* (μέγεθος); cf. note on v. Def. 1. Nicomachus uses the word "submultiple" (ὑποπολλαπλάσιος) also. He defines it in a way corresponding to his definition of multiple (see note on Def. 5 below) as follows (i. 18, 2): "The submultiple, which is by nature first in the division of inequality (called) less, is the number which, when compared with a greater, can measure it more times than once so as to fill it exactly (πληρύντως)." Similarly *sub-double* (ὑποδιπλάσιος) is found in Nicomachus meaning *half*, and so on.

DEFINITION 4.

Μέρη δέ, ὅταν μὴ καταμετρήῃ.

By the expression *parts* (μέρη, the plural of μέρος) Euclid denotes what we should call a *proper fraction*. That is, a *part* being a submultiple, the rather inconvenient term *parts* means any number of such submultiples making up a fraction less than unity. I have not found the word used in this special sense elsewhere, e.g. in Nicomachus, Theon of Smyrna or Iamblichus, except in one place of Theon (p. 79, 26) where it is used of a proper fraction, of which $\frac{2}{3}$ is an illustration.

DEFINITION 5.

Πολλαπλάσιος δὲ ὁ μείζων τοῦ ἐλάσσονος, ὅταν καταμετρήται ὑπὸ τοῦ ἐλάσσονος.

The definition of a *multiple* is identical with that in v. Def. 2, except that the masculine of the adjectives is used agreeing with ἀριθμός understood instead of the neuter agreeing with μέγεθος understood. Nicomachus (i. 18, 1) defines a multiple as being "a species of the greater which is naturally first in order and origin, being the number which, when considered in comparison with another, contains it in itself completely more than once."

DEFINITIONS 6, 7.

6. Ἄρτιος ἀριθμός ἐστιν ὁ δίχα διαιρούμενος.

7. Περισσὸς δὲ ὁ μὴ διαιρούμενος δίχα ἢ [ὁ] μονάδι διαφέρων ἀρτίου ἀριθμοῦ.

Nicomachus (i. 7, 2) somewhat amplifies these definitions of *even* and *odd* numbers thus. "That is *even* which is capable of being divided into two equal parts without an unit falling in the middle, and that is *odd* which cannot be divided into two equal parts because of the aforesaid intervention (μεσσειάν) of the unit." He adds that this definition is derived "from the popular conception" (ἐκ τῆς δημώδους ὑπολήψεως). In contrast to this, he gives (i. 7, 3) the Pythagorean definition, which is, as usual, interesting. "An *even* number is that which admits of being divided, by one and the same operation, into the greatest and the least (parts), greatest in size (πυλικοτήτι) but least in quantity (ποσότητι)...while an *odd* number is that which cannot be so treated, but is divided into two unequal parts." That is, as Iamblichus says (p. 12, 2—9), an even number is divided into parts which are the *greatest* possible "parts," namely halves, and into the *fewest* possible, namely two, two being the first "number" or "collection of units." According to another ancient definition quoted by Nicomachus (i. 7, 4), an even number is that which can be divided both into two equal parts and into two unequal parts (except the first one, the number 2, which is only susceptible of division into equals), but, however it is divided, must have its two parts *of the same kind*, i.e. both even or both odd; while an odd number is that which can only be divided into two unequal parts, and those parts always of *different* kinds, i.e. one odd and one even. Lastly, the definition of odd and even "by means of each other" says that an odd number is that which differs by an unit from an even number on both sides of it, and an even number that which differs by an unit from an odd number on each side. This alternative definition of an odd number is the same thing as the second half of Euclid's definition, "the number which differs by an unit from an even number." This evidently pre-Euclidean definition is condemned by Aristotle as unscientific, because odd and even are coordinate, both being *differentiae* of number, so that one should not be defined by means of the other (*Topics* vi. 4, 142 b 7—10).

DEFINITION 8.

Ἀρτιάκις ἄρτιος ἀριθμός ἐστιν ὁ ὑπὸ ἀρτίου ἀριθμοῦ μετρούμενος κατὰ ἄρτιον ἀριθμόν.

Euclid's definition of an *even-times even* number differs from that given by the later writers, Nicomachus, Theon of Smyrna and Iamblichus; and the inconvenience of it is shown when we come to ix. 34 where it is proved

that a certain sort of number is *both* "even-times even" and "even-times odd." According to the more precise classification of the three other authorities, the "even-times even" and the "even-times odd" are mutually exclusive and are two of three subdivisions into which even numbers fall. Of these three subdivisions the "even-times even" and the "even-times odd" form the extremes, and the "odd-times even" is as it were intermediate, showing the character of both extremes (cf. note on the following definition). The *even-times even* is then the number which has its halves even, the halves of the halves even, and so on, until unity is reached. In short the *even-times even* number is always of the form 2^n . Hence Iamblichus (pp. 20, 21) says Euclid's definition of it as that which is measured by an even number an even number of times is erroneous. In support of this he quotes the number 24 which is four times 6, or six times 4, but yet is not "even-times even" according to Euclid himself (οὐδὲ κατ' αὐτόν), by which he must apparently mean that 24 is also 8 times 3, which does not satisfy Euclid's definition. There can however be no doubt that Euclid meant what he said in his definition as we have it; otherwise IX. 32, which proves that a number of the form 2^n is *even-times even only*, would be quite superfluous and a mere repetition of the definition, while, as already stated, IX. 34 clearly indicates Euclid's view that a number might at the same time be both even-times even and even-times odd. Hence the *μόνος* which some editor of the commentary of Philoponus on Nicomachus found in some copies, making the definition say that the even-times even number is *only* measured by even numbers an even number of times, is evidently an interpolation by some one who wished to reconcile Euclid's definition with the Pythagorean (cf. Heiberg, *Euklid-studien*, p. 200).

A consequential characteristic of the series of even-times even numbers noted by Nicomachus brings in a curious use of the word *δύναμις* (generally *power* in the sense of square, or square root). He says (I. 8, 6—7) that any part, i.e. any submultiple, of an even-times even number is called by an even-times even designation, while it also has an even-times even *value* (it is *ἀρτιάκις ἀρτιοδύναμον*) when expressed as so many actual units. That is, the $\frac{1}{2^m}$ th part of 2^n (where m is less than n) is called after the even-times even number 2^m , while its actual *value* (*δύναμις*) in units is 2^{n-m} , which is also an even-times even number. Thus all the parts, or submultiples, of even-times even numbers, as well as the even-times even numbers themselves, are connected with one kind of number only, the *even*.

DEFINITION 9.

Ἀρτιάκις δὲ περισσός ἐστιν ὁ ὑπὸ ἀρτίου ἀριθμοῦ μετρούμενος κατὰ περισσὸν ἀριθμόν.

Euclid uses the term *even-times odd* (*ἀρτιάκις περισσός*), whereas Nicomachus and the others make it one word, *even-odd* (*ἀρτιοπέριττος*). According to the stricter definition given by the latter (I. 9, 1), the *even-odd* number is related to the *even-times even* as the other extreme. It is such a number as, when once halved, leaves as quotient an odd number; that is, it is of the form $2(2m+1)$. Nicomachus sets the even-odd numbers out as follows,

6, 10, 14, 18, 22, 26, 30, etc.

In this case, as Nicomachus observes, any part, or submultiple, is called by a name *not* corresponding in kind to its actual value (*δύναμις*) in units. Thus,

in the case of 18, the $\frac{1}{3}$ part is called after the even number 2, but its *value* is the odd number 9, and the $\frac{1}{3}$ rd part is called after the odd number 3, while its *value* is the even number 6, and so on.

The third class of even numbers according to the strict subdivision is the *odd-even* (περισσάρτιος). Numbers are of this class when they can be halved twice or more times successively, but the quotient left when they can no longer be halved is an odd number and not unity. They are therefore of the form $2^{n+1}(2m+1)$, where n, m are integers. They are, so to say, intermediate between, or a mixture of, the extreme classes *even-times even* and *even-odd*, for the following reasons. (1) Their subdivision by 2 proceeds for some way like that of the even-times even, but ends in the way that the division of the even-odd by 2 ends. (2) The numbers after which submultiples are called and their *value* (δύναμις) in units may be both of one kind, i.e. both odd or both even (as in the case of the even-times even), or again may be one odd and one even as in the case of the even-odd. For example 24 is an odd-even number; the $\frac{1}{4}$ th, $\frac{1}{12}$ th, $\frac{1}{8}$ th or $\frac{1}{2}$ parts of it are even, but the $\frac{1}{3}$ rd part of it, or 8, is even, and the $\frac{1}{3}$ th part of it, or 3, is odd. (3) Nicomachus shows (I. 10, 6—9) how to form all the numbers of the odd-even class. Set out two lines (a) of odd numbers beginning with 3, (b) of even-times even numbers beginning with 4, thus:

(a) 3, 5, 7, 9, 11, 13, 15 etc.

(b) 4, 8, 16, 32, 64, 128, 256 etc.

Now multiply each of the first numbers into each of the second row. Let the products of one of the first into all the second set make horizontal rows; we then get the rows

12, 24, 48, 96, 192, 384, 768 etc.

20, 40, 80, 160, 320, 640, 1280 etc.

28, 56, 112, 224, 448, 896, 1792 etc.

36, 72, 144, 288, 576, 1152, 2304 etc.

and so on.

Now, says Nicomachus, you will be surprised to see (φανήσεται σοι θαυμαστῶς) that (a) the *vertical* rows have the property of the *even-odd* series, 6, 10, 14, 18, 22 etc., viz. that, if an odd number of successive numbers be taken, the middle number is half the sum of the extremes, and if an even number, the two middle numbers together are equal to the sum of the extremes, (b) the *horizontal* rows have the property of the *even-times even* series 4, 8, 16 etc., viz. that the product of the extremes of any number of successive terms is equal, if their number be odd, to the square of the middle term, or, if their number be even, to the product of the two middle terms.

Let us now return to Euclid. His 9th definition states that an *even-times odd* number is a number which, when divided by an even number, gives an odd number as quotient. Following this definition in our text comes a 10th definition which defines an *odd-times even* number; this is stated to be a number which, when divided by an odd number, gives an even number as quotient. According to these definitions any *even-times odd* number would also be *odd-times even*, and, from the fact that Iamblichus notes this, we may fairly conclude that he found Def. 10 as well as Def. 9 in the text of Euclid which he used. But, if both definitions are genuine, the enunciations of ix. 33 and ix. 34 as we have them present difficulties. ix. 33 says that "If a number have its half odd, it is even-times odd *only*"; but, on the assumption that

both definitions are genuine, this would not be true, for the number would be *odd-times even* as well. IX. 34 says that "If a number neither be one of those which are continually doubled from 2, nor have its half odd, it is both even-times even and even-times odd." The term *odd-times even* (περισσάκις ἄρτιος) not occurring in these propositions, nor anywhere else after the definition, that definition becomes superfluous. Iamblichus however (p. 24, 7—14) quotes these enunciations differently. In the first he has instead of "even-times odd only" the words "*both even-times odd and odd-times even*"; and, in the second, for "both even-times even and even-times odd" he has "is both even-times even and at the same time even-times odd and odd-times even." In both cases therefore "odd-times even" is added to the enunciation as Iamblichus had it; the words cannot have been added by Iamblichus himself because he himself does not use the term *odd-times even*, but the one word *odd-even* (περισσάρτιος). In order to get over the difficulties involved by Def. 10 and these differences of reading we have practically to choose between (1) accepting Iamblichus' reading in all three places and (2) adhering to the reading of our MSS. in IX. 33, 34 and rejecting Def. 10 altogether as an interpolation. Now the readings of our text of IX. 33, 34 are those of the Vatican MS. and the Theonine MSS. as well; hence they must go back to a time before Theon, and must therefore be almost as old as those of Iamblichus. Heiberg considers it improbable that Euclid would wish to maintain a pointless distinction between *even-times odd* and *odd-times even*, and on the whole concludes that Def. 10 was first interpolated by some ignorant person who did not notice the difference between the Euclidean and Pythagorean classification, but merely noticed the absence of a definition of *odd-times even* and fabricated one as a companion to the other. When this was done, it would be easy to see that the statement in IX. 33 that the number referred to is "even-times odd *only*" was not strictly true, and that the addition of the words "and odd-times even" was necessary in IX. 33 and IX. 34 as well.

DEFINITION 10.

Περισσάκις δὲ περισσὸς ἀριθμὸς ἐστὶν ὁ ὑπὸ περισσοῦ ἀριθμοῦ μετρούμενος κατὰ περισσὸν ἀριθμόν.

The *odd-times odd* number is not defined as such by Nicomachus and Iamblichus; for them these numbers would apparently belong to the *composite* subdivision of *odd* numbers. Theon of Smyrna on the other hand says (p. 23, 21) that *odd-times odd* was one of the names applied to *prime* numbers (excluding 2), for these have two odd factors, namely 1 and the number itself. This is certainly a curious use of the term.

DEFINITION 11.

Πρῶτος ἀριθμὸς ἐστὶν ὁ μονάδι μόνῃ μετρούμενος.

A *prime* number (πρῶτος ἀριθμός) is called by Nicomachus, Theon, and Iamblichus a "*prime and incomposite* (ἀσύνθετος) number." Theon (p. 23, 9) defines it practically as Euclid does, viz. as a number "measured by no number, but by an unit only." Aristotle too says that a prime number is not measured by any number (*Anal. post.* II. 13, 96 a 36), an unit not being a number (*Metaph.* 1088 a 6), but only the beginning of number (Theon of Smyrna says the same thing, p. 24, 23). According to Nicomachus (I. 11, 2) the prime number is a

subdivision, not of numbers, but of *odd* numbers; it is "an odd number which admits of no other part except that which is called after its own name (*παρώνυμον ἑαυτῷ*).” The prime numbers are 3, 5, 7 etc., and there is no submultiple of 3 except $\frac{1}{3}$ rd, no submultiple of 11 except $\frac{1}{11}$ th, and so on. In all these cases the only submultiple is an unit. According to Nicomachus 3 is the first prime number, whereas Aristotle (*Topics* VIII. 2, 157 a 39) regards 2 as a prime number: "as the dyad is the only even number which is prime," showing that this divergence from the Pythagorean doctrine was earlier than Euclid. The number 2 also satisfies Euclid's definition of a prime number. Iamblichus (p. 30, 27 sqq.) makes this the ground of another attack upon Euclid. His argument (the text of which, however, leaves much to be desired) appears to be that 2 is the *only* even number which has no other part except an unit, while the subdivisions of the even, as previously explained by him (the *even-times even*, the *even-odd*, and *odd-even*), all exclude primeness, and he has previously explained that 2 is *potentially* even-odd, being obtained by multiplying by 2 the *potentially* odd, i.e. the unit; hence 2 is regarded by him as bound up with the subdivisions of even, which exclude primeness. Theon seems to hold the same view as regards 2, but supports it by an apparent circle. A prime number, he says (p. 23, 14—23), is also called *odd-times odd*; therefore only odd numbers are prime and incomposite. Even numbers are not measured by the unit alone, except 2, which therefore (p. 24, 7) is *odd-like* (*περισσοειδής*) without being prime.

A variety of other names were applied to prime numbers. We have already noted the curious designation of them as *odd-times odd*. According to Iamblichus (p. 27, 3—5) some called them *euthymetric* (*εὐθυμετρικός*), and Thymaridas *rectilinear* (*εὐθυγραμμικός*), the ground being that they can only be set out in one dimension with no breadth (*ἀπλατῆς γὰρ ἐν τῇ ἐκθέσει ἐφ' ἐν μόνον διαστάμενος*). The same aspect of a prime number is also expressed by Aristotle, who (*Metaph.* 1020 b 3) contrasts the composite number with that which is only in one dimension (*μόνον ἐφ' ἐν ὧν*). Theon of Smyrna (p. 23, 12) gives *γραμμικός* (*linear*) as the alternative name instead of *εὐθυγραμμικός*. In either case, to make the word a proper description of a prime number we have to understand the word *only*; a prime number is that which is *linear*, or *rectilinear*, *only*. For Nicomachus, who uses the form *linear*, expressly says (II. 13, 6) that *all* numbers are so, i.e. all can be represented as linear by dots to the required amount placed in a line.

A prime number was called *prime* or *first*, according to Nicomachus (I. 11, 3), because it can only be arrived at by putting together a certain number of units, and the unit is the beginning of number (cf. Aristotle's second sense of *πρῶτος* "as not being composed of numbers," *ὡς μὴ συγκεῖσθαι ἐξ ἀριθμῶν*, *Anal. Post.* II. 13, 96 a 37), and also, according to Iamblichus, because there is no number before it, being a collection of units (*μονάδων σύστημα*), of which it is a multiple, and it appears *first* as a basis for other numbers to be multiples of.

DEFINITION 12.

Πρῶτοι πρὸς ἀλλήλους ἀριθμοὶ εἰσιν οἱ μονάδι μόνῃ μετρούμενοι καὶ κοινῇ μέτρῳ.

By way of further emphasising the distinction between "prime" and "prime to one another," Theon of Smyrna (p. 23, 6—8) calls the former "prime *absolutely*" (*ἀπλῶς*), and the latter "prime to one another and *not*

absolutely" or "*not in themselves*" (οὐ καθ' αὐτούς). The latter (p. 24, 8—10) are "measured by the unit [sc. only] as common measure, even though, taken by themselves (ὡς πρὸς ἑαυτούς), they be measured by some other numbers." From Theon's illustrations it is clear that with him as with Euclid a number prime to another may be even as well as odd. In Nicomachus (i. 11, 1) and Iamblichus (p. 26, 19), on the other hand, the number which is "in itself secondary (δευτερος) and composite (σύνθετος), but in relation to another prime and incomposite," is a subdivision of *odd*. I shall call more particular attention to this difference of classification when we have reached the definitions of "composite" and "composite to one another"; for the present it is to be noted that Nicomachus (i. 13, 1) defines a number prime *to another* after the same manner as the absolutely prime; it is a number which "is measured not only by the unit as the common measure but also by some other measure, and for this reason can also admit of a part or parts called by a different name besides that called by the same name (as itself), but, when examined in comparison with another number of similar character, is found not to be capable of being measured by a common measure in relation to the other, nor to have the same part, called by the same name as (any of) those simply (ἀπλῶς) contained in the other; e.g. 9 in relation to 25, for each of these is in itself secondary and composite, but, in comparison with one another, they have an unit alone as a common measure and no part is called by the same name in both, but the *third* in one is not in the other, nor is the *fifth* in the other found in the first."

DEFINITION 13.

Σύνθετος ἀριθμός ἐστιν ὁ ἀριθμῷ τινι μετρούμενος.

Euclid's definition of *composite* is again the same as Theon's definition of numbers "composite in relation to themselves," which (p. 24, 16) are "numbers measured by any less number," the unit being, as usual, not regarded as a number. Theon proceeds to say that "of composite numbers they call those which are contained by two numbers *plane*, as being investigated in two dimensions and, as it were, contained by a length and a breadth, while (they call) those (which are contained) by three (numbers) *solid*, as having the third dimension added to them." To a similar effect is the remark of Aristotle (*Metaph.* 1020 b 3) that certain numbers are "composite and are not only in one dimension but such as the plane and the solid (figure) are representations of (μίμημα), these numbers being so many times so many (ποσάκις ποσοί), or so many times so many times so many (ποσάκις ποσάκις ποσοί) respectively." These subdivisions of composite numbers are, of course, the subject of Euclid's definitions 17, 18 respectively. Euclid's composite numbers may be either even or odd, like those of Theon, who gives 6 as an instance, 6 being measured by both 2 and 3.

DEFINITION 14.

Σύνθετοι δὲ πρὸς ἀλλήλους ἀριθμοὶ εἰσιν οἱ ἀριθμῷ τινι μετρούμενοι κοινῷ μέτρῳ.

Theon (p. 24, 18), like Euclid, defines numbers *composite to one another* as "those which are measured by any common measure whatever" (excluding unity, as usual). Theon instances 8 and 6, with 2 as common measure, and 6 and 9, with 3 as common measure.

As hinted above, there is a great difference between Euclid's classification of prime and composite numbers, and of numbers prime and composite to one another, and the classification found in Nicomachus (1. 11—13) and Iamblichus. According to the latter, all these kinds of numbers are subdivisions of the class of *odd* numbers only. As the class of *even* numbers is divided into three kinds, (1) the even-times even, (2) the even-odd, which form the extremes, and (3) the odd-even, which is, as it were, intermediate to the other two, so the class of *odd* numbers is divided into three, of which the third is again a mean between two extremes. The three are:

(1) the *prime and incomposite*, which is like Euclid's prime number except that it excludes 2;

(2) the *secondary and composite*, which is "odd because it is a distinct part of one and the same genus (διὰ τὸ ἐξ ἑνὸς καὶ τοῦ αὐτοῦ γένους διακεκρίσθαι) but has in it nothing of the nature of a first principle (ἀρχοειδές); for it arises from adding some other number (to itself), so that, besides having a part called by the same name as itself, it possesses a part or parts called by another name." Nicomachus cites 9, 15, 21, 25, 27, 33, 35, 39. It is made clear that not only must the factors be both odd, but they must all be prime numbers. This is obviously a very inconvenient restriction of the use of the word *composite*, a word of general signification.

(3) is that which is "*secondary and composite in itself but prime and incomposite to another*." The actual words in which this is defined have been given above in the note on Def. 12. Here again all the factors must be odd and prime.

Besides the inconvenience of restricting the term *composite* to *odd* numbers which are composite, there is in this classification the further serious defect, pointed out by Nesselmann (*Die Algebra der Griechen*, 1842, p. 194), that subdivisions (2) and (3) overlap, subdivision (2) including the whole of subdivision (3). The origin of this confusion is no doubt to be found in Nicomachus' perverse anxiety to be symmetrical; by hook or by crook he must divide *odd* numbers into three kinds as he had divided the *even*. Iamblichus (p. 28, 13) carries his desire to be logical so far as to point out why there cannot be a fourth kind of number contrary in character to (3), namely a number which should be "prime and incomposite in itself, but secondary and composite to another"!

DEFINITION 15.

Ἀριθμὸς ἀριθμὸν πολλαπλασιάζειν λέγεται, ὅταν, ὅσαι εἰσὶν ἐν αὐτῷ μονάδες, τοσαυτάκις συντεθῇ ὁ πολλαπλασιαζόμενος, καὶ γένηται τις.

This is the well known primary definition of multiplication as an abbreviation of addition.

DEFINITION 16.

Ὅταν δὲ δύο ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, ὁ γινόμενος ἐπίπεδος καλεῖται, πλευραὶ δὲ αὐτοῦ οἱ πολλαπλασιάσαντες ἀλλήλους ἀριθμοί.

The words *plane* and *solid* applied to numbers are of course adapted from their use with reference to geometrical figures. A number is therefore called *linear* (γραμμικός) when it is regarded as in one dimension, as being a *length*

(μήκος). When it takes another dimension in addition, namely *breadth* (πλάτος), it is in two dimensions and becomes *plane* (ἐπιπέδος). The distinction between a *plane* and a *plane number* is marked by the use of the neuter in the former case, and the masculine, agreeing with ἀριθμός, in the latter case. So with a *square* and a *square number*, and so on. The most obvious form of a plane number is clearly that corresponding to a rectangle in geometry; the number is the product of two linear numbers regarded as *sides* (πλευραί) forming the length and breadth respectively. Such a number is, as Aristotle says, "so many times so many," and a plane is its counterpart (μίμημα). So Plato, in the *Theaetetus* (147 E—148 B), says: "We divided all numbers into two kinds, (1) that which can be expressed as equal multiplied by equal (τὸν δυνάμενον ἴσον ἰσάκις γίγνεσθαι), and which, likening its form to the square, we called *square* and equilateral; (2) that which is intermediate, and includes 3 and 5 and every number which cannot be expressed as equal multiplied by equal, but is either less times more or more times less, being always contained by a greater and a less side, which number we likened to the oblong figure (προμήκει σχήματι) and called an *oblong* number.... Such *lines* therefore as *square the equilateral and plane number* [i.e. which can form a plane number with equal sides, or a square] we defined as *length* (μήκος); but such as *square the oblong* (here *εἰερομήκης*) [i.e. the square of which is equal to the oblong] we called *roots* (δυνάμεις) as not being commensurable with the others in length, but only in the plane areas (ἐπιπέδοις), to which the squares on them are equal (ἂ δύνανται)." This passage seems to make it clear that Plato would have represented numbers as Euclid does, by straight lines proportional in length to the numbers they represent (so far as practicable); for, since 3 and 5 are with Plato oblong numbers, and *lines* with him represent the sides of oblong numbers (since a line represents the "root," the square on which is equal to the oblong), it follows that the *unit* representing the smaller side must have been represented as a line, and 3, the larger side, as a line of three times the length. But there is another possible way of representing numbers, not by lines of a certain length, but by *points* disposed in various ways, in straight lines or otherwise. Iamblichus tells us (p. 56, 27) that "in old days they represented the quantuplicities of number in a more natural way (φυσικώτερον) by splitting them up into units, and not, as in our day, by symbols" (συμβολικῶς). Aristotle too (*Metaph.* 1092 b 10) mentions one Eurytus as having settled what number belonged to what, such a number to a man, such a number to a horse, and so on, "copying their shapes" (reading τοῦτων, with Zeller) "*with pebbles* (ταῖς ψήφοις), just as those do who arrange numbers in the forms of triangles or squares." We accordingly find numbers represented in Nicomachus and Theon of Smyrna by a number of α's ranged like points according to geometrical figures. According to this system, any number could be represented by points in a straight line, in which case, says Iamblichus (p. 56, 26), we shall call it *rectilinear* because it is *without breadth* and only advances in length (ἀπλατῶς ἐπὶ μόνον τὸ μήκος πρόεισι). The prime number was called by Thymaridas *rectilinear par excellence*, because it was without breadth and in one dimension *only* (ἐφ' ἓν μόνον διαστάμενος). By this must be meant the impossibility of representing, say, 3 as a plane number, in Plato's sense, i.e. as a product of two numbers corresponding to a rectangle in geometry; and this view would appear to rest simply upon the representation of a number by *points*, as distinct from *lines*. Three dots in a straight line would have *no breadth*; and if breadth were introduced in the sense of producing a rectangle, i.e. by placing the same

number of dots in a second line below the first line, the first *plane* number would be 4, and 3 would not be a plane number at all, as Plato says it is. It seems therefore to have been the alternative representation of a number by points, and not lines, which gave rise to the different view of a plane number which we find in Nicomachus and the rest. By means of separate points we can represent numbers in geometrical forms other than rectangles and squares. One dot with two others symmetrically arranged below it shows a *triangle*, which is a figure *in two dimensions* as much as a rectangle or parallelogram is. Similarly we can arrange certain numbers in the form of regular *pentagons* or other polygons. According therefore to this mode of representation, 3 is the first *plane* number, being a *triangular* number. The method of formation of triangular, square, pentagonal and other polygonal numbers is minutely described in Nicomachus (II. 8—11), who distinguishes the separate series of *gnomons* belonging to each, i.e. gives the law determining the number which has to be added to a polygonal number with n in a side, in order to make it into a number of the same form but with $n + 1$ in a side (the addend being of course the gnomon). Thus the gnomonic series for triangular numbers is 1, 2, 3, 4, 5...; that for squares 1, 3, 5, 7...; that for pentagonal numbers 1, 4, 7, 10..., and so on. The subject need not detain us longer here, as we are at present only concerned with the different views of what constitutes a *plane* number.

Of *plane* numbers in the Platonic and Euclidean sense we have seen that Plato recognises *two* kinds, the *square* and the *oblong* (προμήκης or ἑτερομήκης). Here again Euclid's successors, at all events, subdivided the class more elaborately. Nicomachus, Theon of Smyrna, and Iamblichus divide *plane* numbers with unequal *sides* into (1) ἑτερομήκεις, the nearest thing to squares, viz. numbers in which the greater side exceeds the less side by 1 only, or numbers of the form $n(n + 1)$, e.g. 1.2, 2.3, 3.4, etc. (according to Nicomachus), and (2) προμήκεις, or those whose sides differ by 2 or more, i.e. are of the form $n(n + m)$, where m is not less than 2 (Nicomachus illustrates by 2.4, 3.6, etc.). Theon of Smyrna (p. 30, 8—14) makes προμήκεις include ἑτερομήκεις, saying that their sides may differ by 1 or more; he also speaks of *parallelogram*-numbers as those which have one side different from the other by 2 or more; I do not find this latter term in Nicomachus or Iamblichus, and indeed it seems superfluous, as *parallelogram* is here only another name for oblong. Iamblichus (p. 74, 23 sqq.), always critical of Euclid, attacks him again here for confusing the subject by supposing that the ἑτερομήκης number is the product of *any* two different numbers multiplied together, and by not distinguishing the oblong (προμήκης) from it: "for his definition declares the same number to be square and also ἑτερομήκης, as for example 36, 16 and many others: which would be equivalent to the odd number being the same thing as the even." No importance need be attached to this exaggerated statement; it is in any case merely a matter of words, and it is curious that Euclid does not in fact use the word ἑτερομήκης of *numbers* at all, but only of geometrical oblong figures as opposed to squares, so that Iamblichus can apparently only have inferred that he used it in an unorthodox manner from the *geometrical* use of the term in the definitions of Book I. and from the fact that he does not give the two subdivisions of *plane numbers* which are not square, but seems only to divide *plane numbers* into square and not-square. The argument that ἑτερομήκεις numbers are a *natural*, and therefore essential, subdivision Iamblichus appears to found on the method of successive addition by which they can be evolved; as square numbers are obtained by successively adding

odd numbers as *gnomons*, so *ἑτερομήκεις* are obtained by adding *even* numbers as *gnomons*. Thus $1 \cdot 2 = 2$, $2 \cdot 3 = 2 + 4$, $3 \cdot 4 = 2 + 4 + 6$, and so on.

DEFINITION 17.

“Ὅταν δὲ τρεῖς ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, ὁ γενόμενος στερεός ἐστιν, πλευραὶ δὲ αὐτοῦ οἱ πολλαπλασιάσαντες ἀλλήλους ἀριθμοί.

What has been said of the two apparently different ways of regarding a *plane* number seems to apply equally, *mutatis mutandis*, to the definitions of a *solid* number. Aristotle regards it as a number which is so many times so many times so many (*ποσάκις ποσάκις ποσόν*). Plato finishes the passage about lines which represent the sides of *square* numbers and lines which are *roots* (*δυνάμεις*), i.e. the squares on which are equal to the rectangle representing a number which is oblong and not square, by adding the words, “And another similar property belongs to solids” (*καὶ περὶ τὰ στερεὰ ἄλλο τοιοῦτον*). That is, apparently, there would be a corresponding term to *root* (*δύναμις*)—practically representing a surd—to denote the side of a cube equal to a parallelepiped representing a solid number which is the product of three factors but not a cube. Such is a solid number when numbers are represented by *straight lines*: it corresponds in general to a parallelepiped and, when all the factors are equal, to a cube.

But again, if numbers be represented by *points*, we may have solid numbers (i.e. numbers in three dimensions) in the form of *pyramids* as well. The first number of this kind is 4, since we may have three points forming an equilateral triangle in one plane and a fourth point placed in another plane. The length of the sides can be increased by 1 successively; and we can have a series of pyramidal numbers, with triangles, squares or polygons as bases, made up of layers of triangles, squares or similar polygons respectively, each of which layers has one less in the side than the layer below it, until the top of the pyramid is reached, which of course is one point representing unity. Nicomachus (II. 13—16), Theon of Smyrna (p. 41—2), and Iamblichus (p. 95, 15 sqq.), all give the different kinds of *pyramidal* solid numbers in addition to the other kinds.

These three writers make the following further distinctions between solid numbers which are the product of three factors.

1. First there is the equal by equal by equal (*ισάκις ἰσάκις ἴσος*), which is, of course, the cube.

2. The other extreme is the unequal by unequal by unequal (*ἀνισάκις ἀνισάκις ἀνισος*), or that in which all the dimensions are different, e.g. the product of 2, 3, 4 or 2, 4, 8 or 3, 5, 12. These were, according to Nicomachus (II. 16), called *scalene*, while some called them *σφηγίσκοι* (*wedge-shaped*), others *σφηγίσκοι* (from *σφήξ*, a *wasp*), and others *βωμίσκοι* (*altar-shaped*). Theon appears to use the last term only, while Iamblichus of course gives all three names.

3. Intermediate to these, as it were, come the numbers “whose *planes* form *ἑτερομήκεις* numbers” (i.e. numbers of the form $n(n+1)$). These, says Nicomachus, are called *parallelepipedal*.

Lastly come two classes of such numbers each of which has two equal dimensions but not more.

4. If the third dimension is less than the others, the number is *equal by equal by less* (ισάκεις ἴσος ἐλαττονάκεις) and is called a *plinth* (πλινθίς), e.g. 8.8.3.

5. If the third dimension is greater than the others, the number is *equal by equal by greater* (ισάκεις ἴσος μείζονάκεις) and is called a *beam* (δοκίς), e.g. 3.3.7. Another name for this latter kind of number (according to Iamblichus) was *στηλὶς* (diminutive of *στήλη*).

Lastly, in connexion with pyramidal numbers, Nicomachus (II. 14, 5) distinguishes numbers corresponding to *frusta* of pyramids. These are *truncated* (κόλουροι), *twice-truncated* (δικολούροι), *thrice-truncated* (τρικόλουροι) pyramids, and so on, the term being used mostly in theoretic treatises (ἐν συγγράμμασι μάλιστα τοῖς θεωρηματικοῖς). The *truncated* pyramid was formed by cutting off the point forming the vertex. The *twice-truncated* was that which lacked the vertex and the next plane, and so on. Theon of Smyrna (p. 42, 4) only mentions the *truncated* pyramid as "that with its vertex cut off" (ἡ τὴν κορυφὴν ἀποτετμημένη), saying that some also called it a trapezium, after the similitude of a plane trapezium formed by cutting the top off a triangle by a straight line parallel to the base.

DEFINITION 18.

Τετράγωνος ἀριθμός ἐστιν ὁ ισάκεις ἴσος ἢ [ὁ] ὑπὸ δύο ἴσων ἀριθμῶν περιεχόμενος.

A particular kind of square distinguished by Nicomachus and the rest was the square number which ended (in the decimal notation) with the same number as its side, e.g. 1, 25, 36, which are the squares of 1, 5 and 6. These square numbers were called *cyclic* (κυκλικοί) on the analogy of circles in geometry which return again to the point from which they started.

DEFINITION 19.

Κύβος δὲ ὁ ισάκεις ἴσος ισάκεις ἢ [ὁ] ὑπὸ τριῶν ἴσων ἀριθμῶν περιεχόμενος.

Similarly cube numbers which ended with the same number as their sides, and the squares of those sides also, were called *spherical* (σφαιρικοί) or *recurrent* (ἀποκαταστατικοί). One might have expected that the term *spherical* would be applicable also to the cubes of numbers which ended with the same digit as the side but not necessarily with the same digit as the square of the side also. E.g. the cube of 4, i.e. 64, ends with the same digit as 4, but not with the same digit as 16. But apparently 64 was not called a spherical number, the only instances given by Nicomachus and the rest being those cubed from numbers ending with 5 or 6, which end with the same digit if *squared*. A *spherical* number is in fact derived from a *circular* number only, and that by adding another equal dimension. Obviously, as Nesselmann says, the names *cyclic* and *spherical* applied to numbers appeal to an entirely different principle from that on which the figured numbers so far dealt with were formed.

DEFINITION 20.

Ἄριθμοι ἀνάλογόν εἰσιν, ὅταν ὁ πρῶτος τοῦ δευτέρου καὶ ὁ τρίτος τοῦ τετάρτου ἰσάκης ἢ πολλαπλάσιος ἢ τὸ αὐτὸ μέρος ἢ τὰ αὐτὰ μέρη ὦσιν.

Euclid does not give in this Book any definition of ratio, doubtless because it could only be the same as that given at the beginning of Book v., with numbers substituted for "homogeneous magnitudes" and "in respect of *size*" (ἡλικότητα) omitted or altered. We do not find that Nicomachus and the rest give any substantially different definition of a *ratio* between numbers. Theon of Smyrna says, in fact (p. 73, 16), that "ratio in the sense of proportion (λόγος ὁ κατ' ἀνάλογον) is a sort of relation of two homogeneous terms to one another, as for example, double, triple." Similarly Nicomachus says (II. 21, 3) that "a ratio is a relation of two terms to one another," the word for "relation" being in both cases the same as Euclid's (σχέσις). Theon of Smyrna goes on to classify ratios as greater, less, or equal, i.e. as ratios of greater inequality, less inequality, or equality, and then to specify certain arithmetical ratios which had special names, for which he quotes the authority of Adrastus. The names were πολλαπλάσιος, ἐπιμόριος, ἐπιμερής, πολλαπλασιεπιμόριος, πολλαπλασιεπιμερής (the first of which is, of course, a multiple, while the rest are the equivalent of certain types of improper fractions as we should call them), and the reciprocals of each of these described by prefixing ὑπό or *sub*. After describing these particular classes of arithmetical ratios, Theon goes on to say that numbers still have ratios to one another even if they are different from all those previously described. We need not therefore concern ourselves with the various types; it is sufficient to observe that any ratio between numbers can be expressed in the manner indicated in Euclid's definition of arithmetical proportion, for the greater is, in relation to the less, either one or a combination of more than one of the three things, (1) a multiple, (2) a submultiple, (3) a proper fraction.

It is when we come to the definition of *proportion* that we begin to find differences between Euclid, Nicomachus, Theon and Iamblichus. "Proportion," says Theon (p. 82, 6), "is similarity or sameness of more ratios than one," which is of course unobjectionable if it is previously understood what a *ratio* is; but confusion was brought in by those (like Thrasyllus) who said that there were *three proportions* (ἀναλογίαι), the arithmetic, geometric, and harmonic, where of course the reference is to arithmetic, geometric and harmonic *means* (μεσότητες). Hence it was necessary to explain, as Adrastus did (Theon, p. 106, 15), that of the several *means* "the geometric was called both proportion *par excellence* and primary...though the other means were also commonly called *proportions* by some writers." Accordingly we have Nicomachus trying to extend the term "proportion" to cover the various *means* as well as a proportion in three or four terms in the ordinary sense. He says (II. 21, 2): "Proportion, *par excellence* (κυρίως), is the bringing together (σύλληψις) to the same (point) of two or more *ratios*; or, more generally, (the bringing together) of two or more *relations* (σχέσεων), even though they be subjected not to the same *ratio* but to a difference or some other (law)." Iamblichus keeps the senses of the word more distinct. He says, like Theon, that "proportion is similarity or sameness of several ratios" (p. 98, 14), and that "it is to be premised that it was the geometrical (proportion) which the ancients called proportion *par excellence*, though it is now common to apply the name generally to all the remaining means as well" (p. 100, 15). Pappus

remarks (III. p. 70, 17), "A mean differs from a proportion in this respect that, if anything is a proportion, it is also a mean, but not conversely. For there are three means, of which one is arithmetic, one geometric and one harmonic." The last remark implies plainly enough that there is only one *proportion* (ἀναλογία) in the proper sense. So, too, says Iamblichus in another place (p. 104, 19): "the second, the geometric, mean has been called *proportion par excellence* because the terms contain the same ratio, being separated according to the same proportion (ἀνὰ τὸν αὐτὸν λόγον διεστῶτες)." The natural conclusion is that of Nesselmann, that originally the geometric proportion was called ἀναλογία, the others, the arithmetic, the harmonic, etc., *means*; but later usage had obliterated the distinction.

Of proportions in the ancient and Euclidean sense Theon (p. 82, 10) distinguished the *continuous* (συνεχής) and the *separated* (διηρημένη), using the same terms as Aristotle (*Eth. Nic.* 1131 a 32). The meaning is of course clear: in the *continuous* proportion the consequent of one ratio is the antecedent of the next; in the *separated* proportion this is not so. Nicomachus (II. 21, 5—6) uses the words *connected* (συνημμένη) and *disjoined* (διεξευγμένη) respectively. Euclid regularly speaks of numbers in continuous proportion as "proportional in order, or successively" (ἐξῆς ἀνάλογον).

DEFINITION 21.

Ὅμοιοι ἐπίπεδοι καὶ στερεοὶ ἀριθμοὶ εἰσιν οἱ ἀνάλογον ἔχοντες τὰς πλευράς.

Theon of Smyrna remarks (p. 36, 12) that, among plane numbers, *all* squares are similar, while of ἑτερομήκεις those are similar "whose sides, that is, the numbers containing them, are proportional." Here ἑτερομήκεις must evidently be used, not in the sense of a number of the form $n(n+1)$, but as synonymous with προμήκης, *any* oblong number; so that on this occasion Theon follows the terminology of Plato and (according to Iamblichus) of Euclid. Obviously, if the strict sense of ἑτερομήκεις is adhered to, no two numbers of that form can be similar unless they are also *equal*. We may compare Iamblichus' elaborate contrast of the square and the ἑτερομήκης. Since the two sides of the square are equal, a square number might, as he says (p. 82, 9), be fitly called ἰδιομήκης (Nicomachus uses ταῦτομήκης) in contrast to ἑτερομήκης; and the ancients, according to him, called square numbers "the same" and "similar" (ταυτούς τε καὶ ὁμοίους), but ἑτερομήκεις numbers "dissimilar and other" (ἀνομοίους καὶ θατέρους).

With regard to solid numbers, Theon remarks in like manner (p. 37, 2) that *all* cube numbers are similar, while of the others those are similar whose sides are proportional; i.e. in which, as length is to length, so is breadth to breadth and height to height.

DEFINITION 22.

Τέλειος ἀριθμὸς ἐστὶν ὁ τοῖς ἑαυτοῦ μέρεσιν ἴσος ὢν.

Theon of Smyrna (p. 45, 9 sqq.) and Nicomachus (I. 16) both give the same definition of a *perfect* number, as well as the law of formation of such numbers which Euclid proves in the later proposition, IX. 36. They add however definitions of two other kinds of numbers in contrast with it, (1) the *over-perfect* (ὑπερτελής in Nicomachus, ὑπερτέλειος in Theon), the

sum of whose parts, i.e. submultiples, is greater than the number itself, e.g. 12, 24 etc., the sum of the parts of 12 being $6 + 4 + 3 + 2 + 1 = 16$, and the sum of the parts of 24 being $12 + 8 + 6 + 4 + 3 + 2 + 1 = 36$, (2) the *defective* (ἐλλειπής), the sum of whose parts is less than the whole, e.g. 8 or 14, the parts in the first case adding up to $4 + 2 + 1$, or 7, and in the second case to $7 + 2 + 1$, or 10. All three classes are however made by Theon subdivisions of numbers in general, but by Nicomachus subdivisions of *even* numbers.

The term *perfect* was used by the Pythagoreans, but in another sense, of 10; while Theon tells us (p. 46, 14) that 3 was also called perfect "because it is the first number that has beginning, middle and extremity; it is also both a *line* and a *plane* (for it is an equilateral triangle having each side made up of two units), and it is the first link and potentiality of the solid (for a solid must be conceived of in three dimensions)."

There are certain unexpressed axioms used in Book VII. as there are in earlier Books.

The following may be noted.

1. If A measures B , and B measures C , A will measure C .
2. If A measures B , and also measures C , A will measure the difference between B and C when they are unequal.
3. If A measures B , and also measures C , A will measure the sum of B and C .

It is clear, from what we know of the Pythagorean theory of numbers, of musical intervals expressed by numbers, of different kinds of *means* etc., that the substance of Euclid Books VII.—IX. was no new thing but goes back, at least, to the Pythagoreans. It is well known that the mathematics of Plato's *Timaeus* is essentially Pythagorean. It is therefore *a priori* probable (if not perhaps quite certain) that Plato *πυθαγορίζει* even in the passage (32 A, B) where he speaks of numbers "whether solid or square" in continued proportion, and proceeds to say that between *planes* one mean suffices, but to connect two *solids* two means are necessary. This passage has been much discussed, but I think that by "planes" and "solids" Plato certainly meant *square* and *solid numbers* respectively, so that the allusion must be to the theorems established in Eucl. VIII. 11, 12, that between two square numbers there is one mean proportional number, and between two cube numbers there are two mean proportional numbers¹.

¹ It is true that *similar* plane and solid numbers have the same property (Eucl. VIII. 18, 19); but, if Plato had meant similar plane and solid numbers generally, I think it would have been necessary to specify that they were "similar," whereas, seeing that the *Timaeus* is as a whole concerned with regular figures, there is nothing unnatural in allowing *regular* or *equilateral* to be understood. Further Plato speaks first of *δυνάμεις* and *δγκοι* and then of "planes" (ἐπιπεδα) and "solids" (στερεά) in such a way as to suggest that *δυνάμεις* correspond to ἐπιπεδα and *δγκοι* to στερεά. Now the regular meaning of *δύναμις* is *square* (or sometimes *square root*), and I think it is here used in the sense of *square*, notwithstanding that Plato seems to speak of *three* squares in continued proportion, whereas, in general, the mean between two squares as extremes would not be square but oblong. And, if *δυνάμεις* are squares, it is reasonable to suppose that the *δγκοι* are also *equilateral*, i.e. the "solids" are cubes. I am aware that Th. Häbler (*Bibliotheca Mathematica*, VIII.3, 1908, pp. 173—4) thinks that the passage is to be explained by reference to the problem of the duplication of the cube, and does not refer to numbers at all. Against this we have to put the evidence of Nicomachus (II. 24, 6) who, in speaking of "a certain Platonic theorem," quotes the very same results of Eucl. VIII. 11, 12. Secondly, it is worth noting that Häbler's explanation is distinctly ruled out by Democritus the Platonist (3rd cent. A.D.) who, according to Proclus

It is no less clear that, in his method and line of argument, Euclid was following earlier models, though no doubt making improvements in the exposition. The tract on the *Sectio Canonis*, *κατατομή κανόνος* (as to the genuineness of which see above, Vol. I, p. 17) is in style and in the form of the propositions generally akin to the *Elements*. In one proposition (2) the author says "*we learned* (*ἐμάθομεν*) that, if as many numbers as we please be in (continued) proportion, and the first measures the last, the first will also measure the intermediate numbers"; here he practically quotes *Elem.* viii. 7. In the 3rd proposition he proves that no 'number can be a mean between two numbers in the ratio known as *ἐπιμόριος*, the ratio, that is, of $n+1$ to n , where n is any integer greater than unity. Now, fortunately, Boethius, *De institutione musica*, iii. 11 (pp. 285—6, ed. Friedlein), has preserved a proof by Archytas of this same proposition; and the proof is substantially identical with that of Euclid. The two proofs are placed side by side in an article by Tannery (*Bibliotheca Mathematica*, vi. 3, 1905/6, p. 227). Archytas writes the smaller term of the proportion first (instead of the greater, as Euclid does). Let, he says, A, B be the "*superparticularis proportio*" (*ἐπιμόριον διάστημα* in Euclid). Take C, DE the smallest numbers which are in the ratio of A to B . [Here DE means $D + E$: and in this respect the notation is different from that of Euclid who, as usual, takes a line DF divided into two parts at G , GF corresponding to E , and DG to D , in Archytas' notation. The step of taking C, DE , the smallest numbers in the ratio of A to B , presupposes Eucl. vii. 33.] Then DE exceeds C by an aliquot part of itself and of C [cf. the definition of *ἐπιμόριος ἀριθμός* in Nicomachus, i. 19, 1]. Let D be the excess [i.e. E is supposed equal to C]. "I say that D is not a number but an unit."

For, if D is a number and a part of DE , it measures DE ; hence it measures E , that is, C . Thus D measures both C and DE , which is impossible; for the smallest numbers which are in the same ratio as any numbers are prime to one another. [This presupposes Eucl. vii. 22.] Therefore D is an unit; that is, DE exceeds C by an unit. Hence no number can be found which is a mean between two numbers C, DE . Therefore neither can any number be a mean between the original numbers A, B which are in the same ratio [this implies Eucl. vii. 20].

We have then here a clear indication of the existence at least as early as the date of Archytas (about 430—365 B.C.) of an *Elements of Arithmetic* in the form which we call Euclidean; and no doubt text-books of the sort existed even before Archytas, which probably Archytas himself and Eudoxus improved and developed in their turn.

(In *Platonis Timaeum commentaria*, 149 C), said that the difficulties of the passage of the *Timaeus* had misled some people into connecting it with the duplication of the cube, whereas it really referred to *similar* planes and solids with sides in *rational numbers*. Thirdly, I do not think that, under the supposition that the Delian problem is referred to, we get the required sense. The problem in that case is not that of finding two mean proportionals *between two cubes* but that of finding a second cube the content of which shall be equal to twice, or k times (where k is any number not a complete cube), the content of a given cube (a^3). Two mean proportionals are found, not between cubes, but between two *straight lines* in the ratio of 1 to k , or between a and ka . Unless k is a cube, there would be no point in saying that two means are necessary to connect 1 and k , and not one mean; for $\sqrt[3]{k}$ is no more natural than $\sqrt{k}$, and would be less natural in the case where k happened to be square. On the other hand, if k is a cube, so that it is a question of finding means between *cube numbers*, the dictum of Plato is perfectly intelligible; nor is any real difficulty caused by the generality of the statement that two means are *always* necessary to connect them, because any property enunciated generally of two cube numbers should obviously be true of cubes *as such*, that is, it must hold in the extreme case of two cubes which are *prime to one another*.

BOOK VII. PROPOSITIONS.

PROPOSITION I.

Two unequal numbers being set out, and the less being continually subtracted in turn from the greater, if the number which is left never measures the one before it until an unit is left, the original numbers will be prime to one another.

For, the less of two unequal numbers AB , CD being continually subtracted from the greater, let the number which is left never measure the one before it until an unit is left ;

I say that AB , CD are prime to one another, that is, that an unit alone measures AB , CD .

For, if AB , CD are not prime to one another, some number will measure them.

Let a number measure them, and let it be E ; let CD , measuring BF , leave FA less than itself,

let AF , measuring DG , leave GC less than itself, and let GC , measuring FH , leave an unit HA .

Since, then, E measures CD , and CD measures BF , therefore E also measures BF .

But it also measures the whole BA ; therefore it will also measure the remainder AF .

But AF measures DG ; therefore E also measures DG .



But it also measures the whole DC ·
therefore it will also measure the remainder CG .

But CG measures FH ;
therefore E also measures FH .

But it also measures the whole FA ;
therefore it will also measure the remainder, the unit AH ,
though it is a number : which is impossible.

Therefore no number will measure the numbers AB , CD ;
therefore AB , CD are prime to one another. [VII. Def. 12]

Q. E. D.

It is proper to remark here that the representation in Books VII. to IX. of numbers by straight lines is adopted by Heiberg from the MSS. The method of those editors who substitute *points* for lines is open to objection because it practically necessitates, in many cases, the use of specific numbers, which is contrary to Euclid's manner.

"Let CD , measuring BF , leave FA less than itself." This is a neat abbreviation for saying, measure along BA successive lengths equal to CD until a point F is reached such that the length FA remaining is less than CD ; in other words, let BF be the largest exact multiple of CD contained in BA .

Euclid's method in this proposition is an application to the particular case of prime numbers of the method of finding the greatest common measure of two numbers not prime to one another, which we shall find in the next proposition. With our notation, the method may be shown thus. Supposing the two numbers to be a , b , we have, say, .

$$\begin{array}{r} b) a(p \\ \underline{pb} \\ c) b(q \\ \underline{qc} \\ d) c(r \\ \underline{rd} \\ 1 \end{array}$$

If now a , b are not prime to one another, they must have a common measure e , where e is some integer, not unity.

And since e measures a , b , it measures $a - pb$, i.e. c .

Again, since e measures b , c , it measures $b - qc$, i.e. d ,
and lastly, since e measures c , d , it measures $c - rd$, i.e. 1:
which is impossible.

Therefore there is no integer, except unity, that measures a , b , which are accordingly prime to one another.

Observe that Euclid assumes as an axiom that, if a , b are both divisible by c , so is $a - pb$. In the next proposition he assumes as an axiom that c will in the case supposed divide $a + pb$.

PROPOSITION 2.

Given two numbers not prime to one another, to find their greatest common measure.

Let AB , CD be the two given numbers not prime to one another.

Thus it is required to find the greatest common measure of AB , CD .

If now CD measures AB —and it also measures itself— CD is a common measure of CD , AB .

And it is manifest that it is also the greatest ; for no greater number than CD will measure CD .

But, if CD does not measure AB , then, the less of the numbers AB , CD being continually subtracted from the greater, some number will be left which will measure the one before it.

For an unit will not be left ; otherwise AB , CD will be prime to one another [VII. 1], which is contrary to the hypothesis.

Therefore some number will be left which will measure the one before it.

Now let CD , measuring BE , leave EA less than itself, let EA , measuring DF , leave FC less than itself, and let CF measure AE .

Since then, CF measures AE , and AE measures DF , therefore CF will also measure DF .

But it also measures itself ;
therefore it will also measure the whole CD .

But CD measures BE ;
therefore CF also measures BE .

But it also measures EA ;
therefore it will also measure the whole BA .

But it also measures CD ;
therefore CF measures AB , CD .

Therefore CF is a common measure of AB , CD .



I say next that it is also the greatest.

For, if CF is not the greatest common measure of AB , CD , some number which is greater than CF will measure the numbers AB , CD .

Let such a number measure them, and let it be G .

Now, since G measures CD , while CD measures BE , G also measures BE .

But it also measures the whole BA ;
therefore it will also measure the remainder AE .

But AE measures DF ;
therefore G will also measure DF .

But it also measures the whole DC ;
therefore it will also measure the remainder CF , that is, the greater will measure the less : which is impossible.

Therefore no number which is greater than CF will measure the numbers AB , CD ;

therefore CF is the greatest common measure of AB , CD .

PORISM. From this it is manifest that, if a number measure two numbers, it will also measure their greatest common measure.

Q. E. D.,

Here we have the exact method of finding the greatest common measure given in the text-books of algebra, including the *reductio ad absurdum* proof that the number arrived at is not only a common measure but the *greatest* common measure. The process of finding the greatest common measure is simply shown thus :

$$\begin{array}{r} b) a (p \\ \underline{pb} \\ c) b (q \\ \underline{qc} \\ d) c (r \\ \underline{rd} \end{array}$$

We shall arrive, says Euclid, at some number, say d , which measures the one before it, i.e. such that $c = rd$. Otherwise the process would go on until we arrived at unity. This is impossible because in that case a , b would be prime to one another, which is contrary to the hypothesis.

Next, like the text-books of algebra, he goes on to show that d will be *some* common measure of a , b . For d measures c ;
therefore it measures $qc + d$, that is, b ,
and hence it measures $pb + c$, that is, a .

Lastly, he proves that d is the *greatest* common measure of a , b as follows.

Suppose that e is a common measure greater than d .

Then e , measuring a , b , must measure $a - pb$, or c .

Similarly e must measure $b - qc$, that is, d : which is impossible, since e is by hypothesis greater than d .

Therefore etc.

Euclid's proposition is thus *identical* with the algebraical proposition as generally given, e.g. in Todhunter's algebra, except that of course Euclid's numbers are integers.

Nicomachus gives the same rule (though without proving it) when he shows how to determine whether two given *odd* numbers are prime or not prime to one another, and, if they are not prime to one another, what is their common measure. We are, he says, to compare the numbers in turn by continually taking the less from the greater as many times as possible, then taking the remainder as many times as possible from the less of the original numbers, and so on; this process "will finish either at a unit or at some one and the same number," by which it is implied that the division of a greater number by a less is done by *separate subtractions* of the less. Thus, with regard to 21 and 49, Nicomachus says, "I subtract the less from the greater; 28 is left; then again I subtract from this the same 21 (for this is possible); 7 is left; I subtract this from 21, 14 is left; from which I again subtract 7 (for this is possible); 7 will be left, but 7 cannot be subtracted from 7." The last phrase is curious, but the meaning of it is obvious enough, as also the meaning of the phrase about ending "at one and the same number."

The proof of the Porism is of course contained in that part of the proposition which proves that G , a common measure different from CF , must measure CF . The supposition, thereby proved to be false, that G is greater than CF does not affect the validity of the proof that G measures CF in any case.

PROPOSITION 3.

Given three numbers not prime to one another, to find their greatest common measure.

Let A, B, C be the three given numbers not prime to one another;

thus it is required to find the greatest common measure of A, B, C .

For let the greatest common measure, D , of the two numbers A, B be taken;

[VII. 2]

then D either measures, or does not measure, C .

First, let it measure it.

But it measures A, B also;

therefore D measures A, B, C ;

therefore D is a common measure of A, B, C .

I say that it is also the greatest.



For, if D is not the greatest common measure of A, B, C , some number which is greater than D will measure the numbers A, B, C .

Let such a number measure them, and let it be E .

Since then E measures A, B, C ,

it will also measure A, B ;

therefore it will also measure the greatest common measure of A, B . [VII. 2, Por.]

But the greatest common measure of A, B is D ;
therefore E measures D , the greater the less: which is impossible.

Therefore no number which is greater than D will measure the numbers A, B, C ;

therefore D is the greatest common measure of A, B, C .

Next, let D not measure C ;

I say first that C, D are not prime to one another.

For, since A, B, C are not prime to one another, some number will measure them.

Now that which measures A, B, C will also measure A, B , and will measure D , the greatest common measure of A, B . [VII. 2, Por.]

But it measures C also;

therefore some number will measure the numbers D, C ;

therefore D, C are not prime to one another.

Let then their greatest common measure E be taken.

[VII. 2]

Then, since E measures D ,

and D measures A, B ,

therefore E also measures A, B .

But it measures C also;

therefore E measures A, B, C ;

therefore E is a common measure of A, B, C .

I say next that it is also the greatest.

For, if E is not the greatest common measure of A, B, C , some number which is greater than E will measure the numbers A, B, C .

Let such a number measure them, and let it be F .

Now, since F measures A, B, C ,
 it also measures A, B ;
 therefore it will also measure the greatest common measure
 of A, B . [VII. 2, Por.]

But the greatest common measure of A, B is D ;
 therefore F measures D .

And it measures C also;
 therefore F measures D, C ;
 therefore it will also measure the greatest common measure
 of D, C . [VII. 2, Por.]

But the greatest common measure of D, C is E ;
 therefore F measures E , the greater the less: which is
 impossible.

Therefore no number which is greater than E will measure
 the numbers A, B, C ;
 therefore E is the greatest common measure of A, B, C .

Q. E. D.

Euclid's proof is here longer than we should make it because he
 distinguishes two cases, the simpler of which is really included in the other.

Having taken the greatest common measure, say d , of a, b , two of the
 three given numbers a, b, c , he distinguishes the cases

- (1) in which d measures c ,
- (2) in which d does not measure c .

In the first case the greatest common measure of d, c is d itself; in the
 second case it has to be found by a repetition of the process of VII. 2. In
 either case the greatest common measure of a, b, c is the greatest common
 measure of d, c .

But, after disposing of the simpler case, Euclid thinks it necessary to
 prove that, if d does not measure c , d and c must necessarily have a greatest
 common measure. This he does by means of the original hypothesis that
 a, b, c are not prime to one another. Since they are not prime to one another,
 they must have a common measure; any common measure of a, b is a measure
 of d , and therefore any common measure of a, b, c is a common measure of
 d, c ; hence d, c must have a common measure, and are therefore not prime to
 one another.

The proofs of cases (1) and (2) repeat exactly the same argument as we
 saw in VII. 2, and it is proved separately for d in case (1) and e in case (2),
 where e is the greatest common measure of d, c ,

- (a) that it is a common measure of a, b, c ,
- (β) that it is the *greatest* common measure.

Heron remarks (an-Nairizi, ed. Curtze, p. 191) that the method does
 not only enable us to find the greatest common measure of *three* numbers;
 it can be used to find the greatest common measure of as many numbers

as we please. This is because any number measuring two numbers also measures their greatest common measure; and hence we can find the G.C.M. of pairs, then the G.C.M. of pairs of these, and so on, until only two numbers are left and we find the G.C.M. of these. Euclid tacitly assumes this extension in VII. 33, where he takes the greatest common measure of *as many numbers as we please*.

PROPOSITION 4.

Any number is either a part or parts of any number, the less of the greater.

Let A, BC be two numbers, and let BC be the less;
I say that BC is either a part, or parts, of A .

For A, BC are either prime to one another or not.

First, let A, BC be prime to one another.

Then, if BC be divided into the units in it, each unit of those in BC will be some part of A ; so that BC is parts of A .

Next let A, BC not be prime to one another; then BC either measures, or does not measure, A .

If now BC measures A , BC is a part of A .

But, if not, let the greatest common measure D of A, BC be taken;

and let BC be divided into the numbers equal to D , namely BE, EF, FC .

Now, since D measures A , D is a part of A .

But D is equal to each of the numbers BE, EF, FC ; therefore each of the numbers BE, EF, FC is also a part of A ; so that BC is parts of A .

Therefore etc.

Q. E. D.

The meaning of the enunciation is of course that, if a, b be two numbers of which b is the less, then b is either a *submultiple* or some *proper fraction* of a .

(1) If a, b be prime to one another, divide each into its units; then b contains b of the same parts of which a contains a . Therefore b is "parts" or a *proper fraction* of a .

(2) If a, b be not prime to one another, either b measures a , in which case b is a submultiple or "part" of a , or, if g be the greatest common measure of a, b , we may put $a = mg$ and $b = ng$, and b will contain n of the same parts (g) of which a contains m , so that b is again "parts," or a *proper fraction*, of a .



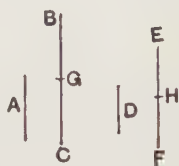
PROPOSITION 5.

If a number be a part of a number, and another be the same part of another, the sum will also be the same part of the sum that the one is of the one.

For let the number A be a part of BC ,
and another, D , the same part of another EF that A is of BC ;
I say that the sum of A , D is also the same
part of the sum of BC , EF that A is of BC .

For since, whatever part A is of BC , D
is also the same part of EF ,

therefore, as many numbers as there are in
 BC equal to A , so many numbers are there
also in EF equal to D .



Let BC be divided into the numbers equal to A , namely
 BG , GC ,

and EF into the numbers equal to D , namely EH , HF ;
then the multitude of BG , GC will be equal to the multitude
of EH , HF .

And, since BG is equal to A , and EH to D ,
therefore BG , EH are also equal to A , D .

For the same reason
 GC , HF are also equal to A , D .

Therefore, as many numbers as there are in BC equal to
 A , so many are there also in BC , EF equal to A , D .

Therefore, whatever multiple BC is of A , the same multiple
also is the sum of BC , EF of the sum of A , D .

Therefore, whatever part A is of BC , the same part also
is the sum of A , D of the sum of BC , EF .

Q. E. D.

If $a = \frac{1}{n} b$, and $c = \frac{1}{n} d$, then

$$a + c = \frac{1}{n} (b + d).$$

The proposition is of course true for any quantity of pairs of numbers
similarly related, as is the next proposition also; and both propositions are
used in the extended form in VII. 9, 10.

PROPOSITION 6.

If a number be parts of a number, and another be the same parts of another, the sum will also be the same parts of the sum that the one is of the one.

For let the number AB be parts of the number C , and another, DE , the same parts of another, F , that AB is of C ;

I say that the sum of AB , DE is also the same parts of the sum of C , F that AB is of C .



For since, whatever parts AB is of C , DE is also the same parts of F ,

therefore, as many parts of C as there are in AB , so many parts of F are there also in DE .

Let AB be divided into the parts of C , namely AG , GB , and DE into the parts of F , namely DH , HE ; thus the multitude of AG , GB will be equal to the multitude of DH , HE .

And since, whatever part AG is of C , the same part is DH of F also,

therefore, whatever part AG is of C , the same part also is the sum of AG , DH of the sum of C , F . [VII. 5]

For the same reason,

whatever part GB is of C , the same part also is the sum of GB , HE of the sum of C , F .

Therefore, whatever parts AB is of C , the same parts also is the sum of AB , DE of the sum of C , F .

Q. E. D.

If $a = \frac{m}{n} b$, and $c = \frac{m}{n} d$,

then $a + c = \frac{m}{n} (b + d)$.

More generally, if

$$a = \frac{m}{n} b, c = \frac{m}{n} d, e = \frac{m}{n} f, \dots$$

then $(a + c + e + g + \dots) = \frac{m}{n} (b + d + f + h + \dots)$.

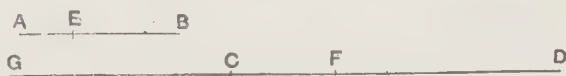
In Euclid's proposition $m < n$, but the generality of the result is of course not affected. This proposition and the last are complementary to v. 1, which proves the corresponding result with *multiple* substituted for "*part*" or "*parts*."

PROPOSITION 7.

If a number be that part of a number, which a number subtracted is of a number subtracted, the remainder will also be the same part of the remainder that the whole is of the whole.

For let the number AB be that part of the number CD which AE subtracted is of CF subtracted;

I say that the remainder EB is also the same part of the remainder FD that the whole AB is of the whole CD .



For, whatever part AE is of CF , the same part also let EB be of CG .

Now since, whatever part AE is of CF , the same part also is EB of CG ,

therefore, whatever part AE is of CF , the same part also is AB of GF . [VII. 5]

But, whatever part AE is of CF , the same part also, by hypothesis, is AB of CD ;

therefore, whatever part AB is of GF , the same part is it of CD also;

therefore GF is equal to CD .

Let CF be subtracted from each;

therefore the remainder GC is equal to the remainder FD .

Now since, whatever part AE is of CF , the same part also is EB of GC ,

while GC is equal to FD ,

therefore, whatever part AE is of CF , the same part also is EB of FD .

But, whatever part AE is of CF , the same part also is AB of CD ;

therefore also the remainder EB is the same part of the remainder FD that the whole AB is of the whole CD .

Q. E. D.

If $a = \frac{1}{n}b$ and $c = \frac{1}{n}d$, we are to prove that

$$a - c = \frac{1}{n}(b - d),$$

a result differing from that of VII. 5 in that *minus* is substituted for *plus*. Euclid's method is as follows.

Suppose that e is taken such that

$$a - c = \frac{1}{n}e. \dots\dots\dots(1)$$

Now

$$c = \frac{1}{n}d.$$

Therefore

$$a = \frac{1}{n}(d + e), \quad [\text{VII. 5}]$$

whence, from the hypothesis, $d + e = b$,

so that $e = b - d$,

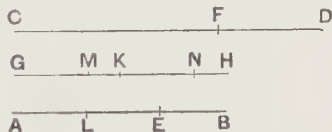
and, substituting this value of e in (1), we have

$$a - c = \frac{1}{n}(b - d).$$

PROPOSITION 8.

If a number be the same parts of a number that a number subtracted is of a number subtracted, the remainder will also be the same parts of the remainder that the whole is of the whole.

For let the number AB be the same parts of the number CD that AE subtracted is of CF subtracted;



I say that the remainder EB is also the same parts of the remainder FD that the whole AB is of the whole CD .

For let GH be made equal to AB .

Therefore, whatever parts GH is of CD , the same parts also is AE of CF .

Let GH be divided into the parts of CD , namely GK, KH , and AE into the parts of CF , namely AL, LE ; thus the multitude of GK, KH will be equal to the multitude of AL, LE .

Now since, whatever part GK is of CD , the same part also is AL of CF ,

while CD is greater than CF ,

therefore GK is also greater than AL .

Let GM be made equal to AL .

Therefore, whatever part GK is of CD , the same part also is GM of CF ;

therefore also the remainder MK is the same part of the remainder FD that the whole GK is of the whole CD . [VII. 7]

Again, since, whatever part KH is of CD , the same part also is EL of CF ,

while CD is greater than CF ,

therefore HK is also greater than EL .

Let KN be made equal to EL .

Therefore, whatever part KH is of CD , the same part also is KN of CF ;

therefore also the remainder NH is the same part of the remainder FD that the whole KH is of the whole CD .

[VII. 7]

But the remainder MK was also proved to be the same part of the remainder FD that the whole GK is of the whole CD ;

therefore also the sum of MK , NH is the same parts of DF that the whole HG is of the whole CD .

But the sum of MK , NH is equal to EB ,
and HG is equal to BA ;

therefore the remainder EB is the same parts of the remainder FD that the whole AB is of the whole CD .

Q. E. D.

$$\text{If} \quad a = \frac{m}{n}b \quad \text{and} \quad c = \frac{m}{n}d, \quad (m < n)$$

$$\text{then} \quad a - c = \frac{m}{n}(b - d).$$

Euclid's proof amounts to the following.

Take e equal to $\frac{1}{n}b$, and f equal to $\frac{1}{n}d$.

Then since, by hypothesis, $b > d$,

$$e > f,$$

and, by VII. 7,

$$e - f = \frac{1}{n}(b - d).$$

Repeat this for all the parts equal to e and f that there are in a, b respectively, and we have, by addition (a, b containing m of such parts respectively),

$$m(e-f) = \frac{m}{n}(b-d).$$

But

$$m(e-f) = a-c.$$

Therefore

$$a-c = \frac{m}{n}(b-d).$$

The propositions vii. 7, 8 are complementary to v. 5 which gives the corresponding result with *multiple* in the place of "part" or "parts."

PROPOSITION 9.

If a number be a part of a number, and another be the same part of another, alternately also, whatever part or parts the first is of the third, the same part, or the same parts, will the second also be of the fourth.

For let the number A be a part of the number BC , and another, D , the same part of another, EF , that A is of BC ;

I say that, alternately also, whatever part or parts A is of D , the same part or parts is BC of EF also.



For since, whatever part A is of BC , the same part also is D of EF ,

therefore, as many numbers as there are in BC equal to A , so many also are there in EF equal to D .

Let BC be divided into the numbers equal to A , namely BG, GC ,

and EF into those equal to D , namely EH, HF ;

thus the multitude of BG, GC will be equal to the multitude of EH, HF .

Now, since the numbers BG, GC are equal to one another, and the numbers EH, HF are also equal to one another, while the multitude of BG, GC is equal to the multitude of EH, HF ,

therefore, whatever part or parts BG is of EH , the same part or the same parts is GC of HF also;

so that, in addition, whatever part or parts BG is of EH , the same part also, or the same parts, is the sum BC of the sum EF .

[vii. 5, 6]

But BG is equal to A , and EH to D ;
therefore, whatever part or parts A is of D , the same part or
the same parts is BC of EF also.

Q. E. D.

If $a = \frac{1}{n} b$ and $c = \frac{1}{n} d$, then, whatever fraction ("part" or "parts") a is of c , the same fraction will b be of d .

Dividing b into each of its parts equal to a , and d into each of its parts equal to c , it is clear that, whatever fraction one of the parts a is of one of the parts c , the same fraction is any other of the parts a of any other of the parts c .

And the number of the parts a is equal to the number of the parts c , viz. n .

Therefore, by VII. 5, 6, na is the same fraction of nc that a is of c , i.e. b is the same fraction of d that a is of c .

PROPOSITION 10.

If a number be parts of a number, and another be the same parts of another, alternately also, whatever parts or part the first is of the third, the same parts or the same part will the second also be of the fourth.

For let the number AB be parts of the number C ,
and another, DE , the same parts of another,
 F ;

I say that, alternately also, whatever parts or
part AB is of DE , the same parts or the
same part is C of F also.

For since, whatever parts AB is of C ,
the same parts also is DE of F ,
therefore, as many parts of C as there are
in AB , so many parts also of F are there in DE .

Let AB be divided into the parts of C , namely AG , GB ,
and DE into the parts of F , namely DH , HE ;
thus the multitude of AG , GB will be equal to the multitude
of DH , HE .

Now since, whatever part AG is of C , the same part also
is DH of F ,

alternately also, whatever part or parts AG is of DH ,
the same part or the same parts is C of F also.

[VII. 9]

For the same reason also,
whatever part or parts GB is of HE , the same part or the
same parts is C of F also;



so that, in addition, whatever parts or part AB is of DE , the same parts also, or the same part, is C of F . [VII. 5, 6]

Q. E. D.

If $a = \frac{m}{n}b$ and $c = \frac{m}{n}d$, then, whatever fraction a is of b , the same fraction is b of d .

To prove this, a is divided into its m parts equal to b/n , and c into its m parts equal to d/n .

Then, by VII. 9, whatever fraction one of the m parts of a is of one of the m parts of c , the same fraction is b of d .

And, by VII. 5, 6, whatever fraction one of the m parts of a is of one of the m parts of c , the same fraction is the sum of the parts of a (that is, a) of the sum of the parts of c (that is, c).

Whence the result follows.

In the Greek text, after the words "so that, in addition" in the last line but one, is an additional explanation making the reference to VII. 5, 6 clearer, as follows: "whatever part or parts AG is of DH , the same part or the same parts is GB of HE also ;

therefore also, whatever part or parts AG is of DH , the same part or the same parts is AB of DE also. [VII. 5, 6]

But it was proved that, whatever part or parts AG is of DH , the same part or the same parts is C of F also ;

therefore also" etc. as in the last two lines of the text.

Heiberg concludes, on the authority of P, which only has the words in the margin in a later hand, that they may be attributed to Theon.

PROPOSITION 11.

If, as whole is to whole, so is a number subtracted to a number subtracted, the remainder will also be to the remainder as whole to whole.

As the whole AB is to the whole CD , so let AE subtracted be to CF subtracted ;

I say that the remainder EB is also to the remainder FD as the whole AB to the whole CD .

Since, as AB is to CD , so is AE to CF , whatever part or parts AB is of CD , the same part or the same parts is AE of CF also ; [VII. Def. 20]

Therefore also the remainder EB is the same part or parts of FD that AB is of CD .



[VII. 7, 8]

Therefore, as EB is to FD , so is AB to CD . [VII. Def. 20]

Q. E. D.

It will be observed that, in dealing with the proportions in Props. 11—13, Euclid only contemplates the case where the first number is "a part" or "parts" of the second, while in Prop. 13 he assumes the first to be "a part"

or "parts" of the third also; that is, the first number is in all three propositions assumed to be less than the second, and in Prop. 13 less than the third also. Yet the figures in Props. 11 and 13 are inconsistent with these assumptions. If the facts are taken to correspond to the figures in these propositions, it is necessary to take account of the other possibilities involved in the definition of proportion (VII. Def. 20), that the first number may also be a multiple, or a multiple *plus* "a part" or "parts" (including *once* as a multiple in this case), of each number with which it is compared. Thus a number of different cases would have to be considered. The remedy is to make the ratio which is in the lower terms the first ratio, and to invert the ratios, if necessary, in order to make "a part" or "parts" literally apply.

If $a : b = c : d$, ($a > c$, $b > d$)
 then $(a - c) : (b - d) = a : b$.

This proposition for numbers corresponds to v. 19 for magnitudes. The enunciation is the same except that the masculine (agreeing with ἀριθμός) takes the place of the neuter (agreeing with μέγεθος).

The proof is no more than a combination of the arithmetical definition of proportion (VII. Def. 20) with the results of VII. 7, 8. The language of proportions is turned into the language of fractions by Def. 20; the results of VII. 7, 8 are then used and the language retransformed by Def. 20 into the language of proportions.

PROPOSITION 12.

If there be as many numbers as we please in proportion, then, as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents.

Let A, B, C, D be as many numbers as we please in proportion, so that,

as A is to B , so is C to D ;

I say that, as A is to B , so are A, C to B, D .

For since, as A is to B , so is C to D ,
 whatever part or parts A is of B , the same part
 or parts is C of D also. [VII. Def. 20]

Therefore also the sum of A, C is the same
 part or the same parts of the sum of B, D that A is of B .

[VII. 5, 6]

Therefore, as A is to B , so are A, C to B, D . [VII. Def. 20]

If $a : a' = b : b' = c : c' = \dots$,
 then each ratio is equal to $(a + b + c + \dots) : (a' + b' + c' + \dots)$.

The proposition corresponds to v. 12, and the enunciation is word for word the same with that of v. 12 except that ἀριθμός takes the place of μέγεθος.

Again the proof merely connects the arithmetical definition of proportion (VII. Def. 20) with the results of VII. 5, 6, which are quoted as true for any number of number of numbers, and not merely for two numbers as in the enunciations of VII. 5, 6.

PROPOSITION 13.

If four numbers be proportional, they will also be proportional alternately.

Let the four numbers A, B, C, D be proportional, so that,
as A is to B , so is C to D ;

I say that they will also be proportional alternately, so that,
as A is to C , so will B be to D .

For since, as A is to B , so is C to D ,
therefore, whatever part or parts A is of B ,
the same part or the same parts is C of D also.

[VII. Def. 20]

Therefore, alternately, whatever part or
parts A is of C , the same part or the same
parts is B of D also.

Therefore, as A is to C , so is B to D .



[VII. 10]

[VII. Def. 20]

Q. E. D.

If $a : b = c : d$,
then, alternately, $a : c = b : d$.

The proposition corresponds to v. 16 for magnitudes, and the proof consists in connecting VII. Def. 20 with the result of VII. 10.

PROPOSITION 14.

If there be as many numbers as we please, and others equal to them in multitude, which taken two and two are in the same ratio, they will also be in the same ratio ex aequali.

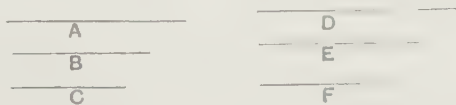
Let there be as many numbers as we please A, B, C ,
and others equal to them in multitude D, E, F , which taken
two and two are in the same ratio, so that,

as A is to B , so is D to E ,

and, as B is to C , so is E to F ;

I say that, *ex aequali*,

as A is to C , so also is D to F .



For, since, as A is to B , so is D to E ,
therefore, alternately,

as A is to D , so is B to E .

[VII. 13]

Again, since, as B is to C , so is E to F ,
therefore, alternately,

as B is to E , so is C to F . [VII. 13]

But, as B is to E , so is A to D ;
therefore also, as A is to D , so is C to F .

Therefore, alternately,
as A is to C , so is D to F . [id.]

If $a : b = d : e$,
and $b : c = e : f$,
then, *ex aequali*, $a : c = d : f$;
and the same is true however many successive numbers are so related.

The proof is simplicity itself.

By VII. 13, alternately, $a : d = b : e$,
and $b : e = c : f$.

Therefore $a : d = c : f$,
and, again alternately, $a : c = d : f$.

Observe that this simple method cannot be used to prove the corresponding proposition for magnitudes, v. 22, although v. 22 has been preceded by the two propositions in that Book corresponding to the propositions used here, viz. v. 16 and v. 11. The reason of this is that this method would only prove v. 22 for six magnitudes *all of the same kind*, whereas the magnitudes in v. 22 are not subject to this limitation.

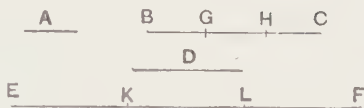
Heiberg remarks in a note on VII. 19 that, while Euclid has proved several propositions of Book v. over again, by a separate proof, for numbers, he has neglected to do so in certain cases; e.g., he often uses v. 11 in these propositions of Book VII., v. 9 in VII. 19, v. 7 in the same proposition, and so on. Thus Heiberg would apparently suppose Euclid to use v. 11 in the last step of the present proof (*Ratios which are the same with the same ratio are also the same with one another*). I think it preferable to suppose that Euclid regarded the last step as axiomatic; since, by the definition of proportion, the first number is the same multiple or the same part or the same parts of the second that the third is of the fourth: the assumption is no more than an assumption that the numbers or proper fractions which are respectively equal to the same number or proper fraction are equal to one another.

Though the proposition is only proved of six numbers, the extension to as many as we please (as expressed in the enunciation) is obvious.

PROPOSITION 15.

If an unit measure any number, and another number measure any other number the same number of times, alternately also, the unit will measure the third number the same number of times that the second measures the fourth.

For let the unit A measure any number BC ,
and let another number D
measure any other number EF the same number of times ;



I say that, alternately also, the unit A measures the number D the same number of times that BC measures EF .

For, since the unit A measures the number BC the same number of times that D measures EF ,
therefore, as many units as there are in BC , so many numbers equal to D are there in EF also.

Let BC be divided into the units in it, BG , GH , HC ,
and EF into the numbers EK , KL , LF equal to D .

Thus the multitude of BG , GH , HC will be equal to the multitude of EK , KL , LF .

And, since the units BG , GH , HC are equal to one another,
and the numbers EK , KL , LF are also equal to one another,
while the multitude of the units BG , GH , HC is equal to the multitude of the numbers EK , KL , LF ,

therefore, as the unit BG is to the number EK , so will the unit GH be to the number KL , and the unit HC to the number LF .

Therefore also, as one of the antecedents is to one of the consequents, so will all the antecedents be to all the consequents ;

[VII. 12]

therefore, as the unit BG is to the number EK , so is BC to EF .

But the unit BG is equal to the unit A ,
and the number EK to the number D .

Therefore, as the unit A is to the number D , so is BC to EF .

Therefore the unit A measures the number D the same number of times that BC measures EF .

Q. E. D.

If there be four numbers 1 , m , a , ma (such that 1 measures m the same number of times that a measures ma), 1 measures a the same number of times that m measures ma .

Except that the first number is unity and the numbers are said to *measure* instead of being a *part* of others, this proposition and its proof do not differ from VII. 9 ; in fact this proposition is a particular case of the other.

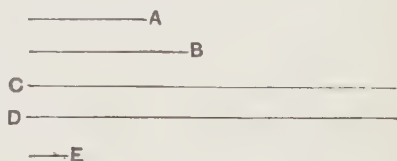
PROPOSITION 16.

If two numbers by multiplying one another make certain numbers, the numbers so produced will be equal to one another.

Let A, B be two numbers, and let A by multiplying B make C , and B by multiplying A make D ;

I say that C is equal to D .

For, since A by multiplying B has made C ,
therefore B measures C according to the units in A .



But the unit E also measures the number A according to the units in it;

therefore the unit E measures A the same number of times that B measures C .

Therefore, alternately, the unit E measures the number B the same number of times that A measures C . [VII. 15]

Again, since B by multiplying A has made D ,
therefore A measures D according to the units in B .

But the unit E also measures B according to the units in it;

therefore the unit E measures the number B the same number of times that A measures D .

But the unit E measured the number B the same number of times that A measures C ;

therefore A measures each of the numbers C, D the same number of times.

Therefore C is equal to D .

Q. E. D.

2. **The numbers so produced.** The Greek has $\alpha\iota\ \gamma\epsilon\gamma\eta\mu\epsilon\upsilon\omicron\iota\ \epsilon\kappa\ \alpha\upsilon\tau\omega\upsilon$, "the (numbers) produced from them." By "from them" Euclid means "from the original numbers," though this is not very clear even in the Greek. I think ambiguity is best avoided by leaving out the words.

This proposition proves that, *if any numbers be multiplied together, the order of multiplication is indifferent*, or $ab = ba$.

It is important to get a clear understanding of what Euclid means when he speaks of *one number multiplying another*. VII. Def. 15 states that the effect of " a multiplying b " is taking a times b . We shall always represent " a times b " by ab and " b times a " by ba . This being premised, the proof that $ab = ba$ may be represented as follows in the language of proportions.

By VII. Def. 20, $1 : a = b : ab.$

Therefore, alternately, $1 : b = a : ab.$

[VII. 13]

Again, by VII. Def. 20, $1 : b = a : ba.$

Therefore $a : ab = a : ba,$
or $ab = ba.$

Euclid does not use the language of proportions but that of fractions or their equivalent measures, quoting VII. 15, a particular case of VII. 13 differently expressed, instead of VII. 13 itself.

PROPOSITION 17.

If a number by multiplying two numbers make certain numbers, the numbers so produced will have the same ratio as the numbers multiplied.

For let the number A by multiplying the two numbers B , C make D , E ;

I say that, as B is to C , so is D to E .

For, since A by multiplying B has made D ,
therefore B measures D according to the units in A .



But the unit F also measures the number A according to the units in it;

therefore the unit F measures the number A the same number of times that B measures D .

Therefore, as the unit F is to the number A , so is B to D .
[VII. Def. 20]

For the same reason,
as the unit F is to the number A , so also is C to E ;
therefore also, as B is to D , so is C to E .

Therefore, alternately, as B is to C , so is D to E . [VII. 13]
Q. E. D.

$$b : c = ab : ac.$$

In this case Euclid translates the language of measures into that of proportions, and the proof is exactly like that set out in the last note.

By VII. Def. 20, $1 : a = b : ab,$

and $1 : a = c : ac.$

Therefore $b : ab = c : ac,$

and, alternately, $b : c = ab : ac.$

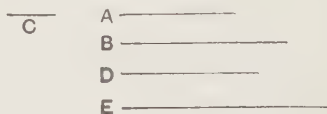
[VII. 13]

PROPOSITION 18.

If two numbers by multiplying any number make certain numbers, the numbers so produced will have the same ratio as the multipliers.

For let two numbers A, B by multiplying any number C make D, E ;

I say that, as A is to B , so is D to E .



For, since A by multiplying C has made D ,

therefore also C by multiplying A has made D . [VII. 16]

For the same reason also

C by multiplying B has made E .

Therefore the number C by multiplying the two numbers A, B has made D, E .

Therefore, as A is to B , so is D to E . [VII. 17]

It is here proved that $a : b = ac : bc$.

The argument is as follows.

$ac = ca$. [VII. 16]

Similarly $bc = cb$.

And $a : b = ca : cb$; [VII. 17]

therefore $a : b = ac : bc$.

PROPOSITION 19.

If four numbers be proportional, the number produced from the first and fourth will be equal to the number produced from the second and third; and, if the number produced from the first and fourth be equal to that produced from the second and third, the four numbers will be proportional.

Let A, B, C, D be four numbers in proportion, so that,
as A is to B , so is C to D ;

and let A by multiplying D make E , and let B by multiplying C make F ;

I say that E is equal to F .

For let A by multiplying C make G .

Since, then, A by multiplying C has made G , and by multiplying D has made E ,
the number A by multiplying the two numbers C, D has made G, E .

Therefore, as C is to D , so is G to E .

[VII. 17]

But, as C is to D , so is A to B ;
therefore also, as A is to B , so is G to E .

Again, since A by multiplying C has made G ,

but, further, B has also by multiplying C made F ,

the two numbers A, B by multiplying a certain number C have made G, F .

Therefore, as A is to B , so is G to F .

[VII. 18]

But further, as A is to B , so is G to E also ;

therefore also, as G is to E , so is G to F .

Therefore G has to each of the numbers E, F the same ratio ;

therefore E is equal to F .

[cf. v. 9]

Again, let E be equal to F ;

I say that, as A is to B , so is C to D .

For, with the same construction,

since E is equal to F ,

therefore, as G is to E , so is G to F .

[cf. v. 7]

But, as G is to E , so is C to D ,

[VII. 17]

and, as G is to F , so is A to B .

[VII. 18]

Therefore also, as A is to B , so is C to D .

Q. E. D.

If

$$a : b = c : d,$$

then

$$ad = bc; \text{ and conversely.}$$

The proof is equivalent to the following.

(1)

$$ac : ad = c : d$$

[VII. 17]

$$= a : b.$$

But

$$a : b = ac : bc.$$

[VII. 18]

Therefore

$$ac : ad = ac : bc,$$

or

$$ad = bc.$$

(2) Since

$$ad = bc,$$

$$ac : ad = ac : bc.$$

But

$$ac : ad = c : d,$$

[VII. 17]

and

$$ac : bc = a : b.$$

[VII. 18]

Therefore

$$a : b = c : d.$$

As indicated in the note on VII. 14 above, Heiberg regards Euclid as basing the inferences contained in the last step of part (1) of this proof and in the first step of part (2) on the propositions V. 9 and V. 7 respectively, since he has not proved those propositions separately for numbers in this Book. I prefer to suppose that he regarded the inferences as obvious and not needing proof, in view of the definition of numbers which are in proportion. E.g., if ac is the same fraction ("part" or "parts") of ad that ac is of bc , it is obvious that ad must be equal to bc .

Heiberg omits from his text here, and relegates to an Appendix, a proposition appearing in the manuscripts V, p, ϕ to the effect that, if *three* numbers be proportional, the product of the extremes is equal to the square of the mean, and conversely. It does not appear in P in the first hand, B has it in the margin only, and Campanus omits it, remarking that Euclid does not give the proposition about *three* proportionals as he does in VI. 17, since it is easily proved by the proposition just given. Moreover an-Nairizi quotes the proposition about three proportionals *as an observation* on VII. 19 probably due to Heron (who is mentioned by name in the preceding paragraph).

PROPOSITION 20.

The least numbers of those which have the same ratio with them measure those which have the same ratio the same number of times, the greater the greater and the less the less.

For let CD, EF be the least numbers of those which have the same ratio with A, B ;

I say that CD measures A the same number of times that EF measures B .

Now CD is not parts of A .

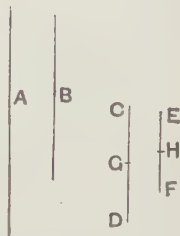
For, if possible, let it be so;

therefore EF is also the same parts of B that CD is of A .

[VII. 13 and Def. 20]

Therefore, as many parts of A as there are in CD , so many parts of B are there also in EF .

Let CD be divided into the parts of A , namely CG, GD , and EF into the parts of B , namely EH, HF ; thus the multitude of CG, GD will be equal to the multitude of EH, HF .



Now, since the numbers CG, GD are equal to one another, and the numbers EH, HF are also equal to one another, while the multitude of CG, GD is equal to the multitude of EH, HF ,

therefore, as CG is to EH , so is GD to HF .

Therefore also, as one of the antecedents is to one of the consequents, so will all the antecedents be to all the consequents. [VII. 12]

Therefore, as CG is to EH , so is CD to EF .

Therefore CG, EH are in the same ratio with CD, EF , being less than they :

which is impossible, for by hypothesis CD, EF are the least numbers of those which have the same ratio with them.

Therefore CD is not parts of A ;

therefore it is a part of it. [VII. 4]

And EF is the same part of B that CD is of A ;

[VII. 13 and Def. 20]

therefore CD measures A the same number of times that EF measures B .

Q. E. D.

If a, b are the least numbers among those which have the same ratio (i.e. if a/b is a fraction in its lowest terms), and c, d are any others in the same ratio, i.e. if

$$a : b = c : d,$$

then $a = \frac{1}{n}c$ and $b = \frac{1}{n}d$, where n is some integer.

The proof is by *reductio ad absurdum*, thus.

[Since $a < c$, a is some proper fraction ("part" or "parts") of c , by VII. 4.]

Now a cannot be equal to $\frac{m}{n}c$, where m is an integer less than n but greater than 1.

For, if $a = \frac{m}{n}c$, $b = \frac{m}{n}d$ also. [VII. 13 and Def. 20]

Take each of the m parts of a with each of the m parts of b , two and two ; the ratio of the members of all pairs is the same ratio $\frac{1}{m}a : \frac{1}{m}b$.

Therefore

$$\frac{1}{m}a : \frac{1}{m}b = a : b. \quad [\text{VII. 12}]$$

But $\frac{1}{m}a$ and $\frac{1}{m}b$ are respectively less than a, b and they are in the same ratio : which contradicts the hypothesis.

Hence a can only be "a part" of c , or

a is of the form $\frac{1}{n}c$,

and therefore

b is of the form $\frac{1}{n}d$.

Here also Heiberg omits a proposition which was no doubt interpolated by Theon (B, V, p, ϕ have it as VII. 22, but P only has it in the margin and in a later hand; Campanus also omits it) proving for numbers the *ex aequali* proposition when "the proportion is perturbed," i.e. (cf. enunciation of v. 22) if

$$a : b = e : f, \dots\dots\dots(1)$$

and

$$b : c = d : e, \dots\dots\dots(2)$$

then

$$a : c = d : f.$$

The proof (see Heiberg's Appendix) depends on VII. 19.

From (1) we have

$$af = be,$$

and from (2)

$$be = cd.$$

[VII. 19]

Therefore

$$af = cd,$$

and accordingly

$$a : c = d : f.$$

[VII. 19]

PROPOSITION 21.

Numbers prime to one another are the least of those which have the same ratio with them.

Let A, B be numbers prime to one another;
I say that A, B are the least of those which have the same ratio with them.

For, if not, there will be some numbers less than A, B which are in the same ratio with A, B .



Let them be C, D .

Since, then, the least numbers of those which have the same ratio measure those which have the same ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent,

[VII. 20]

therefore C measures A the same number of times that D measures B .

Now, as many times as C measures A , so many units let there be in E .

Therefore D also measures B according to the units in E .

And, since C measures A according to the units in E ,
therefore E also measures A according to the units in C .

[VII. 16]

For the same reason

E also measures B according to the units in D . [VII. 16]

Therefore E measures A, B which are prime to one another: which is impossible. [VII. Def. 12]

Therefore there will be no numbers less than A, B which are in the same ratio with A, B .

Therefore A, B are the least of those which have the same ratio with them.

Q. E. D.

In other words, if a, b are prime to one another, the ratio $a : b$ is "in its lowest terms."

The proof is equivalent to the following.

If not, suppose that c, d are the *least* numbers for which

$$a : b = c : d.$$

[Euclid only supposes *some* numbers c, d in the ratio of a to b such that $c < a$, and (consequently) $d < b$. It is however necessary to suppose that c, d are the *least* numbers in that ratio in order to enable VII. 20 to be used in the proof.]

Then [VII. 20] $a = mc$, and $b = md$, where m is some integer.

Therefore $a = cm, b = dm$, [VII. 16]

and m is a common measure of a, b , though these are prime to one another, which is impossible. [VII. Def. 12]

Thus the least numbers in the ratio of a to b cannot be less than a, b themselves.

Where I have quoted VII. 16 Heiberg regards the reference as being to VII. 5. I think the phraseology of the text combined with that of Def. 15 suggests the former rather than the latter.

PROPOSITION 22.

The least numbers of those which have the same ratio with them are prime to one another.

Let A, B be the least numbers of those which have the same ratio with them;

I say that A, B are prime to one another.

For, if they are not prime to one another, some number will measure them.

A _____
B _____
C _____
D _____
E _____

Let some number measure them, and let it be C .

And, as many times as C measures A , so many units let there be in D ,

and, as many times as C measures B , so many units let there be in E

Since C measures A according to the units in D ,
therefore C by multiplying D has made A . [VII. Def. 15]

For the same reason also
 C by multiplying E has made B .

Thus the number C by multiplying the two numbers D , E has made A , B ;

therefore, as D is to E , so is A to B ; [VII. 17]

therefore D , E are in the same ratio with A , B , being less than they : which is impossible.

Therefore no number will measure the numbers A , B .

Therefore A , B are prime to one another.

Q. E. D.

If $a : b$ is "in its lowest terms," a , b are prime to one another.

Again the proof is indirect.

If a , b are not prime to one another, they have some common measure c , and

$$a = mc, \quad b = nc.$$

Therefore $m : n = a : b$. [VII. 17 or 18]

But m , n are less than a , b respectively, so that $a : b$ is not in its lowest terms : which is contrary to the hypothesis.

Therefore etc.

PROPOSITION 23.

If two numbers be prime to one another, the number which measures the one of them will be prime to the remaining number.

Let A , B be two numbers prime to one another, and let any number C measure A ;

I say that C , B are also prime to one another.

For, if C , B are not prime to one another, some number will measure C , B .

Let a number measure them, and let it be D .

Since D measures C , and C measures A ,
therefore D also measures A .

But it also measures B ;



therefore D measures A, B which are prime to one another : which is impossible. [VII. Def. 12]

Therefore no number will measure the numbers C, B .

Therefore C, B are prime to one another.

Q. E. D.

If a, mb are prime to one another, b is prime to a . For, if not, some number d will measure both a and b , and therefore both a and mb : which is contrary to the hypothesis.

Therefore etc.

PROPOSITION 24.

If two numbers be prime to any number, their product also will be prime to the same.

For let the two numbers A, B be prime to any number C , and let A by multiplying B make D ;

I say that C, D are prime to one another.

For, if C, D are not prime to one another, some number will measure C, D .

Let a number measure them, and let it be E .

Now, since C, A are prime to one another,

and a certain number E measures C ,

therefore A, E are prime to one another. [VII. 23]

As many times, then, as E measures D , so many units let there be in F ;

therefore F also measures D according to the units in E .

[VII. 16]

Therefore E by multiplying F has made D . [VII. Def. 15]

But, further, A by multiplying B has also made D ;

therefore the product of E, F is equal to the product of A, B .

But, if the product of the extremes be equal to that of the means, the four numbers are proportional ; [VII. 19]

therefore, as E is to A , so is B to F .

But A, E are prime to one another,

numbers which are prime to one another are also the least of those which have the same ratio, [VII. 21]

and the least numbers of those which have the same ratio with them measure those which have the same ratio the same



number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent ; [VII. 20]

therefore E measures B .

But it also measures C ;

therefore E measures B, C which are prime to one another : which is impossible. [VII. Def. 12]

Therefore no number will measure the numbers C, D .

Therefore C, D are prime to one another.

Q. E. D.

1. their product. $\delta \epsilon \xi \alpha \upsilon \tau \omega \nu \gamma \epsilon \nu \theta \mu \epsilon \nu \omicron \varsigma$, literally "the (number) produced from them," will henceforth be translated as "their product."

If a, b are both prime to c , then ab, c are prime to one another.

The proof is again by *reductio ad absurdum*.

If ab, c are not prime to one another, let them be measured by a and be equal to md, nd , say, respectively.

Now, since a, c are prime to one another and d measures c ,

a, d are prime to one another. [VII. 23]

But, since $ab = md$,

$d : a = b : m$. [VII. 19]

Therefore [VII. 20] d measures b ,

or $b = pd$, say.

But $c = nd$.

Therefore d measures both b and c , which are therefore not prime to one another : which is impossible.

Therefore etc.

PROPOSITION 25.

If two numbers be prime to one another, the product of one of them into itself will be prime to the remaining one.

Let A, B be two numbers prime to one another, and let A by multiplying itself make C ;

I say that B, C are prime to one another.

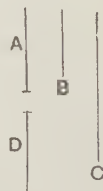
For let D be made equal to A .

Since A, B are prime to one another, and A is equal to D ,

therefore D, B are also prime to one another.

Therefore each of the two numbers D, A is prime to B ;

therefore the product of D, A will also be prime to B . [VII. 24]



But the number which is the product of D , A is C .

Therefore C , B are prime to one another. Q. E. D.

1. the product of one of them into itself. The Greek, $\delta\ \epsilon\kappa\ \tau\omicron\upsilon\ \epsilon\kappa\delta\varsigma\ \alpha\upsilon\tau\omega\kappa\gamma\epsilon\nu\delta\mu\epsilon\nu\omicron\varsigma$, literally "the number produced from the one of them," leaves "multiplied into itself" to be understood.

If a , b are prime to one another,

a^2 is prime to b .

Euclid takes d equal to a , so that d , a are both prime to b .

Hence, by VII. 24, da , i.e. a^2 , is prime to b .

The proposition is a particular case of the preceding proposition: and the method of proof is by substitution of different numbers in the result of that proposition.

PROPOSITION 26.

If two numbers be prime to two numbers, both to each, their products also will be prime to one another.

For let the two numbers A , B be prime to the two numbers C , D ; both to each, and let A by multiplying B make E , and let C by multiplying D make F ;

A _____	C _____
B _____	D _____
E _____	
F _____	

I say that E , F are prime to one another.

For, since each of the numbers A , B is prime to C , therefore the product of A , B will also be prime to C . [VII. 24]

But the product of A , B is E ;
therefore E , C are prime to one another.

For the same reason
 E , D are also prime to one another.

Therefore each of the numbers C , D is prime to E .

Therefore the product of C , D will also be prime to E .

[VII. 24]

But the product of C , D is F .

Therefore E , F are prime to one another. Q. E. D.

If both a and b are prime to each of two numbers c , d , then ab , cd will be prime to one another.

Since a , b are both prime to c ,

ab , c are prime to one another. [VII. 24]

Similarly ab , d are prime to one another.

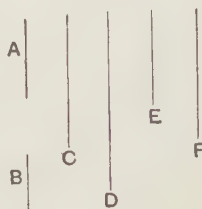
Therefore c , d are both prime to ab ,

and so therefore is cd . [VII. 24]

PROPOSITION 27.

If two numbers be prime to one another, and each by multiplying itself make a certain number, the products will be prime to one another; and, if the original numbers by multiplying the products make certain numbers, the latter will also be prime to one another [and this is always the case with the extremes].

Let A, B be two numbers prime to one another, let A by multiplying itself make C , and by multiplying C make D , and let B by multiplying itself make E , and by multiplying E make F ;
I say that both C, E and D, F are prime to one another.



For, since A, B are prime to one another, and A by multiplying itself has made C , therefore C, B are prime to one another.

[VII. 25]

Since then C, B are prime to one another, and B by multiplying itself has made E , therefore C, E are prime to one another.

[*id.*]

Again, since A, B are prime to one another, and B by multiplying itself has made E , therefore A, E are prime to one another.

[*id.*]

Since then the two numbers A, C are prime to the two numbers B, E , both to each, therefore also the product of A, C is prime to the product of B, E .

[VII. 26]

And the product of A, C is D , and the product of B, E is F .

Therefore D, F are prime to one another.

Q. E. D.

If a, b are prime to one another, so are a^2, b^2 and so are a^3, b^3 ; and, generally, a^n, b^n are prime to one another.

The words in the enunciation which assert the truth of the proposition for *any* powers are suspected and bracketed by Heiberg because (1) in $\pi\epsilon\rho\iota\ \tau\omicron\upsilon\varsigma\ \acute{\alpha}\kappa\rho\omicron\upsilon\varsigma$ the use of $\acute{\alpha}\kappa\rho\omicron\iota$ is peculiar, for it can only mean "the last products," and (2) the words have nothing corresponding to them in the proof, much less is the generalisation proved. Campanus omits the words in the enuncia-

tion, though he adds to the proof a remark that the proposition is true of any, the same or different, powers of a, b . Heiberg concludes that the words are an interpolation of date earlier than Theon.

Euclid's proof amounts to this.

Since a, b are prime to one another, so are a^2, b [VII. 25], and therefore also a^2, b^2 . [VII. 25]

Similarly [VII. 25] a, b^2 are prime to one another.

Therefore a, a^2 and b, b^2 satisfy the description in the enunciation of VII. 26.

Hence a^2, b^2 are prime to one another.

PROPOSITION 28.

If two numbers be prime to one another, the sum will also be prime to each of them; and, if the sum of two numbers be prime to any one of them, the original numbers will also be prime to one another.

For let two numbers AB, BC prime to one another be added;

I say that the sum AC is also prime to each of the numbers AB, BC .



For, if CA, AB are not prime to one another,

some number will measure CA, AB .

Let a number measure them, and let it be D .

Since then D measures CA, AB ,

therefore it will also measure the remainder BC .

But it also measures BA ;

therefore D measures AB, BC which are prime to one another: which is impossible. [VII. Def. 12]

Therefore no number will measure the numbers CA, AB ; therefore CA, AB are prime to one another.

For the same reason

AC, CB are also prime to one another.

Therefore CA is prime to each of the numbers AB, BC .

Again, let CA, AB be prime to one another;

I say that AB, BC are also prime to one another.

For, if AB, BC are not prime to one another,

some number will measure AB, BC .

Let a number measure them, and let it be D .

Now, since D measures each of the numbers AB , BC , it will also measure the whole CA .

But it also measures AB ;

therefore D measures CA , AB which are prime to one another: which is impossible. [VII. Def. 12]

Therefore no number will measure the numbers AB , BC .

Therefore AB , BC are prime to one another.

Q. E. D.

If a , b are prime to one another, $a + b$ will be prime to both a and b ; and conversely.

For suppose $(a + b)$, a are not prime to one another. They must then have some common measure d .

Therefore d also divides the difference $(a + b) - a$, or b , as well as a ; and therefore a , b are not prime to one another: which is contrary to the hypothesis.

Therefore $a + b$ is prime to a .

Similarly $a + b$ is prime to b .

The converse is proved in the same way.

Heiberg remarks on Euclid's assumption that, if c measures both a and b , it also measures $a \pm b$. But it has already (VII. 1, 2) been assumed, more generally, as an axiom that, in the case supposed, c measures $a \pm b$.

PROPOSITION 29.

Any prime number is prime to any number which it does not measure.

Let A be a prime number, and let it not measure B ;
I say that B , A are prime to one another.

For, if B , A are not prime to one another,

some number will measure them.

Let C measure them.

Since C measures B ,

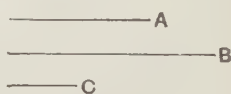
and A does not measure B ,

therefore C is not the same with A .

Now, since C measures B , A ,

therefore it also measures A which is prime, though it is not the same with it:

which is impossible.



Therefore no number will measure B , A .

Therefore A , B are prime to one another.

Q. E. D.

If a is prime and does not measure b , then a , b are prime to one another.
The proof is self-evident.

PROPOSITION 30.

If two numbers by multiplying one another make some number, and any prime number measure the product, it will also measure one of the original numbers.

For let the two numbers A , B by multiplying one another make C , and let any prime number D measure C ;

I say that D measures one of the numbers A , B .

For let it not measure A .

Now D is prime;

therefore A , D are prime to one another.

[vii. 29]

And, as many times as D measures C , so many units let there be in E .

Since then D measures C according to the units in E , therefore D by multiplying E has made C .

[vii. Def. 15]

Further, A by multiplying B has also made C ;

therefore the product of D , E is equal to the product of A , B .

Therefore, as D is to A , so is B to E .

[vii. 19]

But D , A are prime to one another,

primes are also least,

[vii. 21]

and the least measure the numbers which have the same ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent;

[vii. 20]

therefore D measures B .

Similarly we can also show that, if D do not measure B , it will measure A .

Therefore D measures one of the numbers A , B .

Q. E. D.



If c , a prime number, measure ab , c will measure either a or b .

Suppose c does not measure a .

Therefore c, a are prime to one another.

[VII. 29]

Suppose $ab = mc$.

Therefore $c : a = b : m$.

[VII. 19]

Hence [VII. 20, 21] c measures b .

Similarly, if c does not measure b , it measures a .

Therefore it measures one or other of the two numbers a, b .

PROPOSITION 31.

Any composite number is measured by some prime number.

Let A be a composite number ;

I say that A is measured by some prime number.

For, since A is composite,

5 some number will measure it.

A —————

Let a number measure it, and let it
be B .

B —————

C —————

Now, if B is prime, what was en-
joined will have been done.

10 But if it is composite, some number will measure it.

Let a number measure it, and let it be C .

Then, since C measures B ,

and B measures A ,

therefore C also measures A .

15 And, if C is prime, what was enjoined will have been
done.

But if it is composite, some number will measure it.

Thus, if the investigation be continued in this way, some
prime number will be found which will measure the number
20 before it, which will also measure A .

For, if it is not found, an infinite series of numbers will
measure the number A , each of which is less than the other :
which is impossible in numbers.

Therefore some prime number will be found which will
25 measure the one before it, which will also measure A .

Therefore any composite number is measured by some
prime number.

8. if B is prime, what was enjoined will have been done, i.e. the implied *problem* of finding a prime number which measures A .

18. some prime number will be found which will measure. In the Greek the sentence stops here, but it is necessary to add the words "the number before it, which will also measure A ," which are found a few lines further down. It is possible that the words may have fallen out of P here by a simple mistake due to *ομοιοτέλευτον* (Heiberg).

Heiberg relegates to the Appendix an alternative proof of this proposition, to the following effect. Since A is composite, some number will measure it. Let B be the *least* such number. I say that B is prime. For, if not, B is composite, and some number will measure it, say C ; so that C is less than B . But, since C measures B , and B measures A , C must measure A . And C is less than B : which is contrary to the hypothesis.

PROPOSITION 32.

Any number either is prime or is measured by some prime number.

Let A be a number;

I say that A either is prime or is measured by some prime number.

If now A is prime, that which was A —————
enjoined will have been done.

But if it is composite, some prime number will measure it. [VII. 31]

Therefore any number either is prime or is measured by some prime number.

Q. E. D.

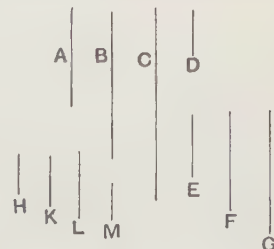
PROPOSITION 33.

Given as many numbers as we please, to find the least of those which have the same ratio with them.

Let A, B, C be the given numbers, as many as we please; thus it is required to find the least of
5 those which have the same ratio with
 A, B, C .

A, B, C are either prime to one another or not.

Now, if A, B, C are prime to one
10 another, they are the least of those
which have the same ratio with them.



[VII. 21]

But, if not, let D the greatest common measure of A, B, C
be taken, [VII. 3]

and, as many times as D measures the numbers A, B, C
 15 respectively, so many units let there be in the numbers
 E, F, G respectively.

Therefore the numbers E, F, G measure the numbers A, B, C respectively according to the units in D . [VII. 16]

Therefore E, F, G measure A, B, C the same number of
 20 times ;

therefore E, F, G are in the same ratio with A, B, C .

[VII. Def. 20]

I say next that they are the least that are in that ratio.

For, if E, F, G are not the least of those which have the
 same ratio with A, B, C ,

25 there will be numbers less than E, F, G which are in the
 same ratio with A, B, C .

Let them be H, K, L ;

therefore H measures A the same number of times that the
 numbers K, L measure the numbers B, C respectively.

30 Now, as many times as H measures A , so many units let
 there be in M ;

therefore the numbers K, L also measure the numbers B, C
 respectively according to the units in M .

And, since H measures A according to the units in M ,

35 therefore M also measures A according to the units in H .
 [VII. 16]

For the same reason

M also measures the numbers B, C according to the units in
 the numbers K, L respectively ;

Therefore M measures A, B, C .

40 Now, since H measures A according to the units in M ,
 therefore H by multiplying M has made A . [VII. Def. 15]

For the same reason also

E by multiplying D has made A .

Therefore the product of E, D is equal to the product of
 45 H, M .

Therefore, as E is to H , so is M to D . [VII. 19]

But E is greater than H ;

therefore M is also greater than D .

And it measures A, B, C :

50 which is impossible, for by hypothesis D is the greatest common measure of A, B, C .

Therefore there cannot be any numbers less than E, F, G which are in the same ratio with A, B, C .

Therefore E, F, G are the least of those which have the
55 same ratio with A, B, C .

Q. E. D.

17. the numbers E, F, G measure the numbers A, B, C respectively, literally (as usual) "each of the numbers E, F, G measures each of the numbers A, B, C ."

Given any numbers $a, b, c, \dots$, to find the least numbers that are in the same ratio.

Euclid's method is the obvious one, and the result is verified by *reductio ad absurdum*.

We will, like Euclid, take three numbers only, a, b, c .

Let g , their greatest common measure, be found [VII. 3], and suppose that

$$a = mg, \text{ i.e. } gm, \quad [\text{VII. 16}]$$

$$b = ng, \text{ i.e. } gn,$$

$$c = pg, \text{ i.e. } gp.$$

It follows, by VII. Def. 20, that

$$m : n : p = a : b : c.$$

m, n, p shall be the numbers required.

For, if not, let x, y, z be the least numbers in the same ratio as a, b, c , being less than m, n, p .

Therefore

$$a = kx \text{ (or } xk, \text{ VII. 16),}$$

$$b = ky \text{ (or } yk),$$

$$c = kz \text{ (or } zk),$$

where k is some integer.

[VII. 20]

Thus

$$mg = a = xk.$$

Therefore

$$m : x = k : g.$$

[VII. 19]

And $m > x$; therefore $k > g$.

Since then k measures a, b, c , it follows that g is not the greatest common measure: which contradicts the hypothesis.

Therefore etc.

It is to be observed that Euclid merely supposes that x, y, z are smaller numbers than m, n, p in the ratio of a, b, c ; but, in order to justify the next inference, which apparently can only depend on VII. 20, x, y, z must also be assumed to be the *least* numbers in the ratio of a, b, c .

The inference from the last proportion that, since $m > x$, $k > g$ is supposed by Heiberg to depend upon VII. 13 and V. 14 together. I prefer to regard Euclid as making the inference quite independently of Book V. E.g., the proportion could just as well be written

$$x : m = g : k,$$

when the definition of proportion in Book VII. (Def. 20) gives all that we want, since, whatever proper fraction x is of m , the same proper fraction is g of k .

PROPOSITION 34.

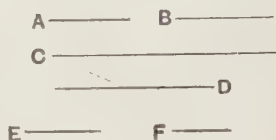
Given two numbers, to find the least number which they measure.

Let A, B be the two given numbers ;
thus it is required to find the least number which they measure.

Now A, B are either prime to one another or not.

First, let A, B be prime to one another, and let A by multiplying B make C ;

therefore also B by multiplying A has made C .



[VII. 16]

Therefore A, B measure C

I say next that it is also the least number they measure.

For, if not, A, B will measure some number which is less than C .

Let them measure D .

Then, as many times as A measures D , so many units let there be in E ,

and, as many times as B measures D , so many units let there be in F ;

therefore A by multiplying E has made D ,

and B by multiplying F has made D ;

[VII. Def. 15]

therefore the product of A, E is equal to the product of B, F .

Therefore, as A is to B , so is F to E .

[VII. 19]

But A, B are prime,

primes are also least,

[VII. 21]

and the least measure the numbers which have the same ratio the same number of times, the greater the greater and the less the less ;

[VII. 20]

therefore B measures E , as consequent consequent.

And, since A by multiplying B, E has made C, D ,

therefore, as B is to E , so is C to D .

[VII. 17]

But B measures E ;

therefore C also measures D , the greater the less :
which is impossible.

Therefore A, B do not measure any number less than C ;
therefore C is the least that is measured by A, B .

Next, let A, B not be prime to one another,
and let F, E , the least numbers of those which have the same
ratio with A, B , be taken ; [vii. 33]

therefore the product of A, E is equal to the product of B, F .
[vii. 19]

And let A by multiplying E
make C ;

therefore also B by multiplying F
has made C ;

therefore A, B measure C .

I say next that it is also the least
number that they measure.

For, if not, A, B will measure some number which is less
than C .

Let them measure D .

And, as many times as A measures D , so many units let
there be in G ,

and, as many times as B measures D , so many units let there
be in H .

Therefore A by multiplying G has made D ,
and B by multiplying H has made D .

Therefore the product of A, G is equal to the product of
 B, H ;

therefore, as A is to B , so is H to G . [vii. 19]

But, as A is to B , so is F to E .

Therefore also, as F is to E , so is H to G .

But F, E are least,

and the least measure the numbers which have the same ratio
the same number of times, the greater the greater and the
less the less ; [vii. 20]

therefore E measures G .

And, since A by multiplying E, G has made C, D ,
therefore, as E is to G , so is C to D . [vii. 17]

But E measures G ;

therefore C also measures D , the greater the less :
which is impossible.

Therefore A, B will not measure any number which is less than C .

Therefore C is the least that is measured by A, B .

Q. E. D.

This is the problem of finding the *least common multiple* of two numbers, as a, b .

I. If a, b be prime to one another, the L.C.M. is ab .

For, if not, let it be d , some number less than ab .

Then $d = ma = nb$, where m, n are integers.

Therefore $a : b = n : m$, [VII. 19]

and hence, a, b being prime to one another,

b measures m . [VII. 20, 21]

But $b : m = ab : am$ [VII. 17]
 $= ab : d$.

Therefore ab measures d : which is impossible.

II. If a, b be not prime to one another, find the numbers which are the least of those having the ratio of a to b , say m, n ; [VII. 33]

then $a : b = m : n$,

and $an = bm$ ($= c$, say); [VII. 19]

c is then the L.C.M.

For, if not, let it be d ($< c$), so that

$ap = bq = d$, where p, q are integers.

Then $a : b = q : p$, [VII. 19]

whence $m : n = q : p$,

so that n measures p . [VII. 20, 21]

And $n : p = an : ap = c : d$,

so that c measures d :

which is impossible.

Therefore etc.

By VII. 33,
$$\left. \begin{aligned} m &= \frac{a}{g} \\ n &= \frac{b}{g} \end{aligned} \right\}, \text{ where } g \text{ is the G.C.M. of } a, b.$$

Hence the L.C.M. is $\frac{ab}{g}$.

PROPOSITION 35.

If two numbers measure any number, the least number measured by them will also measure the same.

For let the two numbers A , B measure any number CD , and let E be the least that they measure ;

I say that E also measures CD .

For, if E does not measure CD , let E , measuring DF , leave CF less than itself.

Now, since A , B measure E , and E measures DF , therefore A , B will also measure DF .

But they also measure the whole CD ; therefore they will also measure the remainder CF which is less than E :

which is impossible.

Therefore E cannot fail to measure CD ; therefore it measures it.

Q. E. D.

The *least* common multiple of any two numbers must measure any other common multiple.

The proof is obvious, depending on the fact that, if any number divides a and b , it also divides $a - pb$.

PROPOSITION 36.

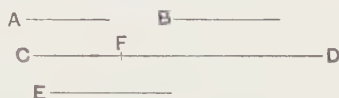
Given three numbers, to find the least number which they measure.

Let A , B , C be the three given numbers ; thus it is required to find the least number which they measure.

Let D , the least number measured by the two numbers A , B , be taken. [VII. 34]

Then C either measures, or does not measure, D .

First, let it measure it.



But A, B also measure D ;
therefore A, B, C measure D .

I say next that it is also the least that they measure.

For, if not, A, B, C will measure some number which is less than D .

Let them measure E .

Since A, B, C measure E ,
therefore also A, B measure E .

Therefore the least number measured by A, B will also measure E . [VII. 35]

But D is the least number measured by A, B ;
therefore D will measure E , the greater the less :
which is impossible.

Therefore A, B, C will not measure any number which is less than D ;

therefore D is the least that A, B, C measure.

Again, let C not measure D ,
and let E , the least number measured by
 C, D , be taken. [VII. 34]

Since A, B measure D ,
and D measures E ,
therefore also A, B measure E .

But C also measures E ;
therefore also A, B, C measure E .

I say next that it is also the least that they measure.

For, if not, A, B, C will measure some number which is less than E .

Let them measure F .

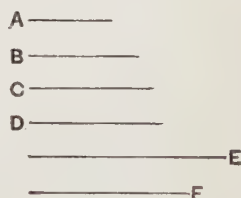
Since A, B, C measure F ,
therefore also A, B measure F ;

therefore the least number measured by A, B will also measure F . [VII. 35]

But D is the least number measured by A, B ;
therefore D measures F .

But C also measures F ;
therefore D, C measure F ,

so that the least number measured by D, C will also measure F .



But E is the least number measured by C, D ;
therefore E measures F , the greater the less :
which is impossible.

Therefore A, B, C will not measure any number which is less than E .

Therefore E is the least that is measured by A, B, C .

Q. E. D.

Euclid's rule for finding the L.C.M. of *three* numbers a, b, c is the rule with which we are familiar. The L.C.M. of a, b is first found, say d , and then the L.C.M. of d and c is found.

Euclid distinguishes the cases (1) in which c measures d , (2) in which c does not measure d . We need only reproduce the proof of the general case (2). The method is that of *reductio ad absurdum*.

Let e be the L.C.M. of d, c .

Since a, b both measure d , and d measures e ,

a, b both measure e .

So does c .

Therefore e is *some* common multiple of a, b, c .

If it is not the *least*, let f be the L.C.M.

Now a, b both measure f ;

therefore d , their L.C.M., also measures f .

[VII. 35]

Thus d, c both measure f ;

therefore e , their L.C.M., measures f :

[VII. 35]

which is impossible, since $f < e$.

Therefore etc.

The process can be continued *ad libitum*, so that we can find the L.C.M., not only of three, but of as many numbers as we please.

PROPOSITION 37.

If a number be measured by any number, the number which is measured will have a part called by the same name as the measuring number.

For let the number A be measured by any number B ;
I say that A has a part called by the same name as B .

A —————

For, as many times as B measures A ,
so many units let there be in C .

B ———

Since B measures A according to the
units in C ,

C ———

and the unit D also measures the number C according to the
units in it,

D —

therefore the unit D measures the number C the same number of times as B measures A .

Therefore, alternately, the unit D measures the number B the same number of times as C measures A ; [VII. 15] therefore, whatever part the unit D is of the number B , the same part is C of A also.

But the unit D is a part of the number B called by the same name as it;

therefore C is also a part of A called by the same name as B , so that A has a part C which is called by the same name as B .

Q. E. D.

If b measures a , then $\frac{1}{b}$ th of a is a whole number.

Let $a = m \cdot b$.

Now $m = m \cdot 1$.

Thus $1, m, b, a$ satisfy the enunciation of VII. 15; therefore m measures a the same number of times that 1 measures b .

But 1 is $\frac{1}{b}$ th part of b ;

therefore m is $\frac{1}{b}$ th part of a .

PROPOSITION 38.

If a number have any part whatever, it will be measured by a number called by the same name as the part.

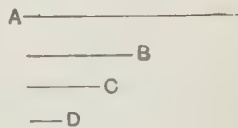
For let the number A have any part whatever, B , and let C be a number called by the same name as the part B ;

I say that C measures A .

For, since B is a part of A called by the same name as C , and the unit D is also a part of C called by the same name as it,

therefore, whatever part the unit D is of the number C , the same part is B of A also;

therefore the unit D measures the number C the same number of times that B measures A .



Therefore, alternately, the unit D measures the number B the same number of times that C measures A . [VII. 15]

Therefore C measures A .

Q. E. D.

This proposition is practically a restatement of the preceding proposition. It asserts that, if b is $\frac{1}{m}$ th part of a ,

i.e., if $b = \frac{1}{m} a$,

then m measures a .

We have $b = \frac{1}{m} a$,

and $1 = \frac{1}{m} m$.

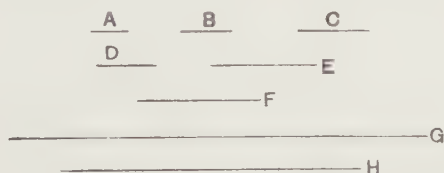
Therefore $1, m, b, a$, satisfy the enunciation of VII. 15, and therefore m measures a the same number of times as 1 measures b , or

$$m = \frac{1}{b} a.$$

PROPOSITION 39.

To find the number which is the least that will have given parts.

Let A, B, C be the given parts ;
thus it is required to find the number which is the least that will have the parts A, B, C .



Let D, E, F be numbers called by the same name as the parts A, B, C ,

and let G , the least number measured by D, E, F , be taken. [VII. 36]

Therefore G has parts called by the same name as D, E, F .

But A, B, C are parts called by the same name as D, E, F ; therefore G has the parts A, B, C . [VII. 37]

I say next that it is also the least number that has.

For, if not, there will be some number less than G which will have the parts A, B, C .

Let it be H .

Since H has the parts A, B, C ,
therefore H will be measured by numbers called by the same name as the parts A, B, C . [VII. 38]

But D, E, F are numbers called by the same name as the parts A, B, C ;

therefore H is measured by D, E, F .

And it is less than G : which is impossible.

Therefore there will be no number less than G that will have the parts A, B, C .

Q. E. D.

This again is practically a restatement in another form of the problem of finding the L.C.M.

To find a number which has $\frac{1}{a}$ th, $\frac{1}{b}$ th and $\frac{1}{c}$ th parts.

Let d be the L.C.M. of a, b, c .

Thus d has $\frac{1}{a}$ th, $\frac{1}{b}$ th and $\frac{1}{c}$ th parts. [VII. 37]

If it is not the least number which has, let the least such number be e .

Then, since e has those parts,

e is measured by a, b, c ; and $e < d$:

which is impossible.

BOOK VIII.

PROPOSITION I.

If there be as many numbers as we please in continued proportion, and the extremes of them be prime to one another, the numbers are the least of those which have the same ratio with them.

Let there be as many numbers as we please, A, B, C, D , in continued proportion,

and let the extremes of them A, D be prime to one another;

I say that A, B, C, D are the least of those which have the same ratio with them.

A ——	E ——
B ——	F ——
C ——	G ——
D ——	H ——

For, if not, let E, F, G, H be less than A, B, C, D , and in the same ratio with them.

Now, since A, B, C, D are in the same ratio with E, F, G, H ,

and the multitude of the numbers A, B, C, D is equal to the multitude of the numbers E, F, G, H ,

therefore, *ex aequali*,

as A is to D , so is E to H . [VII. 14]

But A, D are prime,

primes are also least, [VII. 21]

and the least numbers measure those which have the same ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent. [VII. 20]

Therefore A measures E , the greater the less :
which is impossible.

Therefore E, F, G, H which are less than A, B, C, D are not in the same ratio with them.

Therefore A, B, C, D are the least of those which have the same ratio with them.

Q. E. D.

What we call a *geometrical progression* is with Euclid a series of terms "in continued proportion" (ἐξῆς ἀνάλογον).

This proposition proves that, if $a, b, c, \dots k$ are a series of numbers in geometrical progression, and if a, k are prime to one another, the series is in the lowest terms possible with the same common ratio.

The proof is in *form* by *reductio ad absurdum*. We should no doubt desert this *form* while retaining the substance. If $a', b', c', \dots k'$ be *any* other series of numbers in G.P. with the same common ratio as before, we have, *ex aequali*,

$$a : k = a' : k', \quad [\text{VII. 14}]$$

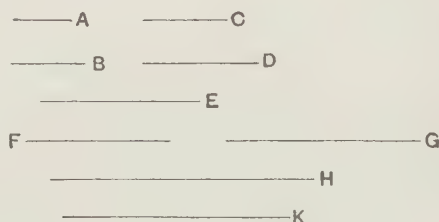
whence, since a, k are prime to one another, a, k measure a', k' respectively, so that a', k' are *greater* than a, k respectively.

PROPOSITION 2.

To find numbers in continued proportion, as many as may be prescribed, and the least that are in a given ratio.

Let the ratio of A to B be the given ratio in least numbers ;

thus it is required to find numbers in continued proportion, as many as may be prescribed, and the least that are in the ratio of A to B .



Let four be prescribed ;

let A by multiplying itself make C , and by multiplying B let it make D ;

let B by multiplying itself make E ;

further, let A by multiplying C, D, E make F, G, H ,

and let B by multiplying E make K .

Now, since A by multiplying itself has made C ,
 and by multiplying B has made D ,
 therefore, as A is to B , so is C to D . [VII. 17]

Again, since A by multiplying B has made D ,
 and B by multiplying itself has made E ,
 therefore the numbers A , B by multiplying B have made the
 numbers D , E respectively.

Therefore, as A is to B , so is D to E . [VII. 18]

But, as A is to B , so is C to D ;

therefore also, as C is to D , so is D to E .

And, since A by multiplying C , D has made F , G ,
 therefore, as C is to D , so is F to G . [VII. 17]

But, as C is to D , so was A to B ;

therefore also, as A is to B , so is F to G .

Again, since A by multiplying D , E has made G , H ,
 therefore, as D is to E , so is G to H . [VII. 17]

But, as D is to E , so is A to B .

Therefore also, as A is to B , so is G to H .

And, since A , B by multiplying E have made H , K ,
 therefore, as A is to B , so is H to K . [VII. 18]

But, as A is to B , so is F to G , and G to H .

Therefore also, as F is to G , so is G to H , and H to K ;
 therefore C , D , E , and F , G , H , K are proportional in the
 ratio of A to B .

I say next that they are the least numbers that are so.

For, since A , B are the least of those which have the
 same ratio with them,

and the least of those which have the same ratio are prime
 to one another, [VII. 22]

therefore A , B are prime to one another.

And the numbers A , B by multiplying themselves re-
 spectively have made the numbers C , E , and by multiplying
 the numbers C , E respectively have made the numbers F , K ;
 therefore C , E and F , K are prime to one another respectively.
 [VII. 27]

But, if there be as many numbers as we please in continued
 proportion, and the extremes of them be prime to one another,

they are the least of those which have the same ratio with them. [VIII. 1]

Therefore C, D, E and F, G, H, K are the least of those which have the same ratio with A, B . Q. E. D.

PORISM. From this it is manifest that, if three numbers in continued proportion be the least of those which have the same ratio with them, the extremes of them are squares, and, if four numbers, cubes.

To find a series of numbers in geometrical progression and in the least terms which have a given common ratio (understanding by that term *the ratio of one term to the next*).

Reduce the given ratio to its lowest terms, say, $a : b$. (This can be done by VII. 33.)

Then $a^n, a^{n-1}b, a^{n-2}b^2, \dots, a^2b^{n-2}, ab^{n-1}, b^n$ is the required series of numbers if $(n + 1)$ terms are required.

That this is a series of terms with the given common ratio is clear from VII. 17, 18.

That the G.P. in the smallest terms possible is proved thus.

a, b are prime to one another, since the ratio $a : b$ is in its lowest terms. [VII. 22]

Therefore a^2, b^2 are prime to one another; so are a^3, b^3 and, generally, a^n, b^n . [VII. 27]

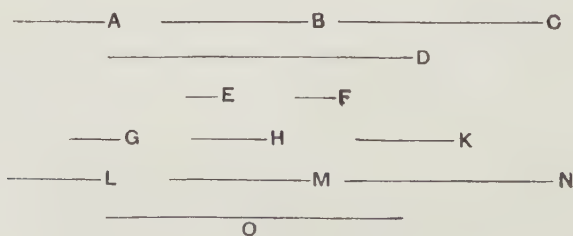
Whence the G.P. is in the smallest possible terms, by VIII. 1.

The Porism observes that, if there are n terms in the series, the extremes are $(n - 1)$ th powers.

PROPOSITION 3.

If as many numbers as we please in continued proportion be the least of those which have the same ratio with them, the extremes of them are prime to one another.

Let as many numbers as we please, A, B, C, D , in continued proportion be the least of those which have the same ratio with them;



I say that the extremes of them A, D are prime to one another.

For let two numbers E, F , the least that are in the ratio of A, B, C, D , be taken, [VII. 33]

then three others G, H, K with the same property ;

and others, more by one continually, [VIII. 2]

until the multitude taken becomes equal to the multitude of the numbers A, B, C, D .

Let them be taken, and let them be L, M, N, O .

Now, since E, F are the least of those which have the same ratio with them, they are prime to one another. [VII. 22]

And, since the numbers E, F by multiplying themselves respectively have made the numbers G, K , and by multiplying the numbers G, K respectively have made the numbers L, O , [VIII. 2, Por.]

therefore both G, K and L, O are prime to one another. [VII. 27]

And, since A, B, C, D are the least of those which have the same ratio with them,

while L, M, N, O are the least that are in the same ratio with A, B, C, D ,

and the multitude of the numbers A, B, C, D is equal to the multitude of the numbers L, M, N, O ,

therefore the numbers A, B, C, D are equal to the numbers L, M, N, O respectively ;

therefore A is equal to L , and D to O .

And L, O are prime to one another.

Therefore A, D are also prime to one another.

Q. E. D.

The proof consists in merely equating the given numbers to the terms of a series found in the manner of VIII. 2.

If $a, b, c, \dots k$ (n terms) be a geometrical progression in the lowest terms having a given common ratio, the terms must respectively be of the form

$$a^{n-1}, a^{n-2}\beta, \dots a^2\beta^{n-3}, a\beta^{n-2}, \beta^{n-1}$$

found by VIII. 2, where $a : \beta$ is the ratio $a : b$ expressed in its lowest terms, so that a, β are prime to one another [VII. 22], and hence a^{n-1}, β^{n-1} are prime to one another [VII. 27].

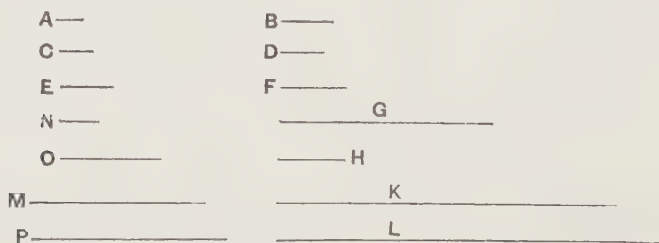
But the two series must be the same, so that

$$a = a^{n-1}, \quad b = \beta^{n-1}$$

PROPOSITION 4.

Given as many ratios as we please in least numbers, to find numbers in continued proportion which are the least in the given ratios.

Let the given ratios in least numbers be that of A to B ,
 5 that of C to D , and that of E to F ;
 thus it is required to find numbers in continued proportion
 which are the least that are in the ratio of A to B , in the
 ratio of C to D , and in the ratio of E to F .



Let G , the least number measured by B , C , be taken.

[VII. 34]

10 And, as many times as B measures G , so many times also
 let A measure H ,
 and, as many times as C measures G , so many times also let
 D measure K .

Now E either measures or does not measure K .

15 First, let it measure it.

And, as many times as E measures K , so many times let
 F measure L also.

Now, since A measures H the same number of times that
 B measures G ,

20 therefore, as A is to B , so is H to G . [VII. Def. 20, VII. 13]

For the same reason also,

as C is to D , so is G to K ,

and further, as E is to F , so is K to L ;

therefore H , G , K , L are continuously proportional in the
 25 ratio of A to B , in the ratio of C to D , and in the ratio of E
 to F .

I say next that they are also the least that have this
 property.

For, if H, G, K, L are not the least numbers continuously
 30 proportional in the ratios of A to B , of C to D , and of E
 to F , let them be N, O, M, P .

Then since, as A is to B , so is N to O ,
 while A, B are least,

and the least numbers measure those which have the same
 35 ratio the same number of times, the greater the greater and
 the less the less, that is, the antecedent the antecedent and the
 consequent the consequent ;

therefore B measures O .

[VII. 20]

For the same reason

40 C also measures O ;

therefore B, C measure O ;

therefore the least number measured by B, C will also
 measure O .

[VII. 35]

But G is the least number measured by B, C ;

45 therefore G measures O , the greater the less :

which is impossible.

Therefore there will be no numbers less than H, G, K, L
 which are continuously in the ratio of A to B , of C to D , and
 of E to F .

50 Next, let E not measure K .



Let M , the least number measured by E, K , be taken.

And, as many times as K measures M , so many times let
 H, G measure N, O respectively,

and, as many times as E measures M , so many times let F
 55 measure P also.

Since H measures N the same number of times that G
 measures O ,

therefore, as H is to G , so is N to O .

[VII. 13 and Def. 20]

But, as H is to G , so is A to B ;

60 therefore also, as A is to B , so is N to O .

For the same reason also,

as C is to D , so is O to M .

Again, since E measures M the same number of times that F measures P ,

65 therefore, as E is to F , so is M to P ; [VII. 13 and Def. 20]

therefore N , O , M , P are continuously proportional in the ratios of A to B , of C to D , and of E to F .

I say next that they are also the least that are in the ratios $A : B$, $C : D$, $E : F$.

70 For, if not, there will be some numbers less than N , O , M , P continuously proportional in the ratios $A : B$, $C : D$, $E : F$.

Let them be Q , R , S , T .

Now since, as Q is to R , so is A to B ,

75 while A , B are least,

and the least numbers measure those which have the same ratio with them the same number of times, the antecedent the antecedent and the consequent the consequent, [VII. 20]
therefore B measures R .

80 For the same reason C also measures R ;
therefore B , C measure R .

Therefore the least number measured by B , C will also measure R . [VII. 35]

But G is the least number measured by B , C ;

85 therefore G measures R .

And, as G is to R , so is K to S : [VII. 13]
therefore K also measures S .

But E also measures S ;

therefore E , K measure S .

90 Therefore the least number measured by E , K will also measure S . [VII. 35]

But M is the least number measured by E , K ;

therefore M measures S , the greater the less :

which is impossible.

95 Therefore there will not be any numbers less than N , O , M , P continuously proportional in the ratios of A to B , of C to D , and of E to F ;

therefore N, O, M, P are the least numbers continuously proportional in the ratios $A : B, C : D, E : F$. Q. E. D.

69, 71, 99. the ratios $A : B, C : D, E : F$. This abbreviated expression is in the Greek of $\text{AB}, \Gamma\Delta, \text{EZ} \lambda\delta\gamma\alpha\iota$.

The term "in continued proportion" is here not used in its proper sense, since a geometrical progression is not meant, but a series of terms each of which bears to the succeeding term a *given*, but not the *same*, ratio.

The proposition furnishes a good example of the cumbrousness of the Greek method of dealing with non-determinate numbers. The proof in fact is not easy to follow without the help of modern symbolical notation. If this be used, the reasoning can be made clear enough.

Euclid takes *three* given ratios and therefore requires to find *four* numbers. We will leave out the simpler particular case which he puts first, that namely in which E accidentally measures K , the multiple of D found in the first few lines; and we will reproduce the general case with *three* ratios.

Let the ratios in their lowest terms be

$$a : b, \quad c : d, \quad e : f$$

Take l_1 , the L.C.M. of b, c , and suppose that

$$l_1 = mb = nc.$$

Form the numbers

$$\left. \begin{array}{l} ma, mb \\ = nc \end{array} \right\}, nd.$$

These are in the ratios of a to b and of c to d respectively.

Next, let l_2 be the L.C.M. of nd, e , and let

$$l_2 = pnd = qe.$$

Now form the numbers

$$\left. \begin{array}{l} pma, pmb \\ = pnc \end{array} \right\}, \quad \left. \begin{array}{l} pnd \\ = qe \end{array} \right\}, \quad gf,$$

and these are the four numbers required.

If they are *not* the least in the given ratios, let

$$x, \quad y, \quad z, \quad u$$

be less numbers in the given ratios.

Since $a : b$ is in its lowest terms, and

$$a : b = x : y,$$

b measures y .

Similarly, since

$$c : d = y : z,$$

c measures y .

Therefore l_1 , the L.C.M. of b, c , measures y .

But

$$l_1 : nd [= c : d] = y : z.$$

Therefore nd measures z .

And, since

$$e : f = z : u,$$

e measures z .

Therefore l_2 , the L.C.M. of nd, e , measures z : which is impossible, since $z < l_2$ or pnd .

The step (line 86) inferring that $G : R = K : S$ is of course *alternando* from $G : K [= C : D] = R : S$.

It will be observed that VIII. 4 corresponds to the portion of VI. 23 which shows how to *compound* two ratios between straight lines.

PROPOSITION 5.

Plane numbers have to one another the ratio compounded of the ratios of their sides.

Let A, B be plane numbers, and let the numbers C, D be the sides of A , and E, F of B ;

I say that A has to B the ratio compounded of the ratios of the sides.

For, the ratios being given which C has to E and D to F , let the least numbers G, H, K that are continuously in the ratios $C : E, D : F$ be taken, so that,

as C is to E , so is G to H ,

and, as D is to F , so is H to K .

[VIII. 4]

And let D by multiplying E make L .

Now, since D by multiplying C has made A , and by multiplying E has made L ,

therefore, as C is to E , so is A to L .

[VII. 17]

But, as C is to E , so is G to H ;

therefore also, as G is to H , so is A to L .

Again, since E by multiplying D has made L , and further by multiplying F has made B ,

therefore, as D is to F , so is L to B .

[VII. 17]

But, as D is to F , so is H to K ;

therefore also, as H is to K , so is L to B .

But it was also proved that,

as G is to H , so is A to L ;

therefore, *ex aequali*,

as G is to K , so is A to B .

[VII. 14]

But G has to K the ratio compounded of the ratios of the sides ;

therefore A also has to B the ratio compounded of the ratios of the sides.

Q. E. D.

1, 5, 29, 31. compounded of the ratios of their sides. As in VI. 23, the Greek has the less exact phrase, "compounded of their sides."

If

$$a = cd, \quad b = ef,$$

then a has to b the ratio compounded of $c : e$ and $d : f$.

Take three numbers the least which are continuously in the given ratios.

If l is the L.C.M. of e , d and $l = me = nd$, the three numbers are

$$\left. \begin{array}{l} mc, \quad me \\ \quad \quad = nd \end{array} \right\}, \quad nf. \quad [\text{VIII. 4}]$$

Now $dc : de = c : e$ [VII. 17]

$$= mc : me = mc : nd.$$

Also $ed : ef = d : f$ [VII. 17]

$$= nd : nf.$$

Therefore, *ex aequali*, $cd : ef = mc : nf$
 $= (\text{ratio compounded of } c : e \text{ and } d : f).$

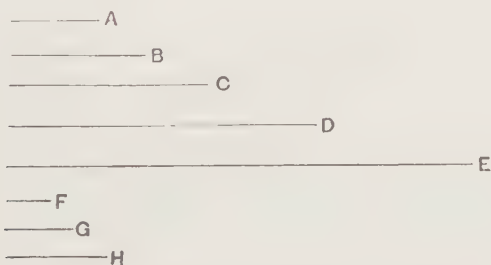
It will be seen that this proof follows exactly the method of VI. 23 for parallelograms.

PROPOSITION 6.

If there be as many numbers as we please in continued proportion, and the first do not measure the second, neither will any other measure any other.

Let there be as many numbers as we please, A, B, C, D, E , in continued proportion, and let A not measure B ;

I say that neither will any other measure any other.



Now it is manifest that A, B, C, D, E do not measure one another in order; for A does not even measure B .

I say, then, that neither will any other measure any other.

For, if possible, let A measure C .

And, however many A, B, C are, let as many numbers F, G, H , the least of those which have the same ratio with A, B, C , be taken. [VII. 33]

Now, since F, G, H are in the same ratio with A, B, C , and the multitude of the numbers A, B, C is equal to the multitude of the numbers F, G, H ,

therefore, *ex aequali*, as A is to C , so is F to H . [VII. 14]

And since, as A is to B , so is F to G ,
 while A does not measure B ,
 therefore neither does F measure G ; [VII. Def. 20]
 therefore F is not an unit, for the unit measures any number.

Now F, H are prime to one another. [VIII. 3]

And, as F is to H , so is A to C ;
 therefore neither does A measure C .

Similarly we can prove that neither will any other measure any other.

Q. E. D.

Let $a, b, c \dots k$ be a geometrical progression in which a does not measure b .
 Suppose, if possible, that a measures some term of the series, as f .
 Take x, y, z, u, v, w the *least* numbers in the ratio a, b, c, d, e, f .

Since $x : y = a : b$,

and a does not measure b ,
 x does not measure y ; therefore x cannot be unity.

And, *ex aequali*, $x : w = a : f$.

Now x, w are prime to one another. [VIII. 3]

Therefore a does not measure f .

We can of course prove that an intermediate term, as b , does not measure a later term f by using the series b, c, d, e, f and remembering that, since $b : c = a : b$, b does not measure c .

PROPOSITION 7.

If there be as many numbers as we please in continued proportion, and the first measure the last, it will measure the second also.

Let there be as many numbers as we please, A, B, C, D ,
 in continued proportion; and
 let A measure D ;

I say that A also measures B .

For, if A does not measure B , neither will any other of the numbers measure any other.

But A measures D .

Therefore A also measures B .

A —
 B —
 C —
 D —

[VIII. 6]

Q. E. D.

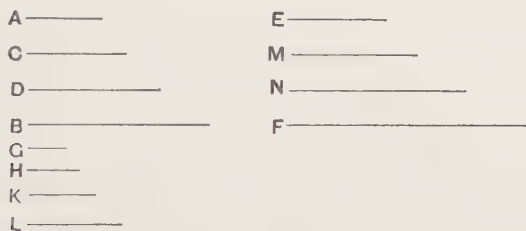
An obvious proof by *reductio ad absurdum* from VIII. 6.

PROPOSITION 8.

If between two numbers there fall numbers in continued proportion with them, then, however many numbers fall between them in continued proportion, so many will also fall in continued proportion between the numbers which have the same ratio with the original numbers.

Let the numbers C, D fall between the two numbers A, B in continued proportion with them, and let E be made in the same ratio to F as A is to B ;

I say that, as many numbers as have fallen between A, B in continued proportion, so many will also fall between E, F in continued proportion.



For, as many as A, B, C, D are in multitude, let so many numbers G, H, K, L , the least of those which have the same ratio with A, C, D, B , be taken ; [VII. 33]

therefore the extremes of them G, L are prime to one another.

[VIII. 3]

Now, since A, C, D, B are in the same ratio with G, H, K, L ,

and the multitude of the numbers A, C, D, B is equal to the multitude of the numbers G, H, K, L ,

therefore, *ex aequali*, as A is to B , so is G to L . [VII. 14]

But, as A is to B , so is E to F ;

therefore also, as G is to L , so is E to F .

But G, L are prime,

primes are also least, [VII. 21]

and the least numbers measure those which have the same ratio the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent. [VII. 20]

Therefore G measures E the same number of times as L measures F .

Next, as many times as G measures E , so many times let H, K also measure M, N respectively ;

therefore G, H, K, L measure E, M, N, F the same number of times.

Therefore G, H, K, L are in the same ratio with E, M, N, F . [VII. Def. 20]

But G, H, K, L are in the same ratio with A, C, D, B ;
therefore A, C, D, B are also in the same ratio with E, M, N, F .

But A, C, D, B are in continued proportion ;
therefore E, M, N, F are also in continued proportion.

Therefore, as many numbers as have fallen between A, B in continued proportion with them, so many numbers have also fallen between E, F in continued proportion.

Q. E. D.

1. fall. The Greek word is *ἐμπίπτειν*, "fall in" = "can be interpolated."

If $a : b = e : f$, and between a, b there are any number of geometric means c, d , there will be as many such means between e, f .

Let $\alpha, \beta, \gamma, \dots, \delta$ be the least possible terms in the same ratio as $a, c, d, \dots b$.

Then α, δ are prime to one another, [VIII. 3]
and, *ex aequali*, $\alpha : \delta = a : b$

$$= e : f.$$

Therefore $e = m\alpha, f = m\delta$, where m is some integer. [VII. 20]

Take the numbers $m\alpha, m\beta, m\gamma, \dots m\delta$.

This is a series in the given ratio, and we have the same number of geometric means between $m\alpha, m\delta$, or e, f , that there are between a, b .

PROPOSITION 9.

If two numbers be prime to one another, and numbers fall between them in continued proportion, then, however many numbers fall between them in continued proportion, so many will also fall between each of them and an unit in continued proportion.

Let A, B be two numbers prime to one another, and let C, D fall between them in continued proportion,
and let the unit E be set out ;

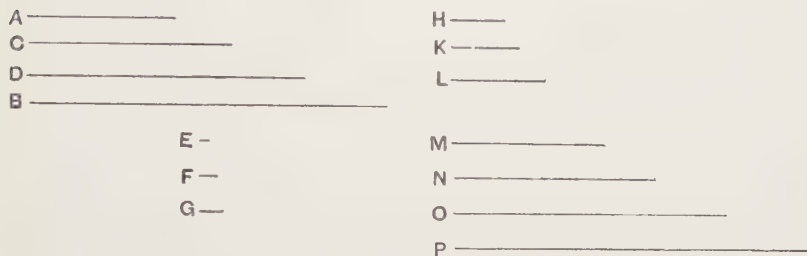
I say that, as many numbers as fall between A, B in con-

tinued proportion, so many will also fall between either of the numbers A, B and the unit in continued proportion.

For let two numbers F, G , the least that are in the ratio of A, C, D, B , be taken,

three numbers H, K, L with the same property,

and others more by one continually, until their multitude is equal to the multitude of A, C, D, B . [VIII. 2]



Let them be taken, and let them be M, N, O, P .

It is now manifest that F by multiplying itself has made H and by multiplying H has made M , while G by multiplying itself has made L and by multiplying L has made P .

[VIII. 2, Por.]

And, since M, N, O, P are the least of those which have the same ratio with F, G ,

and A, C, D, B are also the least of those which have the same ratio with F, G , [VIII. 1]

while the multitude of the numbers M, N, O, P is equal to the multitude of the numbers A, C, D, B ,

therefore M, N, O, P are equal to A, C, D, B respectively;

therefore M is equal to A , and P to B .

Now, since F by multiplying itself has made H , therefore F measures H according to the units in F .

But the unit E also measures F according to the units in it; therefore the unit E measures the number F the same number of times as F measures H .

Therefore, as the unit E is to the number F , so is F to H . [VII. Def. 20]

Again, since F by multiplying H has made M , therefore H measures M according to the units in F .

But the unit E also measures the number F according to the units in it ;
therefore the unit E measures the number F the same number of times as H measures M .

Therefore, as the unit E is to the number F , so is H to M .

But it was also proved that, as the unit E is to the number F , so is F to H ;

therefore also, as the unit E is to the number F , so is F to H , and H to M .

But M is equal to A ;

therefore, as the unit E is to the number F , so is F to H , and H to A .

For the same reason also,

as the unit E is to the number G , so is G to L and L to B .

Therefore, as many numbers as have fallen between A , B in continued proportion, so many numbers also have fallen between each of the numbers A , B and the unit E in continued proportion.

Q. E. D.

Suppose there are n geometric means between a , b , two numbers prime to one another ; there are the same number (n) of geometric means between $\mathbf{1}$ and a and between $\mathbf{1}$ and b .

If c , d ... are the n means between a , b ,

$$a, \quad c, \quad d \dots b$$

are the least numbers in that ratio, since a , b are prime to one another. [VIII. 1]

The terms are therefore respectively identical with

$$a^{n+1}, \quad a^n\beta, \quad a^{n-1}\beta^2 \dots a\beta^n, \quad \beta^{n+1},$$

where a , β is the common ratio in its lowest terms.

[VIII. 2, Por.]

Thus

$$a = a^{n+1}, \quad b = \beta^{n+1}.$$

Now

$$\mathbf{1} : a = a : a^2 = a^2 : a^3 \dots = a^n : a^{n+1},$$

and

$$\mathbf{1} : \beta = \beta : \beta^2 = \beta^2 : \beta^3 \dots = \beta^n : \beta^{n+1} ;$$

whence there are n geometric means between $\mathbf{1}$, a , and between $\mathbf{1}$, b .

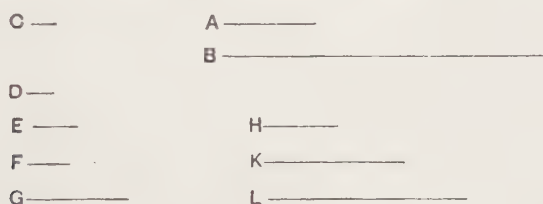
PROPOSITION 10.

If numbers fall between each of two numbers and an unit in continued proportion, however many numbers fall between each of them and an unit in continued proportion, so many also will fall between the numbers themselves in continued proportion.

For let the numbers D , E and F , G respectively fall between the two numbers A , B and the unit C in continued proportion ;

I say that, as many numbers as have fallen between each of the numbers A , B and the unit C in continued proportion, so many numbers will also fall between A , B in continued proportion.

For let D by multiplying F make H , and let the numbers D , F by multiplying H make K , L respectively.



Now, since, as the unit C is to the number D , so is D to E , therefore the unit C measures the number D the same number of times as D measures E . [VII. Def. 20]

But the unit C measures the number D according to the units in D ;

therefore the number D also measures E according to the units in D ;

therefore D by multiplying itself has made E .

Again, since, as C is to the number D , so is E to A , therefore the unit C measures the number D the same number of times as E measures A .

But the unit C measures the number D according to the units in D ;

therefore E also measures A according to the units in D ;

therefore D by multiplying E has made A .

For the same reason also

F by multiplying itself has made G , and by multiplying G has made B .

And, since D by multiplying itself has made E and by multiplying F has made H ,

therefore, as D is to F , so is E to H .

[VII. 17]

For the same reason also,

as D is to F , so is H to G . [VII. 18]

Therefore also, as E is to H , so is H to G .

Again, since D by multiplying the numbers E , H has made A , K respectively,

therefore, as E is to H , so is A to K . [VII. 17]

But, as E is to H , so is D to F ;

therefore also, as D is to F , so is A to K .

Again, since the numbers D , F by multiplying H have made K , L respectively,

therefore, as D is to F , so is K to L . [VII. 18]

But, as D is to F , so is A to K ;

therefore also, as A is to K , so is K to L .

Further, since F by multiplying the numbers H , G has made L , B respectively,

therefore, as H is to G , so is L to B . [VII. 17]

But, as H is to G , so is D to F ;

therefore also, as D is to F , so is L to B .

But it was also proved that,

as D is to F , so is A to K and K to L ;

therefore also, as A is to K , so is K to L and L to B .

Therefore A , K , L , B are in continued proportion.

Therefore, as many numbers as fall between each of the numbers A , B and the unit C in continued proportion, so many also will fall between A , B in continued proportion.

Q. E. D.

If there be n geometric means between 1 and a , and also between 1 and b , there will be n geometric means between a and b .

The proposition is the converse of the preceding.

The n means with the extremes form two geometric series of the form

$$1, \quad a, \quad a^2 \dots a^n, \quad a^{n+1},$$

$$1, \quad \beta, \quad \beta^2 \dots \beta^n, \quad \beta^{n+1},$$

where

$$a^{n+1} = a, \quad \beta^{n+1} = b.$$

By multiplying the last term in the first line by the first in the second, the last but one in the first line by the second in the second, and so on, we get the series

$$a^{n+1}, \quad a^n \beta, \quad a^{n-1} \beta^2 \dots a^2 \beta^{n-1}, \quad a \beta^n, \quad \beta^{n+1}$$

and we have the n means between a and b .

It will be observed that, when Euclid says "*For the same reason* also, as D is to F , so is H to G ," the reference is really to VII. 18 instead of VII. 17.

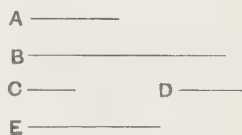
He infers namely that $D \times F : F \times F = D : F$. But since, by VII. 16, the order of multiplication is indifferent, he is practically justified in saying "for the same reason." The same thing occurs in later propositions.

PROPOSITION 11.

Between two square numbers there is one mean proportional number, and the square has to the square the ratio duplicate of that which the side has to the side.

Let A, B be square numbers,
and let C be the side of A , and D of B ;

I say that between A, B there is one mean proportional number, and A has to B the ratio duplicate of that which C has to D .



For let C by multiplying D make E .

Now, since A is a square and C is its side,

therefore C by multiplying itself has made A .

For the same reason also

D by multiplying itself has made B .

Since then C by multiplying the numbers C, D has made A, E respectively,

therefore, as C is to D , so is A to E . [VII. 17]

For the same reason also,

as C is to D , so is E to B . [VII. 18]

Therefore also, as A is to E , so is E to B .

Therefore between A, B there is one mean proportional number.

I say next that A also has to B the ratio duplicate of that which C has to D .

For, since A, E, B are three numbers in proportion,
therefore A has to B the ratio duplicate of that which A has to E . [v. Def. 9]

But, as A is to E , so is C to D .

Therefore A has to B the ratio duplicate of that which the side C has to D . Q. E. D.

According to Nicomachus the theorems in this proposition and the next, that two squares have *one* geometric mean, and two cubes *two* geometric means, between them are Platonic. Cf. *Timaeus*, 32 A seqq. and the note thereon, p. 294 above.

a^2 , b^2 being two squares, it is only necessary to form the product ab and to prove that

$$a^2, ab, b^2$$

are in geometrical progression. Euclid proves that

$$a^2 : ab = ab : b^2$$

by means of VII. 17, 18, as usual.

In assuming that, since a^2 is to b^2 in the duplicate ratio of a^2 to ab , a^2 is to b^2 in the duplicate ratio of a to b , Euclid assumes that ratios which are the duplicates of equal ratios are equal. This, an obvious inference from v. 22, can be inferred just as easily for numbers from VII. 14.

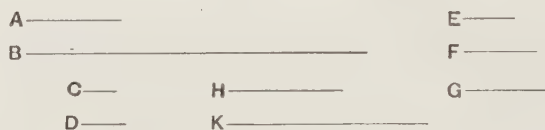
PROPOSITION 12.

Between two cube numbers there are two mean proportional numbers, and the cube has to the cube the ratio triplicate of that which the side has to the side.

Let A , B be cube numbers,

and let C be the side of A , and D of B ;

I say that between A , B there are two mean proportional numbers, and A has to B the ratio triplicate of that which C has to D .



For let C by multiplying itself make E , and by multiplying D let it make F ;

let D by multiplying itself make G ,

and let the numbers C , D by multiplying F make H , K respectively.

Now, since A is a cube, and C its side,

and C by multiplying itself has made E ,

therefore C by multiplying itself has made E and by multiplying E has made A .

For the same reason also

D by multiplying itself has made G and by multiplying G has made B .

And, since C by multiplying the numbers C , D has made E , F respectively,

therefore, as C is to D , so is E to F .

[VII. 17]

For the same reason also,

as C is to D , so is F to G . [VII. 18]

Again, since C by multiplying the numbers E , F has made A , H respectively,

therefore, as E is to F , so is A to H . [VII. 17]

But, as E is to F , so is C to D .

Therefore also, as C is to D , so is A to H .

Again, since the numbers C , D by multiplying F have made H , K respectively,

therefore, as C is to D , so is H to K . [VII. 18]

Again, since D by multiplying each of the numbers F , G has made K , B respectively,

therefore, as F is to G , so is K to B . [VII. 17]

But, as F is to G , so is C to D ;

therefore also, as C is to D , so is A to H , H to K , and K to B .

Therefore H , K are two mean proportionals between A , B .

I say next that A also has to B the ratio triplicate of that which C has to D .

For, since A , H , K , B are four numbers in proportion, therefore A has to B the ratio triplicate of that which A has to H . [v. Def. 10]

But, as A is to H , so is C to D ;

therefore A also has to B the ratio triplicate of that which C has to D .

Q. E. D.

The cube numbers a^3 , b^3 being given, Euclid forms the products a^2b , ab^2 and then proves, as usual, by means of VII. 17, 18 that

$$a^3, a^2b, ab^2, b^3$$

are in continued proportion.

He assumes that, since a^3 has to b^3 the ratio triplicate of $a^3 : a^2b$, the ratio $a^3 : b^3$ is triplicate of the ratio $a : b$ which is equal to $a^3 : a^2b$. This is again an obvious inference from VII. 14.

PROPOSITION 13.

If there be as many numbers as we please in continued proportion, and each by multiplying itself make some number, the products will be proportional; and, if the original numbers by multiplying the products make certain numbers, the latter will also be proportional.

Let there be as many numbers as we please, A, B, C , in continued proportion, so that, as A is to B , so is B to C ;

let A, B, C by multiplying themselves make D, E, F , and by multiplying D, E, F let them make G, H, K ;

I say that D, E, F and G, H, K are in continued proportion.

A ———	G —————
B ———	H —————
C ———	K —————
D ———	
E ———	M —————
F ———	N —————
L ———	P —————
O ———	Q —————

For let A by multiplying B make L ,
and let the numbers A, B by multiplying L make M, N respectively.

And again let B by multiplying C make O ,
and let the numbers B, C by multiplying O make P, Q respectively.

Then, in manner similar to the foregoing, we can prove that

D, L, E and G, M, N, H are continuously proportional in the ratio of A to B ,

and further E, O, F and H, P, Q, K are continuously proportional in the ratio of B to C .

Now, as A is to B , so is B to C ;
therefore D, L, E are also in the same ratio with E, O, F ,
and further G, M, N, H in the same ratio with H, P, Q, K .

And the multitude of D, L, E is equal to the multitude of E, O, F , and that of G, M, N, H to that of H, P, Q, K ;
therefore, *ex aequali*,

as D is to E , so is E to F ,
and, as G is to H , so is H to K . [VII. 14]

If $a, b, c \dots$ be a series in geometrical progression, then

and $\left. \begin{array}{l} a^2, \quad b^2, \quad c^2 \dots \\ a^3, \quad b^3, \quad c^3 \dots \end{array} \right\}$ are also in geometrical progression.

Heiberg brackets the words added to the enunciation which extend the theorem to any powers. The words are "and this always occurs with the extremes" (καὶ ἀεὶ περὶ τοὺς ἄκρους τοῦτο συμβαίνει). They seem to be rightly suspected on the same grounds as the same words added to the enunciation of VII. 27. There is no allusion to them in the proof, much less any proof of the extension.

Euclid forms, besides the squares and cubes of the given numbers, the products $ab, a^2b, ab^2, bc, b^2c, bc^2$. When he says that "we prove in manner similar to the foregoing," he indicates successive uses of VII. 17, 18 as in VIII. 12.

With our notation the proof is as easy to see for *any* powers as for squares and cubes.

To prove that $a^n, b^n, c^n \dots$ are in geometrical progression.

Form all the means between a^n, b^n , and set out the series

$$a^n, \quad a^{n-1}b, \quad a^{n-2}b^2 \dots ab^{n-1}, \quad b^n.$$

The common ratio of one term to the next is $a : b$.

Next take the geometrical progression

$$b^n, \quad b^{n-1}c, \quad b^{n-2}c^2 \dots bc^{n-1}, \quad c^n,$$

the common ratio of which is $b : c$.

Proceed thus for all pairs of consecutive terms.

Now $a : b = b : c = \dots$

Therefore any pair of succeeding terms in one series are in the same ratio as any pair of succeeding terms in any other of the series.

And the number of terms in each is the same, namely $(n + 1)$.

Therefore, *ex aequali*,

$$a^n : b^n = b^n : c^n = c^n : d^n \dots$$

PROPOSITION 14.

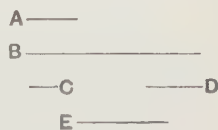
If a square measure a square, the side will also measure the side; and, if the side measure the side, the square will also measure the square.

Let A, B be square numbers, let C, D be their sides, and let A measure B ;

I say that C also measures D .

For let C by multiplying D make E ;
therefore A, E, B are continuously proportional in the ratio of C to D . [VIII. 11]

And, since A, E, B are continuously proportional, and A measures B ,
therefore A also measures E .



[VIII. 7]

And, as A is to E , so is C to D ;
therefore also C measures D .

[VII. Def. 20]

Again, let C measure D ;
I say that A also measures B .

For, with the same construction, we can in a similar manner prove that A , E , B are continuously proportional in the ratio of C to D .

And since, as C is to D , so is A to E ,
and C measures D ,

therefore A also measures E .

[VII. Def. 20]

And A , E , B are continuously proportional;
therefore A also measures B .

Therefore etc.

Q. E. D.

If a^2 measures b^2 , a measures b ; and, if a measures b , a^2 measures b^2 .

(1) a^2 , ab , b^2 are in continued proportion in the ratio of a to b .

Therefore, since a^2 measures b^2 ,

a^2 measures ab .

[VIII. 7]

But $a^2 : ab = a : b$.

Therefore a measures b .

(2) Since a measures b , a^2 measures ab .

And a^2 , ab , b^2 are continuously proportional.

Thus ab measures b^2 .

And a^2 measures ab .

Therefore a^2 measures b^2 .

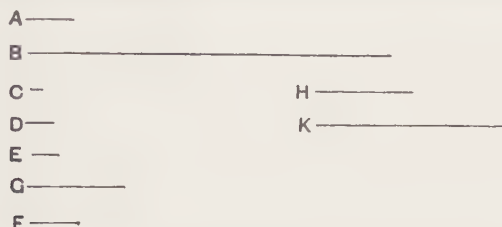
It will be seen that Euclid puts the last step shortly, saying that, since a^2 measures ab , and a^2 , ab , b^2 are in continued proportion, a^2 measures b^2 . The same thing happens in VIII. 15, where the series of terms is one more than here.

PROPOSITION 15.

If a cube number measure a cube number, the side will also measure the side; and, if the side measure the side, the cube will also measure the cube.

For let the cube number A measure the cube B ,
and let C be the side of A and D of B ;
I say that C measures D .

For let C by multiplying itself make E ,
 and let D by multiplying itself make G ;
 further, let C by multiplying D make F ,
 and let C, D by multiplying F make H, K respectively.



Now it is manifest that E, F, G and A, H, K, B are continuously proportional in the ratio of C to D . [VIII. 11, 12]

And, since A, H, K, B are continuously proportional,
 and A measures B ,
 therefore it also measures H . [VIII. 7]

And, as A is to H , so is C to D ;
 therefore C also measures D . [VII. Def. 20]

Next, let C measure D ;
 I say that A will also measure B .

For, with the same construction, we can prove in a similar manner that A, H, K, B are continuously proportional in the ratio of C to D .

And, since C measures D ,
 and, as C is to D , so is A to H ,
 therefore A also measures H , [VII. Def. 20]
 so that A measures B also.

Q. E. D.

If a^3 measures b^3 , a measures b ; and *vice versa*. The proof is, *mutatis mutandis*, the same as for squares.

(1) a^3, a^2b, ab^2, b^3 are continuously proportional in the ratio of a to b ;
 and a^3 measures b^3 .

Therefore a^3 measures a^2b ; [VIII. 7]
 and hence a measures b .

(2) Since a measures b , a^3 measures a^2b .

And, a^3, a^2b, ab^2, b^3 being continuously proportional, each term measures the succeeding term;
 therefore a^3 measures b^3 .

PROPOSITION 16.

If a square number do not measure a square number, neither will the side measure the side; and, if the side do not measure the side, neither will the square measure the square.

Let A, B be square numbers, and let C, D be their sides; and let A not measure B ;

I say that neither does C measure D .

A ———

For, if C measures D , A will also measure B .

B —————

[VIII. 14]

C ———

But A does not measure B ;

D ———

therefore neither will C measure D .

Again, let C not measure D ;

I say that neither will A measure B .

For, if A measures B , C will also measure D . [VIII. 14]

But C does not measure D ;

therefore neither will A measure B .

Q. E. D.

If a^2 does not measure b^2 , a will not measure b ; and, if a does not measure b , a^2 will not measure b^2 .

The proof is a mere *reductio ad absurdum* using VIII. 14.

PROPOSITION 17.

If a cube number do not measure a cube number, neither will the side measure the side; and, if the side do not measure the side, neither will the cube measure the cube.

For let the cube number A not measure the cube number B ,

and let C be the side of A , and D of B ;

A ———

I say that C will not measure D .

B —————

C ———

For if C measures D , A will also measure B .

D ———

[VIII. 15]

But A does not measure B ;

therefore neither does C measure D .

Again, let C not measure D ;

I say that neither will A measure B .

For, if A measures B , C will also measure D . [VIII. 15]

But C does not measure D ;

therefore neither will A measure B .

Q. E. D.

If a^2 does not measure b^2 , a will not measure b ; and *vice versa*.

Proved by *reductio ad absurdum* employing VIII. 15.

PROPOSITION 18.

Between two similar plane numbers there is one mean proportional number; and the plane number has to the plane number the ratio duplicate of that which the corresponding side has to the corresponding side.

Let A, B be two similar plane numbers, and let the numbers C, D be the sides of A , and E, F of B .



Now, since similar plane numbers are those which have their sides proportional, [VII. Def. 21]

therefore, as C is to D , so is E to F .

I say then that between A, B there is one mean proportional number, and A has to B the ratio duplicate of that which C has to E , or D to F , that is, of that which the corresponding side has to the corresponding side.

Now since, as C is to D , so is E to F , therefore, alternately, as C is to E , so is D to F . [VII. 13]

And, since A is plane, and C, D are its sides, therefore D by multiplying C has made A .

For the same reason also E by multiplying F has made B .

Now let D by multiplying E make G .

Then, since D by multiplying C has made A , and by multiplying E has made G , therefore, as C is to E , so is A to G . [VII. 17]

But, as C is to E , so is D to F ;
therefore also, as D is to F , so is A to G .

Again, since E by multiplying D has made G , and by multiplying F has made B ,

therefore, as D is to F , so is G to B . [VII. 17]

But it was also proved that,
as D is to F , so is A to G ;
therefore also, as A is to G , so is G to B .

Therefore A , G , B are in continued proportion.

Therefore between A , B there is one mean proportional number.

I say next that A also has to B the ratio duplicate of that which the corresponding side has to the corresponding side, that is, of that which C has to E or D to F .

For, since A , G , B are in continued proportion,
 A has to B the ratio duplicate of that which it has to G .

[v. Def. 9]

And, as A is to G , so is C to E , and so is D to F .

Therefore A also has to B the ratio duplicate of that which C has to E or D to F .

Q. E. D.

If ab , cd be "similar plane numbers," i.e. products of factors such that

$$a : b = c : d,$$

there is one mean proportional between ab and cd ; and ab is to cd in the duplicate ratio of a to c or of b to d .

Form the product bc (or ad , which is equal to it, by VII. 19).

Then
$$\left. \begin{array}{l} ab, \quad bc \\ \quad \quad = ad \end{array} \right\}, \quad cd$$

is a series of terms in geometrical progression.

For $a : b = c : d$.

Therefore $a : c = b : d$.

[VII. 13]

Therefore $ab : bc = bc : cd$.

[VII. 17 and 16]

Thus bc (or ad) is a geometric mean between ab , cd .

And ab is to cd in the duplicate ratio of ab to bc or of bc to cd , that is, of a to c or of b to d .

PROPOSITION 19.

Between two similar solid numbers there fall two mean proportional numbers; and the solid number has to the similar solid number the ratio triplicate of that which the corresponding side has to the corresponding side.

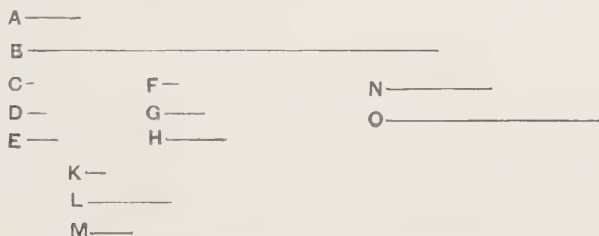
Let A, B be two similar solid numbers, and let C, D, E be the sides of A , and F, G, H of B .

Now, since similar solid numbers are those which have their sides proportional, [VII. Def. 21]

therefore, as C is to D , so is F to G ,

and, as D is to E , so is G to H .

I say that between A, B there fall two mean proportional numbers, and A has to B the ratio triplicate of that which C has to F , D to G , and also E to H .



For let C by multiplying D make K , and let F by multiplying G make L .

Now, since C, D are in the same ratio with F, G , and K is the product of C, D , and L the product of F, G , K, L are similar plane numbers; [VII. Def. 21]

therefore between K, L there is one mean proportional number. [VIII. 18]

Let it be M

Therefore M is the product of D, F , as was proved in the theorem preceding this. [VIII. 18]

Now, since D by multiplying C has made K , and by multiplying F has made M ,

therefore, as C is to F , so is K to M . [VII. 17]

But, as K is to M , so is M to L .

Therefore K, M, L are continuously proportional in the ratio of C to F .

And since, as C is to D , so is F to G ,
alternately therefore, as C is to F , so is D to G . [VII. 13]

For the same reason also,

as D is to G , so is E to H .

Therefore K, M, L are continuously proportional in the ratio of C to F , in the ratio of D to G , and also in the ratio of E to H .

Next, let E, H by multiplying M make N, O respectively.

Now, since A is a solid number, and C, D, E are its sides, therefore E by multiplying the product of C, D has made A .

But the product of C, D is K ;

therefore E by multiplying K has made A .

For the same reason also

H by multiplying L has made B .

Now, since E by multiplying K has made A , and further also by multiplying M has made N ,

therefore, as K is to M , so is A to N . [VII. 17]

But, as K is to M , so is C to F, D to G , and also E to H ;
therefore also, as C is to F, D to G , and E to H , so is A to N .

Again, since E, H by multiplying M have made N, O respectively,

therefore, as E is to H , so is N to O . [VII. 18]

But, as E is to H , so is C to F and D to G ;

therefore also, as C is to F, D to G , and E to H , so is A to N and N to O .

Again, since H by multiplying M has made O , and further also by multiplying L has made B ,

therefore, as M is to L , so is O to B . [VII. 17]

But, as M is to L , so is C to F, D to G , and E to H .

Therefore also, as C is to F, D to G , and E to H , so not only is O to B , but also A to N and N to O .

Therefore A, N, O, B are continuously proportional in the aforesaid ratios of the sides.

I say that A also has to B the ratio triplicate of that which the corresponding side has to the corresponding side, that is, of the ratio which the number C has to F , or D to G , and also E to H .

For, since A, N, O, B are four numbers in continued proportion,

therefore A has to B the ratio triplicate of that which A has to N . [v. Def. 10]

But, as A is to N , so it was proved that C is to F, D to G , and also E to H .

Therefore A also has to B the ratio triplicate of that which the corresponding side has to the corresponding side, that is, of the ratio which the number C has to F, D to G , and also E to H . Q. E. D.

In other words, if $a : b : c = d : e : f$, then there are two geometric means between abc, def ; and abc is to def in the triplicate ratio of a to d , or of b to e , or of c to f .

Euclid first takes the plane numbers ab, de (leaving out c, f) and forms the product bd . Thus, as in VIII. 18,

$$\begin{array}{c} ab, bd, de \\ = ea \end{array}$$

are three terms in geometrical progression in the ratio of a to d , or of b to e .

He next forms the products of c, f respectively into the mean bd .

Then abc, cbd, fbd, def

are in geometrical progression in the ratio of a to d etc.

$$\begin{array}{l} \text{For} \\ \left. \begin{array}{l} abc : cbd = ab : bd = a : d \\ bd : fbd = c : f \\ fbd : def = bd : de = b : e \end{array} \right\} \end{array} \quad [\text{VII. 17}]$$

And $a : d = b : e = c : f$.

The ratio of abc to def is the ratio triplicate of that of abc to cbd , i.e. of that of a to d etc.

PROPOSITION 20.

If one mean proportional number fall between two numbers, the numbers will be similar plane numbers.

For let one mean proportional number C fall between the two numbers A, B ;

5 I say that A, B are similar plane numbers.

Let D, E , the least numbers of those which have the same ratio with A, C , be taken; [VII. 33]

therefore D measures A the same number of times that E measures C . [VII. 20]

10 Now, as many times as D measures A , so many units let there be in F ;

therefore F by multiplying D has made A ,
so that A is plane, and D, F are its sides.

Again, since D, E are the least of the numbers which have
 15 the same ratio with C, B ,
 therefore D measures C the same number of times that E
 measures B . [VII. 20]



As many times, then, as E measures B , so many units let
 there be in G ;
 20 therefore E measures B according to the units in G ;
 therefore G by multiplying E has made B .

Therefore B is plane, and E, G are its sides.

Therefore A, B are plane numbers.

I say next that they are also similar.

25 For, †since F by multiplying D has made A , and by
 multiplying E has made C ,
 therefore, as D is to E , so is A to C , that is, C to B . [VII. 17]

Again, † since E by multiplying F, G has made C, B
 respectively,

30 therefore, as F is to G , so is C to B . [VII. 17]

But, as C is to B , so is D to E ;

therefore also, as D is to E , so is F to G .

And alternately, as D is to F , so is E to G . [VII. 13]

Therefore A, B are similar plane numbers; for their sides
 35 are proportional. Q. E. D.

25. For, since $F \dots 27. C$ to B . The text has clearly suffered corruption here. It is not necessary to *infer* from other facts that, as D is to E , so is A to C ; for this is part of the hypotheses (ll. 6, 7). Again, there is no explanation of the statement (l. 25) that F by multiplying E has made C . It is the statement and explanation of this latter fact which are alone wanted; after which the proof proceeds as in l. 28. We might therefore substitute for ll. 25—28 the following.

“For, since E measures C the same number of times that D measures A [l. 8], that is, according to the units in F [l. 10], therefore F by multiplying E has made C .

And, since E by multiplying F, G ,” etc. etc.

This proposition is the converse of VIII. 18. If a, c, b are in geometrical progression, a, b are “similar plane numbers.”

Let $a : \beta$ be the ratio $a : c$ (and therefore also the ratio $c : b$) in its lowest terms:

Then [VII. 20]

$$\begin{aligned} a &= m\alpha, & c &= m\beta, & \text{where } m &\text{ is some integer,} \\ c &= n\alpha, & b &= n\beta, & \text{where } n &\text{ is some integer.} \end{aligned}$$

Thus a, b are both products of two factors, i.e. plane.

Again, $a : \beta = a : c = c : b$
 $= m : n.$

[VII. 18]

Therefore, alternately, $a : m = \beta : n,$ [VII. 13]
 and hence $ma, n\beta$ are *similar* plane numbers.

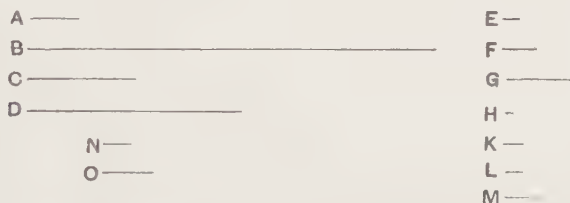
[Our notation makes the second part still more obvious, for $c = m\beta = na.$]

PROPOSITION 21.

If two mean proportional numbers fall between two numbers, the numbers are similar solid numbers.

For let two mean proportional numbers C, D fall between the two numbers A, B ;

I say that A, B are similar solid numbers.



For let three numbers E, F, G , the least of those which have the same ratio with A, C, D , be taken ; [VII. 33 or VIII. 2]
 therefore the extremes of them E, G are prime to one another. [VIII. 3]

Now, since one mean proportional number F has fallen between E, G ,
 therefore E, G are similar plane numbers. [VIII. 20]

Let, then, H, K be the sides of E , and L, M of G .

Therefore it is manifest from the theorem before this that E, F, G are continuously proportional in the ratio of H to L and that of K to M .

Now, since E, F, G are the least of the numbers which have the same ratio with A, C, D ,
 and the multitude of the numbers E, F, G is equal to the multitude of the numbers A, C, D ,
 therefore, *ex aequali*, as E is to G , so is A to D . [VII. 14]

But E, G are prime,
 primes are also least, [VII. 21]
 and the least measure those which have the same ratio with

them the same number of times, the greater the greater and the less the less, that is, the antecedent the antecedent and the consequent the consequent; [VII. 20]

therefore E measures A the same number of times that G measures D .

Now, as many times as E measures A , so many units let there be in N .

Therefore N by multiplying E has made A .

But E is the product of H, K ;

therefore N by multiplying the product of H, K has made A .

Therefore A is solid, and H, K, N are its sides.

Again, since E, F, G are the least of the numbers which have the same ratio as C, D, B ,

therefore E measures C the same number of times that G measures B .

Now, as many times as E measures C , so many units let there be in O .

Therefore G measures B according to the units in O ;

therefore O by multiplying G has made B .

But G is the product of L, M ;

therefore O by multiplying the product of L, M has made B .

Therefore B is solid, and L, M, O are its sides;

therefore A, B are solid.

I say that they are also similar.

For since N, O by multiplying E have made A, C ,

therefore, as N is to O , so is A to C , that is, E to F . [VII. 18]

But, as E is to F , so is H to L and K to M ;

therefore also, as H is to L , so is K to M and N to O .

And H, K, N are the sides of A , and O, L, M the sides of B .

Therefore A, B are similar solid numbers. Q. E. D.

The converse of VIII. 19. If a, c, d, b are in geometrical progression, a, b are "similar solid numbers."

Let α, β, γ be the least numbers in the ratio of a, c, d (and therefore also of c, d, b). [VII. 33 or VIII. 2]

Therefore α, γ are prime to one another: [VIII. 3]

They are also "similar plane numbers." [VIII. 20]

Let

$$\alpha = mn, \quad \gamma = pq,$$

where

$$m : n = p : q.$$

Then, by the proof of VIII. 20,

$$\alpha : \beta = m : p = n : q.$$

Now, *ex aequali*,

$$\alpha : d = \alpha : \gamma,$$

[VII. 14]

and, since α, γ are prime to one another,

$$a = r\alpha, \quad d = r\gamma, \quad \text{where } r \text{ is an integer.}$$

But

$$a = mn :$$

therefore $a = rmn$, and therefore a is "solid."

Again, *ex aequali*,

$$c : b = \alpha : \gamma,$$

and therefore $c = s\alpha, \quad b = s\gamma$, where s is an integer.

Thus $b = spq$, and b is therefore "solid."

Now

$$\alpha : \beta = \alpha : c = r\alpha : s\alpha$$

$$= r : s.$$

[VII. 18]

And, from above,

$$\alpha : \beta = m : p = n : q.$$

Therefore

$$r : s = m : p = n : q,$$

and hence a, b are *similar* solid numbers.

PROPOSITION 22.

If three numbers be in continued proportion, and the first be square, the third will also be square.

Let A, B, C be three numbers in continued proportion, and let A the first be square ;

I say that C the third is also square.

A _____

B _____

C _____

For, since between A, C there is one mean proportional number, B , therefore A, C are similar plane numbers.

[VIII. 20]

But A is square ;

therefore C is also square.

Q. E. D.

A mere application of VIII. 20 to the particular case where one of the "similar plane numbers" is square.

PROPOSITION 23.

If four numbers be in continued proportion, and the first be cube, the fourth will also be cube.

Let A, B, C, D be four numbers in continued proportion, and let A be cube ;

I say that D is also cube.

A _____

B _____

C _____

D _____

For, since between A, D there are two mean proportional numbers B, C ,

therefore A, D are similar solid numbers.

[VIII. 21]

But A is cube ;
therefore D is also cube.

Q. E. D.

A mere application of VIII. 21 to the case where one of the “similar solid numbers” is a cube.

PROPOSITION 24.

If two numbers have to one another the ratio which a square number has to a square number, and the first be square, the second will also be square.

For let the two numbers A, B have to one another the ratio which the square number C has to the square number D , and let A be square ;

A _____
B _____
C _____
D _____

I say that B is also square.

For, since C, D are square,
 C, D are similar plane numbers.

Therefore one mean proportional number falls between C, D . [VIII. 18]

And, as C is to D , so is A to B ;
therefore one mean proportional number falls between A, B also. [VIII. 8]

And A is square ;
therefore B is also square. [VIII. 22]

Q. E. D.

If $a : b = c^2 : d^2$, and a is a square, then b is also a square.

For c^2, d^2 have one mean proportional cd . [VIII. 18]

Therefore a, b , which are in the same ratio, have one mean proportional. [VIII. 8]

And, since a is square, b must also be a square. [VIII. 22]

PROPOSITION 25.

If two numbers have to one another the ratio which a cube number has to a cube number, and the first be cube, the second will also be cube.

For let the two numbers A, B have to one another the ratio which the cube number C has to the cube number D , and let A be cube ;

I say that B is also cube.

For, since C, D are cube,
 C, D are similar solid numbers.

Therefore two mean proportional numbers fall between
 C, D . [VIII. 19]



And, as many numbers as fall between C, D in continued proportion, so many will also fall between those which have the same ratio with them; [VIII. 8]
 so that two mean proportional numbers fall between A, B also.

Let E, F so fall.

Since, then, the four numbers A, E, F, B are in continued proportion,
 and A is cube,
 therefore B is also cube. [VIII. 23]

Q. E. D.

If $a : b = c^3 : d^3$, and a is a cube, then b is also a cube.

For c^3, d^3 have two mean proportionals.

Therefore a, b also have two mean proportionals.

And a is a cube:

therefore b is a cube.

[VIII. 19]

[VIII. 8]

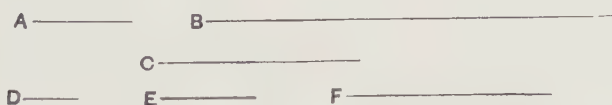
[VIII. 23]

PROPOSITION 26.

Similar plane numbers have to one another the ratio which a square number has to a square number.

Let A, B be similar plane numbers;

I say that A has to B the ratio which a square number has to a square number.



For, since A, B are similar plane numbers,
 therefore one mean proportional number falls between A, B . [VIII. 18]

Let it so fall, and let it be C ;
 and let D, E, F , the least numbers of those which have the
 same ratio with A, C, B , be taken; [VII. 33 or VIII. 2]
 therefore the extremes of them D, F are square. [VIII. 2, Por.]

And since, as D is to F , so is A to B ,
 and D, F are square,
 therefore A has to B the ratio which a square number has to
 a square number.

Q. E. D.

If a, b are similar "plane numbers," let c be the mean proportional
 between them. [VIII. 18]

Take α, β, γ the smallest numbers in the ratio of a, c, b . [VII. 33 or VIII. 2]

Then α, γ are squares. [VIII. 2, Por.]

Therefore a, b are in the ratio of a square to a square.

PROPOSITION 27.

*Similar solid numbers have to one another the ratio which
 a cube number has to a cube number.*

Let A, B be similar solid numbers;
 I say that A has to B the ratio which a cube number has to
 a cube number.



For, since A, B are similar solid numbers,
 therefore two mean proportional numbers fall between A, B . [VIII. 19]

Let C, D so fall,
 and let E, F, G, H , the least numbers of those which have
 the same ratio with A, C, D, B , and equal with them in
 multitude, be taken; [VII. 33 or VIII. 2]
 therefore the extremes of them E, H are cube. [VIII. 2, Por.]

And, as E is to H , so is A to B ;
 therefore A also has to B the ratio which a cube number has
 to a cube number.

Q. E. D.

The same thing as VIII. 26 with cubes. It is proved in the same way except that VIII. 19 is used instead of VIII. 18.

The last note of an-Nairizi in which the name of Heron is mentioned is on this proposition. Heron is there stated (p. 194—5, ed. Curtze) to have added the two propositions that,

1. *If two numbers have to one another the ratio of a square to a square, the numbers are similar plane numbers;*
2. *If two numbers have to one another the ratio of a cube to a cube, the numbers are similar solid numbers.*

The propositions are of course the converses of VIII. 26, 27 respectively. They are easily proved.

(1) If $a : b = c^2 : d^2$,

then, since there is one mean proportional (cd) between c^2, d^2 , [VIII. 11 or 18]

there is also one mean proportional between a, b . [VIII. 8]

Therefore a, b are similar plane numbers. [VIII. 20]

(2) is similarly proved by the use of VIII. 12 or 19, VIII. 8, VIII. 21.

The insertion by Heron of the first of the two propositions, the converse of VIII. 26, is perhaps an argument in favour of the correctness of the text of IX. 10, though (as remarked in the note on that proposition) it does not give the easiest proof Cf. Heron's extension of VII. 3 tacitly assumed by Euclid in VII. 33.

BOOK IX.

PROPOSITION I.

If two similar plane numbers by multiplying one another make some number, the product will be square.

Let A, B be two similar plane numbers, and let A by multiplying B make C ;

I say that C is square.

For let A by multiplying itself make D .

Therefore D is square.

Since then A by multiplying itself has made D , and by multiplying B has made C ,

therefore, as A is to B , so is D to C .

[VII. 17]

And, since A, B are similar plane numbers,

therefore one mean proportional number falls between A, B .

[VIII. 18]

But, if numbers fall between two numbers in continued proportion, as many as fall between them, so many also fall between those which have the same ratio;

[VIII. 8]

so that one mean proportional number falls between D, C also.

And D is square;

therefore C is also square.

[VIII. 22]

Q. E. D.

The product of two similar plane numbers is a square.

Let a, b be two similar plane numbers.

Now $a : b = a^2 : ab$.

[VII. 17]

And between a, b there is one mean proportional.

[VIII. 18]

Therefore between $a^2 : ab$ there is one mean proportional.

[VIII. 8]

And a^2 is square;

therefore ab is square.

[VIII. 22]

PROPOSITION 2.

If two numbers by multiplying one another make a square number, they are similar plane numbers.

Let A, B be two numbers, and let A by multiplying B make the square number C ;

I say that A, B are similar plane numbers.

A —————

B —————

C —————

D —————

For let A by multiplying itself make D ;

therefore D is square.

Now, since A by multiplying itself has made D , and by multiplying B has made C ,

therefore, as A is to B , so is D to C .

[VII. 17]

And, since D is square, and C is so also, therefore D, C are similar plane numbers.

Therefore one mean proportional number falls between D, C .

[VIII. 18]

And, as D is to C , so is A to B ;

therefore one mean proportional number falls between A, B also.

[VIII. 8]

But, if one mean proportional number fall between two numbers, they are similar plane numbers;

[VIII. 20]

therefore A, B are similar plane numbers.

Q. E. D.

If ab is a square number, a, b are similar plane numbers. (The converse of IX. 1.)

For $a : b = a^2 : ab$.

[VII. 17]

And a^2, ab being square numbers, and therefore similar plane numbers, they have one mean proportional.

[VIII. 18]

Therefore a, b also have one mean proportional.

[VIII. 8]

whence a, b are similar plane numbers.

[VIII. 20]

PROPOSITION 3.

If a cube number by multiplying itself make some number, the product will be cube.

For let the cube number A by multiplying itself make B ;
I say that B is cube.

For let C , the side of A , be taken, and let C by multiplying itself make D .

It is then manifest that C by multiplying A —
 D has made A . B —————

Now, since C by multiplying itself has C — D —
 made D ,

therefore C measures D according to the units in itself.

But further the unit also measures C according to the units in it;

therefore, as the unit is to C , so is C to D . [VII. Def. 20]

Again, since C by multiplying D has made A ,
 therefore D measures A according to the units in C .

But the unit also measures C according to the units in it ;
 therefore, as the unit is to C , so is D to A .

But, as the unit is to C , so is C to D ;
 therefore also, as the unit is to C , so is C to D , and D to A .

Therefore between the unit and the number A two mean
 proportional numbers C, D have fallen in continued proportion.

Again, since A by multiplying itself has made B ,
 therefore A measures B according to the units in itself.

But the unit also measures A according to the units in it ;
 therefore, as the unit is to A , so is A to B . [VII. Def. 20]

But between the unit and A two mean proportional numbers
 have fallen ;

therefore two mean proportional numbers will also fall between
 A, B . [VIII. 8]

But, if two mean proportional numbers fall between two
 numbers, and the first be cube, the second will also be cube.

[VIII. 23]

And A is cube ;

therefore B is also cube.

Q. E. D.

The product of a^3 into itself, or $a^3 \cdot a^3$, is a cube.

For $1 : a = a : a^2 = a^2 : a^3$.

Therefore between 1 and a^3 there are two mean proportionals.

Also $1 : a^3 = a^3 : a^3 \cdot a^3$.

Therefore two mean proportionals fall between a^3 and $a^3 \cdot a^3$. [VIII. 8]
 (It is true that VIII. 8 is only enunciated of two pairs of numbers, but the
 proof is equally valid if one number of one pair is unity.)

And a^3 is a cube number :

therefore $a^3 \cdot a^3$ is also cube.

[VIII. 23]

PROPOSITION 4.

If a cube number by multiplying a cube number make some number, the product will be cube.

For let the cube number A by multiplying the cube number B make C ;

I say that C is cube.

A _____

For let A by multiplying itself make D ;

B _____

therefore D is cube. [IX. 3]

C _____

D _____

And, since A by multiplying itself has made D , and by multiplying B has made C , therefore, as A is to B , so is D to C .

[VII. 17]

And, since A , B are cube numbers, A , B are similar solid numbers.

Therefore two mean proportional numbers fall between A , B ;

[VIII. 19]

so that two mean proportional numbers will fall between D , C also.

[VIII. 8]

And D is cube;

therefore C is also cube

[VIII. 23]

Q. E. D.

The product of two cubes, say $a^3 \cdot b^3$, is a cube.

For $a^3 : b^3 = a^3 \cdot a^3 : a^3 \cdot b^3$. [VII. 17]

And two mean proportionals fall between a^3 , b^3 which are similar solid numbers.

[VIII. 19]

Therefore two mean proportionals fall between $a^3 \cdot a^3$, $a^3 \cdot b^3$

[VIII. 8]

But $a^3 \cdot a^3$ is a cube:

[IX. 3]

therefore $a^3 \cdot b^3$ is a cube.

[VIII. 23]

PROPOSITION 5.

If a cube number by multiplying any number make a cube number, the multiplied number will also be cube.

For let the cube number A by multiplying any number B make the cube number C ;

I say that B is cube.

A _____

For let A by multiplying itself make D ;

B _____

therefore D is cube. [IX. 3]

C _____

D _____

Now, since A by multiplying itself has made D , and by multiplying B has made C ,

therefore, as A is to B , so is D to C . [VII. 17]

And since D , C are cube,
they are similar solid numbers.

Therefore two mean proportional numbers fall between D , C . [VIII. 19]

And, as D is to C , so is A to B ;
therefore two mean proportional numbers fall between A , B
also. [VIII. 8]

And A is cube ;
therefore B is also cube. [VIII. 23]

If the product a^3b is a cube number, b is cube.

By IX. 3, the product $a^3 \cdot a^3$ is a cube.

And $a^3 \cdot a^3 : a^3b = a^3 : b$. [VII. 17]

The first two terms are cubes, and therefore "similar solids"; therefore there are two mean proportionals between them. [VIII. 19]

Therefore there are two mean proportionals between a^3 , b . [VIII. 8]

And a^3 is a cube :
therefore b is a cube number. [VIII. 23]

PROPOSITION 6.

If a number by multiplying itself make a cube number, it will itself also be cube.

For let the number A by multiplying itself make the cube number B ;

I say that A is also cube.

For let A by multiplying B make C .

Since, then, A by multiplying itself has made B , and by multiplying B has made C ,

therefore C is cube.

And, since A by multiplying itself has made B ,
therefore A measures B according to the units in itself.

But the unit also measures A according to the units in it.

Therefore, as the unit is to A , so is A to B . [VII. Def. 20]

And, since A by multiplying B has made C ,
therefore B measures C according to the units in A .

But the unit also measures A according to the units in it.

A _____
B _____
C _____

Therefore, as the unit is to A , so is B to C . [VII. Def. 20]

But, as the unit is to A , so is A to B ;

therefore also, as A is to B , so is B to C .

And, since B , C are cube,

they are similar solid numbers.

Therefore there are two mean proportional numbers between B , C . [VIII. 19]

And, as B is to C , so is A to B .

Therefore there are two mean proportional numbers between A , B also. [VIII. 8]

And B is cube ;

therefore A is also cube. [cf. VIII. 23]

Q. E. D.

If a^4 is a cube number, a is also a cube.

For $1 : a = a : a^2 = a^2 : a^3$.

Now a^2 , a^3 are both cubes; and therefore "similar solids"; therefore there are two mean proportionals between them. [VIII. 19]

Therefore there are two mean proportionals between a , a^2 . [VIII. 8]

And a^2 is a cube :

therefore a is also a cube number. [VIII. 23]

It will be noticed that the last step is not an exact quotation of the result of VIII. 23, because it is there the *first* of four terms which is known to be a cube, and the *last* which is proved to be a cube ; here the case is reversed. But there is no difficulty. Without inverting the proportions, we have only to refer to VIII. 21 which proves that a , a^2 , having two mean proportionals between them, are two similar solid numbers ; whence, since a^2 is a cube, a is also a cube.

PROPOSITION 7.

If a composite number by multiplying any number make some number, the product will be solid.

For let the composite number A by multiplying any number B make C ;

I say that C is solid.

For, since A is composite, it will be measured by some number. [VII. Def. 13]

Let it be measured by D ;

and, as many times as D measures A , so many units let there be in E .

A _____
 B _____
 C _____
 D _____ E _____

Since then D measures A according to the units in E ,
therefore E by multiplying D has made A . [VII. Def. 15]

And, since A by multiplying B has made C ,
and A is the product of D, E ,
therefore the product of D, E by multiplying B has made C .

Therefore C is solid, and D, E, B are its sides.

Q. E. D.

Since a composite number is the product of two factors, the result of multiplying it by another number is to produce a number which is the product of three factors, i.e. a "solid number."

PROPOSITION 8.

If as many numbers as we please beginning from an unit be in continued proportion, the third from the unit will be square, as will also those which successively leave out one; the fourth will be cube, as will also all those which leave out two; and the seventh will be at once cube and square, as will also those which leave out five.

Let there be as many numbers as we please, A, B, C, D, E, F , beginning from an unit and in continued proportion;

I say that B , the third from the unit, is square, as are also all those which leave out one; C , the fourth, is cube, as are also all those which leave out two; and F , the seventh, is at once cube and square, as are also all those which leave out five.

A ———
B ———
C ———
D ———
E ———
F ———

For since, as the unit is to A , so is A to B ,
therefore the unit measures the number A the same number of times that A measures B . [VII. Def. 20]

But the unit measures the number A according to the units in it;

therefore A also measures B according to the units in A .

Therefore A by multiplying itself has made B ;
therefore B is square.

And, since B, C, D are in continued proportion, and B is square,
therefore D is also square. [VIII. 22]

For the same reason

F is also square.

Similarly we can prove that all those which leave out one are square.

I say next that C , the fourth from the unit, is cube, as are also all those which leave out two.

For since, as the unit is to A , so is B to C , therefore the unit measures the number A the same number of times that B measures C .

But the unit measures the number A according to the units in A ;

therefore B also measures C according to the units in A .

Therefore A by multiplying B has made C .

Since then A by multiplying itself has made B , and by multiplying B has made C , therefore C is cube.

And, since C, D, E, F are in continued proportion, and C is cube,

therefore F is also cube. [VIII. 23]

But it was also proved square;

therefore the seventh from the unit is both cube and square.

Similarly we can prove that all the numbers which leave out five are also both cube and square.

Q. E. D.

If $1, a, a_2, a_3, \dots$ be a geometrical progression, then $a_2, a_4, a_6, \dots$ are squares;

$a_3, a_6, a_9, \dots$ are cubes;

$a_6, a_{12}, \dots$ are both squares and cubes.

Since

$$1 : a = a : a_2,$$

$$a_2 = a^2.$$

And, since a_2, a_3, a_4 are in geometrical progression and $a_2 (= a^2)$ is a square,

a_4 is a square. [VIII. 22]

Similarly $a_6, a_9, \dots$ are squares.

Next,

$$1 : a = a_2 : a_3$$

$$= a^2 : a_3,$$

whence $a_3 = a^3$, a cube number.

And, since a_3, a_4, a_5, a_6 are in geometrical progression, and a_3 is a cube,

a_6 is a cube. [VIII. 23]

Similarly $a_9, a_{12}, \dots$ are cubes.

Clearly then $a_6, a_{12}, a_{18}, \dots$ are both squares and cubes.

The whole result is of course obvious if the geometrical progression is written, with our notation, 'as

$$1, a, a^2, a^3, a^4, \dots a^n.$$

PROPOSITION 9.

If as many numbers as we please beginning from an unit be in continued proportion, and the number after the unit be square, all the rest will also be square. And, if the number after the unit be cube, all the rest will also be cube.

Let there be as many numbers as we please, A, B, C, D, E, F , beginning from an unit and in continued proportion, and let A , the number after the unit, be square ;

I say that all the rest will also be square.

Now it has been proved that B , the third from the unit, is square, as are also all those which leave out one ; [IX. 8]

I say that all the rest are also square.

For, since A, B, C are in continued proportion, and A is square, therefore C is also square. [VIII. 22]

Again, since B, C, D are in continued proportion, and B is square, D is also square. [VIII. 22]

Similarly we can prove that all the rest are also square.

Next, let A be cube ;

I say that all the rest are also cube.

Now it has been proved that C , the fourth from the unit, is cube, as also are all those which leave out two ; [IX. 8]

I say that all the rest are also cube.

For, since, as the unit is to A , so is A to B , therefore the unit measures A the same number of times as A measures B .

But the unit measures A according to the units in it ; therefore A also measures B according to the units in itself ; therefore A by multiplying itself has made B .

A ———
 B ———
 C ———
 D ———
 E ———
 F ———

And A is cube.

But, if a cube number by multiplying itself make some number, the product is cube. [IX. 3]

Therefore B is also cube.

And, since the four numbers A, B, C, D are in continued proportion,

and A is cube,

D also is cube.

[VIII. 23]

For the same reason

E is also cube, and similarly all the rest are cube.

Q. E. D.

If $1, a^2, a_2, a_3, a_4, \dots$ are in geometrical progression, $a_2, a_3, a_4, \dots$ are all squares;

and, if $1, a^3, a_2, a_3, a_4, \dots$ are in geometrical progression, $a_2, a_3, \dots$ are all cubes.

(1) By IX. 8, $a_2, a_4, a_6, \dots$ are all squares.

And, a^2, a_2, a_3 being in geometrical progression, and a^2 being a square,

a_3 is a square.

[VIII. 22]

For the same reason $a_5, a_7, \dots$ are all squares.

(2) By IX. 8, $a_3, a_6, a_9, \dots$ are all cubes.

Now

$$1 : a^3 = a^3 : a_2.$$

Therefore

$$a_2 = a^3 \cdot a^3, \text{ which is a cube, by IX. 3.}$$

And, a^3, a_2, a_3, a_4 being in geometrical progression, and a^3 being cube,

a_4 is cube.

[VIII. 23]

Similarly we prove that a_5 is cube, and so on.

The results are of course obvious in our notation, the series being

$$(1) \quad 1, a^2, a^4, a^6, \dots a^{2n},$$

$$(2) \quad 1, a^3, a^6, a^{12}, \dots a^{3n}.$$

PROPOSITION 10.

If as many numbers as we please beginning from an unit be in continued proportion, and the number after the unit be not square, neither will any other be square except the third from the unit and all those which leave out one. And, if the number after the unit be not cube, neither will any other be cube except the fourth from the unit and all those which leave out two.

Let there be as many numbers as we please, A, B, C, D, E, F , beginning from an unit and in continued proportion, and let A , the number after the unit, not be square;

I say that neither will any other be square except the third from the unit < and those which leave out one > .

For, if possible, let C be square.

But B is also square ; [IX. 8]

[therefore B, C have to one another the ratio which a square number has to a square number].

A ———
 B ———
 C ———
 D ———
 E ———
 F ———

And, as B is to C , so is A to B ;

therefore A, B have to one another the ratio which a square number has to a square number ;

[so that A, B are similar plane numbers]. [VIII. 26, converse]

And B is square ;

therefore A is also square :

which is contrary to the hypothesis.

Therefore C is not square.

Similarly we can prove that neither is any other of the numbers square except the third from the unit and those which leave out one.

Next, let A not be cube.

I say that neither will any other be cube except the fourth from the unit and those which leave out two.

For, if possible, let D be cube.

Now C is also cube ; for it is fourth from the unit. [IX. 8]

And, as C is to D , so is B to C ;

therefore B also has to C the ratio which a cube has to a cube.

And C is cube ;

therefore B is also cube.

[VIII. 25]

And since, as the unit is to A , so is A to B ,

and the unit measures A according to the units in it,

therefore A also measures B according to the units in itself ;

therefore A by multiplying itself has made the cube number B .

But, if a number by multiplying itself make a cube number, it is also itself cube. [IX. 6]

Therefore A is also cube :

which is contrary to the hypothesis.

Therefore D is not cube.

Similarly we can prove that neither is any other of the numbers cube except the fourth from the unit and those which leave out two.

Q. E. D.

If 1, a , a_2 , a_3 , a_4 , ... be a geometrical progression, then (1), if a is not a square, none of the terms will be square except a_2 , a_4 , a_6 , ...; and (2), if a is not a cube, none of the terms will be cube except a_3 , a_6 , a_9 , ...

With reference to the first part of the proof, viz. that which proves that, if a_3 is a square, a must be a square, Heiberg remarks that the words which I have bracketed are perhaps spurious; for it is easier to use VIII. 24 than the *converse* of VIII. 26, and a use of VIII. 24 would correspond better to the use of VIII. 25 in the second part relating to cubes. I agree in this view and have bracketed the words accordingly. (See however note, p. 383, on converses of VIII. 26, 27 given by Heron.) If this change be made, the proof runs as follows.

(1) If possible, let a_3 be square.

Now $a_2 : a_3 = a : a_2$.

But a_2 is a square.

[IX. 8]

Therefore a is to a_2 in the ratio of a square to a square.

And a_2 is square;

therefore a is square [VIII. 24]: which is impossible.

(2) If possible, let a_4 be a cube.

Now $a_3 : a_4 = a_2 : a_3$.

And a_3 is a cube.

[IX. 8]

Therefore a_2 is to a_3 in the ratio of a cube to a cube.

And a_3 is a cube:

therefore a_2 is a cube.

[VIII. 25]

But, since

$$1 : a = a : a_2,$$

$$a_2 = a^2.$$

And, since a^2 is a cube,

a must be a cube [IX. 6]: which is impossible.

The propositions VIII. 24, 25 are here not quoted in their exact form in that the *first* and *second* squares, or cubes, change places. But there is no difficulty, since the method by which the theorems are proved shows that either inference is equally correct.

PROPOSITION 11.

If as many numbers as we please beginning from an unit be in continued proportion, the less measures the greater according to some one of the numbers which have place among the proportional numbers.

Let there be as many numbers as we please, B, C, D, E , beginning from the unit A and in continued proportion ;

I say that B , the least of the numbers B, C, D, E , measures E according to some one of the numbers C, D .

A ———
B ———
C ———
D ———
E ———

For since, as the unit A is to B , so is D to E ,

therefore the unit A measures the number B the same number of times as D measures E ;

therefore, alternately, the unit A measures D the same number of times as B measures E . [VII. 15]

But the unit A measures D according to the units in it ; therefore B also measures E according to the units in D ; so that B the less measures E the greater according to some number of those which have place among the proportional numbers.—

PORISM. And it is manifest that, whatever place the measuring number has, reckoned from the unit, the same place also has the number according to which it measures, reckoned from the number measured, in the direction of the number before it.—

Q. E. D.

The proposition and the porism together assert that, if $1, a, a_2, \dots a_n$ be a geometrical progression, a_r measures a_n and gives the quotient a_{n-r} ($r < n$).

Euclid only proves that $a_n = a \cdot a_{n-1}$, as follows.

$$1 : a = a_{n-1} : a_n.$$

Therefore 1 measures a the same number of times as a_{n-1} measures a_n .

Hence 1 measures a_{n-1} the same number of times as a measures a_n ; [VII. 15]

that is, $a_n = a \cdot a_{n-1}$.

We can supply the proof of the porism as follows.

$$1 : a = a_r : a_{r+1},$$

$$a : a_2 = a_{r+1} : a_{r+2},$$

$$\dots\dots\dots$$

$$a_{n-r-1} : a_{n-r} = a_{n-1} : a_n,$$

whence, *ex aequali*,

$$1 : a_{n-r} = a_r : a_n.$$

[VII. 14]

It follows, by the same argument as before, that

$$a_n = a_r \cdot a_{n-r}.$$

With our notation, we have the theorem of indices that

$$a^{m+n} = a^m \cdot a^n.$$

PROPOSITION 12.

If as many numbers as we please beginning from an unit be in continued proportion, by however many prime numbers the last is measured, the next to the unit will also be measured by the same.

Let there be as many numbers as we please, A, B, C, D , beginning from an unit, and in continued proportion;
I say that, by however many prime numbers D is measured, A will also be measured by the same.

A —	F —————
B —	G —————
C ———	H ———
D —————	
E —	

For let D be measured by any prime number E ;
I say that E measures A .

For suppose it does not;
now E is prime, and any prime number is prime to any which it does not measure; [VII. 29]
therefore E, A are prime to one another.

And, since E measures D , let it measure it according to F ,
therefore E by multiplying F has made D .

Again, since A measures D according to the units in C , [IX. 11 and Por.]
therefore A by multiplying C has made D .

But, further, E has also by multiplying F made D ;
therefore the product of A, C is equal to the product of E, F .

Therefore, as A is to E , so is F to C . [VII. 19]

But A, E are prime,
primes are also least, [VII. 21]
and the least measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent; [VII. 20]
therefore E measures C .

Let it measure it according to G ;
therefore E by multiplying G has made C .

But, further, by the theorem before this,
 A has also by multiplying B made C . [IX. 11 and Por.]

Therefore the product of A, B is equal to the product of E, G .

Therefore, as A is to E , so is G to B . [VII. 19]

But A, E are prime,

primes are also least, [VII. 21]

and the least numbers measure those which have the same ratio with them the same number of times, the antecedent the antecedent and the consequent the consequent : [VII. 20]

therefore E measures B .

Let it measure it according to H ;

therefore E by multiplying H has made B .

But further A has also by multiplying itself made B ; [IX. 8]

therefore the product of E, H is equal to the square on A .

Therefore, as E is to A , so is A to H . [VII. 19]

But A, E are prime,

primes are also least, [VII. 21]

and the least measure those which have the same ratio the same number of times, the antecedent the antecedent and the consequent the consequent ; [VII. 20]

therefore E measures A , as antecedent antecedent.

But, again, it also does not measure it :

which is impossible.

Therefore E, A are not prime to one another.

Therefore they are composite to one another.

But numbers composite to one another are measured by some number. [VII. Def. 14]

And, since E is by hypothesis prime,

and the prime is not measured by any number other than itself, therefore E measures A, E ,

so that E measures A .

[But it also measures D ;

therefore E measures A, D .]

Similarly we can prove that, by however many prime numbers D is measured, A will also be measured by the same.

Q. E. D.

If $1, a, a_2, \dots a_n$ be a geometrical progression, and a_n be measured by any prime number p , a will also be measured by p .

For, if possible, suppose that p does not measure a ; then, p being prime, p, a are prime to one another. [VII. 29]

Suppose $a_n = m \cdot p$.

Now $a_n = a \cdot a_{n-1}$. [IX. 11]

Therefore $a \cdot a_{n-1} = m \cdot p$,

and $a : p = m : a_{n-1}$. [VII. 19]

Hence, a, p being prime to one another,

p measures a_{n-1} . [VII. 20, 21]

By a repetition of the same process, we can prove that p measures a_{n-2} and therefore a_{n-3} , and so on, and finally that p measures a .

But, by hypothesis, p does not measure a : which is impossible.

Hence p, a are not prime to one another:

therefore they have some common factor. [VII. Def. 14]

But p is the only number which measures p ;

therefore p measures a .

Heiberg remarks that, as, in the $\epsilon\kappa\theta\epsilon\sigma\iota\varsigma$, Euclid sets himself to prove that E measures A , the words bracketed above are unnecessary and therefore perhaps interpolated.

PROPOSITION 13.

If as many numbers as we please beginning from an unit be in continued proportion, and the number after the unit be prime, the greatest will not be measured by any except those which have a place among the proportional numbers.

Let there be as many numbers as we please, A, B, C, D , beginning from an unit and in continued proportion, and let A , the number after the unit, be prime;

I say that D , the greatest of them, will not be measured by any other number except A, B, C .

A —————	E ———
B —————	F —————
C —————	G ———
D —————	H —————

For, if possible, let it be measured by E , and let E not be the same with any of the numbers A, B, C .

It is then manifest that E is not prime.

For, if E is prime and measures D ,

it will also measure A [IX. 12], which is prime, though it is not the same with it:

which is impossible.

Therefore E is not prime.

Therefore it is composite.

But any composite number is measured by some prime number; [VII. 31]

therefore E is measured by some prime number.

I say next that it will not be measured by any other prime except A .

For, if E is measured by another,

and E measures D ,

that other will also measure D ;

so that it will also measure A [IX. 12], which is prime, though it is not the same with it :

which is impossible.

Therefore A measures E .

And, since E measures D , let it measure it according to F .

I say that F is not the same with any of the numbers A, B, C .

For, if F is the same with one of the numbers A, B, C , and measures D according to E ,

therefore one of the numbers A, B, C also measures D according to E .

But one of the numbers A, B, C measures D according to some one of the numbers A, B, C ; [IX. 11]

therefore E is also the same with one of the numbers A, B, C : which is contrary to the hypothesis.

Therefore F is not the same as any one of the numbers A, B, C .

Similarly we can prove that F is measured by A , by proving again that F is not prime.

For, if it is, and measures D ,

it will also measure A [IX. 12], which is prime, though it is not the same with it :

which is impossible ;

therefore F is not prime.

Therefore it is composite.

But any composite number is measured by some prime number; [VII. 31]

therefore F is measured by some prime number.

I say next that it will not be measured by any other prime except A .

For, if any other prime number measures F ,
and F measures D ,

that other will also measure D ;

so that it will also measure A [IX. 12], which is prime, though it is not the same with it:

which is impossible.

Therefore A measures F .

And, since E measures D according to F ,
therefore E by multiplying F has made D .

But, further, A has also by multiplying C made D ; [IX. 11]
therefore the product of A , C is equal to the product of E , F .

Therefore, proportionally, as A is to E , so is F to C .

[VII. 19]

But A measures E ;

therefore F also measures C .

Let it measure it according to G .

Similarly, then, we can prove that G is not the same with any of the numbers A , B , and that it is measured by A .

And, since F measures C according to G

therefore F by multiplying G has made C .

But, further, A has also by multiplying B made C ; [IX. 11]
therefore the product of A , B is equal to the product of F , G .

Therefore, proportionally, as A is to F , so is G to B .

[VII. 19]

But A measures F ;

therefore G also measures B .

Let it measure it according to H .

Similarly then we can prove that H is not the same with A .

And, since G measures B according to H ,

therefore G by multiplying H has made B .

But further A has also by multiplying itself made B ;

[IX. 8]

therefore the product of H , G is equal to the square on A .

Therefore, as H is to A , so is A to G .

[VII. 19]

But A measures G ;
 therefore H also measures A , which is prime, though it is not
 the same with it :
 which is absurd.

Therefore D the greatest will not be measured by any
 other number except A, B, C .

Q. E. D.

If $1, a, a_2, \dots a_n$ be a geometrical progression, and if a is prime, a_n will not
 be measured by any numbers except the preceding terms of the series.

If possible, let a_n be measured by b , a number different from all the
 preceding terms.

Now b cannot be prime, for, if it were, it would measure a . [IX. 12]

Therefore b is composite, and hence will be measured by *some* prime
 number [VII. 31], say p .

Thus p must measure a_n and therefore a [IX. 12]; so that p cannot be
 different from a , and b is not measured by any prime number except a .

Suppose that $a_n = b \cdot c$.

Now c cannot be identical with any of the terms $a, a_2, \dots a_{n-1}$; for, if it
 were, b would be identical with another of them : [IX. 11]
 which is contrary to the hypothesis.

We can now prove (just as for b) that c cannot be prime and cannot be
 measured by any prime number except a .

Since $b \cdot c = a_n = a \cdot a_{n-1}$, [IX. 11]

$$a : b = c : a_{n-1},$$

whence, since a measures b ,

$$c \text{ measures } a_{n-1}.$$

Let $a_{n-1} = c \cdot d$.

We now prove in the same way that d is not identical with any of the terms
 $a, a_2, \dots a_{n-2}$, is not prime, and is not measured by any prime except a , and
 also that

$$d \text{ measures } a_{n-2}.$$

Proceeding in this way, we get a last factor, say k , which measures a
 though different from it :

which is absurd, since a is prime.

Thus the original supposition that a_n can be measured by a number b
 different from all the terms $a, a_2, \dots a_{n-1}$ must be incorrect.

Therefore etc.

PROPOSITION 14.

*If a number be the least that is measured by prime numbers,
 it will not be measured by any other prime number except those
 originally measuring it.*

For let the number A be the least that is measured by the
 prime numbers B, C, D ;

I say that A will not be measured by any other prime number except B, C, D .

For, if possible, let it be measured by the prime number E , and let E not be the same with any one of the numbers B, C, D .

A —————	B ———
E —————	C ———
F —————	D ———

Now, since E measures A , let it measure it according to F ;

therefore E by multiplying F has made A .

And A is measured by the prime numbers B, C, D .

But, if two numbers by multiplying one another make some number, and any prime number measure the product, it will also measure one of the original numbers ; [VII. 30]

therefore B, C, D will measure one of the numbers E, F .

Now they will not measure E ;

for E is prime and not the same with any one of the numbers B, C, D .

Therefore they will measure F , which is less than A : which is impossible, for A is by hypothesis the least number measured by B, C, D .

Therefore no prime number will measure A except B, C, D .

Q. E. D.

In other words, a number can be resolved into prime factors in only one way.

Let a be the least number measured by each of the prime numbers $b, c, d, \dots k$.

If possible, suppose that a has a prime factor p different from $b, c, d, \dots k$.

Let $a = p \cdot m$.

Now $b, c, d, \dots k$, measuring a , must measure one of the two factors p, m . [VII. 30]

They do not, by hypothesis, measure p ;

therefore they must measure m , a number less than a :

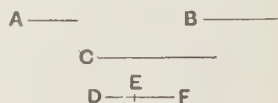
which is contrary to the hypothesis.

Therefore a has no prime factors except $b, c, d, \dots k$.

PROPOSITION 15.

If three numbers in continued proportion be the least of those which have the same ratio with them, any two whatever added together will be prime to the remaining number.

Let A, B, C , three numbers in continued proportion, be the least of those which have the same ratio with them ;



I say that any two of the numbers A, B, C whatever added together are prime to the remaining number, namely A, B to C ; B, C to A ; and further A, C to B .

For let two numbers DE, EF , the least of those which have the same ratio with A, B, C , be taken. [VIII. 2]

It is then manifest that DE by multiplying itself has made A , and by multiplying EF has made B , and, further, EF by multiplying itself has made C . [VIII. 2]

Now, since DE, EF are least, they are prime to one another. [VII. 22]

But, if two numbers be prime to one another, their sum is also prime to each ; [VII. 28]
therefore DF is also prime to each of the numbers DE, EF .

But further DE is also prime to EF ;
therefore DF, DE are prime to EF .

But, if two numbers be prime to any number, their product is also prime to the other ; [VII. 24]
so that the product of FD, DE is prime to EF ;
hence the product of FD, DE is also prime to the square on EF . [VII. 25]

But the product of FD, DE is the square on DE together with the product of DE, EF ; [II. 3]
therefore the square on DE together with the product of DE, EF is prime to the square on EF .

And the square on DE is A ,
the product of DE, EF is B ,
and the square on EF is C ;
therefore A, B added together are prime to C .

Similarly we can prove that B, C added together are prime to A .

I say next that A, C added together are also prime to B .

For, since DF is prime to each of the numbers DE, EF , the square on DF is also prime to the product of DE, EF .

[VII. 24, 25]

But the squares on DE, EF together with twice the product of DE, EF are equal to the square on DF ; [II. 4]

therefore the squares on DE, EF together with twice the product of DE, EF are prime to the product of DE, EF .

Separando, the squares on DE, EF together with once the product of DE, EF are prime to the product of DE, EF .

Therefore, *separando* again, the squares on DE, EF are prime to the product of DE, EF .

And the square on DE is A ,

the product of DE, EF is B ,

and the square on EF is C .

Therefore A, C added together are prime to B .

Q. E. D.

If a, b, c be a geometrical progression in the least terms which have a given common ratio, $(b+c), (c+a), (a+b)$ are respectively prime to a, b, c .

Let $a : \beta$ be the common ratio in its lowest terms, so that the geometrical progression is

$$a^2, a\beta, \beta^2. \quad [\text{VIII. 2}]$$

Now, a, β being prime to one another,

$a + \beta$ is prime to both a and β . [VII. 28]

Therefore $(a + \beta), a$ are both prime to β .

Hence $(a + \beta) a$ is prime to β , [VII. 24]

and therefore to β^2 ; [VII. 25]

i.e. $a^2 + a\beta$ is prime to β^2 ,

or $a + b$ is prime to c .

Similarly, $a\beta + \beta^2$ is prime to a^2 ,

or $b + c$ is prime to a .

Lastly, $a + \beta$ being prime to both a and β ,

$(a + \beta)^2$ is prime to $a\beta$, [VII. 24, 25]

or $a^2 + \beta^2 + 2a\beta$ is prime to $a\beta$:

whence $a^2 + \beta^2$ is prime to $a\beta$.

The latter inference, made in two steps, may be proved by *reductio ad absurdum* as Commandinus proves it.

If $a^2 + \beta^2$ is not prime to $a\beta$, let x measure them;

therefore x measures $a^2 + \beta^2 + 2a\beta$ as well as $a\beta$;

hence $a^2 + \beta^2 + 2a\beta$ and $a\beta$ are not prime to one another, which is contrary to the hypothesis.

PROPOSITION 16.

If two numbers be prime to one another, the second will not be to any other number as the first is to the second.

For let the two numbers A, B be prime to one another ;
I say that B is not to any other number as
 A is to B .

For, if possible, as A is to B , so let B be
to C .

A _____
B _____
C _____

Now A, B are prime,
primes are also least,

[VII. 21]

and the least numbers measure those which have the same
ratio the same number of times, the antecedent the antecedent
and the consequent the consequent ;

[VII. 20]

therefore A measures B as antecedent antecedent.

But it also measures itself ;
therefore A measures A, B which are prime to one another :
which is absurd.

Therefore B will not be to C , as A is to B .

Q. E. D.

If a, b are prime to one another, they can have no integral third
proportional.

If possible, let $a : b = b : x$.

Therefore [VII. 20, 21] a measures b ; and a, b have the common measure
 a , which is contrary to the hypothesis.

PROPOSITION 17.

*If there be as many numbers as we please in continued
proportion, and the extremes of them be prime to one another,
the last will not be to any other number as the first to the
second.*

For let there be as many numbers as we please, A, B, C, D ,
in continued proportion,
and let the extremes of them, A ,
 D , be prime to one another ;
I say that D is not to any other
number as A is to B .

A _____ B _____
C _____
D _____
E _____

For, if possible, as A is to B , so let D be to E ;
therefore, alternately, as A is to D , so is B to E .

[VII. 13]

But A, D are prime,
 primes are also least, [VII. 21]
 and the least numbers measure those which have the same
 ratio the same number of times, the antecedent the antecedent
 and the consequent the consequent. [VII. 20]

Therefore A measures B .

And, as A is to B , so is B to C .

Therefore B also measures C ;

so that A also measures C .

And since, as B is to C , so is C to D ,
 and B measures C ,
 therefore C also measures D .

But A measured C ;

so that A also measures D .

But it also measures itself;

therefore A measures A, D which are prime to one another :
 which is impossible.

Therefore D will not be to any other number as A is to B .

Q. E. D.

If $a, a_2, a_3, \dots a_n$ be a geometrical progression, and a, a_n are prime to one another, then a, a_2, a_n can have no integral fourth proportional.

For, if possible, let $a : a_2 = a_n : x$.

Therefore $a : a_n = a_2 : x$,

and hence [VII. 20, 21] a measures a_2 .

Therefore a_2 measures a_3 ,

[VII. Def. 20]

and hence a measures a_3 , and therefore also ultimately a_n .

Thus a, a_n are both measured by a : which is contrary to the hypothesis.

PROPOSITION 18.

Given two numbers, to investigate whether it is possible to find a third proportional to them.

Let A, B be the given two numbers, and let it be required to investigate whether it is possible to find a third proportional to them.

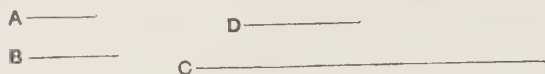
Now A, B are either prime to one another or not.

And, if they are prime to one another, it has been proved that it is impossible to find a third proportional to them.

[IX. 16]

Next, let A, B not be prime to one another,
and let B by multiplying itself make C .

Then A either measures C or does not measure it.



First, let it measure it according to D ;
therefore A by multiplying D has made C .

But, further, B has also by multiplying itself made C ;
therefore the product of A, D is equal to the square on B .

Therefore, as A is to B , so is B to D ; [VII. 19]
therefore a third proportional number D has been found to
 A, B .

Next, let A not measure C ;
I say that it is impossible to find a third proportional number
to A, B .

For, if possible, let D , such third proportional, have been
found.

Therefore the product of A, D is equal to the square on B .

But the square on B is C ;

therefore the product of A, D is equal to C .

Hence A by multiplying D has made C ;
therefore A measures C according to D .

But, by hypothesis, it also does not measure it:
which is absurd.

Therefore it is not possible to find a third proportional
number to A, B when A does not measure C . Q. E. D.

Given two numbers a, b , to find the condition that they may have an
integral third proportional.

(1) a, b must not be prime to one another.

[IX. 16]

(2) a must measure b^2 .

For, if a, b, c be in continued proportion,

$$ac = b^2.$$

Therefore a measures b^2 .

Condition (1) is included in condition (2) since, if $b^2 = ma$, a and b cannot
be prime to one another.

The result is of course easily seen if the three terms in continued
proportion be written

$$a, \quad a \frac{b}{a}, \quad a \left(\frac{b}{a} \right)^2.$$

PROPOSITION 19.

Given three numbers, to investigate when it is possible to find a fourth proportional to them.

Let A, B, C be the given three numbers, and let it be required to investigate when it is possible to find a fourth proportional to them.

A _____
 B _____
 C _____
 D _____
 E _____

Now either they are not in continued proportion, and the extremes of them are prime to one another ;
 or they are in continued proportion, and the extremes of them are not prime to one another ;
 or they are not in continued proportion, nor are the extremes of them prime to one another ;
 or they are in continued proportion, and the extremes of them are prime to one another.

If then A, B, C are in continued proportion, and the extremes of them A, C are prime to one another, it has been proved that it is impossible to find a fourth proportional number to them. [IX. 17]

†Next, let A, B, C not be in continued proportion, the extremes being again prime to one another ;
 I say that in this case also it is impossible to find a fourth proportional to them.

For, if possible, let D have been found, so that,
 as A is to B , so is C to D ,
 and let it be contrived that, as B is to C , so is D to E .

Now, since, as A is to B , so is C to D ,
 and, as B is to C , so is D to E ,
 therefore, *ex aequali*, as A is to C , so is C to E . [VII. 14]

But A, C are prime,
 primes are also least, [VII. 21]
 and the least numbers measure those which have the same ratio, the antecedent the antecedent and the consequent the consequent. [VII. 20]

Therefore A measures C as antecedent antecedent.

But it also measures itself ;
therefore A measures A, C which are prime to one another :
which is impossible.

Therefore it is not possible to find a fourth proportional to A, B, C .†

Next, let A, B, C be again in continued proportion,
but let A, C not be prime to one another.

I say that it is possible to find a fourth proportional to them.

For let B by multiplying C make D ;
therefore A either measures D or does not measure it.

First, let it measure it according to E ;
therefore A by multiplying E has made D .

But, further, B has also by multiplying C made D ;
therefore the product of A, E is equal to the product of B, C ;

therefore, proportionally, as A is to B , so is C to E ; [VII. 19]
therefore E has been found a fourth proportional to A, B, C .

Next, let A not measure D ;
I say that it is impossible to find a fourth proportional number to A, B, C .

For, if possible, let E have been found ;
therefore the product of A, E is equal to the product of B, C .
[VII. 19]

But the product of B, C is D ;
therefore the product of A, E is also equal to D .

Therefore A by multiplying E has made D ;
therefore A measures D according to E ,
so that A measures D .

But it also does not measure it :
which is absurd.

Therefore it is not possible to find a fourth proportional number to A, B, C when A does not measure D .

Next, let A, B, C not be in continued proportion, nor the extremes prime to one another.

And let B by multiplying C make D .

Similarly then it can be proved that, if A measures D ,
it is possible to find a fourth proportional to them, but, if it
does not measure it, impossible.

Q. E. D.

Given three numbers a, b, c , to find the condition that they may have an integral fourth proportional.

The Greek text of part of this proposition is hopelessly corrupt. According to it Euclid takes four cases.

- (1) a, b, c not in continued proportion, and a, c prime to one another.
- (2) a, b, c in continued proportion, and a, c not prime to one another.
- (3) a, b, c not in continued proportion, and a, c not prime to one another.
- (4) a, b, c in continued proportion, and a, c prime to one another.

(4) is the case dealt with in IX. 17, where it is shown that on hypothesis (4) a fourth proportional cannot be found.

The text now takes case (1) and asserts that a fourth proportional cannot be found in this case either. We have only to think of 4, 6, 9 in order to see that there is something wrong here. The supposed proof is also wrong. If possible, says the text, let d be a fourth proportional to a, b, c , and let e be taken such that

$$b : c = d : e.$$

Then, *ex aequali*,

$$a : c = c : e,$$

whence a measures c :

[VII. 20, 21]

which is impossible, since a, c are prime to one another.

But this does not prove that a fourth proportional d cannot be found ; it only proves that, if d is a fourth proportional, no integer e can be found to satisfy the equation

$$b : c = d : e.$$

Indeed it is obvious from IX. 16 that in the equation

$$a : c = c : e$$

e cannot be integral.

The cases (2) and (3) are correctly given, the first in full, and the other as a case to be proved "similarly" to it.

These two cases really give all that is necessary.

Let the product bc be taken.

Then, if a measures bc , suppose $bc = ad$;

therefore

$$a : b = c : d,$$

and d is a fourth proportional.

But, if a does *not* measure bc , no fourth proportional can be found. For, if x were a fourth proportional, ax would be equal to bc , and a would measure bc .

The sufficient condition in any case for the possibility of finding a fourth proportional to a, b, c is that a should measure bc .

Theon appears to have corrected the proof by leaving out the incorrect portion which I have included between daggers and the last case (3) dealt with in the last lines. Also, in accordance with this arrangement, he does not distinguish four cases at the beginning but only two. "Either A, B, C are in continued proportion and the extremes of them A, C are prime to one another ; or not." Then, instead of introducing case (2) by the words "Next let A, B, C ...to find a fourth proportional to them," immediately following the second dagger above, Theon merely says "*But, if not,*" [i.e. if it is not the case that a, b, c are in G.P. and a, c prime to one another] "let B by multiplying C make D ," and so on.

August adopts Theon's form of the proof. Heiberg does not feel able to do this, in view of the superiority of the authority for the text as given above (P); he therefore retains the latter without any attempt to emend it.

PROPOSITION 20.

Prime numbers are more than any assigned multitude of prime numbers.

Let A, B, C be the assigned prime numbers ;

I say that there are more prime numbers than A, B, C .

For let the least number measured by A, B, C be taken,



and let it be DE ;

let the unit DF be added to DE .

Then EF is either prime or not.

First, let it be prime ;

then the prime numbers A, B, C, EF have been found which are more than A, B, C .

Next, let EF not be prime ;

therefore it is measured by some prime number. [VII. 31]

Let it be measured by the prime number G .

I say that G is not the same with any of the numbers A, B, C .

For, if possible, let it be so.

Now A, B, C measure DE ;

therefore G also will measure DE .

But it also measures EF .

Therefore G , being a number, will measure the remainder, the unit DF :

which is absurd.

Therefore G is not the same with any one of the numbers A, B, C .

And by hypothesis it is prime.

Therefore the prime numbers A, B, C, G have been found which are more than the assigned multitude of A, B, C .

Q. E. D.

We have here the important proposition that *the number of prime numbers is infinite*.

The proof will be seen to be the same as that given in our algebraical text-books. Let $a, b, c, \dots k$ be any prime numbers.

Take the product $abc \dots k$ and add unity.

Then $(abc \dots k + 1)$ is either a prime number or not a prime number.

(1) If it *is*, we have added another prime number to those given.

(2) If it is *not*, it must be measured by some prime number [vii. 31], say p .

Now p cannot be identical with any of the prime numbers $a, b, c, \dots k$.

For, if it is, it will divide $abc \dots k$.

Therefore, since it divides $(abc \dots k + 1)$ also, it will measure the difference, or unity :

which is impossible.

Therefore in any case we have obtained one fresh prime number.

And the process can be carried on to any extent.

PROPOSITION 21.

If as many even numbers as we please be added together, the whole is even.

For let as many even numbers as we please, AB, BC, CD, DE , be added together ;

I say that the whole AE  AE is even.

For, since each of the numbers AB, BC, CD, DE is even, it has a half part ; [vii. Def. 6]

so that the whole AE also has a half part.

But an even number is that which is divisible into two equal parts ; [id.]

therefore AE is even.

Q. E. D.

In this and the following propositions up to ix. 34 inclusive we have a number of theorems about odd, even, "even-times even" and "even-times odd" numbers respectively. They are all simple and require no explanation in order to enable them to be followed easily.

PROPOSITION 22.

If as many odd numbers as we please be added together, and their multitude be even, the whole will be even.

For let as many odd numbers as we please, AB, BC, CD, DE , even in multitude, be added together ;

I say that the whole AE is even.

For, since each of the numbers AB , BC , CD , DE is odd, if an unit be subtracted from each, each of the remainders will be even; [VII. Def. 7]

so that the sum of them will be even.

[IX. 21]



But the multitude of the units is also even.
Therefore the whole AE is also even.

[IX. 21]

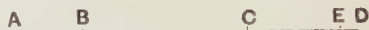
Q. E. D.

PROPOSITION 23.

If as many odd numbers as we please be added together, and their multitude be odd, the whole will also be odd.

For let as many odd numbers as we please, AB , BC , CD , the multitude of which is odd, be added together;

I say that the whole AD is also odd.



Let the unit DE be subtracted from CD ;
therefore the remainder CE is even.

[VII. Def. 7]

But CA is also even;
therefore the whole AE is also even.

[IX. 22]

[IX. 21]

And DE is an unit.

Therefore AD is odd.

[VII. Def. 7]

Q. E. D.

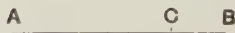
3. Literally "let there be as many numbers as we please, of which *let* the multitude be odd." This form, natural in Greek, is awkward in English.

PROPOSITION 24.

If from an even number an even number be subtracted, the remainder will be even.

For from the even number AB let the even number BC be subtracted:

I say that the remainder CA is even.



For, since AB is even, it has a half part.

[VII. Def. 6]

For the same reason BC also has a half part ;
so that the remainder [CA also has a half part, and] AC is
therefore even.

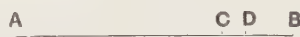
Q. E. D.

PROPOSITION 25.

*If from an even number an odd number be subtracted, the
remainder will be odd.*

For from the even number AB let the odd number BC be
subtracted ;

I say that the remainder CA is odd.



For let the unit CD be sub-
tracted from BC ;

therefore DB is even.

[VII. Def. 7]

But AB is also even ;

therefore the remainder AD is also even.

[IX. 24]

And CD is an unit ;

therefore CA is odd.

[VII. Def. 7]

Q. E. D.

PROPOSITION 26.

*If from an odd number an odd number be subtracted, the
remainder will be even.*

For from the odd number AB let the odd number BC be
subtracted ;

I say that the remainder CA is even.



For, since AB is odd, let the unit
 BD be subtracted ;

therefore the remainder AD is even.

[VII. Def. 7]

For the same reason CD is also even ;

[VII. Def. 7]

so that the remainder CA is also even.

[IX. 24]

Q. E. D.

PROPOSITION 27.

If from an odd number an even number be subtracted, the remainder will be odd.

For from the odd number AB let the even number BC be subtracted ;

I say that the remainder CA is odd.

Let the unit AD be subtracted ;
therefore DB is even. [VII. Def. 7]

But BC is also even ;
therefore the remainder CD is even.

Therefore CA is odd.

$A \quad D \qquad \qquad C \qquad \qquad B$
|-----|-----|

[IX. 24]

[VII. Def. 7]

Q. E. D.

PROPOSITION 28.

If an odd number by multiplying an even number make some number, the product will be even.

For let the odd number A by multiplying the even number B make C ;

I say that C is even.

For, since A by multiplying B has made C ,

therefore C is made up of as many numbers equal to B as there are units in A . [VII. Def. 15]

And B is even ;
therefore C is made up of even numbers.

But, if as many even numbers as we please be added together, the whole is even.

Therefore C is even.

A —
 B —
 C —————

[IX. 21]

Q. E. D.

PROPOSITION 29.

If an odd number by multiplying an odd number make some number, the product will be odd.

For let the odd number A by multiplying the odd number B make C ;

I say that C is odd.

For, since A by multiplying B has made C ,

A —
 B —
 C —————

therefore C is made up of as many numbers equal to B as there are units in A . [VII. Def. 15]

And each of the numbers A , B is odd ;
therefore C is made up of odd numbers the multitude of which is odd.

Thus C is odd.

[IX. 23]

Q. E. D.

PROPOSITION 30.

If an odd number measure an even number, it will also measure the half of it.

For let the odd number A measure the even number B ;
I say that it will also measure the half
of it.

For, since A measures B ,
let it measure it according to C ;
I say that C is not odd.

A —
B —————
C —

For, if possible, let it be so.
Then, since A measures B according to C ,
therefore A by multiplying C has made B .

Therefore B is made up of odd numbers the multitude
of which is odd.

Therefore B is odd : [IX. 23]
which is absurd, for by hypothesis it is even.

Therefore C is not odd ;
therefore C is even.

Thus A measures B an even number of times.
For this reason then it also measures the half of it.

Q. E. D.

PROPOSITION 31.

If an odd number be prime to any number, it will also be prime to the double of it.

For let the odd number A be prime to any number B ,
and let C be double of B ;
I say that A is prime to C .

For, if they are not prime
to one another, some number
will measure them.

A —
B —————
C —————
D —

Let a number measure them, and let it be D .

Now A is odd ;

therefore D is also odd.

And since D which is odd measures C ,
and C is even,

therefore $[D]$ will measure the half of C also. [IX. 30]

But B is half of C ;

therefore D measures B .

But it also measures A ;

therefore D measures A, B which are prime to one another :
which is impossible.

Therefore A cannot but be prime to C .

Therefore A, C are prime to one another.

Q. E. D.

PROPOSITION 32.

Each of the numbers which are continually doubled beginning from a dyad is even-times even only.

For let as many numbers as we please, B, C, D , have been continually doubled beginning from the dyad A ;

I say that B, C, D are even-times even only.

A —
B ———
C —————
D —————

Now that each of the numbers B, C, D is even-times even is manifest ; for it is doubled from a dyad.

I say that it is also even-times even only.

For let an unit be set out.

Since then as many numbers as we please beginning from an unit are in continued proportion,
and the number A after the unit is prime,
therefore D , the greatest of the numbers A, B, C, D , will not be measured by any other number except A, B, C . [IX. 13]

And each of the numbers A, B, C is even ;
therefore D is even-times even only. [VII. Def. 8]

Similarly we can prove that each of the numbers B, C is even-times even only.

Q. E. D.

See the notes on VII. Deff. 8 to 11 for a discussion of the difficulties shown by Iamblichus to be involved by the Euclidean definitions of "even-times even," "even-times odd" and "odd-times even."

PROPOSITION 33.

If a number have its half odd, it is even-times odd only.

For let the number A have its half odd ;

I say that A is even-times odd only.

Now that it is even-times odd is manifest ; for the half of it, being odd, measures it an even number of times. [VII. Def. 9]

I say next that it is also even-times odd only.

For, if A is even-times even also, it will be measured by an even number according to an even number ; [VII. Def. 8]
so that the half of it will also be measured by an even number though it is odd :
which is absurd.

Therefore A is even-times odd only. Q. E. D.

PROPOSITION 34.

If a number neither be one of those which are continually doubled from a dyad, nor have its half odd, it is both even-times even and even-times odd.

For let the number A neither be one of those doubled from a dyad, nor have its half odd ;

I say that A is both even-times even and even-times odd.

Now that A is even-times even is manifest ; for it has not its half odd. [VII. Def. 8]

I say next that it is also even-times odd.

For, if we bisect A , then bisect its half, and do this continually, we shall come upon some odd number which will measure A according to an even number.

For, if not, we shall come upon a dyad, and A will be among those which are doubled from a dyad : which is contrary to the hypothesis.

Thus A is even-times odd.

But it was also proved even-times even.

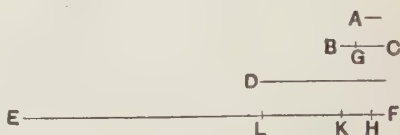
Therefore A is both even-times even and even-times odd.

Q. E. D.

PROPOSITION 35.

If as many numbers as we please be in continued proportion, and there be subtracted from the second and the last numbers equal to the first, then, as the excess of the second is to the first, so will the excess of the last be to all those before it.

Let there be as many numbers as we please in continued proportion, A, BC, D, EF , beginning from A as least, and let there be subtracted from BC and EF the numbers BG, FH , each equal to A ;
 I say that, as GC is to A , so is EH to A, BC, D .



For let FK be made equal to BC , and FL equal to D .

Then, since FK is equal to BC ,

and of these the part FH is equal to the part BG ,
 therefore the remainder HK is equal to the remainder GC .

And since, as EF is to D , so is D to BC , and BC to A ,
 while D is equal to FL , BC to FK , and A to FH ,
 therefore, as EF is to FL , so is LF to FK , and FK to FH .

Separando, as EL is to LF , so is LK to FK , and KH to FH . [VII. 11, 13]

Therefore also, as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents; [VII. 12]

therefore, as KH is to FH , so are EL, LK, KH to LF, FK, HF .

But KH is equal to GC , FH to A , and LF, FK, HF to D, BC, A ;

therefore, as GC is to A , so is EH to D, BC, A .

Therefore, as the excess of the second is to the first, so is the excess of the last to all those before it.

Q. E. D.

This proposition is perhaps the most interesting in the arithmetical Books, since it gives a method, and a very elegant one, of *summing any series of terms in geometrical progression*.

Let $a_1, a_2, a_3, \dots, a_n, a_{n+1}$ be a series of terms in geometrical progression. Then Euclid's proposition proves that

$$(a_{n+1} - a_1) : (a_1 + a_2 + \dots + a_n) = (a_2 - a_1) : a_1.$$

For clearness' sake we will on this occasion use the fractional notation of algebra to represent proportions.

Euclid's method then comes to this.

Since

$$\frac{a_{n+1}}{a_n} = \frac{a_n}{a_{n-1}} = \dots = \frac{a_2}{a_1},$$

we have, *separando*,

$$\frac{a_{n+1} - a_n}{a_n} = \frac{a_n - a_{n-1}}{a_{n-1}} = \dots = \frac{a_2 - a_1}{a_1},$$

whence, since, as one of the antecedents is to one of the consequents, so is the sum of all the antecedents to the sum of all the consequents, [VII. 12]

$$\frac{a_{n+1} - a_1}{a_n + a_{n-1} + \dots + a_1} = \frac{a_2 - a_1}{a_1},$$

which gives $a_1 + a_2 + \dots + a_n$, or S_n .

If, to compare the result with that arrived at in algebraical text-books, we write the series in the form

$$a, ar, ar^2, \dots, ar^{n-1} \quad (n \text{ terms}),$$

we have

$$\frac{ar^n - a}{S_n} = \frac{ar - a}{a},$$

or

$$S_n = \frac{a(r^n - 1)}{r - 1}.$$

PROPOSITION 36.

If as many numbers as we please beginning from an unit be set out continuously in double proportion, until the sum of all becomes prime, and if the sum multiplied into the last make some number, the product will be perfect.

For let as many numbers as we please, A, B, C, D , beginning from an unit be set out in double proportion, until the sum of all becomes prime,

let E be equal to the sum, and let E by multiplying D make FG ;

I say that FG is perfect.

For, however many A, B, C, D are in multitude, let so many E, HK, L, M be taken in double proportion beginning from E ;

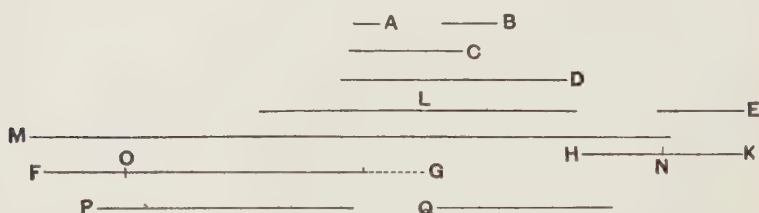
therefore, *ex aequali*, as A is to D , so is E to M . [VII. 14]

Therefore the product of E, D is equal to the product of A, M . [VII. 19]

And the product of E, D is FG ;
therefore the product of A, M is also FG .

Therefore A by multiplying M has made FG ;
therefore M measures FG according to the units in A .

And A is a dyad;
therefore FG is double of M .



But M, L, HK, E are continuously double of each other;
therefore E, HK, L, M, FG are continuously proportional in double proportion.

Now let there be subtracted from the second HK and the last FG the numbers HN, FO , each equal to the first E ;
therefore, as the excess of the second is to the first, so is the excess of the last to all those before it. [IX. 35]

Therefore, as NK is to E , so is OG to M, L, KH, E .

And NK is equal to E ;

therefore OG is also equal to M, L, HK, E .

But FO is also equal to E ,

and E is equal to A, B, C, D and the unit.

Therefore the whole FG is equal to E, HK, L, M and A, B, C, D and the unit;
and it is measured by them.

I say also that FG will not be measured by any other number except A, B, C, D, E, HK, L, M and the unit.

For, if possible, let some number P measure FG ,
and let P not be the same with any of the numbers A, B, C, D, E, HK, L, M .

And, as many times as P measures FG , so many units let there be in Q ;
therefore Q by multiplying P has made FG .

But, further, E has also by multiplying D made FG ;
therefore, as E is to Q , so is P to D . [VII. 19]

And, since A, B, C, D are continuously proportional
beginning from an unit,
therefore D will not be measured by any other number except
 A, B, C . [IX. 13]

And, by hypothesis, P is not the same with any of the
numbers A, B, C ;
therefore P will not measure D .

But, as P is to D , so is E to Q ;
therefore neither does E measure Q . [VII. Def. 20]

And E is prime ;
and any prime number is prime to any number which it does
not measure. [VII. 29]

Therefore E, Q are prime to one another.

But primes are also least, [VII. 21]
and the least numbers measure those which have the same
ratio the same number of times, the antecedent the antecedent
and the consequent the consequent ; [VII. 20]
and, as E is to Q , so is P to D ;
therefore E measures P the same number of times that Q
measures D .

But D is not measured by any other number except
 A, B, C ;
therefore Q is the same with one of the numbers A, B, C .

Let it be the same with B .

And, however many B, C, D are in multitude, let so many
 E, HK, L be taken beginning from E .

Now E, HK, L are in the same ratio with B, C, D ;
therefore, *ex aequali*, as B is to D , so is E to L . [VII. 14]

Therefore the product of B, L is equal to the product of
 D, E . [VII. 19]

But the product of D, E is equal to the product of Q, P ;
therefore the product of Q, P is also equal to the product of
 B, L .

Therefore, as Q is to B , so is L to P . [VII. 19]

And Q is the same with B ;
therefore L is also the same with P :

which is impossible, for by hypothesis P is not the same with any of the numbers set out.

Therefore no number will measure FG except A, B, C, D, E, HK, L, M and the unit.

And FG was proved equal to A, B, C, D, E, HK, L, M and the unit;

and a perfect number is that which is equal to its own parts;
[VII. Def. 22]
therefore FG is perfect.

Q. E. D.

If the sum of any number of terms of the series

$$1, 2, 2^2, \dots, 2^{n-1}$$

be prime, and the said sum be multiplied by the last term, the product will be a "perfect" number, i.e. equal to the sum of all its factors.

Let $1 + 2 + 2^2 + \dots + 2^{n-1} (= S_n)$ be prime;
then shall $S_n \cdot 2^{n-1}$ be "perfect."

Take $(n-1)$ terms of the series

$$S_n, 2S_n, 2^2S_n, \dots, 2^{n-2}S_n.$$

These are then terms proportional to the terms

$$2, 2^2, 2^3, \dots, 2^{n-1}.$$

Therefore, *ex aequali*,

$$2 : 2^{n-1} = S_n : 2^{n-2}S_n, \quad [\text{VII. 14}]$$

or

$$2 \cdot 2^{n-2}S_n = 2^{n-1} \cdot S_n. \quad [\text{VII. 19}]$$

(This is of course obvious algebraically, but Euclid's notation requires him to prove it.)

Now, by IX. 35, we can sum the series $S_n + 2S_n + \dots + 2^{n-2}S_n$,

and $(2S_n - S_n) : S_n = (2^{n-1}S_n - S_n) : (S_n + 2S_n + \dots + 2^{n-2}S_n)$.

Therefore $S_n + 2S_n + 2^2S_n + \dots + 2^{n-2}S_n = 2^{n-1}S_n - S_n$,

or

$$\begin{aligned} 2^{n-1}S_n &= S_n + 2S_n + 2^2S_n + \dots + 2^{n-2}S_n + S_n \\ &= S_n + 2S_n + \dots + 2^{n-2}S_n + (1 + 2 + 2^2 + \dots + 2^{n-1}), \end{aligned}$$

and $2^{n-1}S_n$ is measured by every term of the right hand expression.

It is now necessary to prove that $2^{n-1}S_n$ cannot have any factor except those terms.

Suppose, if possible, that it has a factor x different from all of them,
and let

$$2^{n-1}S_n = x \cdot m.$$

Therefore

$$S_n : m = x : 2^{n-1}. \quad [\text{VII. 19}]$$

Now 2^{n-1} can only be measured by the preceding terms of the series
 $1, 2, 2^2, \dots, 2^{n-1}$,
[IX. 13]

and x is different from all of these;

therefore x does not measure 2^{n-1} ,

so that S_n does not measure m .

[VII. Def. 20]

And S_n is prime; therefore it is prime to m .

It follows [VII. 20, 21] that

[VII. 29]

$$m \text{ measures } 2^{n-1}.$$

Suppose that

$$m = 2^r.$$

Now, *ex aequali*,

$$2^r : 2^{n-1} = S_n : 2^{n-r-1} S_n.$$

Therefore

$$2^r \cdot 2^{n-r-1} S_n = 2^{n-1} S_n \quad [\text{VII. 19}]$$

$$= x \cdot m, \text{ from above.}$$

And $m = 2^r$;

therefore $x = 2^{n-r-1} S_n$, one of the terms of the series $S_n, 2S_n, 2^2S_n, \dots, 2^{n-2}S_n$: which contradicts the hypothesis.

There $2^{n-1}S_n$ has no factors except

$$S_n, 2S_n, 2^2S_n, \dots, 2^{n-2}S_n, 1, 2, 2^2, \dots, 2^{n-1}.$$

Theon of Smyrna and Nicomachus both define a "perfect" number and give the law of its formation. Nicomachus gives four perfect numbers and no more, namely 6, 28, 496, 8128. He says they are formed in "ordered" fashion, there being one among the units (i.e. less than 10), one among the tens (less than 100), one among the hundreds (less than 1000) and one among the thousands (less than 10000); he adds that they terminate in 6 or 8 alternately. They do all terminate in 6 or 8, as can easily be proved by means of the formula $(2^n - 1)2^{n-1}$ (cf. Loria, *Le scienze esatte nell' antica Grecia*, pp. 840—1), but not alternately, for the fifth and sixth perfect numbers both end in 6, and the seventh and eighth both end in 8. Iamblichus adds a tentative suggestion that perhaps there may be, in like manner, one perfect number among the "first myriads" (less than 10000²), one among the "second myriads" (less than 10000³), and so on. This is, as we shall see, incorrect.

It is natural that the subject of perfect numbers should, ever since Euclid's time, have had a fascination for mathematicians. Fermat (1601—1655), in a letter to Mersenne (*Œuvres de Fermat*, ed. Tannery and Henry, Vol. II., 1894, pp. 197—9), enunciated three propositions which much facilitate the investigation whether a given number of the form $2^n - 1$ is prime or not. If we write in one line the exponents 1, 2, 3, 4, etc. of the successive powers of 2 and underneath them respectively the numbers representing the corresponding powers of 2 diminished by 1, thus,

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad \dots n$$

$$1 \quad 3 \quad 7 \quad 15 \quad 31 \quad 63 \quad 127 \quad 255 \quad 511 \quad 1023 \quad 2047 \dots 2^n - 1,$$

the following relations are found to subsist between the numbers in the first line and those directly below them in the second line.

1. If the exponent is not a prime number, the corresponding number is not a prime number either (since $a^{pq} - 1$ is always divisible by $a^p - 1$ as well as by $a^q - 1$).

2. If the exponent is a prime number, the corresponding number diminished by 1 is divisible by twice the exponent. [$(2^n - 2)/2n = (2^{n-1} - 1)/n$; so that this is a special case of "Fermat's theorem" that, if p is a prime number and a is prime to p , then a^{p-1} is divisible by p .]

3. If the exponent n is a prime number, the corresponding number is only divisible by numbers of the form $(2mn + 1)$. If therefore the corresponding number in the second line has no factors of this form, it has no integral factor.

The first and third of these propositions are those which are specially useful for the purpose in question. As usual, Fermat does not give his proofs but merely adds: "Voilà trois fort belles propositions que j'ay trouvées et prouvées non sans peine. Je les puis appeller les fondemens de l'invention des nombres parfaits."

I append a few details of discoveries of further perfect numbers after the first four. The next are as follows:

$$\text{fifth, } 2^{12} (2^{13} - 1) = 33\,550\,336$$

$$\text{sixth, } 2^{16} (2^{17} - 1) = 8\,589\,869\,056$$

$$\text{seventh, } 2^{18} (2^{19} - 1) = 137\,438\,691\,328$$

$$\text{eighth, } 2^{30} (2^{31} - 1) = 2\,305\,843\,008\,139\,952\,128$$

$$\text{ninth, } 2^{60} (2^{61} - 1) = 2\,658\,455\,991\,569\,831\,744\,654\,692\,615\,953\,842\,176$$

$$\text{tenth, } 2^{88} (2^{89} - 1).$$

It has further been proved that $2^{107} - 1$ is prime, and so is $2^{127} - 1$. Hence $2^{106} (2^{107} - 1)$ and $2^{126} (2^{127} - 1)$ are two more perfect numbers.

The fifth perfect number may have been known to Iamblichus, though he does not give it; it was however known, with all its factors, in the fifteenth century, as appears from a tract written in German which was discovered by Curtze (Cod. lat. Monac. 14908). The first eight perfect numbers were calculated by Jean Prestet (d. 1670). Fermat had stated, and Euler proved, that $2^{31} - 1$ is prime. The ninth perfect number was found by P. Seelhoff (*Zeitschrift für Math. u. Physik*, xxxi., 1886, pp. 174—8) and verified by E. Lucas (*Mathesis*, vii., 1887, pp. 45—6). The tenth was discovered by R. E. Powers (see *Bulletin of the American Mathematical Society*, xviii., 1912, p. 162). $2^{107} - 1$ was proved to be prime by E. Fauquembergue and R. E. Powers (1914), while Fauquembergue proved that $2^{127} - 1$ is prime.

There have been attempts, so far unsuccessful, to solve the question whether there exist other “perfect numbers” than those of Euclid, and, in particular, perfect numbers which are *odd*. (Cf. several notes by Sylvester in *Comptes rendus*, cvi., 1888; Catalan, “Mélanges mathématiques” in *Mém. de la Soc. de Liège*, 2^e Série, xv., 1888, pp. 205—7; C. Servais in *Mathesis*, vii., pp. 228—30 and viii., pp. 92—93, 135; E. Cesàro in *Mathesis*, vii., pp. 245—6; E. Lucas in *Mathesis*, x., pp. 74—6).

For the detailed history of the whole subject see L. E. Dickson, *History of the Theory of Numbers*, Vol. I., 1919, pp. iii—iv, 3—33.

BOOK X.

INTRODUCTORY NOTE.

We have seen (Vol. I., p. 351 etc.) that the discovery of the irrational is due to the Pythagoreans. The first scholium on Book x. of the *Elements* states that the Pythagoreans were the first to address themselves to the investigation of commensurability, having discovered it by means of their observation of numbers. They discovered, the scholium continues, that not all magnitudes have a common measure. "They called all magnitudes measurable by the same measure commensurable, but those which are not subject to the same measure incommensurable, and again such of these as are measured by some other common measure commensurable with one another, and such as are not, incommensurable with the others. And thus by *assuming* their measures they referred everything to *different* commensurabilities, but, though they were different, even so (they proved that) not all magnitudes are commensurable with any. (They showed that) all magnitudes can be rational (ῥητά) and all irrational (ἄλογα) in a relative sense (ὡς πρὸς τι); hence the commensurable and the incommensurable would be for them *natural* (kinds) (φύσει), while the rational and irrational would rest on *assumption* or *convention* (θέσει)." The scholium quotes further the legend according to which "the first of the Pythagoreans who made public the investigation of these matters perished in a shipwreck," conjecturing that the authors of this story "perhaps spoke allegorically, hinting that everything irrational and formless is properly concealed, and, if any soul should rashly invade this region of life and lay it open, it would be carried away into the sea of becoming and be overwhelmed by its unresting currents." There would be a reason also for keeping the discovery of irrationals secret for the time in the fact that it rendered unstable so much of the groundwork of geometry as the Pythagoreans had based upon the imperfect theory of proportions which applied only to numbers. We have already, after Tannery, referred to the probability that the discovery of incommensurability must have necessitated a great recasting of the whole fabric of elementary geometry, pending the discovery of the general theory of proportion applicable to incommensurable as well as to commensurable magnitudes.

It seems certain that it was with reference to the length of the diagonal of a square or the hypotenuse of an isosceles right-angled triangle that the irrational was discovered. Plato (*Theætetus*, 147 D) tells us that Theodorus of Cyrene wrote about square roots (δυνάμεις), proving that the square roots of

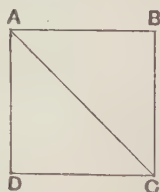
three square feet and five square feet are not commensurable with that of one square foot, and so on, selecting each such square root up to that of 17 square feet, at which for some reason he stopped. No mention is here made of $\sqrt{2}$, doubtless for the reason that its incommensurability had been proved before. Now we are told that Pythagoras invented a formula for finding right-angled triangles in rational numbers, and in connexion with this it was inevitable that the Pythagoreans should investigate the relations between sides and hypotenuse in other right-angled triangles. They would naturally give special attention to the isosceles right-angled triangle; they would try to measure the diagonal, would arrive at successive approximations, in rational fractions, to the value of $\sqrt{2}$, and would find that successive efforts to obtain an exact expression for it failed. It was however an enormous step to conclude that such exact expression was *impossible*, and it was this step which the Pythagoreans made. We now know that the formation of the *side*- and *diagonal*-numbers explained by Theon of Smyrna and others was Pythagorean, and also that the theorems of Eucl. II. 9, 10 were used by the Pythagoreans in direct connexion with this method of approximating to the value of $\sqrt{2}$. The very method by which Euclid proves these propositions is itself an indication of their connexion with the investigation of $\sqrt{2}$, since he uses a figure made up of two isosceles right-angled triangles.

The actual method by which the Pythagoreans proved the incommensurability of $\sqrt{2}$ with unity was no doubt that referred to by Aristotle (*Anal. prior.* I. 23, 41 a 26—7), a *reductio ad absurdum* by which it is proved that, if the diagonal is commensurable with the side, it will follow that the same number is both odd and even. The proof formerly appeared in the texts of Euclid as x. 117, but it is undoubtedly an interpolation, and August and Heiberg accordingly relegate it to an Appendix. It is in substance as follows.

Suppose AC , the diagonal of a square, to be commensurable with AB , its side. Let $\alpha : \beta$ be their ratio expressed in the smallest numbers.

Then $\alpha > \beta$ and therefore necessarily > 1 .

Now $AC^2 : AB^2 = \alpha^2 : \beta^2$,
and, since $AC^2 = 2AB^2$, [Eucl. I. 47]
 $\alpha^2 = 2\beta^2$.



Therefore α^2 is even, and therefore α is even.

Since $\alpha : \beta$ is in its lowest terms, it follows that β must be *odd*.

Put $\alpha = 2\gamma$;
therefore $4\gamma^2 = 2\beta^2$,
or $\beta^2 = 2\gamma^2$,
so that β^2 , and therefore β , must be *even*.

But β was also *odd*;
which is impossible.

This proof only enables us to prove the incommensurability of the diagonal of a square with its side, or of $\sqrt{2}$ with unity. In order to prove the incommensurability of the sides of squares, one of which has *three* times the area of another, an entirely different procedure is necessary; and we find in fact that, even a century after Pythagoras' time, it was still necessary to use *separate* proofs (as the passage of the *Theaetetus* shows that Theodorus did) to establish the incommensurability with unity of $\sqrt{3}$, $\sqrt{5}$, ... up to $\sqrt{17}$.

This fact indicates clearly that the general theorem in Eucl. x. 9 that *squares which have not to one another the ratio of a square number to a square number have their sides incommensurable in length* was not arrived at all at once, but was, in the manner of the time, developed out of the separate consideration of special cases (Hankel, p. 103).

The proposition x. 9 of Euclid is definitely ascribed by the scholiast to Theaetetus. Theaetetus was a pupil of Theodorus, and it would seem clear that the theorem was not known to Theodorus. Moreover the Platonic passage itself (*Theaet.* 147D sqq.) represents the young Theaetetus as striving after a general conception of what we call a *surd*. "The idea occurred to me, seeing that *square roots* (δυνάμεις) appeared to be unlimited in multitude, to try to arrive at one collective term by which we could designate all these square roots.... I divided number in general into two classes. The number which can be expressed as equal multiplied by equal (ἴσον ἰσάκεις) I likened to a square in form, and I called it square and equilateral.... The intermediate number, such as three, five, and any number which cannot be expressed as equal multiplied by equal, but is either less times more or more times less, so that it is always contained by a greater and less side, I likened to an oblong figure and called an oblong number.... Such straight lines then as square the equilateral and plane number I defined as length (μῆκος), and such as square the oblong *square roots* (δυνάμεις), as not being commensurable with the others in length but only in the plane areas to which their squares are equal."

There is further evidence of the contributions of Theaetetus to the theory of incommensurables in a commentary on Eucl. x. discovered, in an Arabic translation, by Wöepcke (*Mémoires présentés à l'Académie des Sciences*, xiv., 1856, pp. 658—720). It is certain that this commentary is of Greek origin. Wöepcke conjectures that it was by Vettius Valens, an astronomer, apparently of Antioch, and a contemporary of Claudius Ptolemy (2nd cent. A.D.). Heiberg, with greater probability, thinks that we have here a fragment of the commentary of Pappus (*Euklid-studien*, pp. 169—71), and this is rendered practically certain by Suter (*Die Mathematiker und Astronomen der Araber und ihre Werke*, pp. 49 and 211). This commentary states that the theory of irrational magnitudes "had its origin in the school of Pythagoras. It was considerably developed by Theaetetus the Athenian, who gave proof, in this part of mathematics, as in others, of ability which has been justly admired. He was one of the most happily endowed of men, and gave himself up, with a fine enthusiasm, to the investigation of the truths contained in these sciences, as Plato bears witness for him in the work which he called after his name. As for the exact distinctions of the above-named magnitudes and the rigorous demonstrations of the propositions to which this theory gives rise, I believe that they were chiefly established by this mathematician; and, later, the great Apollonius, whose genius touched the highest point of excellence in mathematics, added to these discoveries a number of remarkable theories after many efforts and much labour."

"For Theaetetus had distinguished square roots [*puissances* must be the δυνάμεις of the Platonic passage] commensurable in length from those which are incommensurable, and had divided the well-known species of irrational lines after the different means, assigning the *medial* to geometry, the *binomial* to arithmetic, and the *apotome* to harmony, as is stated by Eudemus the Peripatetic."

"As for Euclid, he set himself to give rigorous rules, which he established,

relative to commensurability and incommensurability in general; he made precise the definitions and the distinctions between rational and irrational magnitudes, he set out a great number of orders of irrational magnitudes, and finally he clearly showed their whole extent."

The allusion in the last words must apparently be to x. 115, where it is proved that from the *medial* straight line an unlimited number of other irrationals can be derived, all different from it and from one another.

The connexion between the *medial* straight line and the geometric mean is obvious, because it is in fact the mean proportional between two rational straight lines "commensurable in square only." Since $\frac{1}{2}(x+y)$ is the *arithmetic* mean between x , y , the reference to it of the binomial can be understood. The connexion between the apotome and the harmonic mean is explained by some propositions in the second book of the Arabic commentary. The harmonic mean between x , y is $\frac{2xy}{x+y}$, and propositions of which Woepcke

quotes the enunciations prove that, if a rational or a medial area has for one of its sides a *binomial* straight line, the other side will be an *apotome* of corresponding order (these propositions are generalised from Eucl. x. 111—4); the fact is that $\frac{2xy}{x+y} = \frac{2xy}{x^2-y^2} \cdot (x-y)$.

One other predecessor of Euclid appears to have written on irrationals, though we know no more of the work than its title as handed down by Diogenes Laertius¹. According to this tradition, Democritus wrote *περὶ ἀλόγων γραμμῶν καὶ ναστῶν β'*, *two Books on irrational straight lines and solids* (or *atoms*). Hultsch (*Neue Jahrbücher für Philologie und Pädagogik*, 1881, pp. 578—9) conjectures that the true reading may be *περὶ ἀλόγων γραμμῶν κλαστῶν*, "on irrational broken lines." Hultsch seems to have in mind *straight* lines divided into two parts one of which is rational and the other irrational ("Aus einer Art von Umkehr des Pythagoreischen Lehrsatzes über das rechtwinklige Dreieck gieng zunächst mit Leichtigkeit hervor, dass man eine Linie construiren könne, welche als irrational zu bezeichnen ist, aber durch Brechung sich darstellen lässt als die Summe einer rationalen und einer irrationalen Linie"). But I doubt the use of *κλαστός* in the sense of breaking one straight line into parts; it should properly mean a bent line, i.e. two straight lines forming an angle or *broken short off* at their point of meeting. It is also to be observed that *ναστόν* is quoted as a Democritean word (opposite to *κενόν*) in a fragment of Aristotle (202). I see therefore no reason for questioning the correctness of the title of Democritus' book as above quoted².

I will here quote a valuable remark of Zeuthen's relating to the classification of irrationals. He says (*Geschichte der Mathematik im Altertum und Mittelalter*, p. 56) "Since such roots of equations of the second degree as are incommensurable with the given magnitudes cannot be expressed by means of the latter and of numbers, it is conceivable that the Greeks, in exact investigations, introduced no approximate values but worked on with the magnitudes they had found, which were represented by straight lines obtained by the construction corresponding to the solution of the equation. That is exactly the same thing which happens when we do not evaluate roots but content ourselves with expressing them by radical signs and other algebraical symbols. But, inasmuch as one straight line looks like another, the Greeks did not get

¹ Diog. Laert. ix. 47, p. 239 (ed. Cobet).

² Cf. *ante*, Vol. I., p. 413.

the same clear view of what they denoted (i.e. by simple inspection) as our system of symbols assures to us. For this reason it was necessary to undertake a classification of the irrational magnitudes which had been arrived at by successive solution of equations of the second degree." To much the same effect Tannery wrote in 1882 (*De la solution géométrique des problèmes du second degré avant Euclide* in *Mémoires de la Société des sciences physiques et naturelles de Bordeaux*, 2^e Série, IV. pp. 395—416). Accordingly Book X. formed a repository of results to which could be referred problems which depended on the solution of certain types of equations, quadratic and biquadratic but reducible to quadratics.

Consider the quadratic equations

$$x^2 \pm 2ax \cdot \rho \pm \beta \cdot \rho^2 = 0,$$

where ρ is a rational straight line, and a, β are coefficients. Our quadratic equations in algebra leave out the ρ ; but I put it in, because it has always to be remembered that Euclid's x is a *straight line*, not an algebraical quantity, and is therefore to be found in terms of, or in relation to, a certain assumed *rational straight line*, and also because with Euclid ρ may be not only of the

form a , where a represents a units of length, but also of the form $\sqrt{\frac{m}{n}} \cdot a$, which represents a length "commensurable in square only" with the unit of length, or $\sqrt{A}$ where A represents a number (not square) of units of *area*. The use therefore of ρ in our equations makes it unnecessary to multiply different cases according to the relation of ρ to the unit of length, and has the further advantage that, e.g., the expression $\rho \pm \sqrt{k} \cdot \rho$ is just as general as the expression $\sqrt{k} \cdot \rho \pm \sqrt{\lambda} \cdot \rho$, since ρ covers the form $\sqrt{k} \cdot \rho$, both expressions covering a length either commensurable in length, or "commensurable in square only," with the unit of length.

Now the *positive* roots of the quadratic equations

$$x^2 \pm 2ax \cdot \rho \pm \beta \cdot \rho^2 = 0$$

can only have the following forms

$$\left. \begin{aligned} x_1 &= \rho (a + \sqrt{a^2 - \beta}), \quad x_1' = \rho (a - \sqrt{a^2 - \beta}) \\ x_2 &= \rho (\sqrt{a^2 + \beta} + a), \quad x_2' = \rho (\sqrt{a^2 + \beta} - a) \end{aligned} \right\}.$$

The negative roots do not come in, since x must be a *straight line*. The omission however to bring in negative roots constitutes no loss of generality, since the Greeks would write the equation leading to negative roots in another form so as to make them positive, i.e. they would change the sign of x in the equation.

Now the positive roots x_1, x_1', x_2, x_2' may be classified according to the character of the coefficients a, β and their relation to one another.

I. Suppose that a, β do not contain any surds, i.e. are either integers or of the form m/n , where m, n are integers.

Now in the expressions for x_1, x_1' it may be that

$$(1) \beta \text{ is of the form } \frac{m^2}{n^2} a^2.$$

Euclid expresses this by saying that the square on $a\rho$ exceeds the square on $\rho\sqrt{a^2 - \beta}$ by the square on a straight line commensurable in length with $a\rho$.

In this case x_1 is, in Euclid's terminology, a *first binomial* straight line,

and x_1' a *first apotome*.

(2) In general, β not being of the form $\frac{m^2}{n^2}a^2$,

x_1 is a *fourth binomial*,

x_1' a *fourth apotome*.

Next, in the expressions for x_2, x_2' it may be that

(1) β is equal to $\frac{m^2}{n^2}(a^2 + \beta)$, where m, n are integers, i.e. β is of the form

$$\frac{m^2}{n^2 - m^2}a^2.$$

Euclid expresses this by saying that the square on $\rho\sqrt{a^2 + \beta}$ exceeds the square on $a\rho$ by the square on a straight line commensurable in length with $\rho\sqrt{a^2 + \beta}$.

In this case x_2 is, in Euclid's terminology, a *second binomial*,

x_2' a *second apotome*.

(2) In general, β not being of the form $\frac{m^2}{n^2 - m^2}a^2$,

x_2 is a *fifth binomial*,

x_2' a *fifth apotome*.

II. Now suppose that a is of the form $\sqrt{\frac{m}{n}}$, where m, n are integers, and let us denote it by $\sqrt{\lambda}$.

Then in this case

$$x_1 = \rho(\sqrt{\lambda} + \sqrt{\lambda - \beta}), \quad x_1' = \rho(\sqrt{\lambda} - \sqrt{\lambda - \beta}),$$

$$x_2 = \rho(\sqrt{\lambda} + \beta + \sqrt{\lambda}), \quad x_2' = \rho(\sqrt{\lambda} + \beta - \sqrt{\lambda}).$$

Thus x_1, x_1' are of the same form as x_3, x_3' .

If $\sqrt{\lambda - \beta}$ in x_1, x_1' is not surd but of the form m/n , and if $\sqrt{\lambda + \beta}$ in x_2, x_2' is not surd but of the form m/n , the roots are comprised among the forms already shown, the first, second, fourth and fifth binomials and apotomes.

If $\sqrt{\lambda - \beta}$ in x_1, x_1' is surd, then

(1) we may have β of the form $\frac{m^2}{n^2}\lambda$, and in this case

x_1 is a *third binomial* straight line,

x_1' a *third apotome*;

(2) in general, β not being of the form $\frac{m^2}{n^2}\lambda$,

x_1 is a *sixth binomial* straight line,

x_1' a *sixth apotome*.

With the expressions for x_2, x_2' the distinction between the third and sixth binomials and apotomes is of course the distinction between the cases

(1) in which $\beta = \frac{m^2}{n^2}(\lambda + \beta)$, or β is of the form $\frac{m^2}{n^2 - m^2}\lambda$,

and (2) in which β is not of this form.

If we take the square root of the product of ρ and each of the six binomials and six apotomes just classified, i.e.

$$\rho^2(a \pm \sqrt{a^2 - \beta}), \quad \rho^2(\sqrt{a^2 + \beta} \pm a),$$

in the six different forms that each may take, we find six new irrationals with a positive sign separating the two terms, and six corresponding irrationals with a negative sign. These are of course roots of the equations

$$x^4 \pm 2\alpha x^2 \cdot \rho^2 \pm \beta \cdot \rho^4 = 0.$$

These irrationals really come before the others in Euclid's order (x. 36—41 for the positive sign and x. 73—78 for the negative sign). As we shall see in due course, the straight lines actually found by Euclid are

1. $\rho \pm \sqrt{k} \cdot \rho$, the *binomial* (ἡ ἐκ δύο ὀνομάτων)
and the *apotome* (ἀποτομή),

which are the positive roots of the biquadratic (reducible to a quadratic)

$$x^4 - 2(1+k)\rho^2 \cdot x^2 + (1-k)^2 \rho^4 = 0.$$

2. $k^{\frac{1}{4}}\rho \pm k^{\frac{3}{4}}\rho$, the *first bimedral* (ἐκ δύο μέσων πρώτη)
and the *first apotome of a medial* (μέσης ἀποτομή πρώτη),

which are the positive roots of

$$x^4 - 2\sqrt{k}(1+k)\rho^2 \cdot x^2 + k(1-k)^2 \rho^4 = 0.$$

3. $k^{\frac{1}{4}}\rho \pm \frac{\sqrt{\lambda}}{k^{\frac{1}{4}}}\rho$, the *second bimedral* (ἐκ δύο μέσων δευτέρα)
and the *second apotome of a medial* (μέσης ἀποτομή δευτέρα),

which are the positive roots of the equation

$$x^4 - 2\frac{k+\lambda}{\sqrt{k}}\rho^2 \cdot x^2 + \frac{(k-\lambda)^2}{k}\rho^4 = 0.$$

4. $\frac{\rho}{\sqrt{2}}\sqrt{1 + \frac{k}{1+k^2}} \pm \frac{\rho}{\sqrt{2}}\sqrt{1 - \frac{k}{1+k^2}}$,

the *major* (irrational straight line) (μείζων)

and the *minor* (irrational straight line) (ελάσσων),

which are the positive roots of the equation

$$x^4 - 2\rho^2 \cdot x^2 + \frac{k^2}{1+k^2}\rho^4 = 0.$$

5. $\frac{\rho}{\sqrt{2}(1+k^2)}\sqrt{\sqrt{1+k^2}+k} \pm \frac{\rho}{\sqrt{2}(1+k^2)}\sqrt{\sqrt{1+k^2}-k}$,

the “*side*” of a rational plus a medial (area) (ῥητὸν καὶ μέσον δυναμένη)

and the “*side*” of a medial minus a rational area (in the Greek ἡ μετὰ ῥητοῦ μέσον τὸ ὅλον ποιούσα),

which are the positive roots of the equation

$$x^4 - \frac{2}{\sqrt{1+k^2}}\rho^2 \cdot x^2 + \frac{k^2}{(1+k^2)^2}\rho^4 = 0,$$

6. $\frac{\lambda^{\frac{1}{4}}\rho}{\sqrt{2}}\sqrt{1 + \frac{k}{1+k^2}} \pm \frac{\lambda^{\frac{1}{4}}\rho}{\sqrt{2}}\sqrt{1 - \frac{k}{1+k^2}}$,

the “*side*” of the sum of two medial areas (ἡ δύο μέσα δυναμένη)

and the “*side*” of a medial minus a medial area (in the Greek ἡ μετὰ μέσον μέσον τὸ ὅλον ποιούσα),

which are the positive roots of the equation

$$x^4 - 2\sqrt{\lambda} \cdot x^2 \rho^2 + \lambda \frac{k^2}{1+k^2}\rho^4 = 0.$$

The above facts and formulae admit of being stated in a great variety of ways according to the notation and the particular letters used. Consequently the summaries which have been given of Eucl. x. by various writers differ much in appearance while expressing the same thing in substance. The first summary in algebraical form (and a very elaborate one) seems to have been that of Cossali (*Origine, trasporto in Italia, primi progressi in essa dell' Algebra*, Vol. II., pp. 242—65) who takes credit accordingly (p. 265). In 1794 Meier Hirsch published at Berlin an *Algebraischer Commentar über das zehnte Buch der Elemente des Euklides* which gives the contents in algebraical form but fails to give any indication of Euclid's methods, using modern forms of proof only. In 1834 Poselger wrote a paper, *Ueber das zehnte Buch der Elemente des Euklides*, in which he pointed out the defects of Hirsch's reproduction and gave a summary of his own, which however, though nearer to Euclid's form, is difficult to follow in consequence of an elaborate system of abbreviations, and is open to the objection that it is not algebraical enough to enable the character of Euclid's irrationals to be seen at a glance. Other summaries will be found (1) in Nesselmann, *Die Algebra der Griechen*, pp. 165—84; (2) in Loria, *Le scienze esatte nell' antica Grecia*, 1914, pp. 221—34; (3) in Christensen's article "Ueber Gleichungen vierten Grades im zehnten Buch der Elemente Euklids" in the *Zeitschrift für Math. u. Physik (Historisch-litterarische Abtheilung)*, xxxiv. (1889), pp. 201—17. The only summary in English that I know is that in the *Penny Cyclopaedia*, under "Irrational quantity," by De Morgan, who yielded to none in his admiration of Book x. "Euclid investigates," says De Morgan, "every possible variety of lines which can be represented by $\sqrt{(\sqrt{a} \pm \sqrt{b})}$, a and b representing two commensurable lines.... This book has a completeness which none of the others (not even the fifth) can boast of; and we could almost suspect that Euclid, having arranged his materials in his own mind, and having completely elaborated the 10th Book, wrote the preceding books after it and did not live to revise them thoroughly."

Much attention was given to Book x. by the early algebraists. Thus Leonardo of Pisa (fl. about 1200 A.D.) wrote in the 14th section of his *Liber Abaci* on the theory of irrationalities (*de tractatu binomiorum et recisorum*), without however (except in treating of irrational trinomials and cubic irrationalities) adding much to the substance of Book x.; and, in investigating the equation

$$x^3 + 2x^2 + 10x = 20,$$

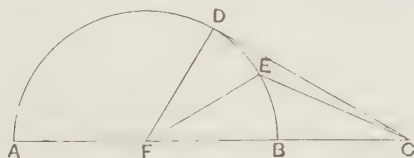
propounded by Johannes of Palermo, he proved that none of the irrationals in Eucl. x. would satisfy it (Hankel, pp. 344—6, Cantor, II., p. 43). Luca Paciolo (about 1445—1514 A.D.) in his algebra based himself largely, as he himself expressly says, on Euclid x. (Cantor, II., p. 293). Michael Stifel (1486 or 1487 to 1567) wrote on irrational numbers in the second Book of his *Arithmetica integra*, which Book may be regarded, says Cantor (II., p. 402), as an elucidation of Eucl. x. The works of Cardano (1501—76) abound in speculations regarding the irrationals of Euclid, as may be seen by reference to Cossali (Vol. II., especially pp. 268—78 and 382—99); the character of the various odd and even powers of the binomials and apotomes is therein investigated, and Cardano considers in detail of what particular forms of equations, quadratic, cubic, and biquadratic, each class of Euclidean irrationals can be roots. Simon Stevin (1548—1620) gave an *Appendice des incommensurables grandeurs en laquelle est sommairement déclaré le contenu du Dixiesme Livre d'Euclide (Oeuvres mathématiques*, Leyde, 1634, pp. 218—22); he speaks thus

of the book: "La difficulté du dixiesme Livre d'Euclide est à plusieurs devenue en horreur, voire jusque à l'appeler la croix des mathématiciens, matière trop dure à digérer, et en la quelle n'aperçoivent aucune utilité," a passage quoted by Loria (*op. cit.*, p. 222).

It will naturally be asked, what use did the Greek geometers actually make of the theory of irrationals developed at such length in Book x.? The answer is that Euclid himself, in Book XIII., makes considerable use of the second portion of Book x. dealing with the irrationals affected with a negative sign, the *apotomes* etc. One object of Book XIII. is to investigate the relation of the sides of a pentagon inscribed in a circle and of an icosahedron and dodecahedron inscribed in a sphere to the diameter of the circle or sphere respectively, supposed rational. The connexion with the regular pentagon of a straight line cut in extreme and mean ratio is well known, and Euclid first proves (XIII. 6) that, if a *rational* straight line is so divided, the parts are the irrationals called *apotomes*, the lesser part being a *first apotome*. Then, on the assumption that the diameters of a circle and sphere respectively are rational, he proves (XIII. 11) that the side of the inscribed regular pentagon is the irrational straight line called *minor*, as is also the side of the inscribed icosahedron (XIII. 16), while the side of the inscribed dodecahedron is the irrational called an *apotome* (XIII. 17).

Of course the investigation in Book x. would not have been complete if it had dealt only with the irrationals affected with a *negative* sign. Those affected with the positive sign, the *binomials* etc., had also to be discussed, and we find both portions of Book x., with its nomenclature, made use of by Pappus in two propositions, of which it may be of interest to give the enunciations here.

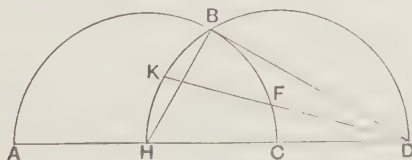
If, says Pappus (iv. p. 178), AB be the rational diameter of a semicircle, and if AB be produced to C so that BC is equal to the radius, if CD be a tangent,



if E be the middle point of the arc BD , and if CE be joined, then CE is the irrational straight line called *minor*. As a matter of fact, if ρ is the radius,

$$CE^2 = \rho^2 (5 - 2\sqrt{3}) \text{ and } CE = \sqrt{\frac{5 + \sqrt{13}}{2}} - \sqrt{\frac{5 - \sqrt{13}}{2}}.$$

If, again (p. 182), CD be equal to the radius of a semicircle supposed



rational, and if the tangent DB be drawn and the angle ADB be bisected by DF meeting the circumference in F , then DF is the excess by which the *binomial* exceeds the *straight line which produces with a rational area a medial*

whole (see Eucl. x. 77). (In the figure DK is the *binomial* and KF the other irrational straight line.) As a matter of fact, if ρ be the radius,

$$KD = \rho \cdot \frac{\sqrt{3+1}}{\sqrt{2}}, \text{ and } KF = \rho \cdot \sqrt{\sqrt{3}-1} = \rho \left(\sqrt{\frac{\sqrt{3+1}}{2}} - \sqrt{\frac{\sqrt{3-1}}{2}} \right).$$

Proclus tells us that Euclid left out, as alien to a selection of *elements*, the discussion of the more complicated irrationals, “the unordered irrationals which Apollonius worked out more fully” (Proclus, p. 74, 23), while the scholiast to Book x. remarks that Euclid does not deal with all rationals and irrationals but only the simplest kinds by the combination of which an infinite number of irrationals are obtained, of which Apollonius also gave some. The author of the commentary on Book x. found by Woepcke in an Arabic translation, and above alluded to, also says that “it was Apollonius who, beside the *ordered* irrational magnitudes, showed the existence of the *unordered* and by accurate methods set forth a great number of them.” It can only be vaguely gathered, from such hints as the commentator proceeds to give, what the character of the extension of the subject given by Apollonius may have been. See note at end of Book.

DEFINITIONS.

1. Those magnitudes are said to be **commensurable** which are measured by the same measure, and those **incommensurable** which cannot have any common measure.

2. Straight lines are **commensurable in square** when the squares on them are measured by the same area, and **incommensurable in square** when the squares on them cannot possibly have any area as a common measure.

3. With these hypotheses, it is proved that there exist straight lines infinite in multitude which are commensurable and incommensurable respectively, some in length only, and others in square also, with an assigned straight line. Let then the assigned straight line be called **rational**, and those straight lines which are commensurable with it, whether in length and in square or in square only, **rational**, but those which are incommensurable with it **irrational**.

4. And let the square on the assigned straight line be called **rational** and those areas which are commensurable with it **rational**, but those which are incommensurable with it **irrational**, and the straight lines which produce them **irrational**, that is, in case the areas are squares, the sides themselves, but in case they are any other rectilineal figures, the straight lines on which are described squares equal to them.

DEFINITION 1.

Σύμμετρα μεγέθη λέγεται τὰ τῷ αὐτῷ μέτρῳ μετρούμενα, ἀσύμμετρα δέ, ὡν μὴδὲν ἐνδέχεται κοινὸν μέτρον γενέσθαι.

DEFINITION 2.

Εὐθεΐαι δυνάμει σύμμετροί εἰσιν, ὅταν τὰ ἀπ' αὐτῶν τετράγωνα τῷ αὐτῷ χωρίῳ μετρηῇται, ἀσύμμετροι δέ, ὅταν τοῖς ἀπ' αὐτῶν τετραγώνοις μὴδὲν ἐνδέχεται χωρίον κοινὸν μέτρον γενέσθαι.

Commensurable in square is in the Greek δυνάμει σύμμετρος. In earlier translations (e.g. Williamson's) δυνάμει has been translated "in power," but, as the particular *power* represented by δύναμις in Greek geometry is *square*, I have thought it best to use the latter word throughout. It will be observed that Euclid's expression *commensurable in square only* (used in Def. 3 and constantly) corresponds to what Plato makes Theaetetus call a *square root* (δύναμις) in the sense of a *surd*. If a is any straight line, a and $a\sqrt{m}$, or $a\sqrt{m}$ and $a\sqrt{n}$ (where m, n are integers or arithmetical fractions in their lowest terms, proper or improper, but not square) are *commensurable in square only*. Of course (as explained in the Porism to x. 10) all straight lines *commensurable in length* (μήκει), in Euclid's phrase, are *commensurable in square* also; but not all straight lines which are *commensurable in square* are *commensurable in length* as well. On the other hand, straight lines *incommensurable in square* are necessarily *incommensurable in length* also; but not all straight lines which are *incommensurable in length* are *incommensurable in square*. In fact, straight lines which are *commensurable in square only* are *incommensurable in length*, but obviously not *incommensurable in square*.

DEFINITION 3.

Τούτων ὑποκειμένων δέικνται, ὅτι τῇ προτεθείσῃ εὐθείᾳ ὑπάρχουσιν εὐθεΐαι πλήθει ἄπειροι σύμμετροί τε καὶ ἀσύμμετροι αἱ μὲν μήκει μόνον, αἱ δὲ καὶ δυνάμει. καλείσθω οὖν ἡ μὲν προτεθείσα εὐθεΐα ῥητή, καὶ αἱ ταύτῃ σύμμετροι εἴτε μήκει καὶ δυνάμει εἴτε δυνάμει μόνον ῥηταί, αἱ δὲ ταύτῃ ἀσύμμετροι ἀλογοὶ καλείσθωσαν.

The first sentence of the definition is decidedly elliptical. It should, strictly speaking, assert that "with a given straight line there are an infinite number of straight lines which are (1) commensurable either (a) in square only or (b) in square and in length also, and (2) incommensurable, either (a) in length only or (b) in length and in square also."

The relativity of the terms *rational* and *irrational* is well brought out in this definition. We may set out *any straight line* and call it rational, and it is then with reference to this assumed rational straight line that others are called *rational* or *irrational*.

We should carefully note that the signification of *rational* in Euclid is wider than in our terminology. With him, not only is a straight line *commensurable in length* with a rational straight line rational, but a straight line is rational which is commensurable with a rational straight line *in square only*. That is, if p is a rational straight line, not only is $\frac{m}{n}p$ rational, where m, n are integers and

m/n in its lowest terms is not square, but $\sqrt{\frac{m}{n}} \cdot \rho$ is *rational* also. We should

in this case call $\sqrt{\frac{m}{n}} \cdot \rho$ *irrational*. It would appear that Euclid's termino-

logy here differed as much from that of his predecessors as it does from ours. We are familiar with the phrase ἀρρητος διάμετρος τῆς πεμπτὰδος by which Plato (evidently after the Pythagoreans) describes the diagonal of a square on a straight line containing 5 units of length. This "inexpressible diameter of five (squared)" means $\sqrt{50}$, in contrast to the ῥητὴ διάμετρος, the "expressible diameter" of the same square, by which is meant the approxi-

mation $\sqrt{50-1}$, or 7. Thus for Euclid's predecessors $\sqrt{\frac{m}{n}} \cdot \rho$ would apparently not have been rational but ἀρρητος, "inexpressible," i.e. irrational.

I shall throughout my notes on this Book denote a *rational* straight line in Euclid's sense by ρ , and by ρ and σ when two different rational straight lines are required. Wherever then I use ρ or σ , it must be remembered that ρ , σ may have either of the forms a , $\sqrt{k} \cdot a$, where a represents a units of length, a being either an integer or of the form m/n , where m , n are both integers, and k is an integer or of the form m/n (where both m , n are integers) but not square. In other words, ρ , σ may have either of the forms a or $\sqrt{A}$, where A represents A units of *area* and A is integral or of the form m/n , where m , n are both integers. It has been the habit of writers to give a and $\sqrt{a}$ as the alternative forms of ρ , but I shall always use $\sqrt{A}$ for the second in order to keep the dimensions right, because it must be borne in mind throughout that ρ is an *irrational straight line*.

As Euclid extends the signification of *rational* (ῥητός, literally *expressible*), so he limits the scope of the term ἄλογος (literally *having no ratio*) as applied to straight lines. That this limitation was started by himself may perhaps be inferred from the form of words "let straight lines incommensurable with it be called irrational." Irrational straight lines then are with Euclid straight lines commensurable *neither in length nor in square* with the assumed rational straight line. $\sqrt{k} \cdot a$ where k is not square is not irrational; $\sqrt[3]{k} \cdot a$ is irrational, and so (as we shall see later on) is $(\sqrt{k} \pm \sqrt{\lambda}) a$.

DEFINITION 4.

Καὶ τὸ μὲν ἀπὸ τῆς προτεθείσης εὐθείας τετράγωνον ῥητόν, καὶ τὰ τούτῳ σύμμετρα ῥητά, τὰ δὲ τούτῳ ἀσύμμετρα ἄλογα καλείσθω, καὶ αἱ δυνάμεναι αὐτὰ ἄλογοι, εἰ μὲν τετράγωνα εἴη, αὐταὶ αἱ πλευραὶ, εἰ δὲ ἕτερα ἅτινα εὐθύγραμμα, αἱ ἴσα αὐτοῖς τετράγωνα ἀναγράφουσαι.

As applied to *areas*, the terms *rational* and *irrational* have, on the other hand, the same sense with Euclid as we should attach to them. According to Euclid, if ρ is a rational straight line in *his* sense, ρ^2 is *rational* and any area commensurable with it, i.e. of the form $k\rho^2$ (where k is an integer, or of the form m/n , where m , n are integers), is rational; but any area of the form $\sqrt{k} \cdot \rho^2$ is *irrational*. Euclid's *rational area* thus contains A units of *area*, where A is an integer or of the form m/n , where m , n are integers; and his *irrational area* is of the form $\sqrt{k} \cdot A$. His irrational *area* is then connected with his irrational *straight line* by making the latter the square root of the

former. This would give us for the irrational *straight line* $\sqrt[4]{k}$. $\sqrt{A}$, which of course includes $\sqrt[4]{k} \cdot a$.

αἱ δυνάμεναι αὐτά are the straight lines the squares on which are equal to the areas, in accordance with the regular meaning of δύνασθαι. It is scarcely possible, in a book written in geometrical language, to translate δυνάμενη as the *square root* (of an *area*) and δύνασθαι as *to be the square root* (of an *area*), although I can use the term "square root" when in my notes I am using an algebraical expression to represent an area; I shall therefore hereafter use the word "side" for δυνάμενη and "to be the side of" for δύνασθαι, so that "side" will in such expressions be a short way of expressing the "side of a *square equal to (an area)*." In this particular passage it is not quite practicable to use the words "side of" or "straight line the square on which is equal to," for these expressions occur just afterwards for two alternatives which the word δυνάμενη covers. I have therefore exceptionally translated "the straight lines which produce them" (i.e. if squares are described upon them as sides).

αἱ ἴσα αὐτοῖς τετράγωνα ἀναγράφονται, literally "the (straight lines) which *describe* squares equal to them": a peculiar use of the active of ἀναγράφειν, the meaning being of course "the straight lines on which *are described* the squares" which are equal to the rectilineal figures.

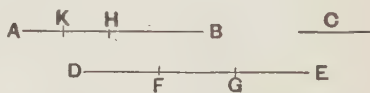
BOOK X. PROPOSITIONS.

PROPOSITION I.

Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out.

Let AB, C be two unequal magnitudes of which AB is the greater :

I say that, if from AB there be subtracted a magnitude greater than its half, and from that which



is left a magnitude greater than its half, and if this process be repeated continually, there will be left some magnitude which will be less than the magnitude C .

For C if multiplied will sometime be greater than AB .

[cf. v. Def. 4]

Let it be multiplied, and let DE be a multiple of C , and greater than AB ;

let DE be divided into the parts DF, FG, GE equal to C ,

from AB let there be subtracted BH greater than its half,

and, from AH, HK greater than its half,

and let this process be repeated continually until the divisions in AB are equal in multitude with the divisions in DE .

Let, then, AK, KH, HB be divisions which are equal in multitude with DF, FG, GE .

Now, since DE is greater than AB ,

and from DE there has been subtracted EG less than its half,

and, from AB, BH greater than its half,

therefore the remainder GD is greater than the remainder HA .

And, since GD is greater than HA ,
and there has been subtracted, from GD , the half GF ,
and, from HA , HK greater than its half,
therefore the remainder DF is greater than the remainder AK .

But DF is equal to C ;
therefore C is also greater than AK .

Therefore AK is less than C .

Therefore there is left of the magnitude AB the magnitude AK which is less than the lesser magnitude set out, namely C .

Q. E. D.

And the theorem can be similarly proved even if the parts subtracted be halves.

This proposition will be remembered because it is the lemma required in Euclid's proof of xii. 2 to the effect that circles are to one another as the squares on their diameters. Some writers appear to be under the impression that xii. 2 and the other propositions in Book xii. in which the method of exhaustion is used are the only places where Euclid makes use of x. 1; and it is commonly remarked that x. 1 might just as well have been deferred till the beginning of Book xii. Even Cantor (*Gesch. d. Math.* 13, p. 269) remarks that "Euclid draws no inference from it [x. 1], not even that which we should more than anything else expect, namely that, if two magnitudes are incommensurable, we can always form a magnitude commensurable with the first which shall differ from the second magnitude by as little as we please." But, so far from making no use of x. 1 before xii. 2, Euclid actually uses it in the very next proposition, x. 2. This being so, as the next note will show, it follows that, since x. 2 gives the criterion for the incommensurability of two magnitudes (a very necessary preliminary to the study of incommensurables), x. 1 comes exactly where it should be.

Euclid uses x. 1 to prove not only xii. 2 but xii. 5 (that pyramids with the same height and triangular bases are to one another as their bases), by means of which he proves (xii. 7 and Por.) that any pyramid is a third part of the prism which has the same base and equal height, and xii. 10 (that any cone is a third part of the cylinder which has the same base and equal height), besides other similar propositions. Now xii. 7 Por. and xii. 10 are theorems specifically attributed to Eudoxus by Archimedes (*On the Sphere and Cylinder*, Preface), who says in another place (*Quadrature of the Parabola*, Preface) that the first of the two, and the theorem that circles are to one another as the squares on their diameters, were proved by means of a certain lemma which he states as follows: "Of unequal lines, unequal surfaces, or unequal solids, the greater exceeds the less by such a magnitude as is capable, if added [continually] to itself, of exceeding any magnitude of those which are comparable with one another," i.e. of magnitudes of the same kind as the original magnitudes. Archimedes also says (*loc. cit.*) that the second of the two theorems which he attributes to Eudoxus (Eucl. xii. 10) was proved by means of "a lemma similar to the aforesaid." The lemma stated thus by Archimedes is decidedly different from x. 1, which, however, Archimedes himself uses several times, while he refers to the use of it

in XII. 2 (*On the Sphere and Cylinder*, I. 6). As I have before suggested (*The Works of Archimedes*, p. xlviii), the apparent difficulty caused by the mention of *two* lemmas in connexion with the theorem of Eucl. XII. 2 may be explained by reference to the proof of X. 1. Euclid there takes the lesser magnitude and says that it is possible, by multiplying it, to make it some time exceed the greater, and this statement he clearly bases on the 4th definition of Book V., to the effect that "magnitudes are said to bear a ratio to one another which can, if multiplied, exceed one another." Since then the smaller magnitude in X. 1 may be regarded as the difference between some two unequal magnitudes, it is clear that the lemma stated by Archimedes is in substance used to prove the lemma in X. 1, which appears to play so much larger a part in the investigations of quadrature and cubature which have come down to us.

Besides being employed in Eucl. X. 1, the "Axiom of Archimedes" appears in Aristotle, who also practically quotes the result of X. 1 itself. Thus he says, *Physics* VIII. 10, 266 b 2, "By continually adding to a finite (magnitude) I shall exceed any definite (magnitude), and similarly by continually subtracting from it I shall arrive at something less than it," and *ibid.* III. 7, 207 b 10 "For bisections of a magnitude are endless." It is thus somewhat misleading to use the term "Archimedes' Axiom" for the "lemma" quoted by him, since he makes no claim to be the discoverer of it, and it was obviously much earlier.

Stolz (see G. Vitali in *Questioni riguardanti le matematiche elementari*, I., pp. 129—30) showed how to prove the so-called Axiom or Postulate of Archimedes by means of the Postulate of Dedekind, thus. Suppose the two magnitudes to be straight lines. It is required to prove that, *given two straight lines, there always exists a multiple of the smaller which is greater than the other.*

Let the straight lines be so placed that they have a common extremity and the smaller lies along the other on the same side of the common extremity.

If AC be the greater and AB the smaller, we have to prove that there exists an integral number n such that $n \cdot AB > AC$.

Suppose that this is not true but that there are some points, like B , not coincident with the extremity A , and such that, n being any integer however great, $n \cdot AB < AC$; and we have to prove that this assumption leads to an absurdity.



The points of AC may be regarded as distributed into two "parts," namely

- (1) points H for which there exists no integer n such that $n \cdot AH > AC$,
- (2) points K for which an integer n does exist such that $n \cdot AK > AC$.

This division into parts satisfies the conditions for the application of Dedekind's Postulate, and therefore there exists a point M such that the points of AM belong to the first part and those of MC to the second part.

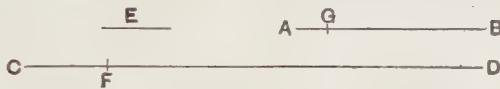
Take now a point Y on MC such that $MY < AM$. The middle point (X) of AY will fall between A and M and will therefore belong to the first part; but, since there exists an integer n such that $n \cdot AY > AC$, it follows that $2n \cdot AX > AC$: which is contrary to the hypothesis.

PROPOSITION 2.

If, when the less of two unequal magnitudes is continually subtracted in turn from the greater, that which is left never measures the one before it, the magnitudes will be incommensurable.

For, there being two unequal magnitudes AB , CD , and AB being the less, when the less is continually subtracted in turn from the greater, let that which is left over never measure the one before it;

I say that the magnitudes AB , CD are incommensurable.



For, if they are commensurable, some magnitude will measure them.

Let a magnitude measure them, if possible, and let it be E ; let AB , measuring FD , leave CF less than itself, let CF measuring BG , leave AG less than itself, and let this process be repeated continually, until there is left some magnitude which is less than E .

Suppose this done, and let there be left AG less than E .

Then, since E measures AB ,

while AB measures DF ,

therefore E will also measure FD .

But it measures the whole CD also;

therefore it will also measure the remainder CF .

But CF measures BG ;

therefore E also measures BG .

But it measures the whole AB also;

therefore it will also measure the remainder AG , the greater the less:

which is impossible.

Therefore no magnitude will measure the magnitudes AB , CD ;

therefore the magnitudes AB , CD are incommensurable.

[x. Def. 1]

Therefore etc.

This proposition states the test for incommensurable magnitudes, founded on the usual operation for finding the greatest common measure. The sign of the incommensurability of two magnitudes is that this operation never comes to an end, while the successive remainders become smaller and smaller until they are less than any assigned magnitude.

Observe that Euclid says "let this process be repeated continually until there is left some magnitude which is less than E ." Here he evidently assumes that the process *will* some time produce a remainder less than any assigned magnitude E . Now this is by no means self-evident, and yet Heiberg (though so careful to supply references) and Lorenz do not refer to the basis of the assumption, which is in reality x. 1, as Billingsley and Williamson were shrewd enough to see. The fact is that, if we set off a smaller magnitude once or oftener along a greater which it does not exactly measure, until the remainder is less than the smaller magnitude, we take away from the greater *more than its half*. Thus, in the figure, FD is more than the half of CD , and BG more than the half of AB . If we continued the process, AG marked off along CF as many times as possible would cut off more than its half; next, more than half AG would be cut off, and so on. Hence along CD , AB *alternately* the process would cut off more than half, then more than half the remainder and so on, so that on *both* lines we should ultimately arrive at a remainder less than any assigned length.

The method of finding the greatest common measure exhibited in this proposition and the next is of course again the same as that which we use and which may be shown thus:

$$\begin{array}{r} b) a (p \\ \underline{pb} \\ c) b (q \\ \underline{qc} \\ d) c (r \\ \underline{rd} \\ e \end{array}$$

The proof too is the same as ours, taking just the same form, as shown in the notes to the similar propositions VII. 1, 2 above. In the present case the hypothesis is that the process never stops, and it is required to prove that a , b cannot in that case have any common measure, as f . For suppose that f is a common measure, and suppose the process to be continued until the remainder e , say, is less than f .

Then, since f measures a , b , it measures $a - pb$, or c .

Since f measures b , c , it measures $b - qc$, or d ; and, since f measures c , d , it measures $c - rd$, or e : which is impossible, since $e < f$.

Euclid assumes as axiomatic that, if f measures a , b , it measures $ma \pm nb$.

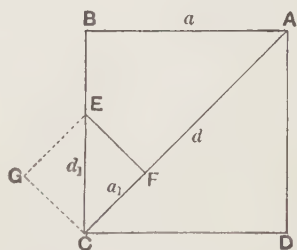
In practice, of course, it is often unnecessary to carry the process far in order to see that it will never stop, and consequently that the magnitudes are incommensurable. A good instance is pointed out by Allman (*Greek Geometry from Thales to Euclid*, pp. 42, 137-8). Euclid proves in XIII. 5 that, if AB be cut in extreme and mean ratio at C , and if DA equal to AC be added, then DB is also cut in extreme and mean ratio at A . This is indeed obvious from the proof of II. 11. It follows conversely that, if BD is cut into extreme and mean ratio at A , and AC , equal to the lesser segment AD , be subtracted from the greater AB , AB is similarly divided at C . We can then



mark off from AC a portion equal to CB , and AC will then be similarly divided, and so on. Now the greater segment in a line thus divided is greater than half the line, but it follows from XIII. 3 that it is less than twice the lesser segment, i.e. the lesser segment can never be marked off more than *once* from the greater. Our process of marking off the lesser segment from the greater continually is thus exactly that of finding the greatest common measure. If, therefore, the segments were commensurable, the process would stop. But it clearly does not; therefore the segments are incommensurable.

Allman expresses the opinion that it was rather in connexion with the line cut in extreme and mean ratio than with reference to the diagonal and side of a square that the Pythagoreans discovered the incommensurable. But the evidence seems to put it beyond doubt that the Pythagoreans did discover the incommensurability of $\sqrt{2}$ and devoted much attention to this particular case. The view of Allman does not therefore commend itself to me, though it is likely enough that the Pythagoreans were aware of the incommensurability of the segments of a line cut in extreme and mean ratio. At all events the Pythagoreans could hardly have carried their investigations into the incommensurability of the segments of this line very far, since Theaetetus is said to have made the first classification of irrationals, and to him is also, with reasonable probability, attributed the substance of the first part of Eucl. XIII., in the sixth proposition of which occurs the proof that the segments of a rational straight line cut in extreme and mean ratio are *apotomes*.

Again, the incommensurability of $\sqrt{2}$ can be proved by a method practically equivalent to that of x. 2, and without carrying the process very far. This method is given in Chrystal's *Text-book of Algebra* (I. p. 270). Let d, a be the diagonal and side respectively of a square $ABCD$. Mark off AF along AC equal to a . Draw FE at right angles to AC meeting BC in E .



It is easily proved that

$$BE = EF = FC,$$

$$CF = AC - AB = d - a \dots\dots\dots(1).$$

$$CE = CB - CF = a - (d - a) \\ = 2a - d \dots\dots\dots(2).$$

Suppose, if possible, that d, a are commensurable. If d, a are both commensurably expressible in terms of any finite unit, each must be an integral multiple of a certain finite unit.

But from (1) it follows that CF , and from (2) it follows that CE , is an integral multiple of the same unit.

And CF, CE are the side and diagonal of a square $CFEG$, the side of which is *less than half the side of the original square*. If a_1, d_1 are the side and diagonal of this square,

$$\left. \begin{aligned} a_1 &= d - a \\ d_1 &= 2a - d \end{aligned} \right\}.$$

Similarly we can form a square with side a_2 and diagonal d_2 which are less than half a_1, d_1 respectively, and a_2, d_2 must be integral multiples of the same unit, where

$$\left. \begin{aligned} a_2 &= d_1 - a_1, \\ d_2 &= 2a_1 - d_1; \end{aligned} \right\}$$

and this process may be continued indefinitely until (x. 1) we have a square as small as we please, the side and diagonal of which are integral multiples of a finite unit: which is absurd.

Therefore a , d are incommensurable.

It will be observed that this method is the opposite of that shown in the Pythagorean series of side- and diagonal-numbers, the squares being successively smaller instead of larger.

PROPOSITION 3.

Given two commensurable magnitudes, to find their greatest common measure.

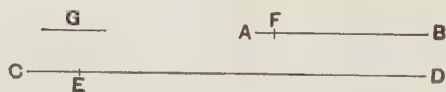
Let the two given commensurable magnitudes be AB , CD of which AB is the less;

thus it is required to find the greatest common measure of AB , CD .

Now the magnitude AB either measures CD or it does not.

If then it measures it—and it measures itself also— AB is a common measure of AB , CD .

And it is manifest that it is also the greatest;
for a greater magnitude than the magnitude AB will not measure AB .



Next, let AB not measure CD .

Then, if the less be continually subtracted in turn from the greater, that which is left over will sometime measure the one before it, because AB , CD are not incommensurable;

[cf. x. 2]

let AB , measuring ED , leave EC less than itself,

let EC , measuring FB , leave AF less than itself,

and let AF measure CE .

Since, then, AF measures CE ,

while CE measures FB ,

therefore AF will also measure FB .

But it measures itself also;

therefore AF will also measure the whole AB .

But AB measures DE ;
therefore AF will also measure ED .

But it measures CE also;
therefore it also measures the whole CD .

Therefore AF is a common measure of AB , CD .

I say next that it is also the greatest.

For, if not, there will be some magnitude greater than AF which will measure AB , CD .

Let it be G .

Since then G measures AB ,
while AB measures ED ,
therefore G will also measure ED .

But it measures the whole CD also;
therefore G will also measure the remainder CE .

But CE measures FB ;
therefore G will also measure FB .

But it measures the whole AB also,
and it will therefore measure the remainder AF , the greater
the less:
which is impossible.

Therefore no magnitude greater than AF will measure
 AB , CD ;

therefore AF is the greatest common measure of AB , CD .

Therefore the greatest common measure of the two given
commensurable magnitudes AB , CD has been found.

Q. E. D.

PORISM. From this it is manifest that, if a magnitude
measure two magnitudes, it will also measure their greatest
common measure.

This proposition for two commensurable *magnitudes* is, *mutatis mutandis*,
exactly the same as VII. 2 for numbers. We have the process

$$\begin{array}{r} b) a (p \\ \underline{pb} \\ c) b (q \\ \underline{qc} \\ d) c (r \\ \underline{rd} \end{array}$$

where c is equal to rd and therefore there is no remainder,

It is then proved that d is a common measure of a, b ; and next, by a *reductio ad absurdum*, that it is the *greatest* common measure, since any common measure must measure d , and no magnitude greater than d can measure d . The *reductio ad absurdum* is of course one of form only.

The Porism corresponds exactly to the Porism to VII. 2.

The process of finding the greatest common measure is probably given in this Book, not only for the sake of completeness, but because in x. 5 a common measure of two magnitudes A, B is assumed and used, and therefore it is important to show that such a measure can be *found* if not already known.

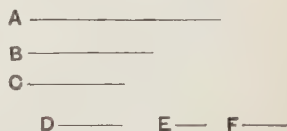
PROPOSITION 4.

Given three commensurable magnitudes, to find their greatest common measure.

Let A, B, C be the three given commensurable magnitudes; thus it is required to find the greatest common measure of A, B, C .

Let the greatest common measure of the two magnitudes A, B be taken, and let it be D ;

then D either measures C , or does not measure it.



First, let it measure it.

Since then D measures C ,
while it also measures A, B ,
therefore D is a common measure of A, B, C .

And it is manifest that it is also the greatest;
for a greater magnitude than the magnitude D does not measure A, B .

Next, let D not measure C .

I say first that C, D are commensurable.

For, since A, B, C are commensurable,
some magnitude will measure them,
and this will of course measure A, B also;
so that it will also measure the greatest common measure of
 A, B , namely D .

[x. 3, Por.]

But it also measures C ;
so that the said magnitude will measure C, D ;
therefore C, D are commensurable.

Now let their greatest common measure be taken, and let it be E . [x. 3]

Since then E measures D ,
while D measures A, B ,
therefore E will also measure A, B .

But it measures C also ;
therefore E measures A, B, C ;
therefore E is a common measure of A, B, C .

I say next that it is also the greatest.

For, if possible, let there be some magnitude F greater than E , and let it measure A, B, C .

Now, since F measures A, B, C ,
it will also measure A, B ,
and will measure the greatest common measure of A, B .

But the greatest common measure of A, B is D ; [x. 3, Por.]
therefore F measures D .

But it measures C also ;
therefore F measures C, D ;
therefore F will also measure the greatest common measure of C, D . [x. 3, Por.]

But that is E ;
therefore F will measure E , the greater the less :
which is impossible.

Therefore no magnitude greater than the magnitude E will measure A, B, C ;

therefore E is the greatest common measure of A, B, C if D do not measure C ,

and, if it measure it, D is itself the greatest common measure.

Therefore the greatest common measure of the three given commensurable magnitudes has been found.

PORISM. From this it is manifest that, if a magnitude measure three magnitudes, it will also measure their greatest common measure.

Similarly too, with more magnitudes, the greatest common measure can be found, and the porism can be extended.

Q. E. D.

This proposition again corresponds exactly to VII. 3 for numbers. As there Euclid thinks it necessary to prove that, a, b, c not being prime to one another, d and c are also not prime to one another, so here he thinks it necessary to prove that d, c are commensurable, as they must be since any common measure of a, b must be a measure of their greatest common measure d (x. 3, Por.).

The argument in the proof that e , the greatest common measure of d, c , is the greatest common measure of a, b, c , is the same as that in VII. 3 and x. 3.

The Porism contains the extension of the process to the case of four or more magnitudes, corresponding to Heron's remark with regard to the similar extension of VII. 3 to the case of four or more *numbers*.

PROPOSITION 5.

Commensurable magnitudes have to one another the ratio which a number has to a number.

Let A, B be commensurable magnitudes ;

I say that A has to B the ratio which a number has to a number.

For, since A, B are commensurable, some magnitude will measure them.

Let it measure them, and let it be C .

$$\begin{array}{r} \text{A} \\ \hline \text{D} \\ \hline \text{E} \end{array} \qquad \text{B} \qquad \text{C}$$

And, as many times as C measures A , so many units let there be in D ;

and, as many times as C measures B , so many units let there be in E .

Since then C measures A according to the units in D , while the unit also measures D according to the units in it, therefore the unit measures the number D the same number of times as the magnitude C measures A ;

therefore, as C is to A , so is the unit to D ; [VII. Def. 20]

therefore, inversely, as A is to C , so is D to the unit.

[cf. v. 7, Por.]

Again, since C measures B according to the units in E , while the unit also measures E according to the units in it,

therefore the unit measures E the same number of times as C measures B ;

therefore, as C is to B , so is the unit to E .

But it was also proved that,

as A is to C , so is D to the unit ;

therefore, *ex aequali*,

as A is to B , so is the number D to E . [v. 22]

Therefore the commensurable magnitudes A , B have to one another the ratio which the number D has to the number E .

Q. E. D.

The argument is as follows. If a , b be commensurable magnitudes, they have some common measure c , and

$$a = mc,$$

$$b = nc,$$

where m , n are integers.

It follows that

$$c : a = 1 : m, \dots\dots\dots (1),$$

or, inversely,

$$a : c = m : 1;$$

and also that

$$c : b = 1 : n,$$

so that, *ex aequali*,

$$a : b = m : n.$$

It will be observed that, in stating the proportion (1), Euclid is merely expressing the fact that a is the same multiple of c that m is of 1. In other words, he rests the statement on the definition of proportion in vii. Def. 20. This, however, is applicable only to four *numbers*, and c , a are not numbers but magnitudes. Hence the statement of the proportion is not legitimate unless it is proved that it is true in the sense of v. Def. 5 with regard to magnitudes in general, the numbers 1, m being *magnitudes*. Similarly with regard to the other proportions in the proposition.

There is, therefore, a hiatus. Euclid ought to have proved that magnitudes which are proportional in the sense of vii. Def. 20 are also proportional in the sense of v. Def. 5, or that the proportion of numbers is included in the proportion of magnitudes as a particular case. Simson has proved this in his Proposition C inserted in Book v. (see Vol. II. pp. 126—8). The portion of that proposition which is required here is the proof that,

if
$$\left. \begin{array}{l} a = mb \\ c = md \end{array} \right\},$$

then $a : b = c : d$, in the sense of v. Def. 5.

Take any equimultiples pa , pc of a , c and any equimultiples qb , qd of b , d .

Now
$$\left. \begin{array}{l} pa = pmb \\ pc = pmd \end{array} \right\}.$$

But, according as $pmb > < qb$, $pmd > < qd$.

Therefore, according as $pa > < qb$, $pa > < qd$.

And pa , pc are any equimultiples of a , c , and qb , qd any equimultiples of b , d .

Therefore $a : b = c : d$. [v. Def. 5.]

PROPOSITION 6.

If two magnitudes have to one another the ratio which a number has to a number, the magnitudes will be commensurable.

For let the two magnitudes A , B have to one another the ratio which the number D has to the number E ;

5 I say that the magnitudes A , B are commensurable.



For let A be divided into as many equal parts as there are units in D ,

and let C be equal to one of them ;

and let F be made up of as many magnitudes equal to C as
10 there are units in E .

Since then there are in A as many magnitudes equal to C as there are units in D ,

whatever part the unit is of D , the same part is C of A also ;

therefore, as C is to A , so is the unit to D . [vii. Def. 20]

15 But the unit measures the number D ;

therefore C also measures A .

And since, as C is to A , so is the unit to D ,

therefore, inversely, as A is to C , so is the number D to the unit. [cf. v. 7, Por.]

20 Again, since there are in F as many magnitudes equal to C as there are units in E ,

therefore, as C is to F , so is the unit to E . [vii. Def. 20]

But it was also proved that,

as A is to C , so is D to the unit ;

25 therefore, *ex aequali*, as A is to F , so is D to E . [v. 22]

But, as D is to E , so is A to B ;

therefore also, as A is to B , so is it to F also. [v. 11]

Therefore A has the same ratio to each of the magnitudes B , F ;

30 therefore B is equal to F . [v. 9]

But C measures F ;

therefore it measures B also.

Further it measures A also ;

therefore C measures A , B .

- 35 Therefore A is commensurable with B .
Therefore etc.

PORISM. From this it is manifest that, if there be two numbers, as D , E , and a straight line, as A , it is possible to make a straight line [F] such that the given straight line is to
40 it as the number D is to the number E .

And, if a mean proportional be also taken between A , F , as B ,

as A is to F , so will the square on A be to the square on B , that is, as the first is to the third, so is the figure on the first
45 to that which is similar and similarly described on the second.

[VI. 19, Por.]

But, as A is to F , so is the number D to the number E ; therefore it has been contrived that, as the number D is to the number E , so also is the figure on the straight line A to the figure on the straight line B .
Q. E. D.

15. But the unit measures the number D ; therefore C also measures A . These words are redundant, though they are apparently found in all the MSS.

The same link to connect the proportion of numbers with the proportion of magnitudes as was necessary in the last proposition is necessary here. This being premised, the argument is as follows.

Suppose $a : b = m : n$,

where m , n are (integral) numbers.

Divide a into m parts, each equal to c , say,

so that $a = mc$.

Now take d such that $d = nc$.

Therefore we have $a : c = m : 1$,

and $c : d = 1 : n$,

so that, *ex aequali*, $a : d = m : n$

$= a : b$, by hypothesis.

Therefore $b = d = nc$,

so that c measures b n times, and a , b are commensurable.

The Porism is often used in the later propositions. It follows (1) that, if a be a given straight line, and m , n any numbers, a straight line x can be found such that

$$a : x = m : n.$$

(2) We can find a straight line y such that

$$a^2 : y^2 = m : n.$$

For we have only to take y , a mean proportional between a and x , as previously found, in which case a , y , x are in continued proportion and [v. Def. 9]

$$\begin{aligned} a^2 : y^2 &= a : x \\ &= m : n. \end{aligned}$$

PROPOSITION 7.

Incommensurable magnitudes have not to one another the ratio which a number has to a number.

Let A, B be incommensurable magnitudes ;

I say that A has not to B the ratio which a number has to a number.

For, if A has to B the ratio which a number has to a number, A will be commensurable with B . [x. 6]

But it is not ;

therefore A has not to B the ratio which a number has to a number.

$$\frac{A}{B}$$

Therefore etc.

PROPOSITION 8.

If two magnitudes have not to one another the ratio which a number has to a number, the magnitudes will be incommensurable.

For let the two magnitudes A, B not have to one another the ratio which a number has to a number ;

I say that the magnitudes A, B are incommensurable.

$$\frac{A}{B}$$

For, if they are commensurable, A will have to B the ratio which a number has to a number. [x. 5]

But it has not ;

therefore the magnitudes A, B are incommensurable.

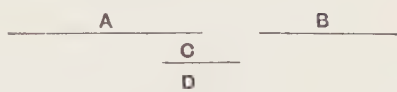
Therefore etc.

PROPOSITION 9.

The squares on straight lines commensurable in length have to one another the ratio which a square number has to a square number ; and squares which have to one another the ratio which a square number has to a square number will also have their sides commensurable in length. But the squares on straight lines incommensurable in length have not to one another the ratio which a square number has to a square number ; and squares which have not to one another the ratio which a square number has to a square number will not have their sides commensurable in length either.

For let A, B be commensurable in length ;

I say that the square on A
has to the square on B the
ratio which a square number
has to a square number.



For, since A is commensurable in length with B ,
therefore A has to B the ratio which a number has to a
number. [x. 5]

Let it have to it the ratio which C has to D .

Since then, as A is to B , so is C to D ,

while the ratio of the square on A to the square on B is
duplicate of the ratio of A to B ,

for similar figures are in the duplicate ratio of their corre-
sponding sides ; [vi. 20, Por.]

and the ratio of the square on C to the square on D is duplicate
of the ratio of C to D ,

for between two square numbers there is one mean proportional
number, and the square number has to the square number the
ratio duplicate of that which the side has to the side ; [viii. 11]
therefore also, as the square on A is to the square on B , so
is the square on C to the square on D .

Next, as the square on A is to the square on B , so let
the square on C be to the square on D ;

I say that A is commensurable in length with B .

For since, as the square on A is to the square on B , so is
the square on C to the square on D ,

while the ratio of the square on A to the square on B is
duplicate of the ratio of A to B ,

and the ratio of the square on C to the square on D is duplicate
of the ratio of C to D ,

therefore also, as A is to B , so is C to D .

Therefore A has to B the ratio which the number C has
to the number D ;

therefore A is commensurable in length with B . [x. 6]

Next, let A be incommensurable in length with B ;

I say that the square on A has not to the square on B the
ratio which a square number has to a square number.

For, if the square on A has to the square on B the ratio

which a square number has to a square number, A will be commensurable with B .

But it is not ;

therefore the square on A has not to the square on B the ratio which a square number has to a square number.

Again, let the square on A not have to the square on B the ratio which a square number has to a square number ;
I say that A is incommensurable in length with B .

For, if A is commensurable with B , the square on A will have to the square on B the ratio which a square number has to a square number.

But it has not ;

therefore A is not commensurable in length with B .

Therefore etc.

PORISM. And it is manifest from what has been proved that straight lines commensurable in length are always commensurable in square also, but those commensurable in square are not always commensurable in length also.

[LEMMA. It has been proved in the arithmetical books that similar plane numbers have to one another the ratio which a square number has to a square number, [VIII. 26] and that, if two numbers have to one another the ratio which a square number has to a square number, they are similar plane numbers. [Converse of VIII. 26]

And it is manifest from these propositions that numbers which are not similar plane numbers, that is, those which have not their sides proportional, have not to one another the ratio which a square number has to a square number.

For, if they have, they will be similar plane numbers : which is contrary to the hypothesis.

Therefore numbers which are not similar plane numbers have not to one another the ratio which a square number has to a square number.]

A scholium to this proposition (Schol. x. No. 62) says categorically that the theorem proved in it was the discovery of Theaetetus.

If a , b be straight lines, and

$$a : b = m : n,$$

where m , n are numbers,

then

$$a^2 : b^2 = m^2 : n^2 ;$$

and conversely.

This inference, which looks so easy when thus symbolically expressed, was by no means so easy for Euclid owing to the fact that a, b are straight lines, and m, n numbers. He has to pass from $a : b$ to $a^2 : b^2$ by means of vi. 20, Por. through the duplicate ratio; the square on a is to the square on b in the duplicate ratio of the corresponding sides a, b . On the other hand, m, n being *numbers*, it is viii. 11 which has to be used to show that $m^2 : n^2$ is the ratio duplicate of $m : n$.

Then, in order to establish his result, Euclid *assumes* that, *if two ratios are equal, the ratios which are their duplicates are also equal*. This is nowhere proved in Euclid, but it is an easy inference from v. 22, as shown in my note on vi. 22.

The converse has to be established in the same careful way, and Euclid *assumes* that *ratios the duplicates of which are equal are themselves equal*. This is much more troublesome to prove than the converse; for proofs I refer to the same note on vi. 22.

The second part of the theorem, deduced by *reductio ad absurdum* from the first, requires no remark.

In the Greek text there is an addition to the Porism which Heiberg brackets as superfluous and not in Euclid's manner. It consists (1) of a sort of proof, or rather explanation, of the Porism and (2) of a statement and explanation to the effect that straight lines incommensurable in length are not necessarily incommensurable in square also, and that straight lines incommensurable in square are, on the other hand, always incommensurable in length also.

The Lemma gives expressions for two numbers which have to one another the ratio of a square number to a square number. *Similar plane numbers* are of the form $pm \cdot pn$ and $qm \cdot qn$, or mnp^2 and mnq^2 , the ratio of which is of course the ratio of p^2 to q^2 .

The converse theorem that, if two numbers have to one another the ratio of a square number to a square number, the numbers are similar plane numbers is not, as a matter of fact, proved in the arithmetical Books. It is the converse of viii. 26 and is used in ix. 10. Heron gave it (see note on viii. 27 above).

Heiberg however gives strong reason for supposing the Lemma to be an interpolation. It has reference to the next proposition, x. 10, and, as we shall see, there are so many objections to x. 10 that it can hardly be accepted as genuine. Moreover there is no reason why, in the Lemma itself, numbers which are *not* similar plane numbers should be brought in as they are.

[PROPOSITION 10.

To find two straight lines incommensurable, the one in length only, and the other in square also, with an assigned straight line.

Let A be the assigned straight line ;
thus it is required to find two straight lines incommensurable, the one in length only, and the other in square also, with A .

Let two numbers B, C be set out which have not to one

another the ratio which a square number has to a square number, that is, which are not similar plane numbers ;

and let it be contrived that,

as B is to C , so is the square on A to
the square on D

A _____
D _____
E _____
B _____
C _____

—for we have learnt how to do this—

[x. 6, Por.]

therefore the square on A is commensurable with the square on D . [x. 6]

And, since B has not to C the ratio which a square number has to a square number,

therefore neither has the square on A to the square on D the ratio which a square number has to a square number ;

therefore A is incommensurable in length with D . [x. 9]

Let E be taken a mean proportional between A , D ;

therefore, as A is to D , so is the square on A to the square on E . [v. Def. 9]

But A is incommensurable in length with D ;

therefore the square on A is also incommensurable with the square on E ; [x. 11]

therefore A is incommensurable in square with E .

Therefore two straight lines D , E have been found incommensurable, D in length only, and E in square and of course in length also, with the assigned straight line A .]

It would appear as though this proposition was intended to supply a justification for the statement in x. Def. 3 that *it is proved* that there are an infinite number of straight lines (*a*) incommensurable in length only, or commensurable in square only, and (*b*) incommensurable in square, with any given straight line.

But in truth the proposition could well be dispensed with ; and the positive objections to its genuineness are considerable.

In the first place, it depends on the following proposition, x. 11 ; for the last step concludes that, since

$$a^2 : y^2 = a : x,$$

and a , x are incommensurable in length, therefore a^2 , y^2 are incommensurable. But Euclid never commits the irregularity of proving a theorem by means of a later one. Gregory sought to get over the difficulty by putting x. 10 after x. 11 ; but of course, if the order were so inverted, the Lemma would still be in the wrong place.

Further, the expression *ἐμάθομεν γάρ*, “for we have learnt (how to do this),” is not in Euclid’s manner and betrays the hand of a learner (though the same

expression is found in the *Sectio Canonis* of Euclid, where the reference is to the *Elements*).

Lastly the manuscript P has the number 10, in the first hand, at the top of x. 11, from which it may perhaps be concluded that x. 10 had at first no number.

It seems best therefore to reject as spurious both the Lemma and x. 10.

The argument of x. 10 is simple. If a be a given straight line and m, n numbers which have not to one another the ratio of square to square, take x such that

$$a^2 : x^2 = m : n, \quad [\text{x. 6, Por.}]$$

whence a, x are incommensurable in length.

[x. 9]

Then take y a mean proportional between a, x , whence

$$a^2 : y^2 = a : x \quad [\text{v. Def. 9}]$$

$$[= \sqrt{m} : \sqrt{n}],$$

and x is incommensurable in length only, while y is incommensurable in square as well as in length, with a .

PROPOSITION 11.

If four magnitudes be proportional, and the first be commensurable with the second, the third will also be commensurable with the fourth; and, if the first be incommensurable with the second, the third will also be incommensurable with the fourth.

Let A, B, C, D be four magnitudes in proportion, so that, as A is to B , so is C to D ,



and let A be commensurable with B ;

I say that C will also be commensurable with D .

For, since A is commensurable with B , therefore A has to B the ratio which a number has to a number. [x. 5]

And, as A is to B , so is C to D ;
therefore C also has to D the ratio which a number has to a number;
therefore C is commensurable with D . [x. 6]

Next, let A be incommensurable with B ;
I say that C will also be incommensurable with D .

For, since A is incommensurable with B , therefore A has not to B the ratio which a number has to a number. [x. 7]

And, as A is to B , so is C to D ;

therefore neither has C to D the ratio which a number has to a number;

therefore C is incommensurable with D . [x. 8]

Therefore etc.

I shall henceforth, for the sake of brevity, use symbols for the terms "commensurable (with)" and "incommensurable (with)" according to the varieties described in x. Deff. 1—4. The symbols are taken from Lorenz and seem convenient.

Commensurable and *commensurable with*, in relation to areas, and *commensurable in length* and *commensurable in length with*, in relation to straight lines, will be denoted by $\sim$.

Commensurable in square only or *commensurable in square only with* (terms applicable only to straight lines) will be denoted by $\simeq$.

Incommensurable (with), of areas, and *incommensurable (with)*, of straight lines will be denoted by $\smile$.

Incommensurable in square (with) (a term applicable to straight lines only) will be denoted by $\smile$.

Suppose a, b, c, d to be four magnitudes such that

$$a : b = c : d.$$

Then (1), if $a \sim b$,

$$a : b = m : n, \text{ where } m, n \text{ are integers, [x. 5]}$$

whence

$$c : d = m : n,$$

and therefore

$$c \sim d. \quad [\text{x. 6}]$$

(2) If $a \simeq b$,

$$a : b \neq m : n, \quad [\text{x. 7}]$$

so that

$$c : d \neq m : n,$$

whence

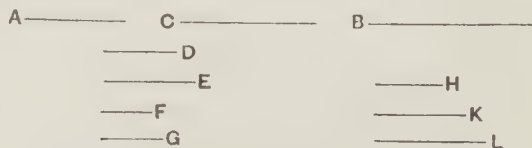
$$c \smile d. \quad [\text{x. 8}]$$

PROPOSITION 12.

Magnitudes commensurable with the same magnitude are commensurable with one another also.

For let each of the magnitudes A, B be commensurable with C ;

I say that A is also commensurable with B .



For, since A is commensurable with C ,
therefore A has to C the ratio which a number has to a number.

[x. 5]

Let it have the ratio which D has to E .

Again, since C is commensurable with B ,
therefore C has to B the ratio which a number has to a
number. [x. 5]

Let it have the ratio which F has to G .

And, given any number of ratios we please, namely the
ratio which D has to E and that which F has to G ,
let the numbers H, K, L be taken continuously in the given
ratios ; [cf. viii. 4]

so that, as D is to E , so is H to K ,

and, as F is to G , so is K to L .

Since, then, as A is to C , so is D to E ,
while, as D is to E , so is H to K ,
therefore also, as A is to C , so is H to K . [v. 11]

Again, since, as C is to B , so is F to G ,
while, as F is to G , so is K to L ,
therefore also, as C is to B , so is K to L . [v. 11]

But also, as A is to C , so is H to K ;
therefore, *ex aequali*, as A is to B , so is H to L . [v. 22]

Therefore A has to B the ratio which a number has to a
number ;
therefore A is commensurable with B . [x. 6]

Therefore etc.

Q. E. D.

We have merely to go through the process of compounding two ratios in
numbers.

Suppose	a, b each $\wedge c$.	
Therefore	$a : c = m : n$, say,	[x. 5]
	$c : b = p : q$, say.	
Now	$m : n = mp : np$,	
and	$p : q = np : nq$.	
Therefore	$a : c = mp : np$,	
	$c : b = np : nq$,	
whence, <i>ex aequali</i> ,	$a : b = mp : nq$,	
so that	$a \wedge b$.	[x. 6]

PROPOSITION 13.

If two magnitudes be commensurable, and the one of them be incommensurable with any magnitude, the remaining one will also be incommensurable with the same.

Let A, B be two commensurable magnitudes, and let one of them, A , be incommensurable with any other magnitude C ;

I say that the remaining one, B , will also be incommensurable with C .

A _____
C _____
B _____

For, if B is commensurable with C , while A is also commensurable with B , A is also commensurable with C .

[x. 12]

But it is also incommensurable with it : which is impossible.

Therefore B is not commensurable with C ; therefore it is incommensurable with it.

Therefore etc.

LEMMA.

Given two unequal straight lines, to find by what square the square on the greater is greater than the square on the less.

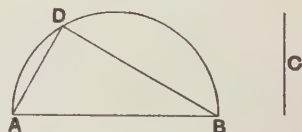
Let AB, C be the given two unequal straight lines, and let AB be the greater of them ; thus it is required to find by what square the square on AB is greater than the square on C .

Let the semicircle ADB be described on AB ,

and let AD be fitted into it equal to C ;

[iv. 1]

let DB be joined.



It is then manifest that the angle ADB is right, [iii. 31] and that the square on AB is greater than the square on AD , that is, C , by the square on DB . [I. 47]

Similarly also, if two straight lines be given, the straight line the square on which is equal to the sum of the squares on them is found in this manner.

Let AD , DB be the given two straight lines, and let it be required to find the straight line the square on which is equal to the sum of the squares on them.

Let them be placed so as to contain a right angle, that formed by AD , DB ;
and let AB be joined.

It is again manifest that the straight line the square on which is equal to the sum of the squares on AD , DB is AB .

[I. 47]

Q. E. D.

The lemma gives an obvious method of finding a straight line (c) equal to $\sqrt{a^2 + b^2}$, where a , b are given straight lines of which a is the greater.

PROPOSITION 14.

If four straight lines be proportional, and the square on the first be greater than the square on the second by the square on a straight line commensurable with the first, the square on the third will also be greater than the square on the fourth by the square on a straight line commensurable with the third.

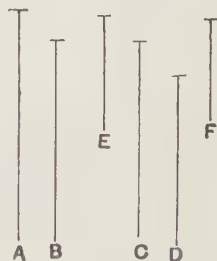
And, if the square on the first be greater than the square on the second by the square on a straight line incommensurable with the first, the square on the third will also be greater than the square on the fourth by the square on a straight line incommensurable with the third.

Let A , B , C , D be four straight lines in proportion, so that, as A is to B , so is C to D ;

and let the square on A be greater than the square on B by the square on E , and
let the square on C be greater than the square on D by the square on F ;

I say that, if A is commensurable with E , C is also commensurable with F ,

and, if A is incommensurable with E , C is also incommensurable with F .



For since, as A is to B , so is C to D ,

therefore also, as the square on A is to the square on B , so is the square on C to the square on D . [VI. 22]

But the squares on E , B are equal to the square on A ,

and the squares on D , F are equal to the square on C .

Therefore, as the squares on E, B are to the square on B , so are the squares on D, F to the square on D ;

therefore, *separando*, as the square on E is to the square on B , so is the square on F to the square on D ; [v. 17]

30 therefore also, as E is to B , so is F to D ; [vi. 22]

therefore, inversely, as B is to E , so is D to F .

But, as A is to B , so also is C to D ;

therefore, *ex aequali*, as A is to E , so is C to F . [v. 22]

Therefore, if A is commensurable with E , C is also com-
35 mensurable with F ,

and, if A is incommensurable with E , C is also incommensurable with F . [x. 11]

Therefore etc.

3, 5, 8, 10. Euclid speaks of the square on the first (third) being greater than the square on the second (fourth) by the square on a straight line commensurable (incommensurable) "with itself (*ἐαυτῇ*)," and similarly in all like phrases throughout the Book. For clearness' sake I substitute "the first," "the third," or whatever it may be, for "itself" in these cases.

Suppose a, b, c, d to be straight lines such that

$$a : b = c : d \dots\dots\dots(1).$$

It follows [vi. 22] that $a^2 : b^2 = c^2 : d^2 \dots\dots\dots(2).$

In order to prove that, *convertendo*,

$$a^2 : (a^2 - b^2) = c^2 : (c^2 - d^2)$$

Euclid has to use a somewhat roundabout method owing to the absence of a *convertendo* proposition in his Book v. (which omission Simson supplied by his Prop. E).

It follows from (2) that

$$\{(a^2 - b^2) + b^2\} : b^2 = \{(c^2 - d^2) + d^2\} : d^2,$$

whence, *separando*, $(a^2 - b^2) : b^2 = (c^2 - d^2) : d^2$, [v. 17]

and, inversely, $b^2 : (a^2 - b^2) = d^2 : (c^2 - d^2).$

From this and (2), *ex aequali*,

$$a^2 : (a^2 - b^2) = c^2 : (c^2 - d^2). \quad [v. 22]$$

Hence $a : \sqrt{a^2 - b^2} = c : \sqrt{c^2 - d^2}$. [vi. 22]

According therefore as $a \cap$ or $\cup \sqrt{a^2 - b^2}$,

$$c \cap \text{ or } \cup \sqrt{c^2 - d^2}. \quad [x. 11]$$

If $a \cap \sqrt{a^2 - b^2}$, we may put $\sqrt{a^2 - b^2} = ka$, where k is of the form m/n and m, n are integers. And if $\sqrt{a^2 - b^2} = ka$, it follows in this case that $\sqrt{c^2 - d^2} = kc$.

PROPOSITION 15.

If two commensurable magnitudes be added together, the whole will also be commensurable with each of them; and, if the whole be commensurable with one of them, the original magnitudes will also be commensurable.

For let the two commensurable magnitudes AB , BC be added together;

I say that the whole AC is also commensurable with each of the magnitudes AB , BC .



For, since AB , BC are commensurable, some magnitude will measure them.

Let it measure them, and let it be D .

Since then D measures AB , BC , it will also measure the whole AC .

But it measures AB , BC also;

therefore D measures AB , BC , AC ;

therefore AC is commensurable with each of the magnitudes AB , BC . [x. Def. 1]

Next, let AC be commensurable with AB ;

I say that AB , BC are also commensurable.

For, since AC , AB are commensurable, some magnitude will measure them.

Let it measure them, and let it be D .

Since then D measures CA , AB , it will also measure the remainder BC .

But it measures AB also;

therefore D will measure AB , BC ;

therefore AB , BC are commensurable. [x. Def. 1]

Therefore etc.

(1) If a , b be any two commensurable magnitudes, they are of the form mc , nc , where c is a common measure of a , b and m , n some integers.

It follows that $a + b = (m + n)c$;

therefore $(a + b)$, being measured by c , is commensurable with both a and b .

(2) If $a + b$ is commensurable with either a or b , say a , we may put $a + b = mc$, $a = nc$, where c is a common measure of $(a + b)$, a , and m , n are integers.

Subtracting, we have $b = (m - n)c$,

whence $b \sim a$.

PROPOSITION 16.

If two incommensurable magnitudes be added together, the whole will also be incommensurable with each of them; and, if the whole be incommensurable with one of them, the original magnitudes will also be incommensurable.

For let the two incommensurable magnitudes AB , BC be added together;

I say that the whole AC is also incommensurable with each of the magnitudes AB , BC .

For, if CA , AB are not incommensurable, some magnitude will measure them.

Let it measure them, if possible, and let it be D .

Since then D measures CA , AB ,
therefore it will also measure the remainder BC .

But it measures AB also;

therefore D measures AB , BC .

Therefore AB , BC are commensurable;

but they were also, by hypothesis, incommensurable:
which is impossible.

Therefore no magnitude will measure CA , AB ;

therefore CA , AB are incommensurable.

[x. Def. 1]

Similarly we can prove that AC , CB are also incommensurable.

Therefore AC is incommensurable with each of the magnitudes AB , BC .

Next, let AC be incommensurable with one of the magnitudes AB , BC .

First, let it be incommensurable with AB ;

I say that AB , BC are also incommensurable.

For, if they are commensurable, some magnitude will measure them.

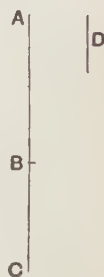
Let it measure them, and let it be D .

Since then D measures AB , BC ,

therefore it will also measure the whole AC .

But it measures AB also;

therefore D measures CA , AB ,



Therefore CA , AB are commensurable ;
but they were also, by hypothesis, incommensurable :
which is impossible.

Therefore no magnitude will measure AB , BC ;
therefore AB , BC are incommensurable. [x. Def. 1]

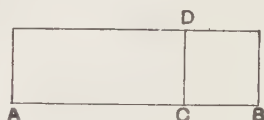
Therefore etc.

LEMMA.

If to any straight line there be applied a parallelogram deficient by a square figure, the applied parallelogram is equal to the rectangle contained by the segments of the straight line resulting from the application.

For let there be applied to the straight line AB the parallelogram AD deficient by the square figure DB ;

I say that AD is equal to the rectangle contained by AC , CB .



This is indeed at once manifest ;

for, since DB is a square,

DC is equal to CB ;

and AD is the rectangle AC , CD , that is, the rectangle AC , CB .

Therefore etc.

If a be the given straight line, and x the side of the square by which the applied rectangle is to be deficient, the rectangle is equal to $ax - x^2$, which is of course equal to $x(a - x)$. The rectangle may be written xy , where $x + y = a$. Given the area $x(a - x)$, or xy (where $x + y = a$), two different applications will give rectangles equal to this area, the sides of the defect being x or $a - x$ (x or y) respectively ; but the second mode of expression shows that the rectangles do not differ in form but only in position.

PROPOSITION 17.

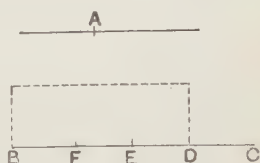
If there be two unequal straight lines, and to the greater there be applied a parallelogram equal to the fourth part of the square on the less and deficient by a square figure, and if it divide it into parts which are commensurable in length, then the square on the greater will be greater than the square on the less by the square on a straight line commensurable with the greater.

And, if the square on the greater be greater than the square on the less by the square on a straight line commensurable with

10 the greater, and if there be applied to the greater a parallelogram equal to the fourth part of the square on the less and deficient by a square figure, it will divide it into parts which are commensurable in length.

Let A , BC be two unequal straight lines, of which BC is
15 the greater,

and let there be applied to BC a parallelogram equal to the fourth part of the square on the less, A , that is, equal to the square on the half of A , and deficient
20 by a square figure. Let this be the rectangle BD , DC , [cf. Lemma]



and let BD be commensurable in length with DC ;

I say that the square on BC is greater than the square on A by the square on a straight line commensurable with BC .

25 For let BC be bisected at the point E ,
and let EF be made equal to DE .

Therefore the remainder DC is equal to BF .

And, since the straight line BC has been cut into equal parts at E , and into unequal parts at D ,

30 therefore the rectangle contained by BD , DC , together with the square on ED , is equal to the square on EC ; [II. 5]

And the same is true of their quadruples;

therefore four times the rectangle BD , DC , together with four times the square on DE , is equal to four times the square
35 on EC .

But the square on A is equal to four times the rectangle BD , DC ;

and the square on DF is equal to four times the square on DE , for DF is double of DE .

40 And the square on BC is equal to four times the square on EC , for again BC is double of CE .

Therefore the squares on A , DF are equal to the square on BC ,

so that the square on BC is greater than the square on A by
45 the square on DF .

It is to be proved that BC is also commensurable with DF .

Since BD is commensurable in length with DC ,

therefore BC is also commensurable in length with CD . [X. 15]

But CD is commensurable in length with CD , BF , for
 50 CD is equal to BF . [x. 6]

Therefore BC is also commensurable in length with BF ,
 CD , [x. 12]

so that BC is also commensurable in length with the remainder
 FD ; [x. 15]

55 therefore the square on BC is greater than the square on A
 by the square on a straight line commensurable with BC .

Next, let the square on BC be greater than the square on
 A by the square on a straight line commensurable with BC ,
 let a parallelogram be applied to BC equal to the fourth part
 60 of the square on A and deficient by a square figure, and let
 it be the rectangle BD , DC .

It is to be proved that BD is commensurable in length
 with DC .

With the same construction, we can prove similarly that
 65 the square on BC is greater than the square on A by the
 square on FD .

But the square on BC is greater than the square on A
 by the square on a straight line commensurable with BC .

Therefore BC is commensurable in length with FD ,
 70 so that BC is also commensurable in length with the remainder,
 the sum of BF , DC . [x. 15]

But the sum of BF , DC is commensurable with DC , [x. 6]
 so that BC is also commensurable in length with CD ; [x. 12]
 and therefore, *separando*, BD is commensurable in length
 75 with DC . [x. 15]

Therefore etc.

45. After saying literally that "the square on BC is greater than the square on A by the square on DF ," Euclid adds the equivalent expression with $\deltaύναται$ in its technical sense, $\eta\ \beta\Gamma\ \acute{\alpha}\rho\alpha\ \tau\eta\varsigma\ A\ \mu\epsilon\iota\zeta\omega\nu\ \deltaύναται\ \tau\eta\ \Delta Z$. As this is untranslatable in English except by a paraphrase in practically the same words as have preceded, I have not attempted to reproduce it.

This proposition gives the condition that the roots of the equation in x ,

$$ax - x^2 = \beta \left(= \frac{b^2}{4}, \text{ say} \right),$$

are commensurable with a , or that x is expressible in terms of a and integral numbers, i.e. is of the form $\frac{m}{n} a$. No better proof can be found for the fact that Euclid and the Greeks used their solutions of quadratic equations for *numerical* problems. On no other assumption could an elaborate discussion of the conditions of incommensurability of the roots with given lengths or

with a given number of units of length be explained. In a purely *geometrical* solution the distinction between commensurable and incommensurable roots has no point, because each can equally easily be represented by straight lines. On the other hand, on the assumption that the *numerical* solution of quadratic equations was an important part of the system of the Greek geometers, the distinction between the cases where the roots are commensurable and incommensurable respectively with a given length or unit becomes of great importance. Since the Greeks had no means of *expressing* what we call an irrational number, the case of an equation with incommensurable roots could *only* be represented by them geometrically; and the geometrical representations had to serve instead of what we can express by formulae involving surds.

Euclid proves in this proposition and the next that, x being determined from the equation

$$x(a-x) = \frac{b^2}{4} \dots\dots\dots (1),$$

$x, (a-x)$ are commensurable in length when $\sqrt{a^2-b^2}$, a are so, and incommensurable in length when $\sqrt{a^2-b^2}$, a are incommensurable; and conversely.

Observe the similarity of his proof to our algebraical method of solving the equation. a being represented in the figure by BC , and x by CD ,

$$EF = ED = \frac{a}{2} - x$$

and

$$x(a-x) + \left(\frac{a}{2} - x\right)^2 = \frac{a^2}{4}, \quad \text{by Eucl. II. 5.}$$

If we multiply throughout by 4,

$$4x(a-x) + 4\left(\frac{a}{2} - x\right)^2 = a^2,$$

whence, by (1),

$$b^2 + (a-2x)^2 = a^2,$$

or

$$a^2 - b^2 = (a-2x)^2,$$

and

$$\sqrt{a^2-b^2} = a-2x.$$

We have to prove in this proposition

(1) that, if $x, (a-x)$ are commensurable in length, so are $a, \sqrt{a^2-b^2}$,

(2) that, if $a, \sqrt{a^2-b^2}$ are commensurable in length, so are $x, (a-x)$.

(1) To prove that $a, a-2x$ are commensurable in length Euclid employs several successive steps, thus.

Since $(a-x) \cap x$, $a \cap x$. [X. 15]

But $x \cap 2x$. [X. 6]

Therefore $a \cap 2x$ [X. 12]

$$\cap (a-2x). \quad [\text{X. 15}]$$

That is, $a \cap \sqrt{a^2-b^2}$.

(2) Since $a \cap \sqrt{a^2-b^2}$, $a \cap a-2x$,

whence $a \cap 2x$. [X. 15]

But $2x \cap x$; [X. 6]

therefore $a \cap x$, [X. 12]

and hence $(a-x) \cap x$. [X. 15]

It is often more convenient to use the symmetrical form of equation in this and similar cases, viz.

$$\left. \begin{aligned} xy &= \frac{b^2}{4} \\ x + y &= a \end{aligned} \right\}.$$

The result with this mode of expression is that

(1) if $x \cap y$, then $a \cap \sqrt{a^2 - b^2}$; and

(2) if $a \cap \sqrt{a^2 - b^2}$, then $x \cap y$.

The truth of the proposition is even easier to see in this case, since $(x - y)^2 = (a^2 - b^2)$.

PROPOSITION 18.

If there be two unequal straight lines, and to the greater there be applied a parallelogram equal to the fourth part of the square on the less and deficient by a square figure, and if it divide it into parts which are incommensurable, the square on the greater will be greater than the square on the less by the square on a straight line incommensurable with the greater.

And, if the square on the greater be greater than the square on the less by the square on a straight line incommensurable with the greater, and if there be applied to the greater a parallelogram equal to the fourth part of the square on the less and deficient by a square figure, it divides it into parts which are incommensurable.

Let A, BC be two unequal straight lines, of which BC is the greater,

and to BC let there be applied a parallelogram equal to the fourth part of the square on the less, A , and deficient by a square figure. Let this be the rectangle BD, DC ,

[cf. Lemma before x. 17]

and let BD be incommensurable in length with DC ;

I say that the square on BC is greater than the square on A by the square on a straight line incommensurable with BC .



For, with the same construction as before, we can prove similarly that the square on BC is greater than the square on A by the square on FD .

It is to be proved that BC is incommensurable in length with DF .

Since BD is incommensurable in length with DC ,
therefore BC is also incommensurable in length with CD . [x. 16]

But DC is commensurable with the sum of BF , DC ; [x. 6]
therefore BC is also incommensurable with the sum of BF ,
 DC ; [x. 13]
so that BC is also incommensurable in length with the remainder
 FD . [x. 16]

And the square on BC is greater than the square on A
by the square on FD ;
therefore the square on BC is greater than the square on A
by the square on a straight line incommensurable with BC .

Again, let the square on BC be greater than the square on
 A by the square on a straight line incommensurable with BC ,
and let there be applied to BC a parallelogram equal to the
fourth part of the square on A and deficient by a square figure.
Let this be the rectangle BD , DC .

It is to be proved that BD is incommensurable in length
with DC .

For, with the same construction, we can prove similarly
that the square on BC is greater than the square on A by
the square on FD .

But the square on BC is greater than the square on A by
the square on a straight line incommensurable with BC ;
therefore BC is incommensurable in length with FD ,
so that BC is also incommensurable with the remainder, the
sum of BF , DC . [x. 16]

But the sum of BF , DC is commensurable in length with
 DC ; [x. 6]
therefore BC is also incommensurable in length with DC ,
[x. 13]
so that, *separando*, BD is also incommensurable in length with
 DC . [x. 16]

Therefore etc.

With the same notation as before, we have to prove in this proposition that

- (1) if $(a - x)$, x are incommensurable in length, so are a , $\sqrt{a^2 - b^2}$, and
(2) if a , $\sqrt{a^2 - b^2}$ are incommensurable in length, so are $(a - x)$, x .

Or, with the equations

$$\left. \begin{aligned} xy &= \frac{b^2}{4}, \\ x + y &= a \end{aligned} \right\},$$

- (1) if $x \cup y$, then $a \cup \sqrt{a^2 - b^2}$, and
 (2) if $a \cup \sqrt{a^2 - b^2}$, then $x \cup y$.

The steps are exactly the same as shown under (1) and (2) of the last note, with $\cup$ instead of $\cap$, except only in the lines " $x \cap 2x$ " and " $2x \cap x$ " which are unaltered, while, in the references, x. 13, 16 take the place of x. 12, 15 respectively.

[LEMMA.

Since it has been proved that straight lines commensurable in length are always commensurable in square also, while those commensurable in square are not always commensurable in length also, but can of course be either commensurable or incommensurable in length, it is manifest that, if any straight line be commensurable in length with a given rational straight line, it is called rational and commensurable with the other not only in length but in square also, since straight lines commensurable in length are always commensurable in square also.

But, if any straight line be commensurable in square with a given rational straight line, then, if it is also commensurable in length with it, it is called in this case also rational and commensurable with it both in length and in square; but, if again any straight line, being commensurable in square with a given rational straight line, be incommensurable in length with it, it is called in this case also rational but commensurable in square only.]

PROPOSITION 19.

The rectangle contained by rational straight lines commensurable in length is rational.

For let the rectangle AC be contained by the rational straight lines AB , BC commensurable in length;

I say that AC is rational.

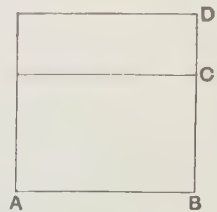
For on AB let the square AD be described;

therefore AD is rational. [x. Def. 4]

And, since AB is commensurable in length with BC ,

while AB is equal to BD ,

therefore BD is commensurable in length with BC .



And, as BD is to BC , so is DA to AC . [vi. 1]

Therefore DA is commensurable with AC . [x. 11]

But DA is rational;

therefore AC is also rational. [x. Def. 4]

Therefore etc.

There is a difficulty in the text of the enunciation of this proposition. The Greek runs τὸ ὑπὸ ῥητῶν μήκει συμμέτρων κατὰ τινα τῶν προειρημένων τρόπων εὐθειῶν περιεχόμενον ὀρθογώνιον ῥητὸν ἐστίν, where the rectangle is said to be contained by "rational straight lines commensurable in length in any of the aforesaid ways." Now straight lines can only be commensurable in length in one way, the degrees of commensurability being commensurability in length and commensurability in square only. But a straight line may be rational in two ways in relation to a given rational straight line, since it may be either commensurable in length, or commensurable in square only, with the latter. Hence Billingsley takes κατὰ τινα τῶν προειρημένων τρόπων with ῥητῶν, translating "straight lines commensurable in length and rational in any of the aforesaid ways," and this agrees with the expression in the next proposition "a straight line once more rational in any of the aforesaid ways"; but the order of words in the Greek seems to be fatal to this way of translating the passage.

The best solution of the difficulty seems to be to reject the words "in any of the aforesaid ways" altogether. They have reference to the Lemma which immediately precedes and which is itself open to the gravest suspicion. It is very prolix, and cannot be called necessary; it appears moreover in connexion with an addition clearly spurious and therefore relegated by Heiberg to the Appendix. The addition does not even pretend to be Euclid's, for it begins with the words "for he calls rational straight lines those...." Hence we should no doubt relegate the Lemma itself to the Appendix. August does so and leaves out the suspected words in the enunciation, as I have done.

Exactly the same arguments apply to the Lemma added (without the heading "Lemma") to x. 23 and the same words "in any of the aforesaid ways" used with "medial straight lines commensurable in length" in the enunciation of x. 24. The said Lemma must stand or fall with that now in question, since it refers to it in terms: "And in the same way as was explained in the case of rationals...."

Hence I have bracketed the Lemma added to x. 23 and left out the objectionable words in the enunciation of x. 24.

If ρ be one of the given rational straight lines (rational of course in the sense of x. Def. 3), the other can be denoted by $k\rho$, where k is, as usual, of the form m/n (where m, n are integers). Thus the rectangle is $k\rho^2$, which is obviously rational since it is commensurable with ρ^2 . [x. Def. 4.]

A rational rectangle may have any of the forms ab , ka^2 , kA or A , where a, b are commensurable with the unit of length, and A with the unit of area.

Since Euclid is not able to use $k\rho$ as a symbol for a straight line commensurable in length with ρ , he has to put his proof in a form corresponding to

$$\rho^2 : k\rho^2 = \rho : k\rho,$$

whence, $\rho, k\rho$ being commensurable, $\rho^2, k\rho^2$ are so also.

[x. 11]

PROPOSITION 20.

If a rational area be applied to a rational straight line, it produces as breadth a straight line rational and commensurable in length with the straight line to which it is applied.

For let the rational area AC be applied to AB , a straight line once more rational in any of the aforesaid ways, producing BC as breadth;

I say that BC is rational and commensurable in length with BA .

For on AB let the square AD be described;

therefore AD is rational. [x. Def. 4]

But AC is also rational;

therefore DA is commensurable with AC .

And, as DA is to AC , so is DB to BC .

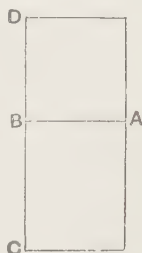
Therefore DB is also commensurable with BC ; [vi. 1] [x. 11]
and DB is equal to BA ;

therefore AB is also commensurable with BC .

But AB is rational;

therefore BC is also rational and commensurable in length with AB .

Therefore etc.



The converse of the last. If ρ is a rational straight line, any rational area is of the form $k\rho^2$. If this be "applied" to ρ , the breadth is $k\rho$ commensurable in length with ρ and therefore rational. We should reach the same result if we applied the area to *another* rational straight line σ . The breadth is then $\frac{k\rho^2}{\sigma} = \frac{k\rho^2}{\sigma^2} \cdot \sigma = \frac{m}{n} k \cdot \sigma$ or $k'\sigma$, say.

PROPOSITION 21.

The rectangle contained by rational straight lines commensurable in square only is irrational, and the side of the square equal to it is irrational. Let the latter be called medial.

For let the rectangle AC be contained by the rational straight lines AB , BC commensurable in square only;

I say that AC is irrational, and the side of the square equal to it is irrational;

and let the latter be called **medial**.

For on AB let the square AD be described; therefore AD is rational. [x. Def. 4]

And, since AB is incommensurable in length with BC ,

for by hypothesis they are commensurable in square only,

while AB is equal to BD ,

therefore DB is also incommensurable in length with BC .

And, as DB is to BC , so is AD to AC ; [vi. 1]

therefore DA is incommensurable with AC . [x. 11]

But DA is rational;

therefore AC is irrational,

so that the side of the square equal to AC is also irrational.

[x. Def. 4]

And let the latter be called **medial**.

Q. E. D.

A *medial* straight line, now defined for the first time, is so called because it is a mean proportional between two rational straight lines commensurable in square only. Such straight lines can be denoted by ρ , $\rho\sqrt{k}$. A medial straight line is therefore of the form $\sqrt{\rho^2\sqrt{k}}$ or $k^{\frac{1}{4}}\rho$. Euclid's proof that this is irrational is equivalent to the following. Take ρ , $\rho\sqrt{k}$ commensurable in square only, so that they are incommensurable in length.

Now $\rho : \rho\sqrt{k} = \rho^2 : \rho^2\sqrt{k}$,

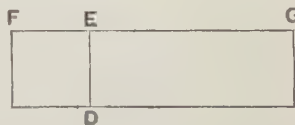
whence [x. 11] $\rho^2\sqrt{k}$ is incommensurable with ρ^2 and therefore irrational [x. Def. 4], so that $\sqrt{\rho^2\sqrt{k}}$ is also irrational [*ibid.*].

A medial straight line may evidently take either of the forms $\sqrt{a\sqrt{B}}$ or $\sqrt[4]{AB}$, where of course B is not of the form k^2A .

LEMMA.

If there be two straight lines, then, as the first is to the second, so is the square on the first to the rectangle contained by the two straight lines.

Let FE , EG be two straight lines.



I say that, as FE is to EG , so is the square on FE to the rectangle FE , EG .

For on FE let the square DF be described,
and let GD be completed.

Since then, as FE is to EG , so is FD to DG , [VI. 1]
and FD is the square on FE ,
and DG the rectangle DE, EG , that is, the rectangle FE, EG ,
therefore, as FE is to EG , so is the square on FE to the
rectangle FE, EG .

Similarly also, as the rectangle GF, EF is to the square
on EF , that is, as GD is to FD , so is GE to EF .

Q. E. D.

If a, b be two straight lines,

$$a : b = a^2 : ab.$$

PROPOSITION 22.

*The square on a medial straight line, if applied to a
rational straight line, produces as breadth a straight line
rational and incommensurable in length with that to which it
is applied.*

Let A be medial and CB rational,
and let a rectangular area BD equal to the square on A be
applied to BC , producing CD as
breadth;

I say that CD is rational and incom-
mensurable in length with CB .

For, since A is medial, the square
on it is equal to a rectangular area
contained by rational straight lines
commensurable in square only.

[X. 21]

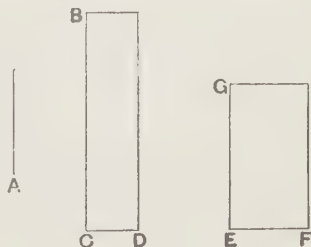
Let the square on it be equal to GF .

But the square on it is also equal to BD ;
therefore BD is equal to GF .

But it is also equiangular with it;

and in equal and equiangular parallelograms the sides about
the equal angles are reciprocally proportional; [VI. 14]
therefore, proportionally, as BC is to EG , so is EF to CD .

Therefore also, as the square on BC is to the square on
 EG , so is the square on EF to the square on CD . [VI. 22]



But the square on CB is commensurable with the square on EG , for each of these straight lines is rational;

therefore the square on EF is also commensurable with the square on CD . [X. 11]

But the square on EF is rational;

therefore the square on CD is also rational; [X. Def. 4]

therefore CD is rational.

And, since EF is incommensurable in length with EG , for they are commensurable in square only,

and, as EF is to EG , so is the square on EF to the rectangle FE, EG , [Lemma]

therefore the square on EF is incommensurable with the rectangle FE, EG . [X. 11]

But the square on CD is commensurable with the square on EF , for the straight lines are rational in square;

and the rectangle DC, CB is commensurable with the rectangle FE, EG , for they are equal to the square on A ;

therefore the square on CD is also incommensurable with the rectangle DC, CB . [X. 13]

But, as the square on CD is to the rectangle DC, CB , so is DC to CB ; [Lemma]

therefore DC is incommensurable in length with CB . [X. 11]

Therefore CD is rational and incommensurable in length with CB .

Q. E. D.

Our algebraical notation makes the result of this proposition almost self-evident. We have seen that the square of a medial straight line is of the form $\sqrt{k} \cdot \rho^2$. If we "apply" this area to another rational straight line σ , the breadth is $\frac{\sqrt{k} \cdot \rho^2}{\sigma}$.

This is equal to $\frac{\sqrt{k} \cdot \rho^2}{\sigma^2} \cdot \sigma = \sqrt{k} \cdot \frac{m}{n} \sigma$, where m, n are integers. The latter straight line, which we may express, if we please, in the form $\sqrt{k'} \cdot \sigma$, is clearly commensurable with σ in square only, and therefore rational but incommensurable in length with σ .

Euclid's proof, necessarily longer, is in two parts.

Suppose that the rectangle $\sqrt{k} \cdot \rho^2 = \sigma \cdot x$.

Then (1)

$$\sigma : \rho = \sqrt{k} : \rho : x, \quad [\text{VI. 14}]$$

whence

$$\sigma^2 : \rho^2 = k\rho^2 : x^2. \quad [\text{VI. 22}]$$

But $\sigma^2 \wedge \rho^2$, and therefore $k\rho^2 \wedge x^2$.

[X. 11]

And $k\rho^2$ is rational,
therefore x^2 , and therefore x , is rational.

[X. Def. 4]

(2) Since $\sqrt{k} \cdot \rho \sim \rho$, $\sqrt{k} \cdot \rho \cup \rho$.

But [Lemma] $\sqrt{k} \cdot \rho : \rho = k\rho^2 : \sqrt{k} \cdot \rho^2$,
whence $k\rho^2 \cup \sqrt{k} \cdot \rho^2$.

[X. 11]

But $\sqrt{k} \cdot \rho^2 = \sigma x$, and $k\rho^2 \cup x^2$ (from above);

therefore $x^2 \cup \sigma x$;

[X. 13]

and, since $x^2 : \sigma x = x : \sigma$,

[Lemma]

$x \cup \sigma$.

PROPOSITION 23.

A straight line commensurable with a medial straight line is medial.

Let A be medial, and let B be commensurable with A ;
I say that B is also medial.

For let a rational straight line CD
be set out,

and to CD let the rectangular area CE
equal to the square on A be applied,
producing ED as breadth;

therefore ED is rational and incommen-
surable in length with CD . [X. 22]

And let the rectangular area CF
equal to the square on B be applied to
 CD , producing DF as breadth.

Since then A is commensurable with B ,
the square on A is also commensurable with the square on B .

But EC is equal to the square on A ,

and CF is equal to the square on B ;

therefore EC is commensurable with CF .

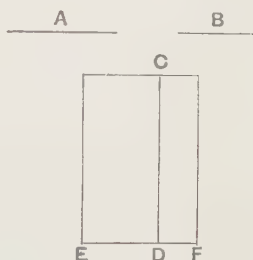
And, as EC is to CF , so is ED to DF ; [VI. 1]

therefore ED is commensurable in length with DF . [X. 11]

But ED is rational and incommensurable in length with
 DC ;

therefore DF is also rational [X. Def. 3] and incommensurable
in length with DC . [X. 13]

Therefore CD , DF are rational and commensurable in
square only.



But the straight line the square on which is equal to the rectangle contained by rational straight lines commensurable in square only is medial ; [x. 21]

therefore the side of the square equal to the rectangle CD , DF is medial.

And B is the side of the square equal to the rectangle CD , DF ;

therefore B is medial.

PORISM. From this it is manifest that an area commensurable with a medial area is medial.

[And in the same way as was explained in the case of rationals [Lemma following x. 18] it follows, as regards medials, that a straight line commensurable in length with a medial straight line is called **medial** and commensurable with it not only in length but in square also, since, in general, straight lines commensurable in length are always commensurable in square also.

But, if any straight line be commensurable in square with a medial straight line, then, if it is also commensurable in length with it, the straight lines are called, in this case too, medial and commensurable in length and in square, but, if in square only, they are called medial straight lines commensurable in square only.]

As explained in the bracketed passage following this proposition, a straight line commensurable with a medial straight line *in square only*, as well as a straight line commensurable with it in length, is medial.

Algebraical notation shows this easily.

If $k^{\frac{1}{2}}\rho$ be the given straight line, $\lambda k^{\frac{1}{2}}\rho$ is a straight line commensurable in length with it and $\sqrt{\lambda} \cdot k^{\frac{1}{2}}\rho$ a straight line commensurable with it in square only.

But $\lambda\rho$ and $\sqrt{\lambda} \cdot \rho$ are both rational [x. Def. 3] and therefore can be expressed by ρ' , and we thus arrive at $k^{\frac{1}{2}}\rho'$, which is clearly medial.

Euclid's proof amounts to the following.

Apply both the areas $\sqrt{k} \cdot \rho^2$ and $\lambda^2\sqrt{k} \cdot \rho^2$ (or $\lambda\sqrt{k} \cdot \rho^2$) to a rational straight line σ .

The breadths $\sqrt{k} \cdot \frac{\rho^2}{\sigma}$ and $\lambda^2\sqrt{k} \cdot \frac{\rho^2}{\sigma}$ (or $\lambda\sqrt{k} \cdot \frac{\rho^2}{\sigma}$) are in the ratio of the areas $\sqrt{k} \cdot \rho^2$ and $\lambda^2\sqrt{k} \cdot \rho^2$ (or $\lambda\sqrt{k} \cdot \rho^2$) themselves and are therefore commensurable.

Now [x. 22] $\sqrt{k} \cdot \frac{\rho^2}{\sigma}$ is rational but incommensurable with σ .

Therefore $\lambda^2\sqrt{k} \cdot \frac{\rho^2}{\sigma}$ (or $\lambda\sqrt{k} \cdot \frac{\rho^2}{\sigma}$) is so also ;

whence the area $\lambda^2 \sqrt[4]{k} \cdot \rho^2$ (or $\lambda \sqrt[4]{k} \cdot \rho^2$) is contained by two rational straight lines commensurable in square only, so that $\lambda k^{\frac{1}{4}} \rho$ (or $\sqrt[4]{\lambda} \cdot k^{\frac{1}{4}} \rho$) is a medial straight line.

It is in the Porism that we have the first mention of a medial *area*. It is the area which is equal to the square on a medial straight line, an area, therefore, of the form $k^{\frac{1}{2}} \rho^2$, which is, as a matter of fact, arrived at, though not named, before the medial *straight line* itself (x. 21).

The Porism states that $\lambda k^{\frac{1}{2}} \rho^2$ is a medial area, which is indeed obvious.

PROPOSITION 24.

The rectangle contained by medial straight lines commensurable in length is medial.

For let the rectangle AC be contained by the medial straight lines AB , BC which are commensurable in length ;

I say that AC is medial.

For on AB let the square AD be described ; therefore AD is medial.

And, since AB is commensurable in length with BC , while AB is equal to BD , therefore DB is also commensurable in length with BC ;

so that DA is also commensurable with AC .



[VI. I, x. 11]

But DA is medial ;

therefore AC is also medial.

[x. 23, Por.]

Q. E. D.

There is the same difficulty in the text of this enunciation as in that of x. 19. The Greek says "medial straight lines commensurable in length in any of the aforesaid ways" ; but straight lines can only be *commensurable in length* in one way, though they can be medial in two ways, as explained in the addition to the preceding proposition, i.e. they can be either commensurable in length or commensurable in square only with a *given* medial straight line. For the same reason as that explained in the note on x. 19 I have omitted "in any of the aforesaid ways" in the enunciation and bracketed the addition to x. 23 to which it refers.

$k^{\frac{1}{4}} \rho$ and $\lambda k^{\frac{1}{4}} \rho$ are medial straight lines commensurable in length. The rectangle contained by them is $\lambda k^{\frac{1}{2}} \rho^2$, which may be written $k^{\frac{1}{2}} \rho'^2$ and is therefore clearly medial.

Euclid's proof proceeds thus. Let x , λx be the two medial straight lines commensurable in length.

Therefore

$$x^2 : x \cdot \lambda x = x : \lambda x.$$

But $x \cdot \lambda x$, so that $x^2 \cdot x \cdot \lambda x$.

[x. 11]

Now x^2 is medial [x. 21];

therefore $x \cdot \lambda x$ is also medial.

[x. 23, Por.]

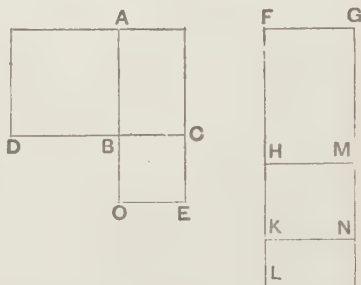
We may of course write two medial straight lines commensurable in length in the forms $mk^{\frac{1}{2}}p$, $nk^{\frac{1}{2}}p$; and these may either be $m\sqrt{a}\sqrt{b}$, $n\sqrt{a}\sqrt{b}$, or $m\sqrt{AB}$, $n\sqrt{AB}$.

PROPOSITION 25.

The rectangle contained by medial straight lines commensurable in square only is either rational or medial.

For let the rectangle AC be contained by the medial straight lines AB , BC which are commensurable in square only; I say that AC is either rational or medial.

For on AB , BC let the squares AD , BE be described; therefore each of the squares AD , BE is medial.



Let a rational straight line FG be set out,

to FG let there be applied the rectangular parallelogram GH equal to AD , producing FH as breadth,

to HM let there be applied the rectangular parallelogram MK equal to AC , producing HK as breadth,

and further to KN let there be similarly applied NL equal to BE , producing KL as breadth;

therefore FH , HK , KL are in a straight line.

Since then each of the squares AD , BE is medial,

and AD is equal to GH , and BE to NL ,

therefore each of the rectangles GH , NL is also medial.

And they are applied to the rational straight line FG ;

therefore each of the straight lines FH , KL is rational and incommensurable in length with FG .

[x. 22]

And, since AD is commensurable with BE ,

therefore GH is also commensurable with NL .

And, as GH is to NL , so is FH to KL ;

[vi. 1]

therefore FH is commensurable in length with KL .

[x. 11]

Therefore FH , KL are rational straight lines commensurable in length;

therefore the rectangle FH , KL is rational. [x. 19]

And, since DB is equal to BA , and OB to BC , therefore, as DB is to BC , so is AB to BO .

But, as DB is to BC , so is DA to AC , [vi. 1]

and, as AB is to BO , so is AC to CO ; [id.]

therefore, as DA is to AC , so is AC to CO .

But AD is equal to GH , AC to MK and CO to NL ;

therefore, as GH is to MK , so is MK to NL ;

therefore also, as FH is to HK , so is HK to KL ; [vi. 1, v. 11]

therefore the rectangle FH , KL is equal to the square on HK . [vi. 17]

But the rectangle FH , KL is rational;

therefore the square on HK is also rational.

Therefore HK is rational.

And, if it is commensurable in length with FG ,

HN is rational; [x. 19]

but, if it is incommensurable in length with FG ,

KH , HM are rational straight lines commensurable in square only, and therefore HN is medial. [x. 21]

Therefore HN is either rational or medial.

But HN is equal to AC ;

therefore AC is either rational or medial.

Therefore etc.

Two medial straight lines commensurable in square only are of the form $k^{\frac{1}{2}}\rho$, $\sqrt{\lambda} \cdot k^{\frac{1}{2}}\rho$

The rectangle contained by them is $\sqrt{\lambda} \cdot k^{\frac{1}{2}}\rho^2$. Now this is in general medial; but, if $\sqrt{\lambda} = k' \sqrt{k}$, the rectangle is $kk'\rho^2$, which is rational.

Euclid's argument is as follows. Let us, for convenience, put x for $k^{\frac{1}{2}}\rho$, so that the medial straight lines are x , $\sqrt{\lambda} \cdot x$.

Form the areas x^2 , $x \cdot \sqrt{\lambda} \cdot x$, λx^2 ,

and let these be respectively equal to σu , σv , σw , where σ is a rational straight line.

Since x^2 , λx^2 are medial areas,

so are σu , σw ,

whence u , w are respectively rational and $\sim \sigma$.

But $x^2 \sim \lambda x^2$,
 so that $\sigma u \sim \sigma v$,
 or $u \sim w$ (1).
 Therefore, u, w being both rational, uw is rational(2).
 Now $x^2 : \sqrt{\lambda} \cdot x^2 = \sqrt{\lambda} \cdot x^2 : \lambda x^2$
 or $\sigma u : \sigma v = \sigma v : \sigma w$,
 so that $u : v = v : w$,
 and $uw = v^2$.
 Hence, by (2), v^2 , and therefore v , is rational(3).
 Now (a) if $v \sim \sigma$, σv or $\sqrt{\lambda} \cdot x^2$ is *rational*;
 (β) if $v \sim \sigma$, so that $v \sim \sigma$, σv or $\sqrt{\lambda} \cdot x^2$ is *medial*.

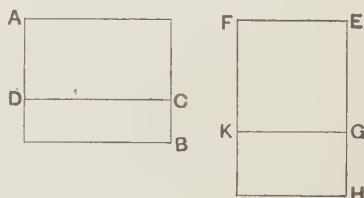
PROPOSITION 26.

A medial area does not exceed a medial area by a rational area.

For, if possible, let the medial area AB exceed the medial area AC by the rational area DB ,

and let a rational straight line EF be set out;

to EF let there be applied the rectangular parallelogram FH equal to AB , producing EH as breadth,



and let the rectangle FG equal to AC be subtracted;
 therefore the remainder BD is equal to the remainder KH .

But DB is rational;
 therefore KH is also rational.

Since, then, each of the rectangles AB, AC is medial,
 and AB is equal to FH , and AC to FG ,
 therefore each of the rectangles FH, FG is also medial.

And they are applied to the rational straight line EF ;
 therefore each of the straight lines HE, EG is rational and
 incommensurable in length with EF . [x. 22]

And, since [DB is rational and is equal to KH ,
 therefore] KH is [also] rational;
 and it is applied to the rational straight line EF ;

therefore GH is rational and commensurable in length with EF . [x. 20]

But EG is also rational, and is incommensurable in length with EF ;

therefore EG is incommensurable in length with GH . [x. 13]

And, as EG is to GH , so is the square on EG to the rectangle EG, GH ;

therefore the square on EG is incommensurable with the rectangle EG, GH . [x. 11]

But the squares on EG, GH are commensurable with the square on EG , for both are rational;

and twice the rectangle EG, GH is commensurable with the rectangle EG, GH , for it is double of it; [x. 6]

therefore the squares on EG, GH are incommensurable with twice the rectangle EG, GH ; [x. 13]

therefore also the sum of the squares on EG, GH and twice the rectangle EG, GH , that is, the square on EH [11. 4], is incommensurable with the squares on EG, GH . [x. 16]

But the squares on EG, GH are rational;

therefore the square on EH is irrational. [x. Def. 4]

Therefore EH is irrational.

But it is also rational:

which is impossible.

Therefore etc.

Q. E. D.

“Apply” the two given medial areas to one and the same rational straight line ρ . They can then be written in the form $\rho : k^{\frac{1}{2}}\rho, \rho : \lambda^{\frac{1}{2}}\rho$.

The difference is then $(\sqrt{k} - \sqrt{\lambda})\rho^2$; and the proposition asserts that this cannot be rational, i.e. $(\sqrt{k} - \sqrt{\lambda})$ cannot be equal to k' . Cf. the proposition corresponding to this in algebraical text-books.

To make Euclid's proof clear we will put x for $k^{\frac{1}{2}}\rho$ and y for $\lambda^{\frac{1}{2}}\rho$.

Suppose $\rho(x - y) = \rho z$,

and, if possible, let ρz be rational, so that z must be rational and $\wedge \rho \dots (1)$.

Since $\rho x, \rho y$ are medial,

x and y are respectively rational and $\vee \rho \dots \dots \dots (2)$.

From (1) and (2), $y \vee z$.

Now $y : z = y^2 : yz$,

so that $y^2 \vee yz$.

But $y^2 + z^2 \sim y^2$,
 and $2yz \sim yz$.
 Therefore $y^2 + z^2 \sim 2yz$,
 whence $(y + z)^2 \sim (y^2 + z^2)$,
 or $x^2 \sim (y^2 + z^2)$.

And $(y^2 + z^2)$ is rational ;
 therefore x^2 , and consequently x , is irrational.

But, by (2), x is rational :
 which is impossible.

Therefore pz is *not* rational.

PROPOSITION 27.

To find medial straight lines commensurable in square only which contain a rational rectangle.

Let two rational straight lines A , B commensurable in square only be set out ;

let C be taken a mean proportional between A , B ,

[vi. 13]

and let it be contrived that,

as A is to B , so is C to D .

[vi. 12]

Then, since A , B are rational and commensurable in square only,

the rectangle A , B , that is, the square on C

[vi. 17], is medial.

[x. 21]

Therefore C is medial.

[x. 21]

And since, as A is to B , so is C to D ,

and A , B are commensurable in square only,

therefore C , D are also commensurable in square only. [x. 11]

And C is medial ;

therefore D is also medial.

[x. 23, addition]

Therefore C , D are medial and commensurable in square only.

I say that they also contain a rational rectangle.

For since, as A is to B , so is C to D ,

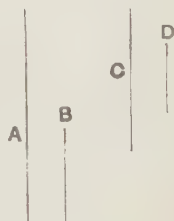
therefore, alternately, as A is to C , so is B to D .

[v. 16]

But, as A is to C , so is C to B ;

therefore also, as C is to B , so is B to D ;

therefore the rectangle C , D is equal to the square on B .



But the square on B is rational ;
therefore the rectangle C, D is also rational.

Therefore medial straight lines commensurable in square only have been found which contain a rational rectangle.

Q. E. D.

Euclid takes two rational straight lines commensurable in square only, say $\rho, k^{\frac{1}{2}}\rho$.

Find the mean proportional, i.e. $k^{\frac{1}{4}}\rho$.

Take x such that $\rho : k^{\frac{1}{2}}\rho = k^{\frac{1}{4}}\rho : x$ (1).

This gives $x = k^{\frac{3}{4}}\rho$,

and the lines required are $k^{\frac{1}{4}}\rho, k^{\frac{3}{4}}\rho$.

For (a) $k^{\frac{1}{4}}\rho$ is medial.

And (β), by (1), since $\rho \sim k^{\frac{1}{2}}\rho$,

$$k^{\frac{1}{4}}\rho \sim k^{\frac{3}{4}}\rho,$$

whence [addition to x. 23], since $k^{\frac{1}{4}}\rho$ is medial,

$k^{\frac{3}{4}}\rho$ is also medial.

The medial straight lines thus found may take either of the forms

$$(1) \sqrt{a}\sqrt{B}, \sqrt{\frac{B\sqrt{B}}{a}} \text{ or } (2) \sqrt[4]{AB}, \sqrt{B\frac{\sqrt{B}}{\sqrt{A}}}.$$

PROPOSITION 28.

To find medial straight lines commensurable in square only which contain a medial rectangle.

Let the rational straight lines A, B, C commensurable in square only be set out ;

let D be taken a mean proportional between A, B , [vi. 13]

and let it be contrived that,

as B is to C , so is D to E . [vi. 12]



Since A, B are rational straight lines commensurable in square only,

therefore the rectangle A, B , that is, the square on D [vi. 17], is medial. [x. 21]

Therefore D is medial.

[x. 21]

And since B, C are commensurable in square only,
and, as B is to C , so is D to E ,

therefore D, E are also commensurable in square only. [x. 11]

But D is medial;

therefore E is also medial.

[x. 23, addition]

Therefore D, E are medial straight lines commensurable in square only.

I say next that they also contain a medial rectangle.

For since, as B is to C , so is D to E ,

therefore, alternately, as B is to D , so is C to E . [v. 16]

But, as B is to D , so is D to A ;

therefore also, as D is to A , so is C to E ;

therefore the rectangle A, C is equal to the rectangle D, E .

[vi. 16]

But the rectangle A, C is medial;

[x. 21]

therefore the rectangle D, E is also medial.

Therefore medial straight lines commensurable in square only have been found which contain a medial rectangle.

Q. E. D.

Euclid takes three straight lines commensurable in square only, i.e. of the form $\rho, k^{\frac{1}{2}}\rho, \lambda^{\frac{1}{2}}\rho$, and proceeds as follows.

Take the mean proportional to $\rho, k^{\frac{1}{2}}\rho$, i.e. $k^{\frac{1}{4}}\rho$.

Then take x such that

$$k^{\frac{1}{2}}\rho : \lambda^{\frac{1}{2}}\rho = k^{\frac{1}{4}}\rho : x \dots\dots\dots (1),$$

so that $x = \lambda^{\frac{1}{2}}\rho / k^{\frac{1}{4}}$.

$k^{\frac{1}{4}}\rho, \lambda^{\frac{1}{2}}\rho / k^{\frac{1}{4}}$ are the required medial straight lines.

For $k^{\frac{1}{4}}\rho$ is medial.

Now, by (1), since $k^{\frac{1}{2}}\rho \sim \lambda^{\frac{1}{2}}\rho$,

$$k^{\frac{1}{4}}\rho \sim x,$$

whence x is also medial [x. 23, addition], while $\sim k^{\frac{1}{4}}\rho$.

Next, by (1),

$$\lambda^{\frac{1}{2}}\rho : x = k^{\frac{1}{2}}\rho : k^{\frac{1}{4}}\rho.$$

$$= k^{\frac{1}{4}}\rho : \rho,$$

whence

$$x \cdot k^{\frac{1}{4}}\rho = \lambda^{\frac{1}{2}}\rho^2, \text{ which is medial.}$$

The straight lines $k^{\frac{1}{4}}\rho, \lambda^{\frac{1}{2}}\rho / k^{\frac{1}{4}}$ of course take different forms according as the original straight lines are of the forms (1) $a, \sqrt{B}, \sqrt{C}$, (2) $\sqrt{A}, \sqrt{B}, \sqrt{C}$, (3) $\sqrt{A}, b, \sqrt{C}$, and (4) $\sqrt{A}, \sqrt{B}, c$.

E.g. in case (1) they are $\sqrt{a\sqrt{B}}$, $\sqrt{\frac{aC}{\sqrt{B}}}$,

in case (2) they are $\sqrt[4]{AB}$, $\sqrt{\frac{C\sqrt{A}}{\sqrt{B}}}$,

and so on.

LEMMA 1.

To find two square numbers such that their sum is also square.

Let two numbers AB , BC be set out, and let them be either both even or both odd.

Then since, whether an even number is subtracted from an even number, or an odd number from an odd number, the remainder is even,



[ix. 24, 26]

therefore the remainder AC is even.

Let AC be bisected at D .

Let AB , BC also be either similar plane numbers, or square numbers, which are themselves also similar plane numbers.

Now the product of AB , BC together with the square on CD is equal to the square on BD .

[ii. 6]

And the product of AB , BC is square, inasmuch as it was proved that, if two similar plane numbers by multiplying one another make some number, the product is square. [ix. 1]

Therefore two square numbers, the product of AB , BC , and the square on CD , have been found which, when added together, make the square on BD .

And it is manifest that two square numbers, the square on BD and the square on CD , have again been found such that their difference, the product of AB , BC , is a square, whenever AB , BC are similar plane numbers.

But when they are not similar plane numbers, two square numbers, the square on BD and the square on DC , have been found such that their difference, the product of AB , BC , is not square.

Q. E. D.

Euclid's method of forming right-angled triangles in integral numbers, already alluded to in the note on i. 47, is as follows.

Take two similar plane numbers, e.g. mnp^2 , mng^2 , which are either both even or both odd, so that their difference is divisible by 2.

Now the product of the two numbers, or $m^2n^2p^2q^2$, is square, [IX. 1]
and, by II. 6,

$$mnp^2 \cdot mnq^2 + \left(\frac{mnp^2 - mnq^2}{2} \right)^2 = \left(\frac{mnp^2 + mnq^2}{2} \right)^2,$$

so that the numbers $mnpq$, $\frac{1}{2}(mnp^2 - mnq^2)$ satisfy the condition that the sum of their squares is also a square number.

It is also clear that $\frac{1}{2}(mnp^2 + mnq^2)$, $mnpq$ are numbers such that the difference of their squares is also square.

LEMMA 2.

To find two square numbers such that their sum is not square.

For let the product of AB , BC , as we said, be square,
and CA even,
and let CA be bisected by D .



It is then manifest that the square product of AB , BC together with the square on CD is equal to the square on BD .

[See Lemma 1]

Let the unit DE be subtracted ;
therefore the product of AB , BC together with the square on CE is less than the square on BD .

I say then that the square product of AB , BC together with the square on CE will not be square.

For, if it is square, it is either equal to the square on BE , or less than the square on BE , but cannot any more be greater, lest the unit be divided.

First, if possible, let the product of AB , BC together with the square on CE be equal to the square on BE ,
and let GA be double of the unit DE .

Since then the whole AC is double of the whole CD ,
and in them AG is double of DE ,
therefore the remainder GC is also double of the remainder EC ;
therefore GC is bisected by E .

Therefore the product of GB , BC together with the square on CE is equal to the square on BE . [II. 6]

But the product of AB , BC together with the square on CE is also, by hypothesis, equal to the square on BE ;

therefore the product of GB , BC together with the square on CE is equal to the product of AB , BC together with the square on CE .

And, if the common square on CE be subtracted, it follows that AB is equal to GB : which is absurd.

Therefore the product of AB , BC together with the square on CE is not equal to the square on BE .

I say next that neither is it less than the square on BE .

For, if possible, let it be equal to the square on BF , and let HA be double of DF .

Now it will again follow that HC is double of CF ; so that CH has also been bisected at F , and for this reason the product of HB , BC together with the square on FC is equal to the square on BF . [II. 6]

But, by hypothesis, the product of AB , BC together with the square on CE is also equal to the square on BF .

Thus the product of HB , BC together with the square on CF will also be equal to the product of AB , BC together with the square on CE :

which is absurd.

Therefore the product of AB , BC together with the square on CE is not less than the square on BE .

And it was proved that neither is it equal to the square on BE .

Therefore the product of AB , BC together with the square on CE is not square.

Q. E. D.

We can, of course, write the identity in the note on Lemma 1 above (p. 64) in the simpler form

$$mp^2 \cdot mq^2 + \left(\frac{mp^2 - mq^2}{2} \right)^2 = \left(\frac{mp^2 + mq^2}{2} \right)^2,$$

where, as before, mp^2 , mq^2 are both odd or both even.

Now, says Euclid,

$$mp^2 \cdot mq^2 + \left(\frac{mp^2 - mq^2}{2} - 1 \right)^2 \text{ is not a square number.}$$

This is proved by *reductio ad absurdum*.

The number is clearly less than $mp^2 \cdot mq^2 + \left(\frac{mp^2 - mq^2}{2}\right)^2$, i.e. less than $\left(\frac{mp^2 + mq^2}{2}\right)^2$.

If then the number is square, its side must be greater than, equal to, or less than $\left(\frac{mp^2 + mq^2}{2} - 1\right)$, the number next less than $\frac{mp^2 + mq^2}{2}$.

But (1) the side cannot be $> \left(\frac{mp^2 + mq^2}{2} - 1\right)$ without being equal to $\frac{mp^2 + mq^2}{2}$, since they are consecutive numbers.

$$(2) \quad (mp^2 - 2)mq^2 + \left(\frac{mp^2 - mq^2}{2} - 1\right)^2 = \left(\frac{mp^2 + mq^2}{2} - 1\right)^2. \quad [11. 6]$$

If then $mp^2 \cdot mq^2 + \left(\frac{mp^2 - mq^2}{2} - 1\right)^2$ is also equal to $\left(\frac{mp^2 + mq^2}{2} - 1\right)^2$,

we must have $(mp^2 - 2)mq^2 = mp^2 \cdot mq^2$,

or $mp^2 - 2 = mp^2$;

which is impossible.

$$(3) \quad \text{If } mp^2 \cdot mq^2 + \left(\frac{mp^2 - mq^2}{2} - 1\right)^2 < \left(\frac{mp^2 + mq^2}{2} - 1\right)^2,$$

suppose it equal to $\left(\frac{mp^2 + mq^2}{2} - r\right)^2$.

$$\text{But } [11. 6] \quad (mp^2 - 2r)mq^2 + \left(\frac{mp^2 - mq^2}{2} - r\right)^2 = \left(\frac{mp^2 + mq^2}{2} - r\right)^2.$$

Therefore

$$(mp^2 - 2r)mq^2 + \left(\frac{mp^2 - mq^2}{2} - r\right)^2 = mp^2 \cdot mq^2 + \left(\frac{mp^2 - mq^2}{2} - 1\right)^2;$$

which is impossible.

Hence all three hypotheses are false, and the sum of the squares $mp^2 \cdot mq^2$ and $\left(\frac{mp^2 - mq^2}{2} - 1\right)^2$ is not square.

PROPOSITION 29.

To find two rational straight lines commensurable in square only and such that the square on the greater is greater than the square on the less by the square on a straight line commensurable in length with the greater.

For let there be set out any rational straight line AB , and two square numbers CD, DE such that their difference CE is not square;

[Lemma 1]

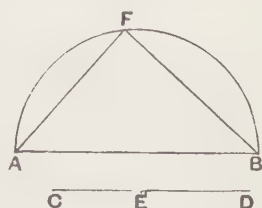
let there be described on AB the semicircle AFB ,

and let it be contrived that,

as DC is to CE , so is the square on BA to the square on AF . [x. 6, Por.]

Let FB be joined.

Since, as the square on BA is to the square on AF , so is DC to CE , therefore the square on BA has to the square on AF the ratio which the number DC has to the number CE ; therefore the square on BA is commensurable with the square on AF .



[x. 6]

But the square on AB is rational; therefore the square on AF is also rational; therefore AF is also rational.

[x. Def. 4]

[*id.*]

And, since DC has not to CE the ratio which a square number has to a square number, neither has the square on BA to the square on AF the ratio which a square number has to a square number; therefore AB is incommensurable in length with AF . [x. 9]

Therefore BA , AF are rational straight lines commensurable in square only.

And since, as DC is to CE , so is the square on BA to the square on AF , therefore, *convertendo*, as CD is to DE , so is the square on AB to the square on BF . [v. 19, Por., III. 31, I. 47]

But CD has to DE the ratio which a square number has to a square number; therefore also the square on AB has to the square on BF the ratio which a square number has to a square number; therefore AB is commensurable in length with BF . [x. 9]

And the square on AB is equal to the squares on AF , FB ; therefore the square on AB is greater than the square on AF by the square on BF commensurable with AB .

Therefore there have been found two rational straight lines BA , AF commensurable in square only and such that the square on the greater AB is greater than the square on the less AF by the square on BF commensurable in length with AB .

Q. E. D.

Take a rational straight line ρ and two numbers m^2, n^2 such that $(m^2 - n^2)$ is not a square.

Take a straight line x such that

$$m^2 : m^2 - n^2 :: \rho^2 : x^2, \dots \dots \dots (1),$$

whence

$$x^2 = \frac{m^2 - n^2}{m^2} \rho^2,$$

and

$$x = \rho \sqrt{1 - k^2}, \quad \text{where } k = \frac{n}{m}.$$

Then $\rho, \rho\sqrt{1 - k^2}$ are the straight lines required.

It follows from (1) that $x^2 \sim \rho^2$,

and x is rational, but $x \sim \rho$.

By (1), *convertendo*, $m^2 : n^2 = \rho^2 : \rho^2 - x^2$,

so that $\sqrt{\rho^2 - x^2} \sim \rho$, and in fact $= k\rho$.

According as ρ is of the form a or $\sqrt{A}$, the straight lines are (1) $a, \sqrt{a^2 - b^2}$ or (2) $\sqrt{A}, \sqrt{A - k^2 A}$.

PROPOSITION 30.

To find two rational straight lines commensurable in square only and such that the square on the greater is greater than the square on the less by the square on a straight line incommensurable in length with the greater.

Let there be set out a rational straight line AB , and two square numbers CE, ED such that their sum CD is not square ;

[Lemma 2]

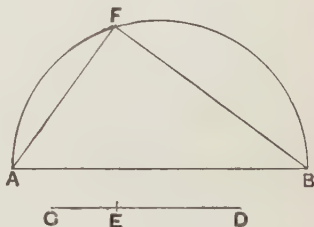
let there be described on AB the semicircle AFB ,

let it be contrived that,

as DC is to CE , so is the square on BA to the square on AF ,

[x. 6, Por.]

and let FB be joined.



Then, in a similar manner to the preceding, we can prove that BA, AF are rational straight lines commensurable in square only.

And since, as DC is to CE , so is the square on BA to the square on AF ,

therefore, *convertendo*, as CD is to DE , so is the square on AB to the square on BF .

[v. 19, Por., III. 31, I. 47]

But CD has not to DE the ratio which a square number has to a square number ;

therefore neither has the square on AB to the square on BF the ratio which a square number has to a square number ; therefore AB is incommensurable in length with BF . [x. 9]

And the square on AB is greater than the square on AF by the square on FB incommensurable with AB .

Therefore AB , AF are rational straight lines commensurable in square only, and the square on AB is greater than the square on AF by the square on FB incommensurable in length with AB .

Q. E. D.

In this case we take m^2 , n^2 such that $m^2 + n^2$ is not square.

Find x such that $m^2 + n^2 : m^2 = \rho^2 : x^2$,

whence

$$x^2 = \frac{m^2}{m^2 + n^2} \rho^2,$$

or

$$x = \frac{\rho}{\sqrt{1 + k^2}}, \quad \text{where } k = \frac{n}{m}.$$

Then ρ , $\frac{\rho}{\sqrt{1 + k^2}}$ satisfy the condition.

The proof is after the manner of the proof of the preceding proposition and need not be repeated.

According as ρ is of the form a or $\sqrt{A}$, the straight lines take the form (1) a , $\sqrt{a^2 - \frac{k^2 a^2}{1 + k^2}}$, that is, a , $\sqrt{a^2 - B}$, or (2) $\sqrt{A}$, $\sqrt{A - B}$ and $\sqrt{A}$, $\sqrt{A - b^2}$.

PROPOSITION 31.

To find two medial straight lines commensurable in square only, containing a rational rectangle, and such that the square on the greater is greater than the square on the less by the square on a straight line commensurable in length with the greater.

Let there be set out two rational straight lines A , B commensurable in square only and such that the square on A , being the greater, is greater than the square on B the less by the square on a straight line commensurable in length with A .

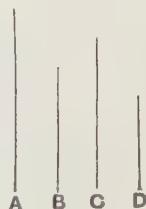
[x. 29]

And let the square on C be equal to the rectangle A , B .

Now the rectangle A , B is medial ; [x. 21]

therefore the square on C is also medial ;

therefore C is also medial.



[x. 21]

Let the rectangle C, D be equal to the square on B .

Now the square on B is rational ;

therefore the rectangle C, D is also rational.

And since, as A is to B , so is the rectangle A, B to the square on B ,

while the square on C is equal to the rectangle A, B ,

and the rectangle C, D is equal to the square on B ,

therefore, as A is to B , so is the square on C to the rectangle C, D .

But, as the square on C is to the rectangle C, D , so is C to D ;

therefore also, as A is to B , so is C to D .

But A is commensurable with B in square only ;

therefore C is also commensurable with D in square only. [x. 11]

And C is medial ;

therefore D is also medial.

[x. 23, addition]

And since, as A is to B , so is C to D ,

and the square on A is greater than the square on B by the square on a straight line commensurable with A ,

therefore also the square on C is greater than the square on D by the square on a straight line commensurable with C .

[x. 14]

Therefore two medial straight lines C, D , commensurable in square only and containing a rational rectangle, have been found, and the square on C is greater than the square on D by the square on a straight line commensurable in length with C .

Similarly also it can be proved that the square on C exceeds the square on D by the square on a straight line incommensurable with C , when the square on A is greater than the square on B by the square on a straight line incommensurable with A .

[x. 30]

I. Take the rational straight lines commensurable in square only found in x. 29, i.e. $\rho, \rho\sqrt{1-k^2}$.

Take the mean proportional $\rho(1-k^2)^{\frac{1}{4}}$ and x such that

$$\rho(1-k^2)^{\frac{1}{4}} : \rho\sqrt{1-k^2} = \rho\sqrt{1-k^2} : x.$$

Then $\rho(1-k^2)^{\frac{1}{4}}, x$, or $\rho(1-k^2)^{\frac{1}{4}}, \rho(1-k^2)^{\frac{3}{4}}$ are straight lines satisfying the given conditions.

For (α) $\rho^2 \sqrt{1-k^2}$ is a medial area, and therefore $\rho(1-k^2)^{\frac{1}{4}}$ is a medial straight line(1); and $x \cdot \rho(1-k^2)^{\frac{1}{4}} = \rho^2(1-k^2)$ and is therefore a rational area.

(β) $\rho, \rho(1-k^2)^{\frac{1}{4}}, \rho \sqrt{1-k^2}, x$ are straight lines in continued proportion, by construction.

Therefore $\rho : \rho \sqrt{1-k^2} = \rho(1-k^2)^{\frac{1}{4}} : x$ (2).

(This Euclid has to prove in a somewhat roundabout way by means of the lemma after x. 21 to the effect that $a : b = ab : b^2$.)

From (2) it follows [x. 11] that $x \sim \rho(1-k^2)^{\frac{1}{4}}$; whence, since $\rho(1-k^2)^{\frac{1}{4}}$ is medial, x or $\rho(1-k^2)^{\frac{3}{4}}$ is medial also.

(γ) From (2), since $\rho, \rho \sqrt{1-k^2}$ satisfy the remaining condition of the problem, $\rho(1-k^2)^{\frac{1}{4}}, \rho(1-k^2)^{\frac{3}{4}}$ do so also [x. 14].

According as ρ is of the form a or $\sqrt{A}$, the straight lines take the forms

$$(1) \quad \sqrt{a\sqrt{a^2-b^2}}, \quad \frac{a^2-b^2}{\sqrt{a\sqrt{a^2-b^2}}},$$

$$\text{or} \quad (2) \quad \sqrt[4]{A(A-k^2A)}, \quad \frac{A-k^2A}{\sqrt[4]{A(A-k^2A)}}.$$

II. To find medial straight lines commensurable in square only containing a rational rectangle, and such that the square on one exceeds the square on the other by the square on a straight line *incommensurable* with the former, we simply begin with the rational straight lines having the corresponding property [x. 30], viz. $\rho, \frac{\rho}{\sqrt{1+k^2}}$, and we arrive at the straight lines

$$\frac{\rho}{(1+k^2)^{\frac{1}{4}}}, \quad \frac{\rho}{(1+k^2)^{\frac{3}{4}}}.$$

According as ρ is of the form a or $\sqrt{A}$, these (if we use the same transformation as at the end of the note on x. 30) may take any of the forms

$$(1) \quad \sqrt{a\sqrt{a^2-B}}, \quad \frac{a^2-B}{\sqrt{a\sqrt{a^2-B}}},$$

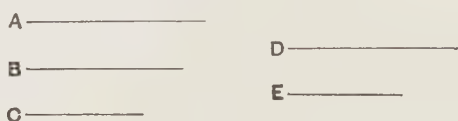
$$\text{or} \quad (2) \quad \sqrt[4]{A(A-B)}, \quad \frac{A-B}{\sqrt[4]{A(A-B)}},$$

$$\text{or} \quad \sqrt[4]{A(A-b^2)}, \quad \frac{A-b^2}{\sqrt[4]{A(A-b^2)}}.$$

PROPOSITION 32.

To find two medial straight lines commensurable in square only, containing a medial rectangle, and such that the square on the greater is greater than the square on the less by the square on a straight line commensurable with the greater.

Let there be set out three rational straight lines A, B, C commensurable in square only, and such that the square on A is greater than the square on C by the square on a straight line commensurable with A , [x. 29]
and let the square on D be equal to the rectangle A, B .



Therefore the square on D is medial;
therefore D is also medial. [x. 21]

Let the rectangle D, E be equal to the rectangle B, C .

Then since, as the rectangle A, B is to the rectangle B, C , so is A to C ,

while the square on D is equal to the rectangle A, B ,
and the rectangle D, E is equal to the rectangle B, C ,
therefore, as A is to C , so is the square on D to the rectangle D, E .

But, as the square on D is to the rectangle D, E , so is D to E ;

therefore also, as A is to C , so is D to E .

But A is commensurable with C in square only;

therefore D is also commensurable with E in square only. [x. 11]

But D is medial;

therefore E is also medial. [x. 23, addition]

And, since, as A is to C , so is D to E ,

while the square on A is greater than the square on C by the square on a straight line commensurable with A ,
therefore also the square on D will be greater than the square on E by the square on a straight line commensurable with D . [x. 14]

I say next that the rectangle D, E is also medial.

For, since the rectangle B, C is equal to the rectangle D, E , while the rectangle B, C is medial, [x. 21]
therefore the rectangle D, E is also medial.

Therefore two medial straight lines D, E , commensurable in square only, and containing a medial rectangle, have been found such that the square on the greater is greater than the

square on the less by the square on a straight line commensurable with the greater.

Similarly again it can be proved that the square on D is greater than the square on E by the square on a straight line incommensurable with D , when the square on A is greater than the square on C by the square on a straight line incommensurable with A . [x. 30]

I. Euclid takes three straight lines of the form ρ , $\rho\sqrt{\lambda}$, $\rho\sqrt{1-k^2}$, takes the mean proportional $\rho\lambda^{\frac{1}{2}}$ between the first two(1), and then finds x such that

$$\rho\lambda^{\frac{1}{2}} : \rho\lambda^{\frac{1}{2}} = \rho\sqrt{1-k^2} : x \text{(2),}$$

whence $x = \rho\lambda^{\frac{1}{2}}\sqrt{1-k^2}$,

and the straight lines $\rho\lambda^{\frac{1}{2}}$, $\rho\lambda^{\frac{1}{2}}\sqrt{1-k^2}$ satisfy the given conditions.

Now (a) $\rho\lambda^{\frac{1}{2}}$ is medial.

(β) We have, from (1) and (2),

$$\rho : \rho\sqrt{1-k^2} = \rho\lambda^{\frac{1}{2}} : x \text{(3),}$$

whence $x \sim \rho\lambda^{\frac{1}{2}}$; and x is therefore medial and $\sim \rho\lambda^{\frac{1}{2}}$.

$$(\gamma) \quad x \cdot \rho\lambda^{\frac{1}{2}} = \rho\sqrt{\lambda} \cdot \rho\sqrt{1-k^2}.$$

But the latter is medial;

[x. 21]

therefore $x \cdot \rho\lambda^{\frac{1}{2}}$, or $\rho\lambda^{\frac{1}{2}} \cdot \rho\lambda^{\frac{1}{2}}\sqrt{1-k^2}$, is medial.

Lastly (δ) ρ , $\rho\sqrt{1-k^2}$ have the remaining property in the enunciation;

therefore $\rho\lambda^{\frac{1}{2}}$, $\rho\lambda^{\frac{1}{2}}\sqrt{1-k^2}$ have it also.

[x. 14]

(Euclid has not the assistance of symbols to prove the proportion (3) above. He therefore uses the lemmas $ab : bc = a : c$ and $d^2 : de = d : e$ to deduce from the relations

$$\left. \begin{aligned} ab &= d^2 \\ d : b &= c : e \\ a : c &= d : e. \end{aligned} \right\}$$

and
that

The straight lines $\rho\lambda^{\frac{1}{2}}$, $\rho\lambda^{\frac{1}{2}}\sqrt{1-k^2}$ may take any of the following forms according as the straight lines first taken are

$$(1) \quad a, \sqrt{B}, \sqrt{a^2-c^2}, \quad (2) \quad \sqrt{A}, \sqrt{B}, \sqrt{A-k^2A}, \quad (3) \quad \sqrt{A}, b, \sqrt{A-k^2A}.$$

$$(1) \quad \sqrt{a}\sqrt{B}, \quad \frac{\sqrt{B(a^2-c^2)}}{\sqrt{a}\sqrt{B}};$$

$$(2) \quad \sqrt[4]{AB}, \quad \frac{\sqrt{B(A-k^2A)}}{\sqrt[4]{AB}};$$

$$(3) \quad \sqrt{b}\sqrt{A}, \quad \frac{b\sqrt{A-k^2A}}{\sqrt{b}\sqrt{A}}.$$

II. If the other conditions are 'the same, but the square on the first medial straight line is to exceed the square on the second by the square on a straight line *incommensurable* with the first, we begin with the three straight lines ρ , $\rho\sqrt{\lambda}$, $\frac{\rho}{\sqrt{1+k^2}}$, and the medial straight lines are

$$\rho\lambda^{\frac{1}{2}}, \quad \frac{\rho\lambda^{\frac{1}{2}}}{\sqrt{1+k^2}}.$$

The possible forms are even more various in this case owing to the more various forms that the original lines may take, e.g.

$$(1) \quad a, \quad \sqrt{B}, \quad \sqrt{a^2 - C};$$

$$(2) \quad \sqrt{A}, \quad b, \quad \sqrt{A - c^2};$$

$$(3) \quad \sqrt{A}, \quad b, \quad \sqrt{A - C};$$

$$(4) \quad \sqrt{A}, \quad \sqrt{B}, \quad \sqrt{A - c^2};$$

$$(5) \quad \sqrt{A}, \quad \sqrt{B}, \quad \sqrt{A - C};$$

the medial straight lines corresponding to these being

$$(1) \quad \sqrt{a}\sqrt{B}, \quad \frac{\sqrt{B}(a^2 - C)}{\sqrt{a}\sqrt{B}};$$

$$(2) \quad \sqrt{b}\sqrt{A}, \quad \frac{b\sqrt{A - c^2}}{\sqrt{b}\sqrt{A}};$$

$$(3) \quad \sqrt{b}\sqrt{A}, \quad \frac{b\sqrt{A - C}}{\sqrt{b}\sqrt{A}};$$

$$(4) \quad \sqrt[4]{AB}, \quad \frac{\sqrt{B}(A - c^2)}{\sqrt[4]{AB}};$$

$$(5) \quad \sqrt[4]{AB}, \quad \frac{\sqrt{B}(A - C)}{\sqrt[4]{AB}}.$$

LEMMA.

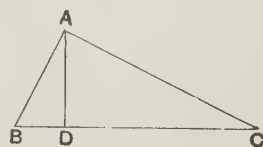
Let ABC be a right-angled triangle having the angle A right, and let the perpendicular AD be drawn;

I say that the rectangle CB , BD is equal to the square on BA ,

the rectangle BC , CD equal to the square on CA ,

the rectangle BD , DC equal to the square on AD ,

and, further, the rectangle BC , AD equal to the rectangle BA , AC .



And first that the rectangle CB, BD is equal to the square on BA .

For, since in a right-angled triangle AD has been drawn from the right angle perpendicular to the base, therefore the triangles ABD, ADC are similar both to the whole ABC and to one another. [vi. 8]

And since the triangle ABC is similar to the triangle ABD , therefore, as CB is to BA , so is BA to BD ; [vi. 4] therefore the rectangle CB, BD is equal to the square on AB . [vi. 17]

For the same reason the rectangle BC, CD is also equal to the square on AC .

And since, if in a right-angled triangle a perpendicular be drawn from the right angle to the base, the perpendicular so drawn is a mean proportional between the segments of the base, [vi. 8, Por.]

therefore, as BD is to DA , so is AD to DC ; therefore the rectangle BD, DC is equal to the square on AD . [vi. 17]

I say that the rectangle BC, AD is also equal to the rectangle BA, AC .

For since, as we said, ABC is similar to ABD , therefore, as BC is to CA , so is BA to AD . [vi. 4]

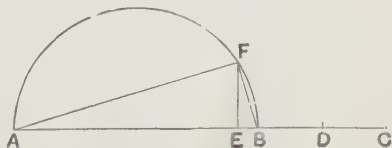
Therefore the rectangle BC, AD is equal to the rectangle BA, AC . [vi. 16]

Q. E. D.

PROPOSITION 33.

To find two straight lines incommensurable in square which make the sum of the squares on them rational but the rectangle contained by them medial.

Let there be set out two rational straight lines AB, BC commensurable in square only and such that the square on the greater AB is greater than the square on the less BC by the square on a straight line incommensurable with AB , [x. 30]



let BC be bisected at D ,

let there be applied to AB a parallelogram equal to the square on either of the straight lines BD , DC and deficient by a square figure, and let it be the rectangle AE , EB ; [vi. 28]

let the semicircle AFB be described on AB ,

let EF be drawn at right angles to AB ,

and let AF , FB be joined.

Then, since AB , BC are unequal straight lines, and the square on AB is greater than the square on BC by the square on a straight line incommensurable with AB , while there has been applied to AB a parallelogram equal to the fourth part of the square on BC , that is, to the square on half of it, and deficient by a square figure, making the rectangle AE , EB , therefore AE is incommensurable with EB . [x. 18]

And, as AE is to EB , so is the rectangle BA , AE to the rectangle AB , BE ,

while the rectangle BA , AE is equal to the square on AF , and the rectangle AB , BE to the square on BF ;

therefore the square on AF is incommensurable with the square on BF ;

therefore AF , FB are incommensurable in square.

And, since AB is rational,

therefore the square on AB is also rational;

so that the sum of the squares on AF , FB is also rational.

[i. 47]

And since, again, the rectangle AE , EB is equal to the square on EF ,

and, by hypothesis, the rectangle AE , EB is also equal to the square on BD ,

therefore FE is equal to BD ;

therefore BC is double of FE ,

so that the rectangle AB , BC is also commensurable with the rectangle AB , EF .

But the rectangle AB , BC is medial;

[x. 21]

therefore the rectangle AB , EF is also medial.

[x. 23, Por.]

But the rectangle AB, EF is equal to the rectangle AF, FB ;
[Lemma]

therefore the rectangle AF, FB is also medial.

But it was also proved that the sum of the squares on these straight lines is rational.

Therefore two straight lines AF, FB incommensurable in square have been found which make the sum of the squares on them rational, but the rectangle contained by them medial.

Q. E. D.

Euclid takes the straight lines found in x. 30, viz. $\rho, \frac{\rho}{\sqrt{1+k^2}}$.

He then solves *geometrically* the equations

$$\left. \begin{aligned} x+y &= \rho \\ xy &= \frac{\rho^2}{4(1+k^2)} \end{aligned} \right\} \dots\dots\dots(1).$$

If x, y are the values found, he takes u, v such that

$$\left. \begin{aligned} u^2 &= \rho x \\ v^2 &= \rho y \end{aligned} \right\} \dots\dots\dots(2),$$

and u, v are straight lines satisfying the conditions of the problem.

Solving algebraically, we get (if $x > y$)

$$x = \frac{\rho}{2} \left(1 + \frac{k}{\sqrt{1+k^2}} \right), \quad y = \frac{\rho}{2} \left(1 - \frac{k}{\sqrt{1+k^2}} \right),$$

whence

$$\left. \begin{aligned} u &= \frac{\rho}{\sqrt{2}} \sqrt{1 + \frac{k}{\sqrt{1+k^2}}} \\ v &= \frac{\rho}{\sqrt{2}} \sqrt{1 - \frac{k}{\sqrt{1+k^2}}} \end{aligned} \right\} \dots\dots\dots(3).$$

Euclid's proof that these straight lines fulfil the requirements is as follows.

(a) The constants in the equations (1) satisfy the conditions of x. 18;
therefore

$$x \sim y.$$

But

$$x : y = u^2 : v^2.$$

Therefore

$$u^2 \sim v^2,$$

and u, v are thus *incommensurable in square*.

(β) $u^2 + v^2 = \rho^2$, which is *rational*.

(γ) By (1),

$$\sqrt{xy} = \frac{\rho}{2\sqrt{1+k^2}}.$$

By (2),

$$\begin{aligned} uv &= \rho \cdot \sqrt{xy} \\ &= \frac{\rho^2}{2\sqrt{1+k^2}}. \end{aligned}$$

But $\frac{\rho^2}{\sqrt{1+k^2}}$ is a medial area,

therefore uv is medial.

Since ρ , $\frac{\rho}{\sqrt{1+k^2}}$ may have any of the three forms

$$(1) a, \sqrt{a^2 - B}, \quad (2) \sqrt{A}, \sqrt{A - B}, \quad (3) \sqrt{A}, \sqrt{A - b^2},$$

u, v may have any of the forms

$$(1) \sqrt{\frac{a^2 + a\sqrt{B}}{2}}, \quad \sqrt{\frac{a^2 - a\sqrt{B}}{2}};$$

$$(2) \sqrt{\frac{A + \sqrt{AB}}{2}}, \quad \sqrt{\frac{A - \sqrt{AB}}{2}};$$

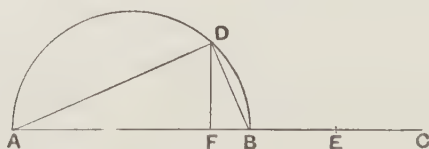
$$(3) \sqrt{\frac{A + b\sqrt{A}}{2}}, \quad \sqrt{\frac{A - b\sqrt{A}}{2}}.$$

PROPOSITION 34.

To find two straight lines incommensurable in square which make the sum of the squares on them medial but the rectangle contained by them rational.

Let there be set out two medial straight lines AB, BC , commensurable in square only, such that the rectangle which they contain is rational, and the square on AB is greater than the square on BC by the square on a straight line incommensurable with AB ;

[x. 31, *ad fin.*]



let the semicircle ADB be described on AB ,

let BC be bisected at E ,

let there be applied to AB a parallelogram equal to the square on BE and deficient by a square figure, namely the rectangle AF, FB ;

[vi. 28]

therefore AF is incommensurable in length with FB .

[x. 18]

Let FD be drawn from F at right angles to AB , and let AD, DB be joined.

Since AF is incommensurable in length with FB ,
therefore the rectangle BA, AF is also incommensurable with
the rectangle AB, BF . [x. 11]

But the rectangle BA, AF is equal to the square on AD ,
and the rectangle AB, BF to the square on DB ;
therefore the square on AD is also incommensurable with the
square on DB .

And, since the square on AB is medial,
therefore the sum of the squares on AD, DB is also medial.
[III. 31, I. 47]

And, since BC is double of DF ,
therefore the rectangle AB, BC is also double of the rectangle
 AB, FD .

But the rectangle AB, BC is rational;
therefore the rectangle AB, FD is also rational. [x. 6]

But the rectangle AB, FD is equal to the rectangle $AD,$
 DB ; [Lemma]

so that the rectangle AD, DB is also rational.

Therefore two straight lines AD, DB incommensurable
in square have been found which make the sum of the squares
on them medial, but the rectangle contained by them rational.

Q. E. D.

In this case we take [x. 31, 2nd part] the medial straight lines

$$\frac{\rho}{(1+k^2)^{\frac{1}{4}}}, \quad \frac{\rho}{(1+k^2)^{\frac{3}{4}}}.$$

Solve the equations

$$\left. \begin{aligned} x+y &= \frac{\rho}{(1+k^2)^{\frac{1}{4}}} \\ xy &= \frac{\rho^2}{4(1+k^2)^{\frac{3}{2}}} \end{aligned} \right\} \dots\dots\dots (1).$$

Take u, v such that, if x, y be the result of the solution,

$$\left. \begin{aligned} u^2 &= \frac{\rho}{(1+k^2)^{\frac{1}{4}}} \cdot x \\ v^2 &= \frac{\rho}{(1+k^2)^{\frac{1}{4}}} \cdot y \end{aligned} \right\} \dots\dots\dots (2),$$

and u, v are straight lines satisfying the given conditions.

Euclid's proof is similar to the preceding.

(a) From (1) it follows [x. 18] that

$$x \sim y,$$

whence

$$u^2 \sim v^2,$$

and u, v are thus incommensurable in square.

(β) $u^2 + v^2 = \frac{\rho^2}{\sqrt{1+k^2}}$, which is a medial area.

(γ) $uv = \frac{\rho}{(1+k^2)^{\frac{3}{4}}} \cdot \sqrt{xy}$
 $= \frac{1}{2} \cdot \frac{\rho^2}{1+k^2}$, which is a rational area.

Therefore uv is rational.

To find the actual form of u , v , we have, by solving the equations (1) (if $x > y$),

$$x = \frac{\rho}{2(1+k^2)^{\frac{3}{4}}} (\sqrt{1+k^2} + k),$$

$$y = \frac{\rho}{2(1+k^2)^{\frac{3}{4}}} (\sqrt{1+k^2} - k);$$

and hence

$$u = \frac{\rho}{\sqrt{2(1+k^2)}} \sqrt{\sqrt{1+k^2} + k},$$

$$v = \frac{\rho}{\sqrt{2(1+k^2)}} \sqrt{\sqrt{1+k^2} - k}.$$

Bearing in mind the forms which $\frac{\rho}{(1+k^2)^{\frac{3}{4}}}$, $\frac{\rho}{(1+k^2)^{\frac{3}{4}}}$ may take (see note on x. 31), we shall find that u , v may have any of the forms

$$\begin{aligned} (1) \quad & \sqrt{\frac{(a + \sqrt{B})\sqrt{a^2 - B}}{2}}, & \sqrt{\frac{(a - \sqrt{B})\sqrt{a^2 - B}}{2}}; \\ (2) \quad & \sqrt{\frac{(\sqrt{A} + \sqrt{B})\sqrt{A - B}}{2}}, & \sqrt{\frac{(\sqrt{A} - \sqrt{B})\sqrt{A - B}}{2}}; \\ (3) \quad & \sqrt{\frac{(\sqrt{A} + b)\sqrt{A - b^2}}{2}}, & \sqrt{\frac{(\sqrt{A} - b)\sqrt{A - b^2}}{2}}. \end{aligned}$$

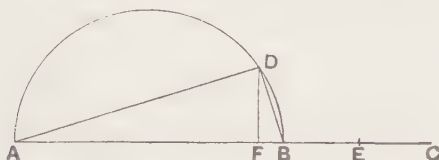
PROPOSITION 35.

To find two straight lines incommensurable in square which make the sum of the squares on them medial and the rectangle contained by them medial and moreover incommensurable with the sum of the squares on them.

Let there be set out two medial straight lines AB , BC commensurable in square only, containing a medial rectangle, and such that the square on AB is greater than the square on BC by the square on a straight line incommensurable with AB ;

[x. 32, *ad fin.*]

let the semicircle ADB be described on AB ,
and let the rest of the construction be as above.



Then, since AF is incommensurable in length with FB ,
[x. 18]

AD is also incommensurable in square with DB .
[x. 11]

And, since the square on AB is medial,
therefore the sum of the squares on AD , DB is also medial.
[III. 31, I. 47]

And, since the rectangle AF , FB is equal to the square
on each of the straight lines BE , DF ,
therefore BE is equal to DF ;
therefore BC is double of FD ,
so that the rectangle AB , BC is also double of the rectangle
 AB , FD .

But the rectangle AB , BC is medial;
therefore the rectangle AB , FD is also medial. [x. 32, Por.]

And it is equal to the rectangle AD , DB ;
[Lemma after x. 32]
therefore the rectangle AD , DB is also medial.

And, since AB is incommensurable in length with BC ,
while CB is commensurable with BE ,
therefore AB is also incommensurable in length with BE ,
[x. 13]

so that the square on AB is also incommensurable with the
rectangle AB , BE . [x. 11]

But the squares on AD , DB are equal to the square on
 AB , [I. 47]

and the rectangle AB , FD , that is, the rectangle AD , DB , is
equal to the rectangle AB , BE ;

therefore the sum of the squares on AD , DB is incommen-
surable with the rectangle AD , DB .

Therefore two straight lines AD , DB incommensurable in square have been found which make the sum of the squares on them medial and the rectangle contained by them medial and moreover incommensurable with the sum of the squares on them.

Q. E. D.

Take the medial straight lines found in x. 32 (2nd part), viz.

$$\rho\lambda^{\frac{1}{2}}, \quad \rho\lambda^{\frac{1}{2}}/\sqrt{1+k^2}.$$

Solve the equations

$$\left. \begin{aligned} x+y &= \rho\lambda^{\frac{1}{2}} \\ xy &= \frac{\rho^2\sqrt{\lambda}}{4(1+k^2)} \end{aligned} \right\} \dots\dots\dots(1),$$

and then put

$$\left. \begin{aligned} u^2 &= \rho\lambda^{\frac{1}{2}} \cdot x \\ v^2 &= \rho\lambda^{\frac{1}{2}} \cdot y \end{aligned} \right\} \dots\dots\dots(2),$$

where x, y are the ascertained values of x, y .

Then u, v are straight lines satisfying the given conditions.

Euclid proves this as follows.

(α) From (1) it follows [x. 18] that $x \cup y$.

Therefore

$$u^2 \cup v^2,$$

and

$$u \sim v.$$

(β)

$$u^2 + v^2 = \rho^2\sqrt{\lambda}, \text{ which is a medial area } \dots\dots\dots(3).$$

(γ)

$$\begin{aligned} uv &= \rho\lambda^{\frac{1}{2}} \cdot \sqrt{xy} \\ &= \frac{1}{2} \frac{\rho^2\sqrt{\lambda}}{\sqrt{1+k^2}}, \text{ which is a medial area } \dots\dots\dots(4); \end{aligned}$$

therefore uv is medial.

(δ)

$$\rho\lambda^{\frac{1}{2}} \cup \frac{1}{2} \frac{\rho\lambda^{\frac{1}{2}}}{\sqrt{1+k^2}},$$

whence

$$\rho^2\sqrt{\lambda} \cup \frac{1}{2} \frac{\rho^2\sqrt{\lambda}}{\sqrt{1+k^2}}.$$

That is, by (3) and (4),

$$(u^2 + v^2) \cup uv.$$

The actual values are found thus. Solving the equations (1), we have

$$x = \frac{\rho\lambda^{\frac{1}{2}}}{2} \left(1 + \frac{k}{\sqrt{1+k^2}} \right),$$

$$y = \frac{\rho\lambda^{\frac{1}{2}}}{2} \left(1 - \frac{k}{\sqrt{1+k^2}} \right),$$

whence

$$u = \frac{\rho\lambda^{\frac{1}{2}}}{\sqrt{2}} \sqrt{1 + \frac{k}{\sqrt{1+k^2}}},$$

$$v = \frac{\rho\lambda^{\frac{1}{2}}}{\sqrt{2}} \sqrt{1 - \frac{k}{\sqrt{1+k^2}}}.$$

According as ρ is of the form a or $\sqrt{A}$, we have a variety of forms for u, v , arrived at by using the same transformations as in the notes on x. 30 and x. 32 (second part), e.g.

$$\begin{aligned} (1) \quad & \sqrt{\frac{(a + \sqrt{C})\sqrt{B}}{2}}, \quad \sqrt{\frac{(a - \sqrt{C})\sqrt{B}}{2}}; \\ (2) \quad & \sqrt{\frac{(\sqrt{A} + \sqrt{C})\sqrt{B}}{2}}, \quad \sqrt{\frac{(\sqrt{A} - \sqrt{C})\sqrt{B}}{2}}; \\ (3) \quad & \sqrt{\frac{(\sqrt{A} + c)\sqrt{B}}{2}}, \quad \sqrt{\frac{(\sqrt{A} - c)\sqrt{B}}{2}}; \end{aligned}$$

and the expressions in (2), (3) with b in place of $\sqrt{B}$.

PROPOSITION 36.

*If two rational straight lines commensurable in square only be added together, the whole is irrational; and let it be called **binomial**.*

For let two rational straight lines AB, BC commensurable in square only be added together;

I say that the whole AC is irrational.



For, since AB is incommensurable in length with BC —
 10 for they are commensurable in square only—

and, as AB is to BC , so is the rectangle AB, BC to the square on BC ,

therefore the rectangle AB, BC is incommensurable with the square on BC . [x. 11]

15 But twice the rectangle AB, BC is commensurable with the rectangle AB, BC [x. 6], and the squares on AB, BC are commensurable with the square on BC —for AB, BC are rational straight lines commensurable in square only— [x. 15]
 therefore twice the rectangle AB, BC is incommensurable
 20 with the squares on AB, BC . [x. 13]

And, *componendo*, twice the rectangle AB, BC together with the squares on AB, BC , that is, the square on AC [ii. 4], is incommensurable with the sum of the squares on AB, BC . [x. 16]

But the sum of the squares on AB, BC is rational;
 25 therefore the square on AC is irrational,
 so that AC is also irrational.

[x. Def. 4]

And let it be called **binomial**.

Q. E. D.

Here begins the first hexad of propositions relating to compound irrational straight lines. The six compound irrational straight lines are formed by *adding* two parts, as the corresponding six in Props. 73—78 are formed by *subtraction*. The relation between the six irrational straight lines in this and the next five propositions with those described in Definitions 11. and the Props. 48—53 following thereon (the *first, second, third, fourth, fifth* and *sixth binomials*) will be seen when we come to Props. 54—59; but it may be stated here that the six compound irrationals in Props. 36—41 can be found by means of the equivalent of *extracting the square root* of the compound irrationals in x. 48—53 (the process being, strictly speaking, the finding of the sides of the squares equal to the rectangles contained by the latter irrationals respectively and a rational straight line as the other side), and it is therefore the further removed compound irrational, so to speak, which is treated first.

In reproducing the proofs of the propositions, I shall for the sake of simplicity call the two parts of the compound irrational straight line x , y , explaining at the outset the forms which x , y really have in each case; x will always be supposed to be the greater segment.

In this proposition x , y are of the form ρ , $\sqrt{k} \cdot \rho$, and $(x+y)$ is proved to be irrational thus.

$x \sim y$, so that $x \cup y$.

Now

$$x : y = x^2 : xy,$$

so that

$$x^2 \cup xy.$$

But $x^2 \sim (x^2 + y^2)$, and $xy \sim 2xy$;

therefore

$$(x^2 + y^2) \cup 2xy,$$

and hence

$$(x^2 + y^2 + 2xy) \cup (x^2 + y^2).$$

But $(x^2 + y^2)$ is rational;

therefore $(x+y)^2$, and therefore $(x+y)$, is *irrational*.

This irrational straight line, $\rho + \sqrt{k} \cdot \rho$, is called a *binomial* straight line.

This and the corresponding *apotome* $(\rho - \sqrt{k} \cdot \rho)$ found in x. 73 are the positive roots of the equation

$$x^4 - 2(1+k)\rho^2 \cdot x^2 + (1-k)^2\rho^4 = 0.$$

PROPOSITION 37.

If two medial straight lines commensurable in square only and containing a rational rectangle be added together, the whole is irrational; and let it be called a first bimedial straight line.

For let two medial straight lines AB , BC commensurable in square only and containing a rational rectangle be added together;



I say that the whole AC is irrational.

For, since AB is incommensurable in length with BC , therefore the squares on AB , BC are also incommensurable with twice the rectangle AB , BC ;

[cf. x. 36, ll. 9—20]

and, *componendo*, the squares on AB , BC together with twice the rectangle AB , BC , that is, the square on AC [II. 4], is incommensurable with the rectangle AB , BC . [x. 16]

But the rectangle AB , BC is rational, for, by hypothesis, AB , BC are straight lines containing a rational rectangle ; therefore the square on AC is irrational ; therefore AC is irrational. [x. Def. 4]

And let it be called a **first bimedral** straight line.

Q. E. D.

Here x , y have the forms $k^{\frac{1}{4}}\rho$, $k^{\frac{3}{4}}\rho$ respectively, as found in x. 27.

Exactly as in the last case we prove that

$$x^2 + y^2 \sim 2xy,$$

whence

$$(x + y)^2 \sim 2xy.$$

But xy is rational ;

therefore $(x + y)^2$, and consequently $(x + y)$, is *irrational*.

The irrational straight line $k^{\frac{1}{4}}\rho + k^{\frac{3}{4}}\rho$ is called a *first bimedral* straight line.

This and the corresponding first *apotome* of a medial ($k^{\frac{1}{4}}\rho - k^{\frac{3}{4}}\rho$) found in x. 74 are the positive roots of the equation

$$x^4 - 2\sqrt{k(1+k)}\rho^2 \cdot x^2 + k(1-k)^2\rho^4 = 0.$$

PROPOSITION 38.

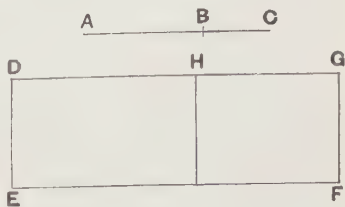
*If two medial straight lines commensurable in square only and containing a medial rectangle be added together, the whole is irrational; and let it be called a **second bimedral** straight line.*

5 For let two medial straight lines AB , BC commensurable in square only and containing a medial rectangle be added together ;

I say that AC is irrational.

10 For let a rational straight line DE be set out, and let the parallelogram DF equal to the square on AC be applied to DE , producing DG as breadth.

15 Then, since the square on AC is equal to the squares on AB , BC and twice the rectangle AB , BC , [I. 44]
[II. 4]
let EH , equal to the squares on AB , BC , be applied to DE ;



therefore the remainder HF is equal to twice the rectangle AB, BC .

- 20 And, since each of the straight lines AB, BC is medial, therefore the squares on AB, BC are also medial.

But, by hypothesis, twice the rectangle AB, BC is also medial.

- And EH is equal to the squares on AB, BC ,
 25 while FH is equal to twice the rectangle AB, BC ;
 therefore each of the rectangles EH, HF is medial.

And they are applied to the rational straight line DE ;
 therefore each of the straight lines DH, HG is rational and
 incommensurable in length with DE . [x. 22]

- 30 Since then AB is incommensurable in length with BC ,
 and, as AB is to BC , so is the square on AB to the rectangle
 AB, BC ,
 therefore the square on AB is incommensurable with the rect-
 angle AB, BC . [x. 11]

- 35 But the sum of the squares on AB, BC is commensurable
 with the square on AB , [x. 15]
 and twice the rectangle AB, BC is commensurable with the
 rectangle AB, BC . [x. 6]

- Therefore the sum of the squares on AB, BC is incom-
 40 mensurable with twice the rectangle AB, BC . [x. 13]

But EH is equal to the squares on AB, BC ,
 and HF is equal to twice the rectangle AB, BC .

Therefore EH is incommensurable with HF ,
 so that DH is also incommensurable in length with HG .
 [vi. 1, x. 11]

- 45 Therefore DH, HG are rational straight lines commen-
 surable in square only ;
 so that DG is irrational. [x. 36]

But DE is rational ;

- and the rectangle contained by an irrational and a rational
 50 straight line is irrational ; [cf. x. 20]

therefore the area DF is irrational,
 and the side of the square equal to it is irrational. [x. Def. 4]

But AC is the side of the square equal to DF ;
therefore AC is irrational.

55 And let it be called a **second bimedial** straight line.

Q. E. D.

After proving (l. 21) that *each* of the squares on AB , BC is medial, Euclid states (ll. 24, 26) that EH , which is equal to the *sum* of the squares, is a medial area, but does not explain why. It is because, by hypothesis, the squares on AB , BC are commensurable, so that the sum of the squares is commensurable with either [x. 15] and is therefore a medial area [x. 23, Por.].

In this case [x. 28, note] x, y are of the forms $k^{\frac{1}{2}}\rho, \lambda^{\frac{1}{2}}\rho/k^{\frac{1}{2}}$ respectively.

Apply each of the areas $(x^2 + y^2)$ and $2xy$ to a rational straight line σ , i.e. suppose

$$x^2 + y^2 = \sigma u,$$

$$2xy = \sigma v.$$

Now it follows from the hypothesis, x. 15 and x. 23, Por. that $(x^2 + y^2)$ is a medial area; and so is $2xy$, by hypothesis;
therefore $\sigma u, \sigma v$ are medial areas.

Therefore each of the straight lines u, v is rational and $u \sim \sigma$ (1).

Again

$$x \sim y;$$

therefore

$$x^2 \sim xy.$$

But

$$x^2 \sim x^2 + y^2 \text{ and } xy \sim 2xy;$$

therefore

$$x^2 + y^2 \sim 2xy,$$

or

$$\sigma u \sim \sigma v;$$

whence

$$u \sim v \text{ (2).}$$

Therefore, by (1), (2), u, v are rational and $\sim$.

It follows, by x. 36, that $(u + v)$ is irrational.

Therefore $(u + v)\sigma$ is an irrational area [this can be deduced from x. 20 by *reductio ad absurdum*].

whence $(x + y)^2$, and consequently $(x + y)$, is irrational.

The irrational straight line $k^{\frac{1}{2}}\rho + \frac{\lambda^{\frac{1}{2}}\rho}{k^{\frac{1}{2}}}$ is called a *second bimedial* straight line.

This and the corresponding *second apotome of a medial* $\left(k^{\frac{1}{2}}\rho - \frac{\sqrt{\lambda}}{k^{\frac{1}{2}}}\rho\right)$ found in x. 75 are the positive roots of the equation

$$x^4 - 2 \frac{k + \lambda}{\sqrt{k}} \rho^2 \cdot x^2 + \frac{(k - \lambda)^2}{k} \rho^4 = 0.$$

PROPOSITION 39.

If two straight lines incommensurable in square which make the sum of the squares on them rational, but the rectangle contained by them medial, be added together, the whole straight line is irrational: and let it be called major.

For let two straight lines AB , BC incommensurable in square, and fulfilling the given conditions [x. 33], be added together ;



I say that AC is irrational.

For, since the rectangle AB , BC is medial, twice the rectangle AB , BC is also medial. [x. 6 and 23, Por.]

But the sum of the squares on AB , BC is rational ; therefore twice the rectangle AB , BC is incommensurable with the sum of the squares on AB , BC ,

so that the squares on AB , BC together with twice the rectangle AB , BC , that is, the square on AC , is also incommensurable with the sum of the squares on AB , BC ; [x. 16]

therefore the square on AC is irrational,

so that AC is also irrational.

[x. Def. 4]

And let it be called **major**.

Q. E. D.

Here x , y are of the form found in x. 33, viz.

$$\frac{\rho}{\sqrt{2}} \sqrt{1 + \frac{k}{\sqrt{1+k^2}}}, \quad \frac{\rho}{\sqrt{2}} \sqrt{1 - \frac{k}{\sqrt{1+k^2}}}.$$

By hypothesis, the rectangle xy is medial ; therefore $2xy$ is medial.

Also $(x^2 + y^2)$ is a rational area.

Therefore

$$x^2 + y^2 \sim 2xy,$$

whence

$$(x + y)^2 \sim (x^2 + y^2),$$

so that $(x + y)^2$, and therefore $(x + y)$, is irrational.

The irrational straight line $\frac{\rho}{\sqrt{2}} \sqrt{1 + \frac{k}{\sqrt{1+k^2}}} + \frac{\rho}{\sqrt{2}} \sqrt{1 - \frac{k}{\sqrt{1+k^2}}}$ is called a *major* (irrational) straight line.

This and the corresponding *minor* irrational found in x. 76 are the positive roots of the equation

$$x^4 - 2\rho^2 \cdot x^2 + \frac{k^2}{1+k^2} \rho^4 = 0.$$

PROPOSITION 40.

If two straight lines incommensurable in square which make the sum of the squares on them medial, but the rectangle contained by them rational, be added together, the whole straight line is irrational ; and let it be called the side of a rational plus a medial area.

For let two straight lines AB, BC incommensurable in square, and fulfilling the given conditions [x. 34], be added together;



I say that AC is irrational.

For, since the sum of the squares on AB, BC is medial, while twice the rectangle AB, BC is rational, therefore the sum of the squares on AB, BC is incommensurable with twice the rectangle AB, BC ;

so that the square on AC is also incommensurable with twice the rectangle AB, BC . [x. 16]

But twice the rectangle AB, BC is rational; therefore the square on AC is irrational.

Therefore AC is irrational. [x. Def. 4]

And let it be called the **side of a rational plus a medial area**.

Q. E. D.

Here x, y have [x. 34] the forms

$$\frac{\rho}{\sqrt{2(1+k^2)}} \sqrt{\sqrt{1+k^2}+k}, \quad \frac{\rho}{\sqrt{2(1+k^2)}} \sqrt{\sqrt{1+k^2}-k}.$$

In this case (x^2+y^2) is a medial, and $2xy$ a rational, area; thus

$$x^2+y^2 \sim 2xy.$$

Therefore

$$(x+y)^2 \sim 2xy,$$

whence, since $2xy$ is rational,

$(x+y)^2$, and consequently $(x+y)$, is irrational.

The irrational straight line

$$\frac{\rho}{\sqrt{2(1+k^2)}} \sqrt{\sqrt{1+k^2}+k} + \frac{\rho}{\sqrt{2(1+k^2)}} \sqrt{\sqrt{1+k^2}-k}$$

is called (for an obvious reason) the "*side*" of a *rational plus a medial* (area).

This and the corresponding irrational with a minus sign found in x. 77 are the positive roots of the equation

$$x^4 - \frac{2}{\sqrt{1+k^2}} \rho^2 \cdot x^2 + \frac{k^2}{(1+k^2)^2} \rho^4 = 0.$$

PROPOSITION 41.

If two straight lines incommensurable in square which make the sum of the squares on them medial, and the rectangle contained by them medial and also incommensurable with the sum of the squares on them, be added together, the whole straight line is irrational; and let it be called the side of the sum of two medial areas.

For let two straight lines AB , BC incommensurable in square and satisfying the given conditions [x. 35] be added together;

I say that AC is irrational.

Let a rational straight line DE be set out, and let there be applied to DE the rectangle DF equal to the squares on AB , BC , and the rectangle GH equal to twice the rectangle AB , BC ;

therefore the whole DH is equal to the square on AC . [II. 4]

Now, since the sum of the squares on AB , BC is medial, and is equal to DF , therefore DF is also medial.

And it is applied to the rational straight line DE ; therefore DG is rational and incommensurable in length with DE . [x. 22]

For the same reason GK is also rational and incommensurable in length with GF , that is, DE .

And, since the squares on AB , BC are incommensurable with twice the rectangle AB , BC , DF is incommensurable with GH ;

so that DG is also incommensurable with GK . [VI. 1, x. 11]

And they are rational; therefore DG , GK are rational straight lines commensurable in square only;

therefore DK is irrational and what is called binomial. [x. 36]

But DE is rational; therefore DH is irrational, and the side of the square which is equal to it is irrational. [x. Def. 4]

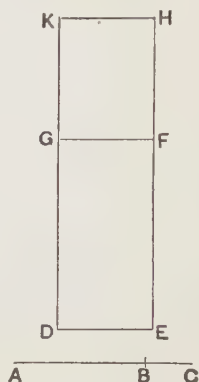
But AC is the side of the square equal to HD ; therefore AC is irrational.

And let it be called the **side of the sum of two medial areas**.

Q. E. D.

In this case x, y are of the form

$$\frac{\rho\lambda^{\frac{1}{2}}}{\sqrt{2}} \sqrt{1 + \frac{k}{\sqrt{1+k^2}}}, \quad \frac{\rho\lambda^{\frac{1}{2}}}{\sqrt{2}} \sqrt{1 - \frac{k}{\sqrt{1+k^2}}}.$$



By hypothesis, $(x^2 + y^2)$ and $2xy$ are medial areas, and

$$x^2 + y^2 \sim 2xy \dots\dots\dots(1).$$

'Apply' these areas respectively to a rational straight line σ , and suppose

$$\left. \begin{array}{l} x^2 + y^2 = \sigma u \\ 2xy = \sigma v \end{array} \right\} \dots\dots\dots(2).$$

Since then σu and σv are both medial areas, u, v are rational and both are $\sim \sigma$ $\dots\dots\dots(3)$.

Now, by (1) and (2),

$$\sigma u \sim \sigma v,$$

so that

$$u \sim v.$$

By this and (3), u, v are rational and $\sim$.

Therefore [x. 36] $(u + v)$ is irrational.

Hence $\sigma(u + v)$ is irrational [deduction from x. 20].

Thus $(x + y)^2$, and therefore $(x + y)$, is irrational.

The irrational straight line

$$\frac{\rho\lambda^{\frac{1}{2}}}{\sqrt{2}} \sqrt{1 + \frac{k}{\sqrt{1+k^2}}} + \frac{\rho\lambda^{\frac{1}{2}}}{\sqrt{2}} \sqrt{1 - \frac{k}{\sqrt{1+k^2}}}$$

is called (again for an obvious reason) the "*side*" of the sum of two medials (medial areas).

This and the corresponding irrational with a minus sign found in x. 78 are the positive roots of the equation

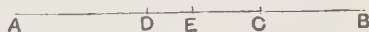
$$x^4 - 2\sqrt{\lambda} \cdot x^2\rho^2 + \lambda \frac{k^2}{1+k^2} \rho^4 = 0.$$

LEMMA.

And that the aforesaid irrational straight lines are divided only in one way into the straight lines of which they are the sum and which produce the types in question, we will now prove after premising the following lemma.

Let the straight line AB be set out, let the whole be cut into unequal parts at each of the points C, D ,

and let AC be supposed greater than DB ;



I say that the squares on AC, CB are greater than the squares on AD, DB .

For let AB be bisected at E .

Then, since AC is greater than DB ,

let DC be subtracted from each;

therefore the remainder AD is greater than the remainder CB .

But AE is equal to EB ;

therefore DE is less than EC ;

therefore the points C, D are not equidistant from the point of bisection.

And, since the rectangle AC, CB together with the square on EC is equal to the square on EB , [II. 5]

and, further, the rectangle AD, DB together with the square on DE is equal to the square on EB , [*id.*]

therefore the rectangle AC, CB together with the square on EC is equal to the rectangle AD, DB together with the square on DE .

And of these the square on DE is less than the square on EC ;

therefore the remainder, the rectangle AC, CB , is also less than the rectangle AD, DB ,

so that twice the rectangle AC, CB is also less than twice the rectangle AD, DB .

Therefore also the remainder, the sum of the squares on AC, CB , is greater than the sum of the squares on AD, DB .

Q. E. D.

3. and which produce the types in question. The Greek is ποιοῦσιν τὰ προκείμενα εἶδη, and I have taken εἶδη to mean "types (of irrational straight lines)," though the expression might perhaps mean "satisfying the conditions in question."

This proves that, if $x + y = u + v$, and if u, v are more nearly equal than x, y (i.e. if the straight line is divided in the second case nearer to the point of bisection), then

$$(x^2 + y^2) > (u^2 + v^2).$$

It is first proved by means of II. 5 that

$$2xy < 2uv,$$

whence, since $(x + y)^2 = (u + v)^2$, the required result follows.

PROPOSITION 42.

A binomial straight line is divided into its terms at one point only.

Let AB be a binomial straight line divided into its terms at C ;

therefore AC, CB are rational straight lines commensurable in square only.



[x. 36]

I say that AB is not divided at another point into two rational straight lines commensurable in square only.

For, if possible, let it be divided at D also, so that AD , DB are also rational straight lines commensurable in square only.

It is then manifest that AC is not the same with DB .

For, if possible, let it be so.

Then AD will also be the same as CB ,

and, as AC is to CB , so will BD be to DA ;

thus AB will be divided at D also in the same way as by the division at C :

which is contrary to the hypothesis.

Therefore AC is not the same with DB .

For this reason also the points C , D are not equidistant from the point of bisection.

Therefore that by which the squares on AC , CB differ from the squares on AD , DB is also that by which twice the rectangle AD , DB differs from twice the rectangle AC , CB ,

because both the squares on AC , CB together with twice the rectangle AC , CB , and the squares on AD , DB together with twice the rectangle AD , DB , are equal to the square on AB . [II. 4]

But the squares on AC , CB differ from the squares on AD , DB by a rational area,

for both are rational ;

therefore twice the rectangle AD , DB also differs from twice the rectangle AC , CB by a rational area, though they are medial [x. 21] :

which is absurd, for a medial area does not exceed a medial by a rational area. [x. 26]

Therefore a binomial straight line is not divided at different points ;

therefore it is divided at one point only.

Q. E. D.

This proposition proves the equivalent of the well-known theorem in surds that,

	if $a + \sqrt{b} = x + \sqrt{y}$,
then	$a = x, \quad b = y,$
and if	$\sqrt{a} + \sqrt{b} = \sqrt{x} + \sqrt{y},$
then	$a = x, \quad b = y \quad (\text{or } a = y, \quad b = x).$

The proposition states that a *binomial* straight line cannot be split up into *terms* (*ὀνόματα*) in two ways. For, if possible, let

$$x + y = x' + y',$$

where x, y , and also x', y' , are the *terms* of a binomial straight line, x', y' being different from x, y (or y', x).

One pair is necessarily more nearly equal than the other. Let x', y' be more nearly equal than x, y .

Then $(x^2 + y^2) - (x'^2 + y'^2) = 2x'y' - 2xy$.

Now by hypothesis $(x^2 + y^2)$, $(x'^2 + y'^2)$ are *rational* areas, being of the form $\rho^2 + k\rho^2$;

but $2x'y'$, $2xy$ are *medial* areas, being of the form $\sqrt{k} \cdot \rho^2$;

therefore the difference of two medial areas is rational:

which is impossible.

[x. 26]

Therefore x', y' cannot be different from x, y (or y', x).

PROPOSITION 43.

A first bimedial straight line is divided at one point only.

Let AB be a first bimedial straight line divided at C , so that AC, CB are medial straight lines commensurable in square only and containing a rational rectangle;



[x. 37]

I say that AB is not so divided at another point.

For, if possible, let it be divided at D also, so that AD, DB are also medial straight lines commensurable in square only and containing a rational rectangle.

Since, then, that by which twice the rectangle AD, DB differs from twice the rectangle AC, CB is that by which the squares on AC, CB differ from the squares on AD, DB ,

while twice the rectangle AD, DB differs from twice the rectangle AC, CB by a rational area—for both are rational—therefore the squares on AC, CB also differ from the squares on AD, DB by a rational area, though they are medial:

which is absurd.

[x. 26]

Therefore a first bimedial straight line is not divided into its terms at different points;

therefore it is so divided at one point only.

Q. E. D.

In this case, with the same hypothesis, viz. that

$$x + y = x' + y',$$

and x', y' are more nearly equal than x, y ,

we have as before $(x^2 + y^2) - (x'^2 + y'^2) = 2x'y' - 2xy$.

But, from the given properties of x, y , and x', y' , it follows that $2xy, 2x'y'$ are *rational*, and $(x^2 + y^2), (x'^2 + y'^2)$ *medial*, areas.

Therefore the difference between two medial areas is rational :

which is impossible:

[x. 26]

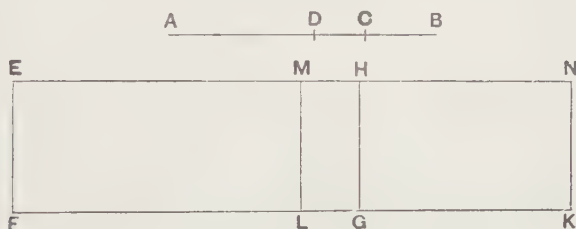
PROPOSITION 44.

A second bimedral straight line is divided at one point only.

Let AB be a second bimedral straight line divided at C , so that AC, CB are medial straight lines commensurable in square only and containing a medial rectangle ; [x. 38]

it is then manifest that C is not at the point of bisection, because the segments are not commensurable in length.

I say that AB is not so divided at another point.



For, if possible, let it be divided at D also, so that AC is not the same with DB , but AC is supposed greater ;

it is then clear that the squares on AD, DB are also, as we proved above [Lemma], less than the squares on AC, CB ;

and suppose that AD, DB are medial straight lines commensurable in square only and containing a medial rectangle.

Now let a rational straight line EF be set out,

let there be applied to EF the rectangular parallelogram EKN equal to the square on AB ,

and let EG equal to the squares on AC, CB be subtracted ; therefore the remainder HK is equal to twice the rectangle AC, CB . [II. 4]

Again, let there be subtracted EL , equal to the squares on AD, DB , which were proved less than the squares on AC, CB [Lemma] ;

therefore the remainder MK is also equal to twice the rectangle AD, DB .

Now, since the squares on AC, CB are medial, therefore EG is medial.

And it is applied to the rational straight line EF ; therefore EH is rational and incommensurable in length with EF . [x. 22]

For the same reason HN is also rational and incommensurable in length with EF .

And, since AC, CB are medial straight lines commensurable in square only, therefore AC is incommensurable in length with CB .

But, as AC is to CB , so is the square on AC to the rectangle AC, CB ; therefore the square on AC is incommensurable with the rectangle AC, CB . [x. 11]

But the squares on AC, CB are commensurable with the square on AC ; for AC, CB are commensurable in square. [x. 15]

And twice the rectangle AC, CB is commensurable with the rectangle AC, CB . [x. 6]

Therefore the squares on AC, CB are also incommensurable with twice the rectangle AC, CB . [x. 13]

But EG is equal to the squares on AC, CB , and HK is equal to twice the rectangle AC, CB ; therefore EG is incommensurable with HK , so that EH is also incommensurable in length with HN . [vi. 1, x. 11]

And they are rational; therefore EH, HN are rational straight lines commensurable in square only.

But, if two rational straight lines commensurable in square only be added together, the whole is the irrational which is called binomial. [x. 36]

Therefore EN is a binomial straight line divided at H .

In the same way EM, MN will also be proved to be rational straight lines commensurable in square only; and EN will be a binomial straight line divided at different points, H and M .

And EH is not the same with MN .

For the squares on AC , CB are greater than the squares on AD , DB .

But the squares on AD , DB are greater than twice the rectangle AD , DB ;

therefore also the squares on AC , CB , that is, EG , are much greater than twice the rectangle AD , DB , that is, MK , so that EH is also greater than MN .

Therefore EH is not the same with MN .

Q. E. D.

As the irrationality of the *second binomial* straight line [x. 38] is proved by means of the irrationality of the binomial straight line [x. 36], so the present theorem is reduced to that of x. 42.

Suppose, if possible, that the second binomial straight line can be divided into its terms as such in two ways, i.e. that

$$x + y = x' + y',$$

where x' , y' are nearer equality than x , y .

Apply $x^2 + y^2$, $2xy$ to a rational straight line σ , i.e. let

$$x^2 + y^2 = \sigma u,$$

$$2xy = \sigma v.$$

Then, as in x. 38, the areas $x^2 + y^2$, $2xy$ are medial, so that σu , σv are medial;

therefore u , v are both rational and $\sim \sigma$ (1).

Again, by hypothesis, x , y are medial straight lines commensurable in square only;

therefore $x \sim y$.

Hence $x^2 \sim xy$.

And $x^2 \sim (x^2 + y^2)$, while $xy \sim 2xy$;

therefore $(x^2 + y^2) \sim 2xy$,

or $\sigma u \sim \sigma v$,

and hence $u \sim v$ (2).

Therefore, by (1) and (2), u , v are rational straight lines commensurable in square only;

therefore $u + v$ is a *binomial* straight line.

Similarly, if $x'^2 + y'^2 = \sigma u'$ and $2x'y' = \sigma v'$,

$u' + v'$ will be proved to be a binomial straight line.

And, since $(x + y)^2 = (x' + y')^2$, and therefore $(u + v) = (u' + v')$, it follows that a binomial straight line is divided as such in two ways:

which is impossible. [x. 42]

Therefore $x + y$, the given second binomial straight line, can only be so divided in one way.

In order to prove that $u + v$, $u' + v'$ represent a *different* division of the same straight line, Euclid assumes that $x^2 + y^2 > 2xy$. This is of course an easy inference from II. 7; but the assumption of it here renders it probable that the Lemma after x. 59 is interpolated.

PROPOSITION 45.

A major straight line is divided at one and the same point only.

Let AB be a major straight line divided at C , so that AC , CB are incommensurable in square and make the sum of the squares on AC , CB rational, but the rectangle AC , CB medial ;



[x. 39]

I say that AB is not so divided at another point.

For, if possible, let it be divided at D also, so that AD , DB are also incommensurable in square and make the sum of the squares on AD , DB rational, but the rectangle contained by them medial.

Then, since that by which the squares on AC , CB differ from the squares on AD , DB is also that by which twice the rectangle AD , DB differs from twice the rectangle AC , CB , while the squares on AC , CB exceed the squares on AD , DB by a rational area—for both are rational—

therefore twice the rectangle AD , DB also exceeds twice the rectangle AC , CB by a rational area, though they are medial : which is impossible.

[x. 26]

Therefore a major straight line is not divided at different points ;

therefore it is only divided at one and the same point.

Q. E. D.

If possible, let the *major* irrational straight line be divided into terms in two ways, viz. as $(x+y)$ and $(x'+y')$, where x' , y' are supposed to be nearer equality than x , y .

We have then, as in x. 42, 43,

$$(x^2 + y^2) - (x'^2 + y'^2) = 2x'y' - 2xy.$$

But, by hypothesis, $(x^2 + y^2)$, $(x'^2 + y'^2)$ are both *rational*, so that their difference is rational.

Also, by hypothesis, $2x'y'$, $2xy$ are both *medial* areas ;

therefore the difference of two medial areas is a rational area : which is impossible.

[x. 26]

Therefore etc.

PROPOSITION 46.

The side of a rational plus a medial area is divided at one point only.

Let AB be the side of a rational plus a medial area divided at C , so that AC , CB are incommensurable in square and make the sum of the squares on AC , CB medial, but twice the rectangle AC , CB rational; [x. 40] I say that AB is not so divided at another point.

For, if possible, let it be divided at D also, so that AD , DB are also incommensurable in square and make the sum of the squares on AD , DB medial, but twice the rectangle AD , DB rational.

Since then that by which twice the rectangle AC , CB differs from twice the rectangle AD , DB is also that by which the squares on AD , DB differ from the squares on AC , CB ,

while twice the rectangle AC , CB exceeds twice the rectangle AD , DB by a rational area,

therefore the squares on AD , DB also exceed the squares on AC , CB by a rational area, though they are medial :

which is impossible. [x. 26]

Therefore the side of a rational plus a medial area is not divided at different points ;
therefore it is divided at one point only.

Q. E. D.

Here, as before, if we use the same notation,

$$(x^2 + y^2) - (x'^2 + y'^2) = 2x'y' - 2xy,$$

and the areas on the left side are, by hypothesis, both medial, while the areas on the right side are both rational.

Thus the result of x. 26 is contradicted, as before.

Therefore etc.

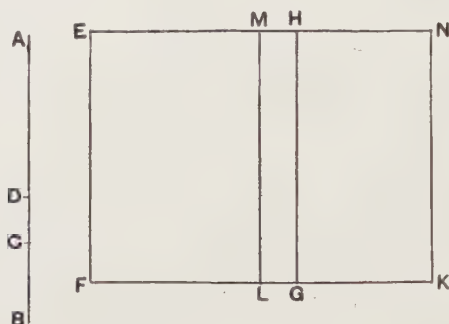
PROPOSITION 47.

The side of the sum of two medial areas is divided at one point only.

Let AB be divided at C , so that AC , CB are incommensurable in square and make the sum of the squares on AC ,

CB medial, and the rectangle AC , CB medial and also incommensurable with the sum of the squares on them;

I say that AB is not divided at another point so as to fulfil the given conditions.



For, if possible, let it be divided at D , so that again AC is of course not the same as BD , but AC is supposed greater; let a rational straight line EF be set out, and let there be applied to EF the rectangle EG equal to the squares on AC , CB ,

and the rectangle HK equal to twice the rectangle AC , CB ; therefore the whole EK is equal to the square on AB . [II. 4]

Again, let EL , equal to the squares on AD , DB , be applied to EF ;

therefore the remainder, twice the rectangle AD , DB , is equal to the remainder MK .

And since, by hypothesis, the sum of the squares on AC , CB is medial, therefore EG is also medial.

And it is applied to the rational straight line EF ; therefore HE is rational and incommensurable in length with EF . [x. 22]

For the same reason HN is also rational and incommensurable in length with EF .

And, since the sum of the squares on AC , CB is incommensurable with twice the rectangle AC , CB , therefore EG is also incommensurable with GN , so that EH is also incommensurable with HN . [VI. I, x. 11]

And they are rational;

therefore EH , HN are rational straight lines commensurable in square only;

therefore EN is a binomial straight line divided at H . [x. 36]

Similarly we can prove that it is also divided at M .

And EH is not the same with MN ;

therefore a binomial has been divided at different points :
which is absurd. [x. 42]

Therefore a side of the sum of two medial areas is not divided at different points ;

therefore it is divided at one point only.

Using the same notation as in the note on x. 44, we suppose that, if possible,

$$x + y = x' + y',$$

and we put

$$\left. \begin{array}{l} x^2 + y^2 = \sigma u \\ 2xy = \sigma v \end{array} \right\} \quad \text{and} \quad \left. \begin{array}{l} x'^2 + y'^2 = \sigma u' \\ 2x'y' = \sigma v' \end{array} \right\}.$$

Then, since $x^2 + y^2$, $2xy$ are medial areas, and σ rational,

$$u, v \text{ are both rational and } \sim \sigma \dots\dots\dots(1).$$

Also, by hypothesis, $x^2 + y^2 \sim 2xy$,

$$\text{whence} \quad u \sim v \dots\dots\dots(2).$$

Therefore, by (1) and (2), u, v are rational and $\sim$.

Hence $u + v$ is a *binomial* straight line.

[x. 36]

Similarly $u' + v'$ is a binomial straight line.

But

$$u + v = u' + v';$$

therefore a binomial straight line is divided into terms in two ways :

which is impossible.

[x. 42]

Therefore etc.

DEFINITIONS II.

1. Given a rational straight line and a binomial, divided into its terms, such that the square on the greater term is greater than the square on the lesser by the square on a straight line commensurable in length with the greater, then, if the greater term be commensurable in length with the rational straight line set out, let the whole be called a **first binomial** straight line ;

2. but if the lesser term be commensurable in length with the rational straight line set out, let the whole be called a **second binomial** ;

3. and if neither of the terms be commensurable in length with the rational straight line set out, let the whole be called a **third binomial**.

4. Again, if the square on the greater term be greater than the square on the lesser by the square on a straight line incommensurable in length with the greater, then, if the greater term be commensurable in length with the rational straight line set out, let the whole be called a **fourth binomial**;

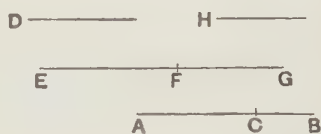
5. if the lesser, a **fifth binomial**;

6. and if neither, a **sixth binomial**.

PROPOSITION 48.

To find the first binomial straight line.

Let two numbers AC , CB be set out such that the sum of them AB has to BC the ratio which a square number has to a square number, but has not to CA the ratio which a square number has to a square number;



[Lemma 1 after x. 28]

let any rational straight line D be set out, and let EF be commensurable in length with D .

Therefore EF is also rational.

Let it be contrived that,

as the number BA is to AC , so is the square on EF to the square on FG .

[x. 6, Por.]

But AB has to AC the ratio which a number has to a number;

therefore the square on EF also has to the square on FG the ratio which a number has to a number,

so that the square on EF is commensurable with the square on FG .

[x. 6]

And EF is rational;

therefore FG is also rational.

And, since BA has not to AC the ratio which a square number has to a square number,

neither, therefore, has the square on EF to the square on FG the ratio which a square number has to a square number ;
therefore EF is incommensurable in length with FG . [x. 9]

Therefore EF , FG are rational straight lines commensurable in square only ;

therefore EG is binomial.

[x. 36]

I say that it is also a first binomial straight line.

For since, as the number BA is to AC , so is the square on EF to the square on FG ,

while BA is greater than AC ,

therefore the square on EF is also greater than the square on FG .

Let then the squares on FG , H be equal to the square on EF .

Now since, as BA is to AC , so is the square on EF to the square on FG ,

therefore, *convertendo*,

as AB is to BC , so is the square on EF to the square on H .

[v. 19, Por.]

But AB has to BC the ratio which a square number has to a square number ;

therefore the square on EF also has to the square on H the ratio which a square number has to a square number.

Therefore EF is commensurable in length with H ; [x. 9]
therefore the square on EF is greater than the square on FG by the square on a straight line commensurable with EF .

And EF , FG are rational, and EF is commensurable in length with D .

Therefore EF is a first binomial straight line.

Q. E. D.

Let $k\rho$ be a straight line commensurable in length with ρ , a given rational straight line.

The two numbers taken may be written $p(m^2 - n^2)$, $p n^2$, where $(m^2 - n^2)$ is not a square.

Take x such that

$$p m^2 : p(m^2 - n^2) = k^2 \rho^2 : x^2 \dots\dots\dots (1),$$

whence

$$x = k\rho \frac{\sqrt{m^2 - n^2}}{m}.$$

Then $k\rho + x$, or $k\rho + k\rho \frac{\sqrt{m^2 - n^2}}{m}$, is a *first binomial* straight line (2).

To prove this we have, from (1),

$$x^2 \sim k^2 p^2,$$

and x is rational, but $x \sim k\rho$;

that is, x is rational and $\sim k\rho$,

so that $k\rho + x$ is a binomial straight line.

Also, $k^2 p^2$ being greater than x^2 , suppose $k^2 p^2 - x^2 = y^2$.

Then, from (1), $p m^2 : p n^2 = k^2 p^2 : y^2$,

whence y is rational and $\sim k\rho$.

Therefore $k\rho + x$ is a *first binomial* straight line [x. Def. 11. 1].

This binomial straight line may be written thus,

$$k\rho + k\rho \sqrt{1 - \lambda^2}.$$

When we come to x. 85, we shall find that the corresponding straight line with a negative sign is the *first apotome*,

$$k\rho - k\rho \sqrt{1 - \lambda^2}.$$

Consider now the equation of which these two expressions are the roots. The equation is

$$x^2 - 2k\rho \cdot x + \lambda^2 k^2 p^2 = 0.$$

In other words, the first binomial and the first apotome correspond to the roots of the equation

$$x^2 - 2ax + \lambda^2 a^2 = 0,$$

where $a = k\rho$.

PROPOSITION 49.

To find the second binomial straight line.

Let two numbers AC , CB be set out such that the sum of them AB has to BC the ratio which a square number has to a square number, but has not to AC the ratio which a square number has to a square number; let a rational straight line D be set out, and let EF be commensurable in length with D ;



therefore EF is rational.

Let it be contrived then that,

as the number CA is to AB , so also is the square on EF to the square on FG ;

[x. 6, Por.]

therefore the square on EF is commensurable with the square on FG .

[x. 6]

Therefore FG is also rational.

Now, since the number CA has not to AB the ratio which a square number has to a square number, neither has the

square on EF to the square on FG the ratio which a square number has to a square number.

Therefore EF is incommensurable in length with FG ;

[x. 9]

therefore EF, FG are rational straight lines commensurable in square only;

therefore EG is binomial.

[x. 36]

It is next to be proved that it is also a second binomial straight line.

For since, inversely, as the number BA is to AC , so is the square on GF to the square on FE ,

while BA is greater than AC ,

therefore the square on GF is greater than the square on FE .

Let the squares on EF, H be equal to the square on GF ; therefore, *convertendo*, as AB is to BC , so is the square on FG to the square on H .

[v. 19, Por.]

But AB has to BC the ratio which a square number has to a square number;

therefore the square on FG also has to the square on H the ratio which a square number has to a square number.

Therefore FG is commensurable in length with H ; [x. 9] so that the square on FG is greater than the square on FE by the square on a straight line commensurable with FG .

And FG, FE are rational straight lines commensurable in square only, and EF , the lesser term, is commensurable in length with the rational straight line D set out.

Therefore EG is a second binomial straight line.

Q. E. D.

Taking a rational straight line $k\rho$ commensurable in length with ρ , and selecting numbers of the same form as before, viz. $p(m^2 - n^2)$, pn^2 , we put

$$p(m^2 - n^2) : pm^2 = k^2\rho^2 : x^2 \dots\dots\dots (1),$$

so that

$$\begin{aligned} x &= k\rho \frac{m}{\sqrt{m^2 - n^2}} \\ &= k\rho \frac{1}{\sqrt{1 - \lambda^2}}, \text{ say } \dots\dots\dots (2). \end{aligned}$$

Just as before, x is rational and $\sim k\rho$, whence $k\rho + x$ is a *binomial* straight line.

By (1), $x^2 > k^2\rho^2$.

Let $x^2 - k^2 p^2 = y^2$,
 whence, from (1), $p^2 m^2 : p^2 n^2 = x^3 : y^2$,
 and y is therefore rational and $\propto x$.

The greater term of the binomial straight line is x and the lesser $k\rho$, and

$$\frac{k\rho}{\sqrt{1-\lambda^2}} + k\rho$$

satisfies the definition of the *second binomial* straight line.

The corresponding *second apotome* [x. 86] is

$$\frac{k\rho}{\sqrt{1-\lambda^2}} - k\rho.$$

The equation of which the two expressions are the roots is

$$x^2 - \frac{2k\rho}{\sqrt{1-\lambda^2}} \cdot x + \frac{\lambda^2}{1-\lambda^2} k^2 p^2 = 0,$$

or
$$x^2 - 2ax + \lambda^2 a^2 = 0,$$

where
$$a = \frac{k\rho}{\sqrt{1-\lambda^2}};$$

PROPOSITION 50.

To find the third binomial straight line.

Let two numbers AC , CB be set out such that the sum of them AB has to BC the ratio which a square number has to a square number, but has not to AC the ratio which a square number has to a square number.



Let any other number D , not square, be set out also, and let it not have to either of the numbers BA , AC the ratio which a square number has to a square number.

Let any rational straight line E be set out, and let it be contrived that, as D is to AB , so is the square on E to the square on FG ; [x. 6, Por.]

therefore the square on E is commensurable with the square on FG . [x. 6]

And E is rational;
 therefore FG is also rational.

And, since D has not to AB the ratio which a square number has to a square number,
 neither has the square on E to the square on FG the ratio which a square number has to a square number ;
 therefore E is incommensurable in length with FG . [x. 9]

Next let it be contrived that, as the number BA is to AC ,
 so is the square on FG to the square on GH ; [x. 6, Por.]
 therefore the square on FG is commensurable with the square on GH . [x. 6]

But FG is rational ;
 therefore GH is also rational.

And, since BA has not to AC the ratio which a square number has to a square number,
 neither has the square on FG to the square on HG the ratio which a square number has to a square number ;
 therefore FG is incommensurable in length with GH . [x. 9]

Therefore FG , GH are rational straight lines commensurable in square only ;
 therefore FH is binomial. [x. 36]

I say next that it is also a third binomial straight line.

For since, as D is to AB , so is the square on E to the square on FG ,

and, as BA is to AC , so is the square on FG to the square on GH ,

therefore, *ex aequali*, as D is to AC , so is the square on E to the square on GH . [v. 22]

But D has not to AC the ratio which a square number has to a square number ;

therefore neither has the square on E to the square on GH the ratio which a square number has to a square number ;
 therefore E is incommensurable in length with GH . [x. 9]

And since, as BA is to AC , so is the square on FG to the square on GH ,

therefore the square on FG is greater than the square on GH .

Let then the squares on GH , K be equal to the square on FG ;

therefore, *convertendo*, as AB is to BC , so is the square on FG to the square on K . [v. 19, Por.]

But AB has to BC the ratio which a square number has to a square number ;

therefore the square on FG also has to the square on K the ratio which a square number has to a square number ;

therefore FG is commensurable in length with K . [x. 9]

Therefore the square on FG is greater than the square on GH by the square on a straight line commensurable with FG .

And FG , GH are rational straight lines commensurable in square only, and neither of them is commensurable in length with E .

Therefore FH is a third binomial straight line.

Q. E. D.

Let ρ be a rational straight line.

Take the numbers $q (m^2 - n^2)$, qn^2 ,

and let p be a third number which is not a square and which has not to qm^2 or $q (m^2 - n^2)$ the ratio of square to square.

Take x such that $p : qm^2 = \rho^2 : x^2$ (1).

Thus x is rational and $\sim \rho$ (2).

Next suppose that $qm^2 : q (m^2 - n^2) = x^2 : y^2$ (3).

It follows that y is rational and $\sim x$ (4).

Thus $(x + y)$ is a *binomial* straight line.

Again, from (1) and (3), *ex aequali*,

$$p : q (m^2 - n^2) = \rho^2 : y^2$$
(5),

whence $y \sim \rho$ (6).

Suppose that $x^2 - y^2 = z^2$.

Then, from (3), *convertendo*,

$$qm^2 : qn^2 = x^2 : z^2,$$

whence $z \sim x$.

Thus $\sqrt{x^2 - y^2} \sim x$,

and x , y are both $\sim \rho$;

therefore $x + y$ is a *third binomial* straight line.

Now, from (1), $x = \rho \cdot \frac{m \sqrt{q}}{\sqrt{p}}$,

and, by (5), $y = \rho \cdot \frac{\sqrt{m^2 - n^2} \cdot \sqrt{q}}{\sqrt{p}}$.

Thus the *third binomial* is

$$\sqrt{\frac{q}{p}} \cdot \rho (m + \sqrt{m^2 - n^2}),$$

which we may write in the form

$$m \sqrt{k} \cdot \rho + m \sqrt{k} \cdot \rho \sqrt{1 - \lambda^2}.$$

The corresponding *third apotome* [x. 87] is

$$m\sqrt{k} \cdot \rho - m\sqrt{k} \cdot \rho \sqrt{1 - \lambda^2}.$$

The two expressions are accordingly the roots of the equation

$$x^2 - 2m\sqrt{k} \cdot \rho x + \lambda^2 m^2 k \rho^2 = 0,$$

or

$$x^2 - 2ax + \lambda^2 a^2 = 0,$$

where

$$a = m\sqrt{k} \cdot \rho.$$

See also note on x. 53 (*ad fin.*).

PROPOSITION 51.

To find the fourth binomial straight line.

Let two numbers AC , CB be set out such that AB neither has to BC , nor yet to AC , the ratio which a square number has to a square number.

Let a rational straight line D be set out, and let EF be commensurable in length with D ; therefore EF is also rational.

Let it be contrived that, as the number BA is to AC , so is the square on EF to the square on FG ;

[x. 6, Por.]

therefore the square on EF is commensurable with the square on FG ;

[x. 6]

therefore FG is also rational.

Now, since BA has not to AC the ratio which a square number has to a square number,

neither has the square on EF to the square on FG the ratio which a square number has to a square number;

therefore EF is incommensurable in length with FG . [x. 9]

Therefore EF , FG are rational straight lines commensurable in square only;

so that EG is binomial.

I say next that it is also a fourth binomial straight line.

For since, as BA is to AC , so is the square on EF to the square on FG ,

therefore the square on EF is greater than the square on FG .

Let then the squares on FG , H be equal to the square on EF ;



therefore, *convertendo*, as the number AB is to BC , so is the square on EF to the square on H . [v. 19, Por.]

But AB has not to BC the ratio which a square number has to a square number;

therefore neither has the square on EF to the square on H the ratio which a square number has to a square number.

Therefore EF is incommensurable in length with H ; [x. 9] therefore the square on EF is greater than the square on GF by the square on a straight line incommensurable with EF .

And EF , FG are rational straight lines commensurable in square only, and EF is commensurable in length with D .

Therefore EG is a fourth binomial straight line.

Q. E. D.

Take numbers m , n such that $(m+n)$ has not to either m or n the ratio of square to square.

Take x such that $(m+n) : m = k^2\rho^2 : x^2$,

whence
$$x = k\rho \sqrt{\frac{m}{m+n}} = \frac{k\rho}{\sqrt{1+\lambda}}, \text{ say.}$$

Then $k\rho + x$, or $k\rho + \frac{k\rho}{\sqrt{1+\lambda}}$, is a *fourth binomial* straight line.

For $\sqrt{k^2\rho^2 - x^2}$ is incommensurable in length with $k\rho$, and $k\rho$ is commensurable in length with ρ .

The corresponding *fourth apotome* [x. 88] is

$$k\rho - \frac{k\rho}{\sqrt{1+\lambda}}.$$

The equation of which the two expressions are the roots is

$$x^2 - 2k\rho \cdot x + \frac{\lambda}{1+\lambda} k^2\rho^2 = 0,$$

or

$$x^2 - 2ax + \frac{\lambda}{1+\lambda} a^2 = 0,$$

where

$$a = k\rho.$$

PROPOSITION 52.

To find the fifth binomial straight line.

Let two numbers AC , CB be set out such that AB has not to either of them the ratio which a square number has to a square number;

let any rational straight line D be set out,

and let EF be commensurable with D ;
therefore EF is rational.

Let it be contrived that, as CA is to AB , so is the square on EF to the square on FG . [x. 6, Por.]

But CA has not to AB the ratio which a square number has to a square number;
therefore neither has the square on EF to the square on FG the ratio which a square number has to a square number.

Therefore EF , FG are rational straight lines commensurable in square only; [x. 9]
therefore EG is binomial. [x. 36]



I say next that it is also a fifth binomial straight line.

For since, as CA is to AB , so is the square on EF to the square on FG ,

inversely, as BA is to AC , so is the square on FG to the square on FE ;

therefore the square on GF is greater than the square on FE .

Let then the squares on EF , H be equal to the square on GF ;

therefore, *convertendo*, as the number AB is to BC , so is the square on GF to the square on H . [v. 19, Por.]

But AB has not to BC the ratio which a square number has to a square number;

therefore neither has the square on FG to the square on H the ratio which a square number has to a square number.

Therefore FG is incommensurable in length with H ; [x. 9]
so that the square on FG is greater than the square on FE by the square on a straight line incommensurable with FG .

And GF , FE are rational straight lines commensurable in square only, and the lesser term EF is commensurable in length with the rational straight line D set out.

Therefore EG is a fifth binomial straight line.

Q. E. D.

If m , n be numbers of the kind taken in the last proposition, take x such that

$$m : (m + n) = k^2 p^2 : x^2.$$

In this case

$$x = k\rho \sqrt{\frac{m+n}{m}}$$

$$= k\rho \sqrt{1+\lambda}, \text{ say,}$$

and $x > k\rho$.

Then $k\rho \sqrt{1+\lambda} + k\rho$ is a *fifth binomial* straight line.

For $\sqrt{x^2 - k^2\rho^2}$, or $\sqrt{\lambda} \cdot k\rho$, is incommensurable in length with $k\rho \sqrt{1+\lambda}$, or x ;

and $k\rho$, but not $k\rho \sqrt{1+\lambda}$, is commensurable in length with ρ .

The corresponding *fifth apotome* [x. 89] is

$$k\rho \sqrt{1+\lambda} - k\rho.$$

The equation of which the fifth binomial and the fifth apotome are the roots is

$$x^2 - 2k\rho \sqrt{1+\lambda} \cdot x + \lambda k^2\rho^2 = 0,$$

or

$$x^2 - 2\alpha x + \frac{\lambda}{1+\lambda} \alpha^2 = 0,$$

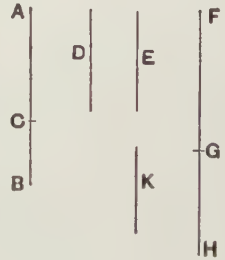
where

$$\alpha = k\rho \sqrt{1+\lambda}.$$

PROPOSITION 53.

To find the sixth binomial straight line.

Let two numbers AC , CB be set out such that AB has not to either of them the ratio which a square number has to a square number; and let there also be another number D which is not square and which has not to either of the numbers BA , AC the ratio which a square number has to a square number.



Let any rational straight line E be set out,

and let it be contrived that, as D is to AB , so is the square on E to the square on FG ;

[x. 6, Por.]

therefore the square on E is commensurable with the square on FG .

[x. 6]

And E is rational;

therefore FG is also rational.

Now, since D has not to AB the ratio which a square number has to a square number,

neither has the square on E to the square on FG the ratio which a square number has to a square number ;

therefore E is incommensurable in length with FG . [x. 9]

Again, let it be contrived that, as BA is to AC , so is the square on FG to the square on GH . [x. 6, Por.]

Therefore the square on FG is commensurable with the square on HG . [x. 6]

Therefore the square on HG is rational ;
therefore HG is rational.

And, since BA has not to AC the ratio which a square number has to a square number,

neither has the square on FG to the square on GH the ratio which a square number has to a square number ;

therefore FG is incommensurable in length with GH . [x. 9]

Therefore FG , GH are rational straight lines commensurable in square only ;

therefore FH is binomial. [x. 36]

It is next to be proved that it is also a sixth binomial straight line.

For since, as D is to AB , so is the square on E to the square on FG ,

and also, as BA is to AC , so is the square on FG to the square on GH ,

therefore, *ex aequali*, as D is to AC , so is the square on E to the square on GH . [v. 22]

But D has not to AC the ratio which a square number has to a square number ;

therefore neither has the square on E to the square on GH the ratio which a square number has to a square number ;

therefore E is incommensurable in length with GH . [x. 9]

But it was also proved incommensurable with FG ;

therefore each of the straight lines FG , GH is incommensurable in length with E .

And, since, as BA is to AC , so is the square on FG to the square on GH ,

therefore the square on FG is greater than the square on GH .

Let then the squares on GH , K be equal to the square on FG ;

therefore, *convertendo*, as AB is to BC , so is the square on FG to the square on K . [v. 19, Por.]

But AB has not to BC the ratio which a square number has to a square number;

so that neither has the square on FG to the square on K the ratio which a square number has to a square number.

Therefore FG is incommensurable in length with K ; [x. 9] therefore the square on FG is greater than the square on GH by the square on a straight line incommensurable with FG .

And FG , GH are rational straight lines commensurable in square only, and neither of them is commensurable in length with the rational straight line E set out.

Therefore FH is a sixth binomial straight line.

Q. E. D.

Take numbers m , n such that $(m+n)$ has not to either of the numbers m , n the ratio of square to square; take also a third number p , which is not square, and which has not to either of the numbers $(m+n)$, m the ratio of square to square.

Let $p : (m+n) = \rho^2 : x^2$ (1)
and $(m+n) : m = x^2 : y^2$ (2).

Then shall $(x+y)$ be a *sixth binomial* straight line.

For, by (1), x is rational and $\sim \rho$.

By (2), since x is rational,

y is rational and $\sim x$.

Hence x , y are rational and commensurable in square only, so that $(x+y)$ is a binomial straight line.

Again, *ex aequali*, from (1) and (2),

$$p : m = \rho^2 : y^2$$
 (3),

whence $y \sim \rho$.

Thus x , y are both incommensurable in length with ρ .

Lastly, from (2), *convertendo*,

$$(m+n) : n = x^2 : (x^2 - y^2),$$

so that $\sqrt{x^2 - y^2} \sim x$.

Therefore $(x+y)$ is a *sixth binomial* straight line.

Now, from (1) and (3),

$$x = \rho \cdot \sqrt{\frac{m+n}{p}} = \rho \sqrt{k}, \text{ say,}$$

$$y = \rho \cdot \sqrt{\frac{m}{p}} = \rho \sqrt{\lambda}, \text{ say,}$$

and the *sixth binomial* straight line may be written

$$\sqrt{k} \cdot \rho + \sqrt{\lambda} \cdot \rho.$$

The corresponding *sixth apotome* is [x. 90]

$$\sqrt{k} \cdot \rho - \sqrt{\lambda} \cdot \rho;$$

and the equation of which the two expressions are the roots is

$$x^2 - 2\sqrt{k} \cdot \rho x + (k - \lambda) \rho^2 = 0,$$

or

$$x^2 - 2ax + \frac{k - \lambda}{k} a^2 = 0,$$

where $a = \sqrt{k} \cdot \rho$.

Tannery remarks ("De la solution géométrique des problèmes du second degré avant Euclide" in *Mémoires de la Société des sciences physiques et naturelles de Bordeaux*, 2^e Série, T. iv.) that Euclid admits as binomials and apotomes the *third* and *sixth* binomials and apotomes which are the square roots of first binomials and apotomes respectively. Hence the third and sixth binomials and apotomes are the positive roots of *biquadratic* equations of the same form as the quadratics which give as roots the first and fourth binomials and apotomes. But this remark seems to be of no value because (as was pointed out a hundred years ago by Cossali, II. p. 260) the squares of *all the six* binomials and apotomes (including the first and fourth) give *first* binomials and apotomes respectively. Hence we may equally well regard them all as roots of biquadratics reducible to quadratics, or generally as roots of equations of the form

$$x^{2n} \pm 2a \cdot x^{2n-1} \pm q = 0;$$

and nothing is gained by raising the degree of the equations in this way.

It is, of course, easy to see that the most general form of binomial and apotome, viz.

$$\rho \cdot \sqrt{k} \pm \rho \cdot \sqrt{\lambda},$$

give *first* binomials and apotomes when squared.

For the square is $\rho \{ (k + \lambda) \rho \pm 2\sqrt{k\lambda} \cdot \rho \}$; and the expression within the bracket is a first binomial or apotome, because

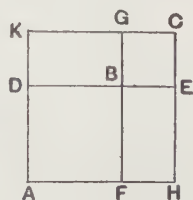
- (1) $k + \lambda > 2\sqrt{k\lambda}$,
- (2) $\sqrt{(k + \lambda)^2 - 4k\lambda} = k - \lambda$, which is $\wedge (k + \lambda)$,
- (3) $(k + \lambda) \rho > \rho$.

LEMMA.

Let there be two squares AB , BC , and let them be placed so that DB is in a straight line with BE ;

therefore FB is also in a straight line with BG .

Let the parallelogram AC be completed; I say that AC is a square, that DG is a mean proportional between AB , BC , and further that DC is a mean proportional between AC , CB .



For, since DB is equal to BF , and BE to BG , therefore the whole DE is equal to the whole FG .

But DE is equal to each of the straight lines AH , KC , and FG is equal to each of the straight lines AK , HC ; [I. 34]

therefore each of the straight lines AH , KC is also equal to each of the straight lines AK , HC .

Therefore the parallelogram AC is equilateral.

And it is also rectangular ;

therefore AC is a square.

And since, as FB is to BG , so is DB to BE ,

while, as FB is to BG , so is AB to DG ,

and, as DB is to BE , so is DG to BC , [vi. 1]

therefore also, as AB is to DG , so is DG to BC . [v. 11]

Therefore DG is a mean proportional between AB , BC .

I say next that DC is also a mean proportional between AC , CB .

For since, as AD is to DK , so is KG to GC —

for they are equal respectively—

and, *componendo*, as AK is to KD , so is KC to CG , [v. 18]

while, as AK is to KD , so is AC to CD ,

and, as KC is to CG , so is DC to CB , [vi. 1]

therefore also, as AC is to DC , so is DC to BC . [v. 11]

Therefore DC is a mean proportional between AC , CB .

Being what it was proposed to prove.

It is here proved that

$$x^2 : xy = xy : y^2,$$

and

$$(x + y)^2 : (x + y)y = (x + y)y : y^2.$$

The first of the two results is proved in the course of x. 25 (lines 6—8 on p. 57 above). This fact may, I think, suggest doubt as to the genuineness of this Lemma.

PROPOSITION 54.

If an area be contained by a rational straight line and the first binomial, the "side" of the area is the irrational straight line which is called binomial.

For let the area AC be contained by the rational straight line AB and the first binomial AD ;

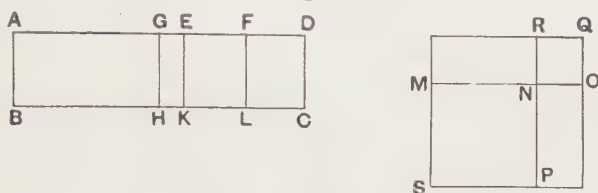
I say that the "side" of the area AC is the irrational straight line which is called binomial.

For, since AD is a first binomial straight line, let it be divided into its terms at E ,

and let AE be the greater term.

It is then manifest that AE , ED are rational straight lines commensurable in square only, the square on AE is greater than the square on ED by the square on a straight line commensurable with AE , and AE is commensurable in length with the rational straight line AB set out. [x. Def. II. 1]

Let ED be bisected at the point F .



Then, since the square on AE is greater than the square on ED by the square on a straight line commensurable with AE ,

therefore, if there be applied to the greater AE a parallelogram equal to the fourth part of the square on the less, that is, to the square on EF , and deficient by a square figure, it divides it into commensurable parts. [x. 17]

Let then the rectangle AG , GE equal to the square on EF be applied to AE ;

therefore AG is commensurable in length with EG .

Let GH , EK , FL be drawn from G , E , F parallel to either of the straight lines AB , CD ;

let the square SN be constructed equal to the parallelogram AH , and the square NQ equal to GK , [II. 14]

and let them be placed so that MN is in a straight line with NO ;

therefore RN is also in a straight line with NP .

And let the parallelogram SQ be completed;

therefore SQ is a square.

[Lemma]

Now, since the rectangle AG , GE is equal to the square on EF ,

therefore, as AG is to EF , so is FE to EG ; [VI. 17]

therefore also, as AH is to EL , so is EL to KG ; [VI. 1]

therefore EL is a mean proportional between AH , GK .

But AH is equal to SN , and GK to NQ ;

therefore EL is a mean proportional between SN , NQ .

But MR is also a mean proportional between the same SN, NQ ; [Lemma]

therefore EL is equal to MR ,
so that it is also equal to PO .

But AH, GK are also equal to SN, NQ ;
therefore the whole AC is equal to the whole SQ , that is, to the square on MO ;
therefore MO is the "side" of AC .

I say next that MO is binomial.

For, since AG is commensurable with GE ,
therefore AE is also commensurable with each of the straight lines AG, GE . [x. 15]

But AE is also, by hypothesis, commensurable with AB ;
therefore AG, GE are also commensurable with AB . [x. 12]

And AB is rational;
therefore each of the straight lines AG, GE is also rational;
therefore each of the rectangles AH, GK is rational, [x. 19]
and AH is commensurable with GK .

But AH is equal to SN , and GK to NQ ;
therefore SN, NQ , that is, the squares on MN, NO , are rational and commensurable.

And, since AE is incommensurable in length with ED , while AE is commensurable with AG , and DE is commensurable with EF ,

therefore AG is also incommensurable with EF , [x. 13]
so that AH is also incommensurable with EL . [vi. 1, x. 11]

But AH is equal to SN , and EL to MR ;
therefore SN is also incommensurable with MR .

But, as SN is to MR , so is PN to NR ; [vi. 1]
therefore PN is incommensurable with NR . [x. 11]

But PN is equal to MN , and NR to NO ;
therefore MN is incommensurable with NO .

And the square on MN is commensurable with the square on NO ,

and each is rational;

therefore MN, NO are rational straight lines commensurable in square only.

Therefore MO is binomial [x. 36] and the "side" of AC .

Q. E. D.

2. "side." I use the word "side" in the sense explained in the note on x. Def. 4 (p. 13 above), i.e. as short for "side of a square equal to." The Greek is $\eta\ \tau\acute{o}\ \chi\omega\rho\acute{o}\nu\ \delta\upsilon\sigma\alpha\mu\acute{\epsilon}\nu\eta$.

A first binomial straight line being, as we have seen in x. 48, of the form

$$k\rho + k\rho\sqrt{1-\lambda^2},$$

the problem solved in this proposition is the equivalent of finding the square root of this expression multiplied by ρ , or of

$$\rho(k\rho + k\rho\sqrt{1-\lambda^2}),$$

and of proving that the said square root represents a binomial straight line as defined in x. 36.

The geometrical method corresponds sufficiently closely to the algebraical one which we should use.

First solve the equations

$$\left. \begin{aligned} u + v &= k\rho \\ uv &= \frac{1}{4}k^2\rho^2(1-\lambda^2) \end{aligned} \right\} \dots\dots\dots(1).$$

Then, if u, v represent the straight lines so found, put

$$\left. \begin{aligned} x^2 &= \rho u \\ y^2 &= \rho v \end{aligned} \right\} \dots\dots\dots(2);$$

and the straight line $(x+y)$ is the square root required.

The actual algebraical solution of (1) gives

$$u - v = k\rho \cdot \lambda,$$

so that

$$u = \frac{1}{2}k\rho(1+\lambda),$$

$$v = \frac{1}{2}k\rho(1-\lambda),$$

and therefore

$$x = \rho\sqrt{\frac{k}{2}(1+\lambda)},$$

$$y = \rho\sqrt{\frac{k}{2}(1-\lambda)},$$

and

$$x + y = \rho\sqrt{\frac{k}{2}(1+\lambda)} + \rho\sqrt{\frac{k}{2}(1-\lambda)}.$$

This is clearly a binomial straight line as defined in x. 36.

Since Euclid has to express his results by straight lines in his figure, and has no symbols to make the result obvious by inspection, he is obliged to prove (1) that $(x+y)$ is the square root of $\rho(k\rho + k\rho\sqrt{1-\lambda^2})$, and (2) that $(x+y)$ is a binomial straight line, in the following manner.

First, he proves, by means of the preceding Lemma, that

$$xy = \frac{k}{2}\rho^2\sqrt{1-\lambda^2} \dots\dots\dots(3);$$

therefore

$$\begin{aligned} (x+y)^2 &= x^2 + y^2 + 2xy \\ &= \rho(u+v) + 2xy \\ &= k\rho^2 + k\rho^2\sqrt{1-\lambda^2}, \text{ by (1) and (3),} \end{aligned}$$

so that

$$x+y = \sqrt{\rho(k\rho + k\rho\sqrt{1-\lambda^2})}.$$

Secondly, it results from (1), [by x. 17], that

$$u \sim v,$$

so that u, v are both $\sim (u+v)$, and therefore $\sim \rho$ (4);

thus u, v are rational,

whence $\rho u, \rho v$ are both rational, and

$$\rho u \sim \rho v.$$

Therefore x^2, y^2 are rational and commensurable(5).

Next, $k\rho \sim k\rho \sqrt{1-\lambda^2}$,

and $k\rho \sim u$, while $k\rho \sqrt{1-\lambda^2} \sim \frac{1}{2}k\rho \sqrt{1-\lambda^2}$;

therefore

$$u \sim \frac{1}{2}k\rho \sqrt{1-\lambda^2},$$

whence

$$\rho u \sim \frac{1}{2}k\rho^2 \sqrt{1-\lambda^2},$$

or

$$x^2 \sim xy,$$

so that

$$x \sim y.$$

By this and (5), x, y are rational and $\sim$, so that $(x+y)$ is a binomial straight line. [x. 36]

x. 91 will prove in like manner that a like theorem holds for *apotomes*, viz. that

$$\rho \sqrt{\frac{k}{2}(1+\lambda)} - \rho \sqrt{\frac{k}{2}(1-\lambda)} = \sqrt{\rho(k\rho - k\rho \sqrt{1-\lambda^2})}.$$

Since the *first binomial* straight line and the *first apotome* are the roots of the equation

$$x^2 - 2k\rho \cdot x + \lambda^2 k^2 \rho^2 = 0,$$

this proposition and x. 91 give us the solution of the biquadratic equation

$$x^4 - 2k\rho^2 \cdot x^2 + \lambda^2 k^2 \rho^4 = 0.$$

PROPOSITION 55.

If an area be contained by a rational straight line and the second binomial, the "side" of the area is the irrational straight line which is called a first bimedial.

For let the area $ABCD$ be contained by the rational straight line AB and the second binomial AD ;

I say that the "side" of the area AC is a first bimedial straight line.

For, since AD is a second binomial straight line, let it be divided into its terms at E , so that AE is the greater term ;
therefore AE, ED are rational straight lines commensurable in square only,

the square on AE is greater than the square on ED by the square on a straight line commensurable with AE ,

and the lesser term ED is commensurable in length with AB .

[x. Def. 11. 2]

Let ED be bisected at F ,

and let there be applied to AE the rectangle AG, GE equal to the square on EF and deficient by a square figure ;

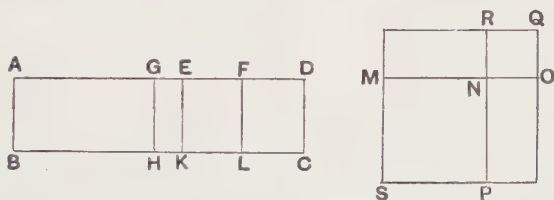
therefore AG is commensurable in length with GE . [x. 17]

Through G, E, F let GH, EK, FL be drawn parallel to
20 AB, CD ,

let the square SN be constructed equal to the parallelogram AH , and the square NQ equal to GK ,

and let them be placed so that MN is in a straight line with NO ;

25 therefore RN is also in a straight line with NP .



Let the square SQ be completed.

It is then manifest from what was proved before that MR is a mean proportional between SN, NQ and is equal to EL , and that MO is the "side" of the area AC .

30 It is now to be proved that MO is a first bimedial straight line.

Since AE is incommensurable in length with ED ,

while ED is commensurable with AB ,

therefore AE is incommensurable with AB . [x. 13]

And, since AG is commensurable with EG ,

35 AE is also commensurable with each of the straight lines AG, GE . [x. 15]

But AE is incommensurable in length with AB ;

therefore AG, GE are also incommensurable with AB . [x. 13]

Therefore BA, AG and BA, GE are pairs of rational
40 straight lines commensurable in square only ;

so that each of the rectangles AH, GK is medial. [x. 21]

Hence each of the squares SN, NQ is medial.

Therefore MN, NO are also medial.

And, since AG is commensurable in length with GE ,

45 AH is also commensurable with GK , [vi. 1, x. 11]

that is, SN is commensurable with NQ ,

that is, the square on MN with the square on NO .

And, since AE is incommensurable in length with ED ,
 while AE is commensurable with AG ,
 50 and ED is commensurable with EF ,
 therefore AG is incommensurable with EF ; [x. 13]
 so that AH is also incommensurable with EL ,
 that is, SN is incommensurable with MR ,
 that is, PN with NR , [vi. 1, x. 11]
 55 that is, MN is incommensurable in length with NO .

But MN , NO were proved to be both medial and commensurable in square;

therefore MN , NO are medial straight lines commensurable in square only.

60 I say next that they also contain a rational rectangle.

For, since DE is, by hypothesis, commensurable with each of the straight lines AB , EF ,

therefore EF is also commensurable with EK . [x. 12]

And each of them is rational;

65 therefore EL , that is, MR is rational, [x. 19]

and MR is the rectangle MN , NO .

But, if two medial straight lines commensurable in square only and containing a rational rectangle be added together, the whole is irrational and is called a first bimedral straight line.

70 Therefore MO is a first bimedral straight line. [x. 37]

Q. E. D.

39. Therefore BA , AG and BA , GE are pairs of rational straight lines commensurable in square only. The text has "Therefore BA , AG , GE are rational straight lines commensurable in square only," which I have altered because it would naturally convey the impression that *any two* of the three straight lines are commensurable in square only, whereas AG , GE are commensurable in length (l. 18), and it is only the other two pairs which are commensurable in square only.

A *second binomial* straight line being [x. 49] of the form

$$\frac{k\rho}{\sqrt{1-\lambda^2}} + k\rho,$$

the present proposition is equivalent to finding the *square root of the expression*

$$\rho \left(\frac{k\rho}{\sqrt{1-\lambda^2}} + k\rho \right):$$

As in the last proposition, Euclid finds u, v from the equations

$$\left. \begin{aligned} u + v &= \frac{k\rho}{\sqrt{1-\lambda^2}} \\ uv &= \frac{1}{2}k^2\rho^2 \end{aligned} \right\} \dots\dots\dots (1),$$

then finds x, y from the equations

$$\left. \begin{aligned} x^2 &= \rho u \\ y^2 &= \rho v \end{aligned} \right\} \dots\dots\dots (2),$$

and then proves (a) that

$$x + y = \sqrt{\rho \left(\frac{k\rho}{\sqrt{1-\lambda^2}} + k\rho \right)},$$

and (β) that $(x + y)$ is a first bimedial straight line [x. 37].

The steps in the proof are as follows.

For (a) reference to the corresponding part of the previous proposition suffices.

(β) By (1) and x. 17,

$$u \sim v;$$

therefore u, v are both rational and $\sim(u+v)$, and therefore $\sim \rho$ [by (1)]... (3).

Hence $\rho u, \rho v$, or x^2, y^2 , are *medial* areas,

so that x, y are also medial (4).

But, since $u \sim v$,

$$x^2 \sim y^2 \dots\dots\dots (5).$$

Again $(u + v)$, or $\frac{k\rho}{\sqrt{1-\lambda^2}}, \sim k\rho$,

so that

$$u \sim \frac{1}{2}k\rho,$$

whence

$$\rho u \sim \frac{1}{2}k\rho^2,$$

or

$$x^2 \sim xy,$$

and

$$x \sim y \dots\dots\dots (6).$$

Thus [(4), (5), (6)] x, y are medial and $\sim$.

Lastly, $xy = \frac{1}{2}k\rho^2$, which is rational.

Therefore $(x + y)$ is a *first bimedial* straight line.

The actual straight lines obtained from (1) are

$$\left. \begin{aligned} u &= \frac{1}{2} \frac{1 + \lambda}{\sqrt{1 - \lambda^2}} k\rho \\ v &= \frac{1}{2} \frac{1 - \lambda}{\sqrt{1 - \lambda^2}} k\rho \end{aligned} \right\},$$

so that

$$x + y = \rho \sqrt{\frac{k}{2} \left(\frac{1 + \lambda}{1 - \lambda} \right)^{\frac{1}{2}}} + \rho \sqrt{\frac{k}{2} \left(\frac{1 - \lambda}{1 + \lambda} \right)^{\frac{1}{2}}}.$$

The corresponding *first apotome of a medial* straight line found in x. 92 being the same thing with a *minus* sign between the terms, the two expressions are the roots of the biquadratic

$$x^4 - \frac{2k\rho^2}{\sqrt{1-\lambda^2}} x^2 + \frac{\lambda^2}{1-\lambda^2} k^2\rho^4 = 0,$$

being the equation in x^2 corresponding to that in x in x. 49.

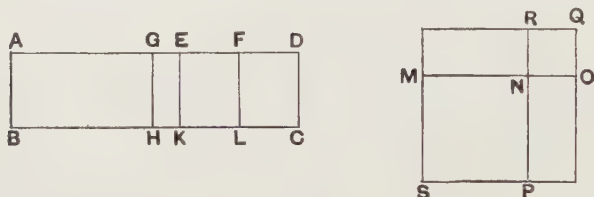
PROPOSITION 56.

If an area be contained by a rational straight line and the third binomial, the "side" of the area is the irrational straight line called a second binomial.

For let the area $ABCD$ be contained by the rational straight line AB and the third binomial AD divided into its terms at E , of which terms AE is the greater;

I say that the "side" of the area AC is the irrational straight line called a second binomial.

For let the same construction be made as before.



Now, since AD is a third binomial straight line, therefore AE , ED are rational straight lines commensurable in square only, the square on AE is greater than the square on ED by the square on a straight line commensurable with AE , and neither of the terms AE , ED is commensurable in length with AB . [x. Def. 11. 3]

Then, in manner similar to the foregoing, we shall prove that MO is the "side" of the area AC , and MN , NO are medial straight lines commensurable in square only; so that MO is binomial.

It is next to be proved that it is also a second binomial straight line.

Since DE is incommensurable in length with AB , that is, with EK , and DE is commensurable with EF , therefore EF is incommensurable in length with EK . [x. 13]

And they are rational;

therefore FE , EK are rational straight lines commensurable in square only.

Therefore EL , that is, MR , is medial. [x. 21]

And it is contained by MN , NO ;

therefore the rectangle MN , NO is medial.

Therefore MO is a second bimedial straight line. [x. 38]

Q. E. D.

This proposition in like manner is the equivalent of finding the square root of the product of ρ and the *third binomial* [x. 50], i.e. of the expression

$$\rho (\sqrt{k} \cdot \rho + \sqrt{k} \cdot \rho \sqrt{1 - \lambda^2}).$$

As before, put

$$\left. \begin{aligned} u + v &= \sqrt{k} \cdot \rho \\ uv &= \frac{1}{4} k \rho^2 (1 - \lambda^2) \end{aligned} \right\} \dots\dots\dots (1).$$

Next, u , v being found, let

$$x^2 = \rho u,$$

$$y^2 = \rho v;$$

then $(x + y)$ is the square root required and is a *second bimedial* straight line.

[x. 38]

For, as in the last proposition, it is proved that $(x + y)$ is the square root, and x , y are medial and ~.

Again, $xy = \frac{1}{2} \sqrt{k} \cdot \rho^2 \sqrt{1 - \lambda^2}$, which is *medial*.

Hence $(x + y)$ is a *second bimedial* straight line.

By solving equations (1), we find

$$u = \frac{1}{2} (\sqrt{k} \cdot \rho + \lambda \sqrt{k} \cdot \rho),$$

$$v = \frac{1}{2} (\sqrt{k} \cdot \rho - \lambda \sqrt{k} \cdot \rho),$$

and

$$x + y = \rho \sqrt{\frac{\sqrt{k}}{2} (1 + \lambda)} + \rho \sqrt{\frac{\sqrt{k}}{2} (1 - \lambda)}.$$

The corresponding *second apotome of a medial* found in x. 93 is the same thing with a minus sign between the terms, and the two are the roots (cf. note on x. 50) of the biquadratic equation

$$x^4 - 2 \sqrt{k} \cdot \rho^2 x^2 + \lambda^2 k \rho^4 = 0.$$

PROPOSITION 57.

If an area be contained by a rational straight line and the fourth binomial, the "side" of the area is the irrational straight line called major.

For let the area AC be contained by the rational straight line AB and the fourth binomial AD divided into its terms at E , of which terms let AE be the greater;

I say that the "side" of the area AC is the irrational straight line called major.

For, since AD is a fourth binomial straight line, therefore AE , ED are rational straight lines commensurable in square only,

the square on AE is greater than the square on ED by the square on a straight line incommensurable with AE , and AE is commensurable in length with AB . [x. Def. II. 4]

Let DE be bisected at F ,

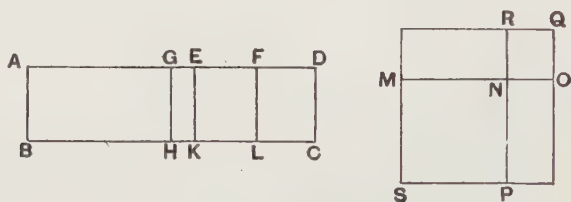
and let there be applied to AE a parallelogram, the rectangle AG , GE , equal to the square on EF ;

therefore AG is incommensurable in length with GE . [x. 18]

Let GH , EK , FL be drawn parallel to AB ,

and let the rest of the construction be as before ;

it is then manifest that MO is the "side" of the area AC .



It is next to be proved that MO is the irrational straight line called major.

Since AG is incommensurable with EG ,

AH is also incommensurable with GK , that is, SN with NQ ; [vi. 1, x. 11]

therefore MN , NO are incommensurable in square.

And, since AE is commensurable with AB ,

AK is rational ; [x. 19]

and it is equal to the squares on MN , NO ;

therefore the sum of the squares on MN , NO is also rational.

And, since DE is incommensurable in length with AB , that is, with EK ,

while DE is commensurable with EF ,

therefore EF is incommensurable in length with EK . [x. 13]

Therefore EK , EF are rational straight lines commensurable in square only ;

therefore LE , that is, MR , is medial.

[x. 21]

And it is contained by MN , NO ;

therefore the rectangle MN , NO is medial.

And the [sum] of the squares on MN , NO is rational, and MN , NO are incommensurable in square.

But, if two straight lines incommensurable in square and making the sum of the squares on them rational, but the rectangle contained by them medial, be added together, the whole is irrational and is called major. [x. 39]

Therefore MO is the irrational straight line called major and is the "side" of the area AC . Q. E. D.

The problem here is to find the square root of the expression [cf. x. 51]

$$\rho \left(k\rho + \frac{k\rho}{\sqrt{1+\lambda}} \right).$$

The procedure is the same.

Find u , v from the equations

$$\left. \begin{aligned} u+v &= k\rho \\ uv &= \frac{1}{4} \frac{k^2 \rho^2}{1+\lambda} \end{aligned} \right\} \dots\dots\dots (1),$$

and, if

$$\left. \begin{aligned} x^2 &= \rho u \\ y^2 &= \rho v \end{aligned} \right\} \dots\dots\dots (2),$$

$(x+y)$ is the required square root.

To prove that $(x+y)$ is the *major* irrational straight line Euclid argues thus.

By x. 18,

$$u \sim v,$$

therefore

$$\rho u \sim \rho v,$$

or

$$x^2 \sim y^2,$$

so that

$$x \sim y \dots\dots\dots (3).$$

Now, since $(u+v) \sim \rho$,

$$(u+v)\rho, \text{ or } (x^2+y^2), \text{ is a rational area} \dots\dots\dots (4).$$

$$\text{Lastly, } xy = \frac{1}{2} \frac{k\rho^2}{\sqrt{1+\lambda}}, \text{ which is a medial area} \dots\dots\dots (5).$$

Thus [(3), (4), (5)] $(x+y)$ is a *major* irrational straight line. [x. 39]

Actual solution gives

$$u = \frac{1}{2} k\rho \left(1 + \sqrt{\frac{\lambda}{1+\lambda}} \right),$$

$$v = \frac{1}{2} k\rho \left(1 - \sqrt{\frac{\lambda}{1+\lambda}} \right),$$

$$\text{and } x+y = \rho \cdot \sqrt{\frac{k}{2} \left(1 + \sqrt{\frac{\lambda}{1+\lambda}} \right)} + \rho \cdot \sqrt{\frac{k}{2} \left(1 - \sqrt{\frac{\lambda}{1+\lambda}} \right)}$$

The corresponding square root found in x. 94 is the *minor* irrational straight line, the terms being separated by a *minus* sign, and the two straight lines are the roots (cf. note on x. 51) of the biquadratic equation

$$x^4 - 2k\rho^2 \cdot x^2 + \frac{\lambda}{1+\lambda} k^2 \rho^4 = 0.$$

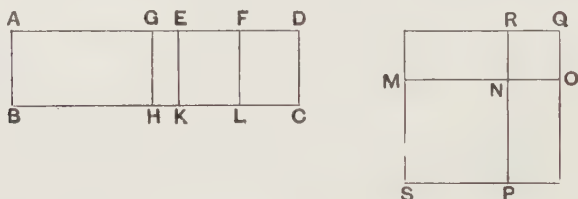
PROPOSITION 58.

If an area be contained by a rational straight line and the fifth binomial, the "side" of the area is the irrational straight line called the side of a rational plus a medial area.

For let the area AC be contained by the rational straight line AB and the fifth binomial AD divided into its terms at E , so that AE is the greater term ;

I say that the "side" of the area AC is the irrational straight line called the side of a rational plus a medial area.

For let the same construction be made as before shown ; it is then manifest that MO is the "side" of the area AC .



It is then to be proved that MO is the side of a rational plus a medial area.

For, since AG is incommensurable with GE , [x. 18]
 therefore AH is also incommensurable with HE , [vi. 1, x. 11]
 that is, the square on MN with the square on NO ;
 therefore MN , NO are incommensurable in square.

And, since AD is a fifth binomial straight line, and ED the lesser segment,
 therefore ED is commensurable in length with AB .

[x. Def. 11. 5]

But AE is incommensurable with ED ;
 therefore AB is also incommensurable in length with AE .

[x. 13]

Therefore AK , that is, the sum of the squares on MN , NO , is medial.

[x. 21]

And, since DE is commensurable in length with AB , that is, with EK ,

while DE is commensurable with EF ,

therefore EF is also commensurable with EK .

[x. 12]

And EK is rational;

therefore EL , that is, MR , that is, the rectangle MN , NO , is also rational. [x. 19]

Therefore MN , NO are straight lines incommensurable in square which make the sum of the squares on them medial, but the rectangle contained by them rational.

Therefore MO is the side of a rational plus a medial area [x. 40] and is the "side" of the area AC .

Q. E. D.

We have here to find the square root of the expression [cf. x. 52]

$$\rho(k\rho\sqrt{1+\lambda}+k\rho).$$

As usual, we put

$$\left. \begin{aligned} u+v &= k\rho\sqrt{1+\lambda} \\ uv &= \frac{1}{2}k^2\rho^2 \end{aligned} \right\} \dots\dots\dots(1).$$

Then, u , v being found, we take

$$\left. \begin{aligned} x^2 &= \rho u \\ y^2 &= \rho v \end{aligned} \right\} \dots\dots\dots(2),$$

and $(x+y)$, so found, is our required square root.

Euclid's proof of the *class* of $(x+y)$ is as follows:

By x. 18, $u \sim v$;

therefore $\rho u \sim \rho v$,

so that $x^2 \sim y^2$,

and $x \sim y$ (3).

Next $u+v \sim k\rho$
 $\sim \rho$,

whence $\rho(u+v)$, or (x^2+y^2) , is a *medial* area(4).

Lastly, $xy = \frac{1}{2}k\rho^2$, which is a *rational* area(5).

Hence [(3), (4), (5)] $(x+y)$ is the *side of a rational plus a medial area*. [x. 40]

If we solve algebraically, we obtain

$$u = \frac{k\rho}{2} (\sqrt{1+\lambda} + \sqrt{\lambda}),$$

$$v = \frac{k\rho}{2} (\sqrt{1+\lambda} - \sqrt{\lambda}),$$

$$\text{and } x+y = \rho \sqrt{\frac{k}{2} (\sqrt{1+\lambda} + \sqrt{\lambda})} + \rho \sqrt{\frac{k}{2} (\sqrt{1+\lambda} - \sqrt{\lambda})}.$$

The corresponding "side" found in x. 95 is a *straight line which produces with a rational area a medial whole*, being of the form $(x-y)$, where x , y have the same values as above.

The two square roots are (cf. note on x. 52) the roots of the biquadratic equation

$$x^4 - 2k\rho^2\sqrt{1+\lambda} \cdot x^2 + \lambda k^2\rho^4 = 0.$$

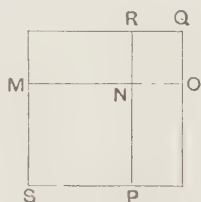
PROPOSITION 59.

If an area be contained by a rational straight line and the sixth binomial, the "side" of the area is the irrational straight line called the side of the sum of two medial areas.

For let the area $ABCD$ be contained by the rational straight line AB and the sixth binomial AD , divided into its terms at E , so that AE is the greater term;

I say that the "side" of AC is the side of the sum of two medial areas.

Let the same construction be made as before shown.



It is then manifest that MO is the "side" of AC , and that MN is incommensurable in square with NO .

Now, since EA is incommensurable in length with AB , therefore EA , AB are rational straight lines commensurable in square only;

therefore AK , that is, the sum of the squares on MN , NO , is medial. [x. 21]

Again, since ED is incommensurable in length with AB , therefore FE is also incommensurable with EK ; [x. 13]

therefore FE , EK are rational straight lines commensurable in square only;

therefore EL , that is, MR , that is, the rectangle MN , NO , is medial. [x. 21]

And, since AE is incommensurable with EF , AK is also incommensurable with EL . [vi. 1, x. 11]

But AK is the sum of the squares on MN , NO , and EL is the rectangle MN , NO ; therefore the sum of the squares on MN , NO is incommensurable with the rectangle MN , NO .

And each of them is medial, and MN , NO are incommensurable in square.

Therefore MO is the side of the sum of two medial areas [x. 41], and is the "side" of AC .

Q. E. D.

Euclid here finds the square root of the expression [cf. x. 53]

$$\rho(\sqrt{k} \cdot \rho + \sqrt{\lambda} \cdot \rho).$$

As usual, we solve the equations

$$\left. \begin{aligned} u + v &= \sqrt{k} \cdot \rho \\ uv &= \frac{1}{4} \lambda \rho^2 \end{aligned} \right\} \dots\dots\dots (1);$$

then, u, v being found, we put

$$\left. \begin{aligned} x^2 &= \rho u \\ y^2 &= \rho v \end{aligned} \right\} \dots\dots\dots (2),$$

and $(x + y)$ is the square root required.

Euclid proves that $(x + y)$ is the *side of (the sum of) two medial areas*, as follows.

As in the last two propositions, x, y are proved to be incommensurable in square.

Now $\sqrt{k} \cdot \rho, \rho$ are commensurable in square only;

therefore $\rho(u + v)$, or $(x^2 + y^2)$, is a *medial area* (3).

Next, $xy = \frac{1}{2} \sqrt{\lambda} \cdot \rho^2$, which is again a *medial area* (4).

Lastly, $\sqrt{k} \cdot \rho \sim \frac{1}{2} \sqrt{\lambda} \cdot \rho,$
so that $\sqrt{k} \cdot \rho^2 \sim \frac{1}{2} \sqrt{\lambda} \cdot \rho^2;$

that is, $(x^2 + y^2) \sim xy$ (5).

Hence [(3), (4), (5)] $(x + y)$ is the *side of the sum of two medial areas*.

Solving the equations algebraically, we have

$$u = \frac{\rho}{2} (\sqrt{k} + \sqrt{k - \lambda}),$$

$$v = \frac{\rho}{2} (\sqrt{k} - \sqrt{k - \lambda}),$$

$$\text{and } x + y = \rho \sqrt{\frac{1}{2} (\sqrt{k} + \sqrt{k - \lambda})} + \rho \sqrt{\frac{1}{2} (\sqrt{k} - \sqrt{k - \lambda})}.$$

The corresponding square root found in x. 96 is $x - y$, where x, y are the same as here.

The two square roots are (cf. note on x. 53) the roots of the biquadratic equation

$$x^4 - 2 \sqrt{k} \cdot \rho^2 x^2 + (k - \lambda) \rho^4 = 0.$$

[LEMMA.

If a straight line be cut into unequal parts, the squares on the unequal parts are greater than twice the rectangle contained by the unequal parts.



Let AB be a straight line, and let it be cut into unequal parts at C , and let AC be the greater;

I say that the squares on AC, CB are greater than twice the rectangle AC, CB .

For let AB be bisected at D .

Since then a straight line has been cut into equal parts at D , and into unequal parts at C ,

therefore the rectangle AC, CB together with the square on CD is equal to the square on AD , [II. 5]

so that the rectangle AC, CB is less than the square on AD ; therefore twice the rectangle AC, CB is less than double of the square on AD .

But the squares on AC, CB are double of the squares on AD, DC ; [II. 9]

therefore the squares on AC, CB are greater than twice the rectangle AC, CB .

Q. E. D.]

We have already remarked (note on x. 44) that the Lemma here proving that

$$x^2 + y^2 > 2xy$$

can hardly be genuine, since the result is used in x. 44.

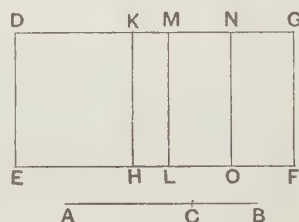
PROPOSITION 60.

The square on the binomial straight line applied to a rational straight line produces as breadth the first binomial.

Let AB be a binomial straight line divided into its terms at C , so that AC is the greater term; let a rational straight line DE be set out,

and let $DEFG$ equal to the square on AB be applied to DE producing DG as its breadth;

I say that DG is a first binomial straight line.



For let there be applied to DE the rectangle DH equal to the square on AC , and KL equal to the square on BC ; therefore the remainder, twice the rectangle AC, CB , is equal to MF .

Let MG be bisected at N , and let NO be drawn parallel [to ML or GF].

Therefore each of the rectangles MO, NF is equal to once the rectangle AC, CB .

Now, since AB is a binomial divided into its terms at C ,

therefore AC, CB are rational straight lines commensurable in square only ; [x. 36]

therefore the squares on AC, CB are rational and commensurable with one another,

so that the sum of the squares on AC, CB is also rational.

And it is equal to DL ; [x. 15]

therefore DL is rational.

And it is applied to the rational straight line DE ;

therefore DM is rational and commensurable in length with DE . [x. 20]

Again, since AC, CB are rational straight lines commensurable in square only,

therefore twice the rectangle AC, CB , that is MF , is medial.

And it is applied to the rational straight line ML ; [x. 21]

therefore MG is also rational and incommensurable in length with ML , that is, DE . [x. 22]

But MD is also rational and is commensurable in length with DE ;

therefore DM is incommensurable in length with MG . [x. 13]

And they are rational ;

therefore DM, MG are rational straight lines commensurable in square only ;

therefore DG is binomial. [x. 36]

It is next to be proved that it is also a first binomial straight line.

Since the rectangle AC, CB is a mean proportional between the squares on AC, CB , [cf. Lemma after x. 53]

therefore MO is also a mean proportional between DH, KL .

Therefore, as DH is to MO , so is MO to KL ,

that is, as DK is to MN , so is MN to MK ; [vi. 1]

therefore the rectangle DK, KM is equal to the square on MN . [vi. 17]

And, since the square on AC is commensurable with the square on CB ,

DH is also commensurable with KL ,

so that DK is also commensurable with KM . [vi. 1, x. 11]

And, since the squares on AC , CB are greater than twice the rectangle AC , CB , [Lemma]

therefore DL is also greater than MF ,

so that DM is also greater than MG . [vi. 1]

And the rectangle DK , KM is equal to the square on MN , that is, to the fourth part of the square on MG , and DK is commensurable with KM .

But, if there be two unequal straight lines, and to the greater there be applied a parallelogram equal to the fourth part of the square on the less and deficient by a square figure, and if it divide it into commensurable parts, the square on the greater is greater than the square on the less by the square on a straight line commensurable with the greater; [x. 17] therefore the square on DM is greater than the square on MG by the square on a straight line commensurable with DM .

And DM , MG are rational,

and DM , which is the greater term, is commensurable in length with the rational straight line DE set out.

Therefore DG is a first binomial straight line. [x. Def. 11. 1]

Q. E. D.

In the hexad of propositions beginning with this we have the solution of the converse problem to that of x. 54—59. We find the *squares* of the irrational straight lines of x. 36—41 and prove that they are respectively equal to the rectangles contained by a rational straight line and the *first, second, third, fourth, fifth and sixth binomials*.

In x. 60 we prove that, $\rho + \sqrt{k} \cdot \rho$ being a *binomial* straight line [x. 36],

$$\frac{(\rho + \sqrt{k} \cdot \rho)^2}{\sigma}$$

is a *first binomial* straight line, and we find it geometrically.

The procedure may be represented thus.

Take x, y, z such that

$$\sigma x = \rho^2,$$

$$\sigma y = k\rho^2$$

$$\sigma \cdot 2z = 2\sqrt{k} \cdot \rho^2,$$

ρ^2 , $k\rho^2$ being of course the squares on the terms of the original binomial, and $2\sqrt{k} \cdot \rho^2$ twice the rectangle contained by them.

Then
$$(x + y) + 2z = \frac{(\rho + \sqrt{k} \cdot \rho)^2}{\sigma},$$

and we have to prove that $(x + y) + 2z$ is a *first binomial* straight line of which $(x + y)$, $2z$ are the terms and $(x + y)$ the greater.

Euclid divides the proof into two parts, showing first that $(x + y) + 2z$ is *some* binomial, and secondly that it is the *first* binomial.

(a) $\rho \sim \sqrt{k} \cdot \rho$, so that $\rho^2, k\rho^2$ are rational and commensurable ;
therefore $\rho^2 + k\rho^2$, or $\sigma(x+y)$, is a rational area,
whence $(x+y)$ is rational and $\sim \sigma$ (1).

Next, $2\rho \cdot \sqrt{k} \cdot \rho$ is a medial area,
so that $\sigma \cdot 2z$ is a medial area,
whence $2z$ is rational but $\sim \sigma$ (2).

Hence [(1), (2)] $(x+y), 2z$ are rational and commensurable in square only(3) ;

thus $(x+y) + 2z$ is a *binomial* straight line. [x. 36]

(β) $\rho^2 : \sqrt{k} \cdot \rho^2 = \sqrt{k} \cdot \rho^2 : k\rho^2$,
so that $\sigma x : \sigma z = \sigma z : \sigma y$,
and $x : z = z : y$,
or $xy = z^2 = \frac{1}{4}(2z)^2$ (4).

Now $\rho^2, k\rho^2$ are commensurable, so that $\sigma x, \sigma y$ are commensurable, and therefore

$$x \sim y \text{(5).}$$

And, since [Lemma] $\rho^2 + k\rho^2 > 2\sqrt{k} \cdot \rho^2$,

$$x + y > 2z.$$

But $(x+y)$ is given, being equal to $\frac{\rho^2 + k\rho^2}{\sigma}$ (6).

Therefore [(4), (5), (6), and x. 17] $\sqrt{(x+y)^2 - (2z)^2} \sim (x+y)$.

And $(x+y), 2z$ are rational and $\sim [(3)]$,

while $(x+y) \sim \sigma [(1)]$.

Hence $(x+y) + 2z$ is a *first binomial*.

The actual value of $(x+y) + 2z$ is, of course,

$$\frac{\rho^2}{\sigma} (1 + k + 2\sqrt{k}).$$

PROPOSITION 61.

The square on the first binomial straight line applied to a rational straight line produces as breadth the second binomial.

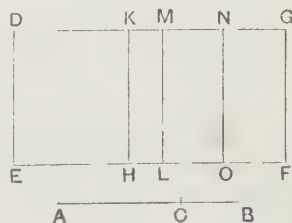
Let AB be a first binomial straight line divided into its medials at C , of which medials AC is the greater ;

let a rational straight line DE be set out,

and let there be applied to DE the parallelogram DF equal to the square on AB , producing DG as its breadth ;

I say that DG is a second binomial straight line.

For let the same construction as before be made.



Then, since AB is a first bimedial divided at C ,
 therefore AC , CB are medial straight lines commensurable in
 square only, and containing a rational rectangle, [x. 37]
 so that the squares on AC , CB are also medial. [x. 21]

Therefore DL is medial. [x. 15 and 23, Por.]

And it has been applied to the rational straight line DE ;
 therefore MD is rational and incommensurable in length
 with DE . [x. 22]

Again, since twice the rectangle AC , CB is rational, MF is
 also rational.

And it is applied to the rational straight line ML ;
 therefore MG is also rational and commensurable in length
 with ML , that is, DE ; [x. 20]
 therefore DM is incommensurable in length with MG . [x. 13]

And they are rational;
 therefore DM , MG are rational straight lines commensurable
 in square only;
 therefore DG is binomial. [x. 36]

It is next to be proved that it is also a second binomial
 straight line.

For, since the squares on AC , CB are greater than twice
 the rectangle AC , CB ,
 therefore DL is also greater than MF ,
 so that DM is also greater than MG . [vi. 1]

And, since the square on AC is commensurable with the
 square on CB ,

DH is also commensurable with KL ,
 so that DK is also commensurable with KM . [vi. 1, x. 11]

And the rectangle DK , KM is equal to the square on MN ;
 therefore the square on DM is greater than the square on
 MG by the square on a straight line commensurable with DM .
 [x. 17]

And MG is commensurable in length with DE .

Therefore DG is a second binomial straight line. [x. Def. 11. 2]

In this case we have to prove that, $(k^{\frac{1}{2}}\rho + k^{\frac{3}{2}}\rho)$ being a *first bimedial*
 straight line, as found in x. 37,

$$\frac{(k^{\frac{1}{2}}\rho + k^{\frac{3}{2}}\rho)^2}{\sigma}$$

is a *second binomial* straight line.

The form of the proposition and the figure being similar to those of x. 60, I can somewhat abbreviate the reproduction of the proof.

Take x, y, z such that

$$\begin{aligned}\sigma x &= k^{\frac{1}{2}}\rho^2, \\ \sigma y &= k^{\frac{3}{2}}\rho^2, \\ \sigma \cdot 2z &= 2k\rho^2.\end{aligned}$$

Then shall $(x + y) + 2z$ be a second binomial.

(a) $k^{\frac{1}{2}}\rho, k^{\frac{3}{2}}\rho$ are medial straight lines commensurable in square only and containing a rational rectangle. [x. 37]

The squares $k^{\frac{1}{2}}\rho^2, k^{\frac{3}{2}}\rho^2$ are medial;

thus the sum, or $\sigma(x + y)$, is medial.

[x. 23, Por.]

Therefore $(x + y)$ is rational and $\sim \sigma$.

And $\sigma \cdot 2z$ is rational;

therefore $2z$ is rational and $\sim \sigma$ (1).

Therefore $(x + y), 2z$ are rational and $\sim$ (2),

so that $(x + y) + 2z$ is a *binomial*.

(β) As before, $(x + y) > 2z$.

Now, $k^{\frac{1}{2}}\rho^2, k^{\frac{3}{2}}\rho^2$ being commensurable,

$$x \sim y.$$

And $xy = z^2$,

$$\text{while } x + y = \frac{k^{\frac{1}{2}}\rho^2 + k^{\frac{3}{2}}\rho^2}{\sigma}.$$

$$\text{Hence [x. 17]} \quad \sqrt{(x + y)^2 - (2z)^2} \sim (x + y) \text{ (3).}$$

But $2z \sim \sigma$, by (1).

Therefore [(1), (2), (3)] $(x + y) + 2z$ is a *second binomial* straight line.

$$\text{Of course } (x + y) + 2z = \frac{\rho^2}{\sigma} \{ \sqrt{k(1 + k)} + 2k \}.$$

PROPOSITION 62.

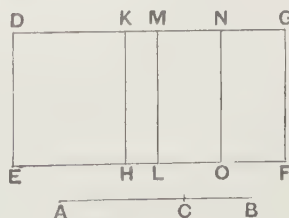
The square on the second binomial straight line applied to a rational straight line produces as breadth the third binomial.

Let AB be a second binomial straight line divided into its medials at C , so that AC is the greater segment;

let DE be any rational straight line, and to DE let there be applied the parallelogram DF equal to the square on AB and producing DG as its breadth;

I say that DG is a third binomial straight line.

Let the same construction be made as before shown.



Then, since AB is a second bimedial divided at C ,
 therefore AC , CB are medial straight lines commensurable in
 square only and containing a medial rectangle, [x. 38]
 so that the sum of the squares on AC , CB is also medial.

[x. 15 and 23 Por.]

And it is equal to DL ;
 therefore DL is also medial.

And it is applied to the rational straight line DE ;
 therefore MD is also rational and incommensurable in length
 with DE . [x. 22]

For the same reason,
 MG is also rational and incommensurable in length with ML ,
 that is, with DE ;
 therefore each of the straight lines DM , MG is rational and
 incommensurable in length with DE .

And, since AC is incommensurable in length with CB ,
 and, as AC is to CB , so is the square on AC to the rectangle
 AC , CB ,
 therefore the square on AC is also incommensurable with the
 rectangle AC , CB . [x. 11]

Hence the sum of the squares on AC , CB is incommen-
 surable with twice the rectangle AC , CB , [x. 12, 13]
 that is, DL is incommensurable with MF ,
 so that DM is also incommensurable with MG . [vi. 1, x. 11]

And they are rational ;
 therefore DG is binomial. [x. 36]

It is to be proved that it is also a third binomial straight line.

In manner similar to the foregoing we may conclude that
 DM is greater than MG ,
 and that DK is commensurable with KM .

And the rectangle DK , KM is equal to the square on
 MN ;
 therefore the square on DM is greater than the square on
 MG by the square on a straight line commensurable with
 DM .

And neither of the straight lines DM , MG is commen-
 surable in length with DE .

Therefore DG is a third binomial straight line. [x. Def. 11. 3]

Q. E. D.

We have to prove that [cf. x. 38]

$$\frac{1}{\sigma} \left(k^{\frac{1}{4}} \rho + \frac{\lambda^{\frac{1}{2}} \rho}{k^{\frac{1}{4}}} \right)^2$$

is a *third binomial* straight line.

Take x, y, z such that

$$\sigma x = k^{\frac{1}{2}} \rho^2,$$

$$\sigma y = \frac{\lambda \rho^2}{k^{\frac{1}{2}}},$$

$$\sigma \cdot 2z = 2 \sqrt{\lambda \cdot \rho^2}.$$

(a) Now $k^{\frac{1}{4}} \rho, \frac{\lambda^{\frac{1}{2}} \rho}{k^{\frac{1}{4}}}$ are medial straight lines commensurable in square only and containing a *medial* rectangle. [x. 38]

The sum of the squares on them, or $\sigma(x+y)$, is *medial*; therefore $(x+y)$ is rational and $\sigma \dots \dots \dots (1)$.

And $\sigma \cdot 2z$ being medial also, $2z$ is rational and $\sigma \dots \dots \dots (2)$.

Now
$$k^{\frac{1}{4}} \rho : \frac{\lambda^{\frac{1}{2}} \rho}{k^{\frac{1}{4}}} = (k^{\frac{1}{4}} \rho)^2 : k^{\frac{1}{4}} \rho \cdot \frac{\lambda^{\frac{1}{2}} \rho}{k^{\frac{1}{4}}}$$
$$= \sigma x : \sigma z,$$

whence $\sigma x \sim \sigma z$.

But $(k^{\frac{1}{4}} \rho)^2 \sim \left\{ (k^{\frac{1}{4}} \rho)^2 + \left(\frac{\lambda^{\frac{1}{2}} \rho}{k^{\frac{1}{4}}} \right)^2 \right\}$, or $\sigma x \sim \sigma(x+y)$, and $\sigma z \sim \sigma \cdot 2z$;

therefore $\sigma(x+y) \sim \sigma \cdot 2z$,
or $(x+y) \sim 2z \dots \dots \dots (3)$.

Hence [(1), (2), (3)] $(x+y) + 2z$ is a binomial straight line. $\dots \dots \dots (4)$.

(β) As before, $(x+y) > 2z$,
and $x \sim y$.

Also $xy = z^2$.

Therefore [x. 17] $\sqrt{(x+y)^2 - (2z)^2} \sim (x+y)$.

And [(1), (2)] neither $(x+y)$ nor $2z$ is $\sim \sigma$.

Therefore $(x+y) + 2z$ is a *third binomial* straight line.

Obviously $(x+y) + 2z = \frac{\rho^2}{\sigma} \left\{ \frac{k+\lambda}{\sqrt{k}} + 2\sqrt{\lambda} \right\}$.

PROPOSITION 63.

The square on the major straight line applied to a rational straight line produces as breadth the fourth binomial.

Let AB be a major straight line divided at C , so that AC is greater than CB ;
let DE be a rational straight line,

and to DE let there be applied the parallelogram DF equal to the square on AB and producing DG as its breadth ;

I say that DG is a fourth binomial straight line.

Let the same construction be made as before shown.

Then, since AB is a major straight line divided at C ,

AC, CB are straight lines incommensurable in square which make the sum of the squares on them rational, but the rectangle contained by them medial. [x. 39]

Since then the sum of the squares on AC, CB is rational, therefore DL is rational ;

therefore DM is also rational and commensurable in length with DE . [x. 20]

Again, since twice the rectangle AC, CB , that is, MF , is medial,

and it is applied to the rational straight line ML ,

therefore MG is also rational and incommensurable in length with DE ; [x. 22]

therefore DM is also incommensurable in length with MG . [x. 13]

Therefore DM, MG are rational straight lines commensurable in square only ;

therefore DG is binomial. [x. 36]



It is to be proved that it is also a fourth binomial straight line.

In manner similar to the foregoing we can prove that DM is greater than MG ,

and that the rectangle DK, KM is equal to the square on MN .

Since then the square on AC is incommensurable with the square on CB ,

therefore DH is also incommensurable with KL ,

so that DK is also incommensurable with KM . [vi. 1, x. 11]

But, if there be two unequal straight lines, and to the greater there be applied a parallelogram equal to the fourth part of the square on the less and deficient by a square figure, and if it divide it into incommensurable parts, then the

square on the greater will be greater than the square on the less by the square on a straight line incommensurable in length with the greater; [x. 18]

therefore the square on DM is greater than the square on MG by the square on a straight line incommensurable with DM .

And DM , MG are rational straight lines commensurable in square only,

and DM is commensurable with the rational straight line DE set out.

Therefore DG is a fourth binomial straight line. [x. Def. II. 4]

Q. E. D.

We have to prove that [cf. x. 39]

$$\frac{1}{\sigma} \left\{ \frac{\rho}{\sqrt{2}} \sqrt{1 + \frac{k}{\sqrt{1+k^2}}} + \frac{\rho}{\sqrt{2}} \sqrt{1 - \frac{k}{\sqrt{1+k^2}}} \right\}^2$$

is a *fourth binomial* straight line.

For brevity we must call this expression

$$\frac{1}{\sigma} (u + v)^2.$$

Take x, y, z such that

$$\left. \begin{aligned} \sigma x &= u^2 \\ \sigma y &= v^2 \\ \sigma \cdot 2z &= 2uv \end{aligned} \right\},$$

wherein it has to be remembered [x. 39] that u, v are incommensurable in square, $(u^2 + v^2)$ is rational, and uv is medial.

(a) $(u^2 + v^2)$, and therefore $\sigma(x + y)$, is rational;

therefore $(x + y)$ is rational and $\cap \sigma \dots \dots \dots (1).$

$2uv$, and therefore $\sigma \cdot 2z$, is medial;

therefore $2z$ is rational and $\cup \sigma \dots \dots \dots (2).$

Thus $(x + y)$, $2z$ are rational and $\sim \dots \dots \dots (3),$

so that $(x + y) + 2z$ is a binomial straight line.

(β) As before, $x + y > 2z$,

and $xy = z^2$.

Now, since $u^2 \cup v^2$,

$\sigma x \cup \sigma y$, or $x \cup y$.

Hence [x. 18] $\sqrt{(x + y)^2 - (2z)^2} \cup (x + y) \dots \dots \dots (4).$

And $(x + y) \cap \sigma$, by (1).

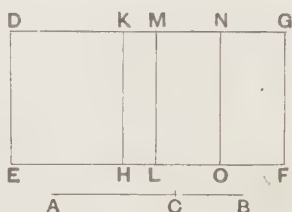
Therefore [(3), (4)] $(x + y) + 2z$ is a *fourth binomial* straight line.

It is of course $\frac{\rho^2}{\sigma} \left\{ 1 + \frac{1}{\sqrt{1+k^2}} \right\}.$

PROPOSITION 64.

The square on the side of a rational plus a medial area applied to a rational straight line produces as breadth the fifth binomial.

Let AB be the side of a rational plus a medial area, divided into its straight lines at C , so that AC is the greater; let a rational straight line DE be set out, and let there be applied to DE the parallelogram DF equal to the square on AB , producing DG as its breadth; I say that DG is a fifth binomial straight line.



Let the same construction as before be made.

Since then AB is the side of a rational plus a medial area, divided at C ,

therefore AC, CB are straight lines incommensurable in square which make the sum of the squares on them medial, but the rectangle contained by them rational. [x. 40]

Since then the sum of the squares on AC, CB is medial, therefore DL is medial, so that DM is rational and incommensurable in length with DE . [x. 22]

Again, since twice the rectangle AC, CB , that is MF , is rational,

therefore MG is rational and commensurable with DE . [x. 20]

Therefore DM is incommensurable with MG ; [x. 13] therefore DM, MG are rational straight lines commensurable in square only; therefore DG is binomial. [x. 36]

I say next that it is also a fifth binomial straight line.

For it can be proved similarly that the rectangle DK, KM is equal to the square on MN , and that DK is incommensurable in length with KM ; therefore the square on DM is greater than the square on MG by the square on a straight line incommensurable with DM . [x. 18]

And DM , MG are commensurable in square only, and the less, MG , is commensurable in length with DE .

Therefore DG is a fifth binomial.

Q. E. D.

To prove that [cf. x. 40]

$$\frac{1}{\sigma} \left\{ \frac{\rho}{\sqrt{2(1+k^2)}} \sqrt{\sqrt{1+k^2}+k} + \frac{\rho}{\sqrt{2(1+k^2)}} \sqrt{\sqrt{1+k^2}-k} \right\}^2$$

is a *fifth binomial* straight line.

For brevity denote it by $\frac{1}{\sigma} (u+v)^2$, and put

$$\sigma v = u^2,$$

$$\sigma y = v^2,$$

$$\sigma \cdot 2z = 2uv.$$

Remembering that [x. 40] $u^2 \sim v^2$, $(u^2 + v^2)$ is medial, and $2uv$ is rational, we proceed thus.

(a) $\sigma(x+y)$ is medial;

therefore $(x+y)$ is rational and $\sim \sigma$ (1).

Next, $\sigma \cdot 2z$ is rational;

therefore $2z$ is rational and $\sim \sigma$ (2).

Thus $(x+y)$, $2z$ are rational and $\sim$ (3),

so that $(x+y) + 2z$ is a binomial straight line.

(β) As before, $x+y > 2z$,

$$xy = z^2,$$

and

$$x \sim y.$$

Therefore [x. 18] $\sqrt{(x+y)^2 - (2z)^2} \sim (x+y)$ (4).

Hence [(2), (3), (4)] $(x+y) + 2z$ is a *fifth binomial* straight line.

It is of course $\frac{\rho^2}{\sigma} \left\{ \frac{1}{\sqrt{1+k^2}} + \frac{1}{1+k^2} \right\}.$

PROPOSITION 65.

The square on the side of the sum of two medial areas applied to a rational straight line produces as breadth the sixth binomial.

Let AB be the side of the sum of two medial areas, divided at C ,

let DE be a rational straight line,

and let there be applied to DE the parallelogram DF equal to the square on AB , producing DG as its breadth;

I say that DG is a sixth binomial straight line.

For let the same construction be made as before.

Then, since AB is the side of the sum of two medial areas, divided at C ,

therefore AC , CB are straight lines incommensurable in square which make the sum of the squares on them medial, the rectangle contained by them medial, and moreover the sum of the squares on them incommensurable with the rectangle contained by them, [x. 41]

so that, in accordance with what was before proved, each of the rectangles DL , MF is medial.

And they are applied to the rational straight line DE ; therefore each of the straight lines DM , MG is rational and incommensurable in length with DE . [x. 22]

And, since the sum of the squares on AC , CB is incommensurable with twice the rectangle AC , CB , therefore DL is incommensurable with MF .

Therefore DM is also incommensurable with MG ; [vi. 1, x. 11]

therefore DM , MG are rational straight lines commensurable in square only;

therefore DG is binomial. [x. 36]

I say next that it is also a sixth binomial straight line.

Similarly again we can prove that the rectangle DK , KM is equal to the square on MN ,

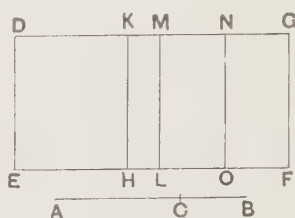
and that DK is incommensurable in length with KM ;

and, for the same reason, the square on DM is greater than the square on MG by the square on a straight line incommensurable in length with DM .

And neither of the straight lines DM , MG is commensurable in length with the rational straight line DE set out.

Therefore DG is a sixth binomial straight line.

Q. E. D.



To prove that [cf. x. 41]

$$\frac{1}{\sigma} \left\{ \frac{\rho\lambda^{\frac{1}{2}}}{\sqrt{2}} \sqrt{1 + \frac{k}{\sqrt{1+k^2}}} + \frac{\rho\lambda^{\frac{1}{2}}}{\sqrt{2}} \sqrt{1 - \frac{k}{\sqrt{1+k^2}}} \right\}^2$$

is a *sixth binomial* straight line.

Denote it by $\frac{1}{\sigma} (u+v)^2$, and put

$$\sigma x = u^2,$$

$$\sigma y = v^2,$$

$$\sigma \cdot 2z = 2uv.$$

Now, by x. 41, $u^2 \cup v^2$, $(u^2 + v^2)$ is medial, $2uv$ is medial, and $(u^2 + v^2) \cup 2uv$.

(a) In this case $\sigma(x+y)$ is medial ;

therefore $(x+y)$ is rational and $\cup \sigma$ (1).

In like manner, $2z$ is rational and $\cup \sigma$ (2).

And, since $\sigma(x+y) \cup \sigma \cdot 2z$,

$$(x+y) \cup 2z$$
(3).

Therefore $(x+y) + 2z$ is a binomial straight line.

(β) As before, $x+y > 2z$,

$$xy = z^2,$$

$$x \cup y ;$$

therefore [x. 18] $\sqrt{(x+y)^2 - (2z)^2} \cup (x+y)$ (4).

Hence [(1), (2), (3), (4)] $(x+y) + 2z$ is a *sixth binomial* straight line.

It is obviously $\frac{\rho^2}{\sigma} \left\{ \sqrt{\lambda} + \frac{\sqrt{\lambda}}{\sqrt{1+k^2}} \right\}.$

PROPOSITION 66.

A straight line commensurable in length with a binomial straight line is itself also binomial and the same in order.

Let AB be binomial, and let CD be commensurable in length with AB ;



I say that CD is binomial and the same in order with AB .

For, since AB is binomial,
let it be divided into its terms at E ,
and let AE be the greater term ;

therefore AE , EB are rational straight lines commensurable in square only. [x. 36]

Let it be contrived that,
as AB is to CD , so is AE to CF ; [vi. 12]
therefore also the remainder EB is to the remainder FD as AB is to CD . [v. 19]

But AB is commensurable in length with CD ;
therefore AE is also commensurable with CF , and EB with FD . [x. 11]

And AE , EB are rational;
therefore CF , FD are also rational.

And, as AE is to CF , so is EB to FD . [v. 11]

Therefore, alternately, as AE is to EB , so is CF to FD . [v. 16]

But AE , EB are commensurable in square only;
therefore CF , FD are also commensurable in square only. [x. 11]

And they are rational;
therefore CD is binomial. [x. 36]

I say next that it is the same in order with AB .

For the square on AE is greater than the square on EB either by the square on a straight line commensurable with AE or by the square on a straight line incommensurable with it.

If then the square on AE is greater than the square on EB by the square on a straight line commensurable with AE , the square on CF will also be greater than the square on FD by the square on a straight line commensurable with CF . [x. 14]

And, if AE is commensurable with the rational straight line set out, CF will also be commensurable with it, [x. 12]
and for this reason each of the straight lines AB , CD is a first binomial, that is, the same in order. [x. Def. II. 1]

But, if EB is commensurable with the rational straight line set out, FD is also commensurable with it, [x. 12]
and for this reason again CD will be the same in order with AB ,

for each of them will be a second binomial. [x. Def. II. 2]

But, if neither of the straight lines AE , EB is commensurable with the rational straight line set out, neither of the straight lines CF , FD will be commensurable with it, [x. 13] and each of the straight lines AB , CD is a third binomial.

[x. Def. 11. 3]

But, if the square on AE is greater than the square on EB by the square on a straight line incommensurable with AE ,

the square on CF is also greater than the square on FD by the square on a straight line incommensurable with CF . [x. 14]

And, if AE is commensurable with the rational straight line set out, CF is also commensurable with it,

and each of the straight lines AB , CD is a fourth binomial.

[x. Def. 11. 4]

But, if EB is so commensurable, so is FD also,

and each of the straight lines AB , CD will be a fifth binomial.

[x. Def. 11. 5]

But, if neither of the straight lines AE , EB is so commensurable, neither of the straight lines CF , FD is commensurable with the rational straight line set out,

and each of the straight lines AB , CD will be a sixth binomial.

[x. Def. 11. 6]

Hence a straight line commensurable in length with a binomial straight line is binomial and the same in order.

Q. E. D.

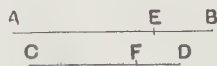
The proofs of this and the following propositions up to x. 70 inclusive are easy and require no elucidation. They are equivalent to saying that, if in each of the preceding irrational straight lines $\frac{m}{n}\rho$ is substituted for ρ , the resulting irrational is of the same kind as that from which it is altered.

PROPOSITION 67.

A straight line commensurable in length with a bimedral straight line is itself also bimedral and the same in order.

Let AB be bimedral, and let CD be commensurable in length with AB ;

I say that CD is bimedral and the same in order with AB .



For, since AB is bimedral, let it be divided into its medials at E ;

therefore AE , EB are medial straight lines commensurable in square only. [x. 37, 38]

And let it be contrived that,
as AB is to CD , so is AE to CF ;
therefore also the remainder EB is to the remainder FD as AB is to CD . [v. 19]

But AB is commensurable in length with CD ;
therefore AE , EB are also commensurable with CF , FD respectively. [x. 11]

But AE , EB are medial;
therefore CF , FD are also medial. [x. 23]

And since, as AE is to EB , so is CF to FD , [v. 11]
and AE , EB are commensurable in square only,
 CF , FD are also commensurable in square only. [x. 11]

But they were also proved medial;
therefore CD is bimedral.

I say next that it is also the same in order with AB .

For since, as AE is to EB , so is CF to FD ,
therefore also, as the square on AE is to the rectangle AE ,
 EB , so is the square on CF to the rectangle CF , FD ;
therefore, alternately,

as the square on AE is to the square on CF , so is the rectangle AE , EB to the rectangle CF , FD . [v. 16]

But the square on AE is commensurable with the square on CF ;
therefore the rectangle AE , EB is also commensurable with the rectangle CF , FD .

If therefore the rectangle AE , EB is rational,
the rectangle CF , FD is also rational,
[and for this reason CD is a first bimedral]; [x. 37]
but if medial, medial, [x. 23, Por.]
and each of the straight lines AB , CD is a second bimedral. [x. 38]

And for this reason CD will be the same in order with AB .

Q. E. D.

PROPOSITION 68.

A straight line commensurable with a major straight line is itself also major.

Let AB be major, and let CD be commensurable with AB ;
I say that CD is major.

Let AB be divided at E ;

therefore AE , EB are straight lines incommensurable in square which make the sum of the squares on them rational, but the rectangle contained by them medial.

[x. 39]



Let the same construction be made as before.

Then since, as AB is to CD , so is AE to CF , and EB to FD ,

therefore also, as AE is to CF , so is EB to FD . [v. 11]

But AB is commensurable with CD ;

therefore AE , EB are also commensurable with CF , FD respectively. [x. 11]

And since, as AE is to CF , so is EB to FD ,
alternately also,

as AE is to EB , so is CF to FD ; [v. 16]

therefore also, *componendo*,

as AB is to BE , so is CD to DF ; [v. 18]

therefore also, as the square on AB is to the square on BE ,
so is the square on CD to the square on DF . [vi. 20]

Similarly we can prove that, as the square on AB is to the square on AE , so also is the square on CD to the square on CF .

Therefore also, as the square on AB is to the squares on AE , EB , so is the square on CD to the squares on CF , FD ;
therefore also, alternately,

as the square on AB is to the square on CD , so are the squares on AE , EB to the squares on CF , FD . [v. 16]

But the square on AB is commensurable with the square on CD ;

therefore the squares on AE , EB are also commensurable with the squares on CF , FD .

And the squares on AE , EB together are rational ;
therefore the squares on CF , FD together are rational.

Similarly also twice the rectangle AE , EB is commensurable with twice the rectangle CF , FD .

And twice the rectangle AE , EB is medial ;
therefore twice the rectangle CF , FD is also medial.

[x. 23, Por.]

Therefore CF , FD are straight lines incommensurable in square which make, at the same time, the sum of the squares on them rational, but the rectangle contained by them medial ;
therefore the whole CD is the irrational straight line called major.

[x. 39]

Therefore a straight line commensurable with the major straight line is major.

Q. E. D.

PROPOSITION 69.

A straight line commensurable with the side of a rational plus a medial area is itself also the side of a rational plus a medial area.

Let AB be the side of a rational plus a medial area,
and let CD be commensurable with AB ;

it is to be proved that CD is also the side of a rational plus a medial area.

Let AB be divided into its straight lines at E ;
therefore AE , EB are straight lines incommensurable in square which make the sum of the squares on them medial, but the rectangle contained by them rational.

[x. 40]



Let the same construction be made as before.

We can then prove similarly that

CF , FD are incommensurable in square,
and the sum of the squares on AE , EB is commensurable with the sum of the squares on CF , FD ,
and the rectangle AE , EB with the rectangle CF , FD ;
so that the sum of the squares on CF , FD is also medial, and the rectangle CF , FD rational.

Therefore CD is the side of a rational plus a medial area.

Q. E. D.

PROPOSITION 70.

A straight line commensurable with the side of the sum of two medial areas is the side of the sum of two medial areas.

Let AB be the side of the sum of two medial areas, and CD commensurable with AB ;

it is to be proved that CD is also the side of the sum of two medial areas.

For, since AB is the side of the sum of two medial areas,

let it be divided into its straight lines at E ;

therefore AE , EB are straight lines incommensurable in square which make the sum of the squares on them medial, the rectangle contained by them medial, and furthermore the sum of the squares on AE , EB incommensurable with the rectangle AE , EB . [x. 41]



Let the same construction be made as before.

We can then prove similarly that

CF , FD are also incommensurable in square,

the sum of the squares on AE , EB is commensurable with the sum of the squares on CF , FD ,

and the rectangle AE , EB with the rectangle CF , FD ;

so that the sum of the squares on CF , FD is also medial,

the rectangle CF , FD is medial,

and moreover the sum of the squares on CF , FD is incommensurable with the rectangle CF , FD .

Therefore CD is the side of the sum of two medial areas.

Q. E. D.

PROPOSITION 71.

If a rational and a medial area be added together, four irrational straight lines arise, namely a binomial or a first bimedial or a major or a side of a rational plus a medial area.

Let AB be rational, and CD medial;

I say that the "side" of the area AD is a binomial or a first bimedial or a major or a side of a rational plus a medial area.

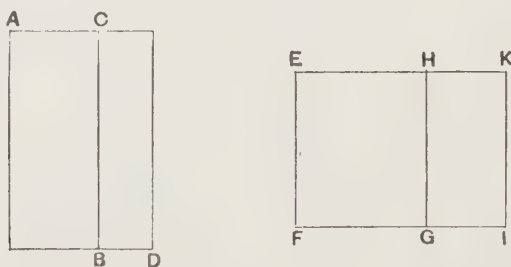
For AB is either greater or less than CD .

First, let it be greater ;

let a rational straight line EF be set out,

let there be applied to EF the rectangle EG equal to AB ,
producing EH as breadth,

and let HI , equal to DC , be applied to EF , producing HK
as breadth.



Then, since AB is rational and is equal to EG ,
therefore EG is also rational.

And it has been applied to EF , producing EH as breadth ;
therefore EH is rational and commensurable in length with
 EF . [x. 20]

Again, since CD is medial and is equal to HI ,
therefore HI is also medial.

And it is applied to the rational straight line EF , pro-
ducing HK as breadth ;
therefore HK is rational and incommensurable in length
with EF . [x. 22]

And, since CD is medial,
while AB is rational,
therefore AB is incommensurable with CD ,
so that EG is also incommensurable with HI .

But, as EG is to HI , so is EH to HK ; [vi. 1]
therefore EH is also incommensurable in length with HK . [x. 11]

And both are rational ;
therefore EH , HK are rational straight lines commensurable
in square only ;
therefore EK is a binomial straight line, divided at H . [x. 36]

And, since AB is greater than CD ,
 while AB is equal to EG and CD to HI ,
 therefore EG is also greater than HI ;
 therefore EH is also greater than HK .

The square, then, on EH is greater than the square on HK either by the square on a straight line commensurable in length with EH or by the square on a straight line incommensurable with it.

First, let the square on it be greater by the square on a straight line commensurable with itself.

Now the greater straight line HE is commensurable in length with the rational straight line EF set out;
 therefore EK is a first binomial. [x. Def. II. 1]

But EF is rational;

and, if an area be contained by a rational straight line and the first binomial, the side of the square equal to the area is binomial. [x. 54]

Therefore the "side" of EI is binomial;
 so that the "side" of AD is also binomial.

Next, let the square on EH be greater than the square on HK by the square on a straight line incommensurable with EH .

Now the greater straight line EH is commensurable in length with the rational straight line EF set out;
 therefore EK is a fourth binomial. [x. Def. II. 4]

But EF is rational;

and, if an area be contained by a rational straight line and the fourth binomial, the "side" of the area is the irrational straight line called major. [x. 57]

Therefore the "side" of the area EI is major;
 so that the "side" of the area AD is also major.

Next, let AB be less than CD ;
 therefore EG is also less than HI ,
 so that EH is also less than HK .

Now the square on HK is greater than the square on EH either by the square on a straight line commensurable with HK or by the square on a straight line incommensurable with it.

First, let the square on it be greater by the square on a straight line commensurable in length with itself.

Now the lesser straight line EH is commensurable in length with the rational straight line EF set out ;
therefore EK is a second binomial. [x. Def. II. 2]

But EF is rational ,

and, if an area be contained by a rational straight line and the second binomial, the side of the square equal to it is a first bimedral ; [x. 55]

therefore the "side" of the area EI is a first bimedral,
so that the "side" of AD is also a first bimedral.

Next, let the square on HK be greater than the square on HE by the square on a straight line incommensurable with HK .

Now the lesser straight line EH is commensurable with the rational straight line EF set out ;
therefore EK is a fifth binomial. [x. Def. II. 5]

But EF is rational ;

and, if an area be contained by a rational straight line and the fifth binomial, the side of the square equal to the area is a side of a rational plus a medial area. [x. 58]

Therefore the "side" of the area EI is a side of a rational plus a medial area,
so that the "side" of the area AD is also a side of a rational plus a medial area.

Therefore etc.

Q. E. D.

A *rational* area being of the form $k\rho^2$, and a *medial* area of the form $\sqrt{\lambda} \cdot \rho^2$, the problem is to classify

$$\sqrt{k\rho^2 + \sqrt{\lambda} \cdot \rho^2}$$

according to the different possible relations between k , λ .

Put

$$\sigma u = k\rho^2,$$

$$\sigma v = \sqrt{\lambda} \cdot \rho^2.$$

Then, since the former rectangle is rational, the latter medial,

$$u \text{ is rational and } \cap \sigma,$$

$$v \text{ is rational and } \cap \sigma.$$

Also the rectangles are incommensurable ;

so that

$$u \sim v.$$

Hence u , v are rational and $\sim$;

whence $(u + v)$ is a binomial straight line.

The possibilities now are as follows :

I. $u > v$.

Then either

$$(1) \sqrt{u^2 - v^2} \sim u,$$

$$\text{or } (2) \sqrt{u^2 - v^2} \sim v,$$

while in both cases $u \sim \sigma$.

In case (1) $(u + v)$ is a *first binomial* straight line,
and in case (2) $(u + v)$ is a *fourth binomial* straight line.

Thus $\sqrt{\sigma(u + v)}$ is either (1) a *binomial* straight line [x. 54] or (2) a *major irrational* straight line [x. 57].

II. $v > u$.

Then either

$$(1) \sqrt{v^2 - u^2} \sim v,$$

$$\text{or } (2) \sqrt{v^2 - u^2} \sim u,$$

while in both cases $v \sim \sigma$, but $u \sim \sigma$.

Hence, in case (1), $(v + u)$ is a *second binomial* straight line,
and, in case (2), $(v + u)$ is a *fifth binomial* straight line.

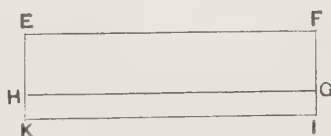
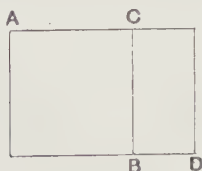
Thus $\sqrt{\sigma(v + u)}$ is either (1) a *first bimedral* straight line [x. 55], or (2) a *side of a rational plus a medial area* [x. 58].

PROPOSITION 72.

If two medial areas incommensurable with one another be added together, the remaining two irrational straight lines arise, namely either a second bimedral or a side of the sum of two medial areas.

For let two medial areas AB , CD incommensurable with one another be added together ;

I say that the "side" of the area AD is either a second bimedral or a side of the sum of two medial areas.



For AB is either greater or less than CD .

First, if it so chance, let AB be greater than CD .

Let the rational straight line EF be set out,

and to EF let there be applied the rectangle EG equal to

AB and producing EH as breadth, and the rectangle HI equal to CD and producing HK as breadth.

Now, since each of the areas AB , CD is medial, therefore each of the areas EG , HI is also medial.

And they are applied to the rational straight line FE , producing EH , HK as breadth;

therefore each of the straight lines EH , HK is rational and incommensurable in length with EF . [x. 22]

And, since AB is incommensurable with CD , and AB is equal to EG , and CD to HI , therefore EG is also incommensurable with HI .

But, as EG is to HI , so is EH to HK ; [vi. 1] therefore EH is incommensurable in length with HK . [x. 11]

Therefore EH , HK are rational straight lines commensurable in square only;

therefore EK is binomial. [x. 36]

But the square on EH is greater than the square on HK either by the square on a straight line commensurable with EH or by the square on a straight line incommensurable with it.

First, let the square on it be greater by the square on a straight line commensurable in length with itself.

Now neither of the straight lines EH , HK is commensurable in length with the rational straight line EF set out;

therefore EK is a third binomial. [x. Def. 11. 3]

But EF is rational;

and, if an area be contained by a rational straight line and the third binomial, the "side" of the area is a second bimedial;

therefore the "side" of EI , that is, of AD , is a second bimedial. [x. 56]

Next, let the square on EH be greater than the square on HK by the square on a straight line incommensurable in length with EH .

Now each of the straight lines EH , HK is incommensurable in length with EF ;

therefore EK is a sixth binomial. [x. Def. 11. 6]

But, if an area be contained by a rational straight line and

the sixth binomial, the "side" of the area is the side of the sum of two medial areas; [x. 59]

so that the "side" of the area AD is also the side of the sum of two medial areas.

Therefore etc.

Q. E. D.

We have to classify, according to the different possible relations between k , λ , the straight line

$$\sqrt{\sqrt{k} \cdot \rho^2 + \sqrt{\lambda} \cdot \rho^2},$$

where $\sqrt{k} \cdot \rho^2$ and $\sqrt{\lambda} \cdot \rho^2$ are incommensurable.

Suppose that

$$\sigma u = \sqrt{k} \cdot \rho^2,$$

$$\sigma v = \sqrt{\lambda} \cdot \rho^2.$$

It is immaterial whether $\sqrt{k} \cdot \rho^2$ or $\sqrt{\lambda} \cdot \rho^2$ is the greater. Suppose, e.g., that the former is.

Now, $\sqrt{k} \cdot \rho^2$, $\sqrt{\lambda} \cdot \rho^2$ being both *medial* areas, and σ rational,

$$u, v \text{ are both rational and } \sim \sigma \dots\dots\dots(1).$$

Again, by hypothesis,

$$\sigma u \sim \sigma v,$$

or

$$u \sim v \dots\dots\dots(2).$$

Hence [(1), (2)] $(u + v)$ is a binomial straight line.

Next, $\sqrt{u^2 - v^2}$ is either commensurable or incommensurable in length with u .

(a) Suppose $\sqrt{u^2 - v^2} \sim u$.

In this case $(u + v)$ is a *third binomial* straight line, and therefore [x. 56]

$\sqrt{\sigma(u + v)}$ is a *second binomial* straight line.

(β) If $\sqrt{u^2 - v^2} \sim u$,

$(u + v)$ is a *sixth binomial* straight line,

and therefore [x. 59]

$\sqrt{\sigma(u + v)}$ is a *side of the sum of two medial areas*.

The binomial straight line and the irrational straight lines after it are neither the same with the medial nor with one another.

For the square on a medial, if applied to a rational straight line, produces as breadth a straight line rational and incommensurable in length with that to which it is applied. [x. 22]

But the square on the binomial, if applied to a rational straight line, produces as breadth the first binomial. [x. 60]

The square on the first binomial, if applied to a rational straight line, produces as breadth the second binomial. [x. 61]

The square on the second bimedral, if applied to a rational straight line, produces as breadth the third binomial. [x. 62]

The square on the major, if applied to a rational straight line, produces as breadth the fourth binomial. [x. 63]

The square on the side of a rational plus a medial area, if applied to a rational straight line, produces as breadth the fifth binomial. [x. 64]

The square on the side of the sum of two medial areas, if applied to a rational straight line, produces as breadth the sixth binomial. [x. 65]

And the said breadths differ both from the first and from one another: from the first because it is rational, and from one another because they are not the same in order; so that the irrational straight lines themselves also differ from one another.

The explanation after x. 72 is for the purpose of showing that all the irrational straight lines treated hitherto are different from one another, viz. the medial, the six irrational straight lines beginning with the binomial, and the six consisting of the first, second, third, fourth, fifth and sixth binomials.

PROPOSITION 73.

If from a rational straight line there be subtracted a rational straight line commensurable with the whole in square only, the remainder is irrational; and let it be called an apotome.

For from the rational straight line AB let the rational straight line BC , commensurable with the whole in square only, be subtracted;

$\overline{A \quad C \quad B}$

I say that the remainder AC is the irrational straight line called **apotome**.

For, since AB is incommensurable in length with BC , and, as AB is to BC , so is the square on AB to the rectangle AB, BC , therefore the square on AB is incommensurable with the rectangle AB, BC . [x. 11]

But the squares on AB, BC are commensurable with the square on AB , [x. 15]
and twice the rectangle AB, BC is commensurable with the rectangle AB, BC . [x. 6]

And, inasmuch as the squares on AB , BC are equal to twice the rectangle AB , BC together with the square on CA ,
[II. 7]

therefore the squares on AB , BC are also incommensurable with the remainder, the square on AC . [x. 13, 16]

But the squares on AB , BC are rational ;
therefore AC is irrational. [x. Def. 4]

And let it be called an **apotome**.

Q. E. D.

Euclid now passes to the irrational straight lines which are the *difference* and not, as before, the sum of two straight lines. *Apotome* ("portion cut off") accordingly takes the place of *binomial* and the other terms follow *mutatis mutandis*. The first hexad of propositions (73 to 78) exhibit the six irrational straight lines which are really the result of extracting the *square root* of the six irrationals in the later propositions 85 to 90 (or, strictly speaking, of finding the sides of squares equal to the rectangles formed by each of those six irrational straight lines respectively with a rational straight line). Thus, just as in the corresponding propositions about the irrational straight lines formed by addition, the further removed irrationals, so to speak, come first.

We shall denote the *apotome* etc. by $(x - y)$, which is formed by subtracting a certain lesser straight line y from a greater x . In x. 79 and later propositions y is called by Euclid the *annex* (η προσαρμόζουσα), being the straight line which, when added to the apotome or other irrational formed by subtraction, makes up the greater x .

The methods of proof are exactly the same as in the preceding propositions about the irrational straight lines formed by *addition*.

In this proposition x , y are rational straight lines commensurable in square only, and we have to prove that $(x - y)$, the *apotome*, is irrational.

$x \sim y$, so that $x \cup y$:

therefore, since

$$x : y = x^2 : xy,$$

$$x^2 \cup xy.$$

But $x^2 \sim (x^2 + y^2)$, and $xy \sim 2xy$;

therefore

$$x^2 + y^2 \cup 2xy,$$

whence

$$(x - y)^2 \cup (x^2 + y^2).$$

But $(x^2 + y^2)$ is rational ,

therefore $(x - y)^2$, and consequently $(x - y)$, is irrational.

The *apotome* $(x - y)$ is of the form $\rho \sim \sqrt{k} \cdot \rho$, just as the binomial straight line is of the form $\rho + \sqrt{k} \cdot \rho$.

PROPOSITION 74.

If from a medial straight line there be subtracted a medial straight line which is commensurable with the whole in square only, and which contains with the whole a rational rectangle, the remainder is irrational. And let it be called a first apotome of a medial straight line.

For from the medial straight line AB let there be subtracted the medial straight line BC which is commensurable with AB in square only and with AB makes the rectangle AB, BC rational;



I say that the remainder AC is irrational; and let it be called a **first apotome of a medial** straight line.

For, since AB, BC are medial,
the squares on AB, BC are also medial.

But twice the rectangle AB, BC is rational;
therefore the squares on AB, BC are incommensurable with twice the rectangle AB, BC ;

therefore twice the rectangle AB, BC is also incommensurable with the remainder, the square on AC , [cf. II. 7]

since, if the whole is incommensurable with one of the magnitudes, the original magnitudes will also be incommensurable.

[x. 16]

But twice the rectangle AB, BC is rational;
therefore the square on AC is irrational;
therefore AC is irrational. [x. Def. 4]

And let it be called a **first apotome of a medial** straight line.

The *first apotome of a medial* straight line is the difference between straight lines of the form $k^{\frac{1}{2}}\rho$, $k^{\frac{3}{2}}\rho$, which are medial straight lines commensurable in square only and forming a rational rectangle.

By hypothesis, x^2, y^2 are *medial* areas.

And, since xy is rational, $(x^2 + y^2) \sim xy$

$$\sim 2xy,$$

whence

$$(x - y)^2 \sim 2xy.$$

But $2xy$ is rational;

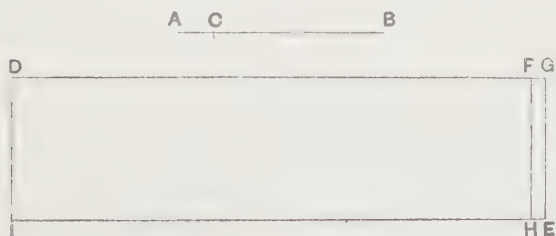
therefore $(x - y)^2$, and consequently $(x - y)$, is irrational.

This irrational, which is of the form $(k^{\frac{1}{2}}\rho \sim k^{\frac{3}{2}}\rho)$, is the *first apotome of a medial* straight line; the term corresponding of course to *first bimedral*, which applies where the sign is positive.

PROPOSITION 75.

If from a medial straight line there be subtracted a medial straight line which is commensurable with the whole in square only, and which contains with the whole a medial rectangle, the remainder is irrational; and let it be called a second apotome of a medial straight line.

For from the medial straight line AB let there be subtracted the medial straight line CB which is commensurable with the whole AB in square only and such that the rectangle AB, BC , which it contains with the whole AB , is medial; [x. 28] I say that the remainder AC is irrational; and let it be called a second apotome of a medial straight line.



For let a rational straight line DI be set out, let DE equal to the squares on AB, BC be applied to DI , producing DG as breadth, and let DH equal to twice the rectangle AB, BC be applied to DI , producing DF as breadth; therefore the remainder FE is equal to the square on AC .

[II. 7]

Now, since the squares on AB, BC are medial and commensurable,

therefore DE is also medial. [x. 15 and 23, Por.]

And it is applied to the rational straight line DI , producing DG as breadth;

therefore DG is rational and incommensurable in length with DI . [x. 22]

Again, since the rectangle AB, BC is medial, therefore twice the rectangle AB, BC is also medial.

[x. 23, Por.]

And it is equal to DH ;
therefore DH is also medial.

And it has been applied to the rational straight line DI ,
producing DF as breadth;
therefore DF is rational and incommensurable in length
with DI . [x. 22]

And, since AB, BC are commensurable in square only,
therefore AB is incommensurable in length with BC ;
therefore the square on AB is also incommensurable with the
rectangle AB, BC . [x. 11]

But the squares on AB, BC are commensurable with the
square on AB , [x. 15]
and twice the rectangle AB, BC is commensurable with the
rectangle AB, BC ; [x. 6]
therefore twice the rectangle AB, BC is incommensurable with
the squares on AB, BC . [x. 13]

But DE is equal to the squares on AB, BC ,
and DH to twice the rectangle AB, BC ;
therefore DE is incommensurable with DH .

But, as DE is to DH , so is GD to DF ; [vi. 1]
therefore GD is incommensurable with DF . [x. 11]

And both are rational;
therefore GD, DF are rational straight lines commensurable
in square only;
therefore FG is an apotome. [x. 73]

But DI is rational,
and the rectangle contained by a rational and an irrational
straight line is irrational, [deduction from x. 20]
and its "side" is irrational.

And AC is the "side" of FE ;
therefore AC is irrational.

And let it be called a **second apotome of a medial
straight line**.

Q. E. D.

We have here the difference between $k^{\frac{1}{2}}\rho$, $\sqrt{\lambda} \cdot \rho/k^{\frac{1}{2}}$, two medial straight
lines commensurable in square only and containing a medial rectangle.

Apply each of the areas $(x^2 + y^2)$, $2xy$ to a rational straight line σ , i.e.
suppose that

$$\begin{aligned}x^2 + y^2 &= \sigma u, \\2xy &= \sigma v.\end{aligned}$$

Then σu , σv are medial areas,
so that u , v are both rational and $\cup \sigma$ (1).

Again, $x \cup y$;
therefore $x^2 \cup xy$,
and consequently $x^2 + y^2 \cup 2xy$,
or $\sigma u \cup \sigma v$,
and $u \cup v$ (2).

Thus [(1), (2)] u , v are rational and $\sim$;
therefore [x. 73] $(u - v)$ is an *apotome*,
and, $(u - v)$ being thus irrational,
 $(u - v)\sigma$ is an irrational area.

Hence $(x - y)^2$, and consequently $(x - y)$, is irrational.

The irrational straight line $k^{\frac{1}{2}}\rho \sim \sqrt{\lambda} \cdot \rho$
 $k^{\frac{1}{2}}$ is called a *second apotome of a*
medial straight line.

PROPOSITION 76.

If from a straight line there be subtracted a straight line which is incommensurable in square with the whole and which with the whole makes the squares on them added together rational, but the rectangle contained by them medial, the remainder is irrational; and let it be called minor.

For from the straight line AB let there be subtracted the straight line BC which is incommensurable in square with the whole and fulfils the given conditions. [x. 33]

I say that the remainder AC is the irrational straight line called **minor**.

For, since the sum of the squares on AB , BC is rational, while twice the rectangle AB , BC is medial, therefore the squares on AB , BC are incommensurable with twice the rectangle AB , BC ;
and, *convertendo*, the squares on AB , BC are incommensurable with the remainder, the square on AC . [II. 7, x. 16]

But the squares on AB , BC are rational;
therefore the square on AC is irrational;
therefore AC is irrational.

And let it be called **minor**.

Q. E. D.

x, y are here of the form found in x. 33, viz.

$$\frac{\rho}{\sqrt{2}} \sqrt{1 + \frac{k}{\sqrt{1+k^2}}}, \quad \frac{\rho}{\sqrt{2}} \sqrt{1 - \frac{k}{\sqrt{1+k^2}}}.$$

By hypothesis $(x^2 + y^2)$ is a *rational*, xy a *medial*, area.

Therefore $(x^2 + y^2) \cup 2xy$,

whence $(x - y)^2 \cup (x^2 + y^2)$.

Therefore $(x - y)^2$, and consequently $(x - y)$, is irrational.

The *minor* (irrational) straight line is thus of the form

$$\frac{\rho}{\sqrt{2}} \sqrt{1 + \frac{k}{\sqrt{1+k^2}}} - \frac{\rho}{\sqrt{2}} \sqrt{1 - \frac{k}{\sqrt{1+k^2}}}.$$

Observe the use of *convertendo* (*ἀναστρέψαντι*) for the inference that, since $(x^2 + y^2) \cup 2xy$, $(x^2 + y^2) \cup (x - y)^2$. The use of the word corresponds exactly to its use in proportions.

PROPOSITION 77.

If from a straight line there be subtracted a straight line which is incommensurable in square with the whole, and which with the whole makes the sum of the squares on them medial, but twice the rectangle contained by them rational, the remainder is irrational: and let it be called that which produces with a rational area a medial whole.

For from the straight line AB let there be subtracted the straight line BC which is incommensurable in square with AB and fulfils the given conditions; [x. 34]

I say that the remainder AC is the irrational straight line aforesaid.

For, since the sum of the squares on AB, BC is medial,

while twice the rectangle AB, BC is rational,

therefore the squares on AB, BC are incommensurable with twice the rectangle AB, BC ;

therefore the remainder also, the square on AC , is incommensurable with twice the rectangle AB, BC . [II. 7, x. 16]

And twice the rectangle AB, BC is rational;

therefore the square on AC is irrational;

therefore AC is irrational.

And let it be called that which produces with a rational area a medial whole.



Here x, y are of the form [cf. x. 34]

$$\frac{\rho}{\sqrt{2(1+k^2)}} \sqrt{\sqrt{1+k^2}+k}, \quad \frac{\rho}{\sqrt{2(1+k^2)}} \sqrt{\sqrt{1+k^2}-k}.$$

By hypothesis, $(x^2 + y^2)$ is a *medial*, xy a *rational*, area ;

thus

$$(x^2 + y^2) \sim 2xy,$$

and therefore

$$(x - y)^2 \sim 2xy,$$

whence $(x - y)^2$, and consequently $(x - y)$, is irrational.

The irrational straight line

$$\frac{\rho}{\sqrt{2(1+k^2)}} \sqrt{\sqrt{1+k^2}+k} - \frac{\rho}{\sqrt{2(1+k^2)}} \sqrt{\sqrt{1+k^2}-k}$$

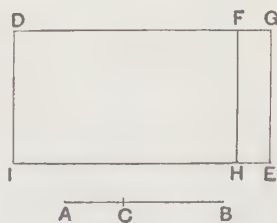
is called *that which produces with a rational area a medial whole* or more literally *that which with a rational area makes the whole medial* (ἡ μετὰ ῥητοῦ μέσων τὸ ὅλον ποιούσα). Here “produces” means “produces when a square is described on it.” A clearer way of expressing the meaning would be to call this straight line the “*side*” of a *medial minus a rational area* corresponding to the “*side*” of a *rational plus a medial area* [x. 40].

PROPOSITION 78.

If from a straight line there be subtracted a straight line which is incommensurable in square with the whole and which with the whole makes the sum of the squares on them medial, twice the rectangle contained by them medial, and further the squares on them incommensurable with twice the rectangle contained by them, the remainder is irrational; and let it be called that which produces with a medial area a medial whole.

For from the straight line AB let there be subtracted the straight line BC incommensurable in square with AB and fulfilling the given conditions ; [x. 35]

I say that the remainder AC is the irrational straight line called *that which produces with a medial area a medial whole*.



For let a rational straight line DI be set out,

to DI let there be applied DE equal to the squares on AB , BC , producing DG as breadth,

and let DH equal to twice the rectangle AB , BC be subtracted.

Therefore the remainder FE is equal to the square on AC , [II. 7]

so that AC is the "side" of FE .

Now, since the sum of the squares on AB, BC is medial and is equal to DE ,

therefore DE is medial.

And it is applied to the rational straight line DI , producing DG as breadth;

therefore DG is rational and incommensurable in length with DI . [x. 22]

Again, since twice the rectangle AB, BC is medial and is equal to DH ,

therefore DH is medial.

And it is applied to the rational straight line DI , producing DF as breadth;

therefore DF is also rational and incommensurable in length with DI . [x. 22]

And, since the squares on AB, BC are incommensurable with twice the rectangle AB, BC ,

therefore DE is also incommensurable with DH .

But, as DE is to DH , so also is DG to DF ; [VI. 1]

therefore DG is incommensurable with DF . [x. 11]

And both are rational;

therefore GD, DF are rational straight lines commensurable in square only.

Therefore FG is an apotome. [x. 73]

And FH is rational;

but the rectangle contained by a rational straight line and an apotome is irrational, [deduction from x. 20]

and its "side" is irrational.

And AC is the "side" of FE ;

therefore AC is irrational.

And let it be called **that which produces with a medial area a medial whole.**

In this case x, y have respectively the forms [cf. x. 35]

$$\frac{\rho\lambda^{\frac{1}{2}}}{\sqrt{2}} \sqrt{1 + \frac{k}{\sqrt{1+k^2}}}, \quad \frac{\rho\lambda^{\frac{1}{2}}}{\sqrt{2}} \sqrt{1 - \frac{k}{\sqrt{1+k^2}}}.$$

Suppose that

$$x^2 + y^2 = \sigma u,$$

$$2xy = \sigma v.$$

By hypothesis, the areas $\sigma u, \sigma v$ are medial;

therefore u, v are both rational and $\sim \sigma$ (1).

Further

$$\sigma u \sim \sigma v,$$

so that

$$u \sim v \text{ (2).}$$

Hence [(1), (2)] u, v are rational and $\sim$,

so that $(u - v)$ is the irrational straight line called *apotome* [x. 73].

Thus $\sigma(u - v)$ is an irrational area,

so that $(x - y)^2$, and consequently $(x - y)$, is irrational.

The irrational straight line

$$\frac{\rho\lambda^{\frac{1}{2}}}{\sqrt{2}} \sqrt{1 + \frac{k}{\sqrt{1+k^2}}} - \frac{\rho\lambda^{\frac{1}{2}}}{\sqrt{2}} \sqrt{1 - \frac{k}{\sqrt{1+k^2}}}$$

is called *that which produces* [i.e. when a square is described on it] *with a medial area a medial whole*, more literally *that which with a medial area makes the whole medial* (η μετὰ μέσου μέσον τὸ ὅλον ποιούσα). A clearer phrase (to us) would be the "*side*" of the difference between two medial areas, corresponding to the "*side*" of (the sum of) two medial areas [x. 41].

PROPOSITION 79.

To an apotome only one rational straight line can be annexed which is commensurable with the whole in square only.

Let AB be an apotome, and BC an annex to it;

therefore AC, CB are rational straight lines commensurable in square only. [x. 73]



I say that no other rational straight line can be annexed to AB which is commensurable with the whole in square only.

For, if possible, let BD be so annexed;

therefore AD, DB are also rational straight lines commensurable in square only. [x. 73]

Now, since the excess of the squares on AD, DB over twice the rectangle AD, DB is also the excess of the squares on AC, CB over twice the rectangle AC, CB ,

for both exceed by the same, the square on AB ,

[II. 7]

therefore, alternately, the excess of the squares on AD , DB over the squares on AC , CB is the excess of twice the rectangle AD , DB over twice the rectangle AC , CB .

But the squares on AD , DB exceed the squares on AC , CB by a rational area,

for both are rational ;

therefore twice the rectangle AD , DB also exceeds twice the rectangle AC , CB by a rational area :

which is impossible,

for both are medial [x. 21], and a medial area does not exceed a medial by a rational area. [x. 26]

Therefore no other rational straight line can be annexed to AB which is commensurable with the whole in square only.

Therefore only one rational straight line can be annexed to an apotome which is commensurable with the whole in square only.

Q. E. D.

This proposition proves the equivalent of the well-known theorem of surds that,

if $a - \sqrt{b} = x - \sqrt{y}$, then $a = x$, $b = y$;

and, if $\sqrt{a} - \sqrt{b} = \sqrt{x} - \sqrt{y}$, then $a = x$, $b = y$.

The method of proof corresponds to that of x. 42 for positive signs.

Suppose, if possible, that an *apotome* can be expressed as $(x - y)$ and also as $(x' - y')$, where x , y are rational straight lines commensurable in square only, and x' , y' are so also.

Of x , x' , let x be the greater.

Now, since

$$x - y = x' - y',$$

$$x^2 + y^2 - (x'^2 + y'^2) = 2xy - 2x'y'.$$

But $(x^2 + y^2)$, $(x'^2 + y'^2)$ are both rational, so that their difference is a rational area.

On the other hand, $2xy$, $2x'y'$ are both *medial* areas, being of the form $\sqrt{k} \cdot \rho^2$;

therefore the difference between two medial areas is rational :

which is impossible [x. 26].

Therefore etc.

PROPOSITION 80.

To a first apotome of a medial straight line only one medial straight line can be annexed which is commensurable with the whole in square only and which contains with the whole a rational rectangle.

For let AB be a first apotome of a medial straight line, and let BC be an annex to AB ;

therefore AC , CB are medial straight lines commensurable in



square only and such that the rectangle AC , CB which they contain is rational; [x. 74]

I say that no other medial straight line can be annexed to AB which is commensurable with the whole in square only and which contains with the whole a rational area.

For, if possible, let DB also be so annexed;

therefore AD , DB are medial straight lines commensurable in square only and such that the rectangle AD , DB which they contain is rational. [x. 74]

Now, since the excess of the squares on AD , DB over twice the rectangle AD , DB is also the excess of the squares on AC , CB over twice the rectangle AC , CB ,

for they exceed by the same, the square on AB , [II. 7]

therefore, alternately, the excess of the squares on AD , DB over the squares on AC , CB is also the excess of twice the rectangle AD , DB over twice the rectangle AC , CB .

But twice the rectangle AD , DB exceeds twice the rectangle AC , CB by a rational area,

for both are rational.

Therefore the squares on AD , DB also exceed the squares on AC , CB by a rational area:

which is impossible,

for both are medial [x. 15 and 23, Por.], and a medial area does not exceed a medial by a rational area. [x. 26]

Therefore etc.

Q. E. D.

Suppose, if possible, that the same *first apotome of a medial* straight line can be expressed in terms of the required character in two ways, so that

$$x - y = x' - y',$$

and suppose that $x > x'$.

In this case $x^2 + y^2$, $(x'^2 + y'^2)$ are both *medial* areas, and $2xy$, $2x'y'$ are both *rational* areas;

and

$$x^2 + y^2 - (x'^2 + y'^2) = 2xy - 2x'y'.$$

Hence x. 26 is contradicted again;

therefore etc.

PROPOSITION 81.

To a second apotome of a medial straight line only one medial straight line can be annexed which is commensurable with the whole in square only and which contains with the whole a medial rectangle.

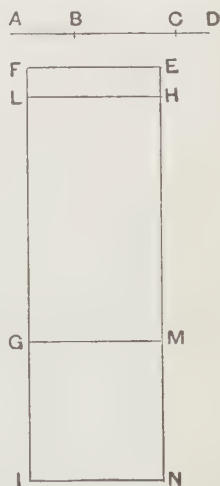
Let AB be a second apotome of a medial straight line and BC an annex to AB ;

therefore AC , CB are medial straight lines commensurable in square only and such that the rectangle AC , CB which they contain is medial. [x. 75]

I say that no other medial straight line can be annexed to AB which is commensurable with the whole in square only and which contains with the whole a medial rectangle.

For, if possible, let BD also be so annexed;

therefore AD , DB are also medial straight lines commensurable in square only and such that the rectangle AD , DB which they contain is medial. [x. 75]



Let a rational straight line EF be set out, let EG equal to the squares on AC , CB be applied to EF , producing EM as breadth, and let HG equal to twice the rectangle AC , CB be subtracted, producing HM as breadth; therefore the remainder EL is equal to the square on AB , so that AB is the "side" of EL . [II. 7]

Again, let EI equal to the squares on AD , DB be applied to EF , producing EN as breadth.

But EL is also equal to the square on AB ; therefore the remainder HI is equal to twice the rectangle AD , DB . [II. 7]

Now, since AC , CB are medial straight lines, therefore the squares on AC , CB are also medial.

And they are equal to EG ;

therefore EG is also medial. [x. 15 and 23, Por.]

And it is applied to the rational straight line EF , producing EM as breadth;

therefore EM is rational and incommensurable in length with EF . [x. 22]

Again, since the rectangle AC, CB is medial, twice the rectangle AC, CB is also medial. [x. 23, Por.]

And it is equal to HG ;

therefore HG is also medial.

And it is applied to the rational straight line EF , producing HM as breadth;

therefore HM is also rational and incommensurable in length with EF . [x. 22]

And, since AC, CB are commensurable in square only, therefore AC is incommensurable in length with CB .

But, as AC is to CB , so is the square on AC to the rectangle AC, CB ;

therefore the square on AC is incommensurable with the rectangle AC, CB . [x. 11]

But the squares on AC, CB are commensurable with the square on AC ,

while twice the rectangle AC, CB is commensurable with the rectangle AC, CB ; [x. 6]

therefore the squares on AC, CB are incommensurable with twice the rectangle AC, CB . [x. 13]

And EG is equal to the squares on AC, CB , while HG is equal to twice the rectangle AC, CB ;

therefore EG is incommensurable with HG .

But, as EG is to HG , so is EM to HM ; [vi. 1]

therefore EM is incommensurable in length with HM . [x. 11]

And both are rational;

therefore EM, MH are rational straight lines commensurable in square only;

therefore EH is an apotome, and HM an annex to it. [x. 73]

Similarly we can prove that HN is also an annex to it ; therefore to an apotome different straight lines are annexed which are commensurable with the wholes in square only : which is impossible. [x. 79]

Therefore etc.

Q. E. D.

As the irrationality of the *second apotome of a medial* straight line was deduced [x. 75] from the irrationality of an apotome, so the present theorem is reduced to x. 79.

Suppose, if possible, that $(x-y)$, $(x'-y')$ are the *same* second apotome of a medial straight line ; and let (say) x be greater than x' .

Apply (x^2+y^2) , $2xy$ and also $(x'^2+y'^2)$, $2x'y'$ to a rational straight line σ , i.e. put

$$\left. \begin{array}{l} x^2+y^2 = \sigma u \\ 2xy = \sigma v \end{array} \right\} \quad \text{and} \quad \left. \begin{array}{l} x'^2+y'^2 = \sigma u' \\ 2x'y' = \sigma v' \end{array} \right\}.$$

Dealing with $(x-y)$ first, we have :

(x^2+y^2) is a medial area, and $2xy$ is also a medial area.

Therefore u, v are both rational and $\sim \sigma$ (1).

Also, since $x \sim y$, $x \sim y$,
so that $x^2 \sim xy$,
whence, as usual, $x^2+y^2 \sim 2xy$,
that is, $\sigma u \sim \sigma v$,
and therefore $u \sim v$ (2).

Thus [(1) and (2)] u, v are rational and $\sim$,
so that $(u-v)$ is an *apotome*.

Similarly $(u'-v')$ is proved to be the *same* apotome.

Hence this apotome is formed in two ways :
which contradicts x. 79.

Therefore the original hypothesis is false, and a *second apotome of a medial* straight line is uniquely formed.

PROPOSITION 82.

To a minor straight line only one straight line can be annexed which is incommensurable in square with the whole and which makes, with the whole, the sum of the squares on them rational but twice the rectangle contained by them medial.

Let AB be the minor straight line, and let BC be an annex to AB ;

therefore AC, CB are straight lines incommensurable in square



which make the sum of the squares on them rational, but twice the rectangle contained by them medial. [x. 76]

I say that no other straight line can be annexed to AB fulfilling the same conditions.

For, if possible, let BD be so annexed ;
therefore AD, DB are also straight lines incommensurable in square which fulfil the aforesaid conditions. [x. 76]

Now, since the excess of the squares on AD, DB over the squares on AC, CB is also the excess of twice the rectangle AD, DB over twice the rectangle AC, CB , while the squares on AD, DB exceed the squares on AC, CB by a rational area, for both are rational, therefore twice the rectangle AD, DB also exceeds twice the rectangle AC, CB by a rational area : which is impossible, for both are medial. [x. 26]

Therefore to a minor straight line only one straight line can be annexed which is incommensurable in square with the whole and which makes the squares on them added together rational, but twice the rectangle contained by them medial.

Q. E. D.

Suppose, if possible, that, with the usual notation,

$$x - y = x' - y' ;$$

and let x (say) be greater than x' .

In this case $(x^2 + y^2), (x'^2 + y'^2)$ are both *rational* areas, and $2xy, 2x'y'$ are both *medial* areas.

But, as before, $(x^2 + y^2) - (x'^2 + y'^2) = 2xy - 2x'y'$, so that the difference between two medial areas is rational : which is impossible [x. 26].

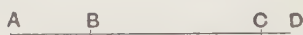
Therefore etc.

PROPOSITION 83.

To a straight line which produces with a rational area a medial whole only one straight line can be annexed which is incommensurable in square with the whole straight line and which with the whole straight line makes the sum of the squares on them medial, but twice the rectangle contained by them rational.

Let AB be the straight line which produces with a rational area a medial whole,

and let BC be an annex to AB ;



therefore AC , CB are straight lines

incommensurable in square which fulfil the given conditions.

[x. 77]

I say that no other straight line can be annexed to AB which fulfils the same conditions.

For, if possible, let BD be so annexed;

therefore AD , DB are also straight lines incommensurable in square which fulfil the given conditions.

[x. 77]

Since then, as in the preceding cases,

the excess of the squares on AD , DB over the squares on AC , CB is also the excess of twice the rectangle AD , DB over twice the rectangle AC , CB ,

while twice the rectangle AD , DB exceeds twice the rectangle AC , CB by a rational area,

for both are rational,

therefore the squares on AD , DB also exceed the squares on AC , CB by a rational area:

which is impossible, for both are medial.

[x. 26]

Therefore no other straight line can be annexed to AB which is incommensurable in square with the whole and which with the whole fulfils the aforesaid conditions;

therefore only one straight line can be so annexed.

Q. E. D.

Suppose, with the same notation, that

$$x - y = x' - y'. \quad (x > x')$$

Here, $(x^2 + y^2)$, $(x'^2 + y'^2)$ being both medial areas, and $2xy$, $2x'y'$ both rational areas,

$$\text{while} \quad (x^2 + y^2) - (x'^2 + y'^2) = 2xy - 2x'y',$$

x. 26 is contradicted again.

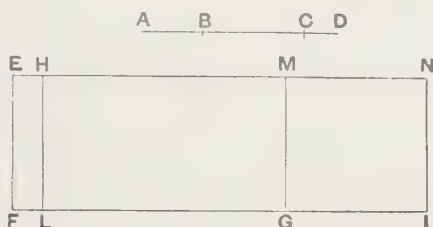
Therefore etc.

PROPOSITION 84.

To a straight line which produces with a medial area a medial whole only one straight line can be annexed which is incommensurable in square with the whole straight line and which with the whole straight line makes the sum of the squares

on them medial and twice the rectangle contained by them both medial and also incommensurable with the sum of the squares on them.

Let AB be the straight line which produces with a medial area a medial whole,
and BC an annex to it;
therefore AC , CB are straight lines incommensurable in square
which fulfil the aforesaid conditions. [x. 78]



I say that no other straight line can be annexed to AB which fulfils the aforesaid conditions.

For, if possible, let BD be so annexed,
so that AD , DB are also straight lines incommensurable in square which make the squares on AD , DB added together medial, twice the rectangle AD , DB medial, and also the squares on AD , DB incommensurable with twice the rectangle AD , DB . [x. 78]

Let a rational straight line EF be set out,
let EG equal to the squares on AC , CB be applied to EF ,
producing EM as breadth,
and let HG equal to twice the rectangle AC , CB be applied
to EF , producing HM as breadth;
therefore the remainder, the square on AB [II. 7], is equal
to EL ;
therefore AB is the "side" of EL .

Again, let EI equal to the squares on AD , DB be applied
to EF , producing EN as breadth.

But the square on AB is also equal to EL ;
therefore the remainder, twice the rectangle AD , DB [II. 7],
is equal to HI .

Now, since the sum of the squares on AC , CB is medial and is equal to EG ,
therefore EG is also medial.

And it is applied to the rational straight line EF , producing EM as breadth;
therefore EM is rational and incommensurable in length with EF . [x. 22]

Again, since twice the rectangle AC , CB is medial and is equal to HG ,
therefore HG is also medial.

And it is applied to the rational straight line EF , producing HM as breadth;
therefore HM is rational and incommensurable in length with EF . [x. 22]

And, since the squares on AC , CB are incommensurable with twice the rectangle AC , CB ,
 EG is also incommensurable with HG ;
therefore EM is also incommensurable in length with MH . [vi. 1, x. 11]

And both are rational;
therefore EM , MH are rational straight lines commensurable in square only;
therefore EH is an apotome, and HM an annex to it. [x. 73]

Similarly we can prove that EH is again an apotome and HN an annex to it.

Therefore to an apotome different rational straight lines are annexed which are commensurable with the wholes in square only:
which was proved impossible. [x. 79]

Therefore no other straight line can be so annexed to AB .

Therefore to AB only one straight line can be annexed which is incommensurable in square with the whole and which with the whole makes the squares on them added together medial, twice the rectangle contained by them medial, and also the squares on them incommensurable with twice the rectangle contained by them.

With the usual notation, suppose that

$$\begin{array}{l} \text{Let} \quad \left. \begin{array}{l} x^2 + y^2 = \sigma u \\ 2xy = \sigma v \end{array} \right\} \text{ and } \left. \begin{array}{l} x - y = x' - y' \quad (x > x') \\ x'^2 + y'^2 = \sigma u' \\ 2x'y' = \sigma v' \end{array} \right\} \end{array}$$

Consider $(x - y)$ first;
it follows, since $(x^2 + y^2)$, $2xy$ are both medial areas, that
 u, v are both rational and $\cup \sigma$ (1).

But $x^2 + y^2 \cup 2xy$,
that is, $\sigma u \cup \sigma v$,
and therefore $u \cup v$ (2).

Therefore [(1) and (2)] u, v are rational and $\sim$;
hence $(u - v)$ is an *apotome*.

Similarly $(u' - v')$ is proved to be the *same* apotome.

Thus the same apotome is formed as such in two ways:
which is impossible [x. 79].

Therefore, etc.

DEFINITIONS III.

1. Given a rational straight line and an apotome, if the square on the whole be greater than the square on the annex by the square on a straight line commensurable in length with the whole, and the whole be commensurable in length with the rational straight line set out, let the apotome be called a **first apotome**.

2. But if the annex be commensurable in length with the rational straight line set out, and the square on the whole be greater than that on the annex by the square on a straight line commensurable with the whole, let the apotome be called a **second apotome**.

3. But if neither be commensurable in length with the rational straight line set out, and the square on the whole be greater than the square on the annex by the square on a straight line commensurable with the whole, let the apotome be called a **third apotome**.

4. Again, if the square on the whole be greater than the square on the annex by the square on a straight line incommensurable with the whole, then, if the whole be commensurable in length with the rational straight line set out, let the apotome be called a **fourth apotome**;

5. if the annex be so commensurable, a **fifth**;

6. and, if neither, a **sixth**.

PROPOSITION 85.

To find the first apotome.

Let a rational straight line A be set out,
and let BG be commensurable in length with A ;
therefore BG is also rational.



Let two square numbers DE , EF be set out, and let their difference FD not be square;
therefore neither has ED to DF the ratio which a square number has to a square number.

Let it be contrived that,
as ED is to DF , so is the square on BG to the square on GC ;
[x. 6, Por.]
therefore the square on BG is commensurable with the square on GC .
[x. 6]

But the square on BG is rational;
therefore the square on GC is also rational;
therefore GC is also rational.

And, since ED has not to DF the ratio which a square number has to a square number,
therefore neither has the square on BG to the square on GC the ratio which a square number has to a square number;
therefore BG is incommensurable in length with GC . [x. 9]

And both are rational;
therefore BG , GC are rational straight lines commensurable in square only;
therefore BC is an apotome. [x. 73]

I say next that it is also a first apotome.

For let the square on H be that by which the square on BG is greater than the square on GC .

Now since, as ED is to FD , so is the square on BG to the square on GC ,
therefore also, *convertendo*, [v. 19, Por.]
as DE is to EF , so is the square on GB to the square on H .

But DE has to EF the ratio which a square number has to a square number,

for each is square ;

therefore the square on GB also has to the square on H the ratio which a square number has to a square number ;

therefore BG is commensurable in length with H . [x. 9]

And the square on BG is greater than the square on GC by the square on H ;

therefore the square on BG is greater than the square on GC by the square on a straight line commensurable in length with BG .

And the whole BG is commensurable in length with the rational straight line A set out.

Therefore BC is a first apotome. [x. Def. iii. 1]

Therefore the first apotome BC has been found.

(Being) that which it was required to find.

Take $k\rho$ commensurable in length with ρ , the given rational straight line.

Let m^2 , n^2 be square numbers such that $(m^2 - n^2)$ is not square.

Take x such that $m^2 : (m^2 - n^2) = k^2\rho^2 : x^2$ (1),

$$\begin{aligned} \text{so that} \quad x &= k\rho \frac{\sqrt{m^2 - n^2}}{m} \\ &= k\rho \sqrt{1 - \lambda^2}, \quad \text{say.} \end{aligned}$$

Then shall $k\rho - x$, or $k\rho - k\rho \sqrt{1 - \lambda^2}$, be a *first apotome*.

For (a) it follows from (1) that x is rational but incommensurable with $k\rho$, whence $k\rho$, x are rational and $\sim$,

so that $(k\rho - x)$ is an apotome.

(\beta) If $y^2 = k^2\rho^2 - x^2$, then, by (1), *convertendo*,

$$m^2 : n^2 = k^2\rho^2 : y^2,$$

whence y , that is, $\sqrt{k^2\rho^2 - x^2}$, is commensurable in length with $k\rho$.

And $k\rho \frown p$;

therefore $k\rho - x$ is a *first apotome*.

As explained in the note to x. 48, the first apotome

$$k\rho - k\rho\sqrt{1 - \lambda^2}$$

is one of the roots of the equation

$$x^2 - 2k\rho \cdot x + \lambda^2 k^2\rho^2 = 0.$$

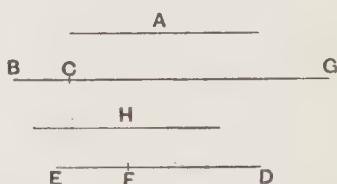
PROPOSITION 86.

To find the second apotome.

Let a rational straight line A be set out, and GC commensurable in length with A ;
therefore GC is rational.

Let two square numbers DE , EF be set out, and let their difference DF not be square.

Now let it be contrived that, as FD is to DE , so is the square on CG to the square on GB .



[x. 6, Por.]

Therefore the square on CG is commensurable with the square on GB . [x. 6]

But the square on CG is rational;
therefore the square on GB is also rational;
therefore GB is rational.

And, since the square on GC has not to the square on GB the ratio which a square number has to a square number, CG is incommensurable in length with GB . [x. 9]

And both are rational;
therefore CG , GB are rational straight lines commensurable in square only;
therefore BC is an apotome. [x. 73]

I say next that it is also a second apotome.

For let the square on H be that by which the square on BG is greater than the square on GC .

Since then, as the square on BG is to the square on GC , so is the number ED to the number DF ,
therefore, *convertendo*,

as the square on BG is to the square on H , so is DE to EF . [v. 19, Por.]

And each of the numbers DE , EF is square;
therefore the square on BG has to the square on H the ratio which a square number has to a square number;
therefore BG is commensurable in length with H . [x. 9]

And the square on BG is greater than the square on GC by the square on H ;
therefore the square on BG is greater than the square on GC

by the square on a straight line commensurable in length with BG .

And CG , the annex, is commensurable with the rational straight line A set out.

Therefore BC is a second apotome.

[x. Def. III. 2]

Therefore the second apotome BC has been found.

Q. E. D.

Take, as before, $k\rho$ commensurable in length with ρ .

Let m^2 , n^2 be again square numbers, but $(m^2 - n^2)$ not square.

Take x such that $(m^2 - n^2) : m^2 = k^2\rho^2 : x^2$ (1),

whence

$$x = k\rho \frac{m}{\sqrt{m^2 - n^2}} \\ = \frac{k\rho}{\sqrt{1 - \lambda^2}}, \text{ say.}$$

Thus x is greater than $k\rho$.

Then $x - k\rho$, or $\frac{k\rho}{\sqrt{1 - \lambda^2}} - k\rho$, is a *second apotome*.

For (a), as before, x is rational and $\sim k\rho$.

(\beta) If $x^2 - k^2\rho^2 = y^2$, we have, from (1),

$$m^2 : n^2 = x^2 : y^2.$$

Thus y , or $\sqrt{x^2 - k^2\rho^2}$, is commensurable in length with x .

And $k\rho$ is $\sim \rho$.

Therefore $x - k\rho$ is a *second apotome*.

As explained in the note on x. 49, the *second apotome*

$$\frac{k\rho}{\sqrt{1 - \lambda^2}} - k\rho$$

is the lesser root of the equation

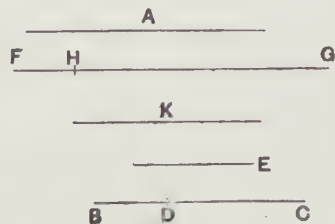
$$x^2 - \frac{2k\rho}{\sqrt{1 - \lambda^2}} \cdot x + \frac{\lambda^2}{1 - \lambda^2} k^2\rho^2 = 0.$$

PROPOSITION 87.

To find the third apotome.

Let a rational straight line A be set out,

let three numbers E , BC , CD be set out which have not to one another the ratio which a square number has to a square number, but let CB have to BD the ratio which a square number has to a square number.



Let it be contrived that, as E is to BC , so is the square on A to the square on FG ,

and, as BC is to CD , so is the square on FG to the square on GH . [x. 6, Por.]

Since then, as E is to BC , so is the square on A to the square on FG ,
therefore the square on A is commensurable with the square on FG . [x. 6]

But the square on A is rational;
therefore the square on FG is also rational;
therefore FG is rational.

And, since E has not to BC the ratio which a square number has to a square number,
therefore neither has the square on A to the square on FG the ratio which a square number has to a square number;
therefore A is incommensurable in length with FG . [x. 9]

Again, since, as BC is to CD , so is the square on FG to the square on GH ,
therefore the square on FG is commensurable with the square on GH . [x. 6]

But the square on FG is rational;
therefore the square on GH is also rational;
therefore GH is rational.

And, since BC has not to CD the ratio which a square number has to a square number,
therefore neither has the square on FG to the square on GH the ratio which a square number has to a square number;
therefore FG is incommensurable in length with GH . [x. 9]

And both are rational;
therefore FG , GH are rational straight lines commensurable in square only;
therefore FH is an apotome. [x. 73]

I say next that it is also a third apotome.

For since, as E is to BC , so is the square on A to the square on FG ,
and, as BC is to CD , so is the square on FG to the square on HG ,
therefore, *ex aequali*, as E is to CD , so is the square on A to the square on HG . [v. 22]

But E has not to CD the ratio which a square number has to a square number ;

therefore neither has the square on A to the square on GH the ratio which a square number has to a square number ;

therefore A is incommensurable in length with GH . [x. 9]

Therefore neither of the straight lines FG , GH is commensurable in length with the rational straight line A set out.

Now let the square on K be that by which the square on FG is greater than the square on GH .

Since then, as BC is to CD , so is the square on FG to the square on GH ,

therefore, *convertendo*, as BC is to BD , so is the square on FG to the square on K . [v. 19, Por.]

But BC has to BD the ratio which a square number has to a square number ;

therefore the square on FG also has to the square on K the ratio which a square number has to a square number.

Therefore FG is commensurable in length with K , [x. 9] and the square on FG is greater than the square on GH by the square on a straight line commensurable with FG .

And neither of the straight lines FG , GH is commensurable in length with the rational straight line A set out ;

therefore FH is a third apotome. [x. Def. iii. 3]

Therefore the third apotome FH has been found.

Q. E. D.

Let ρ be a rational straight line.

Take numbers p , qm^2 , $q(m^2 - n^2)$ which have not to one another the ratio of square to square.

Now let x , y be such that

$$p : qm^2 = \rho^2 : x^2 \dots\dots\dots (1)$$

and

$$qm^2 : q(m^2 - n^2) = x^2 : y^2 \dots\dots\dots (2).$$

Then shall $(x - y)$ be a *third apotome*.

For (a), from (1),

$$x \text{ is rational but } \sim \rho \dots\dots\dots (3).$$

And, from (2), y is rational but $\sim x$.

Therefore x , y are rational and $\sim$,

so that $(x - y)$ is an apotome.

(β) By (1), (2), *ex aequali*,

$$p : q (m^2 - n^2) = p^2 : y^2,$$

whence $y \sim p$.

Thus, by this and (3), x, y are both $\sim p$ (4).

Lastly, let $z^2 = x^2 - y^2$, so that, from (2), *convertendo*,

$$qm^2 : qn^2 = x^2 : z^2;$$

therefore z , or $\sqrt{x^2 - y^2}$, $\sim x$ (5).

Thus [(4) and (5)] $(x - y)$ is a *third apotome*.

To find its form, we have, from (1) and (2),

$$x = p \cdot \frac{m\sqrt{q}}{\sqrt{p}},$$

$$y = p \cdot \frac{\sqrt{m^2 - n^2} \cdot \sqrt{q}}{\sqrt{p}},$$

so that

$$x - y = \sqrt{\frac{q}{p}} \cdot p (m - \sqrt{m^2 - n^2}).$$

This may be written in the form

$$m\sqrt{k} \cdot p - m\sqrt{k} \cdot p\sqrt{1 - \lambda^2}.$$

As explained in the note on x. 50, this is the lesser root of the equation

$$x^2 - 2m\sqrt{k} \cdot px + \lambda^2 m^2 kp^2 = 0.$$

PROPOSITION 88.

To find the fourth apotome.

Let a rational straight line A be set out, and BG commensurable in length with it;

therefore BG is also rational.



Let two numbers DF, FE be set out such that the whole DE has not to either of the numbers DF, EF the ratio which a square number has to a square number.

Let it be contrived that, as DE is to EF , so is the square on BG to the square on GC ;

[x. 6, Por.]

therefore the square on BG is commensurable with the square on GC .

[x. 6]

But the square on BG is rational;

therefore the square on GC is also rational;

therefore GC is rational.

Now, since DE has not to EF the ratio which a square number has to a square number,
 therefore neither has the square on BG to the square on GC the ratio which a square number has to a square number ;
 therefore BG is incommensurable in length with GC . [x. 9]

And both are rational ;
 therefore BG , GC are rational straight lines commensurable in square only ;
 therefore BC is an apotome. [x. 73]

Now let the square on H be that by which the square on BG is greater than the square on GC .

Since then, as DE is to EF , so is the square on BG to the square on GC ,
 therefore also, *convertendo*, as ED is to DF , so is the square on GB to the square on H . [v. 19, Por.]

But ED has not to DF the ratio which a square number has to a square number ;
 therefore neither has the square on GB to the square on H the ratio which a square number has to a square number ;
 therefore BG is incommensurable in length with H . [x. 9]

And the square on BG is greater than the square on GC by the square on H ;
 therefore the square on BG is greater than the square on GC by the square on a straight line incommensurable with BG .

And the whole BG is commensurable in length with the rational straight line A set out.

Therefore BC is a fourth apotome. [x. Def. III. 4]

Therefore the fourth apotome has been found.

Q. E. D.

Beginning with ρ , $k\rho$, as in x. 85, 86, we take numbers m , n such that $(m+n)$ has not to either of the numbers m , n the ratio of a square number to a square number.

Take x such that

$$(m+n) : n = k^2 \rho^2 : x^2 \dots\dots\dots (1),$$

whence

$$x = k\rho \sqrt{\frac{n}{m+n}} \\ = \frac{k\rho}{\sqrt{1+\lambda}}, \text{ say.}$$

Then shall $(k\rho - x)$, or $\left(k\rho - \frac{k\rho}{\sqrt{1+\lambda}}\right)$, be a *fourth apotome*.

For, by (1), x is rational and $\sim kp$.

Also $\sqrt{k^2p^2 - x^2}$ is incommensurable with kp , since

$$(m + n) : m = k^2p^2 : (k^2p^2 - x^2),$$

and the ratio $(m + n) : m$ is not that of a square number to a square number.

And $kp \cap p$.

As explained in the note on x. 51, the *fourth apotome*

$$kp - \frac{kp}{\sqrt{1 + \lambda}}$$

is the lesser root of the quadratic equation

$$x^2 - 2kp \cdot x + \frac{\lambda}{1 + \lambda} k^2p^2 = 0.$$

PROPOSITION 89.

To find the fifth apotome.

Let a rational straight line A be set out, and let CG be commensurable in length with A ;

therefore CG is rational.

Let two numbers DF , FE be set out such that DE again has not to either of the numbers DF , FE the ratio which a square number has to a square number;

and let it be contrived that, as FE is to ED , so is the square on CG to the square on GB .

Therefore the square on GB is also rational;

therefore GB is also rational.

Now since, as DE is to EF , so is the square on BG to the square on GC ,

while DE has not to EF the ratio which a square number has to a square number,

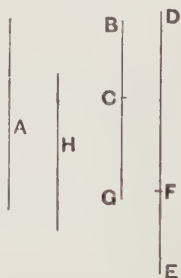
therefore neither has the square on BG to the square on GC the ratio which a square number has to a square number;

therefore BG is incommensurable in length with GC . [x. 9]

And both are rational;

therefore BG , GC are rational straight lines commensurable in square only;

therefore BC is an apotome. [x. 73]



[x. 6]

I say next that it is also a fifth apotome.

For let the square on H be that by which the square on BG is greater than the square on GC .

Since then, as the square on BG is to the square on GC , so is DE to EF ,

therefore, *convertendo*, as ED is to DF , so is the square on BG to the square on H . [v. 19, Por.]

But ED has not to DF the ratio which a square number has to a square number ;

therefore neither has the square on BG to the square on H the ratio which a square number has to a square number ;

therefore BG is incommensurable in length with H . [x. 9]

And the square on BG is greater than the square on GC by the square on H ;

therefore the square on GB is greater than the square on GC by the square on a straight line incommensurable in length with GB .

And the annex CG is commensurable in length with the rational straight line A set out ;

therefore BC is a fifth apotome. [x. Def. iii. 5]

Therefore the fifth apotome BC has been found.

Q. E. D.

Let ρ , $k\rho$ and the numbers m , n of the last proposition be taken.

Take x such that $n : (m + n) = k^2\rho^2 : x^2$ (1).

In this case $x > k\rho$, and
$$x = k\rho\sqrt{\frac{m+n}{n}} \\ = k\rho\sqrt{1+\lambda}, \text{ say.}$$

Then shall $(x - k\rho)$, or $(k\rho\sqrt{1+\lambda} - k\rho)$, be a *fifth apotome*.

For, by (1), x is rational and $\sim k\rho$.

And since, by (1), $(m + n) : m = x^2 : (x^2 - k^2\rho^2)$,

$\sqrt{x^2 - k^2\rho^2}$ is incommensurable with x .

Also $k\rho \sim \rho$.

As explained in the note on x. 52, the *fifth apotome*

$$k\rho\sqrt{1+\lambda} - k\rho$$

is the lesser root of the quadratic

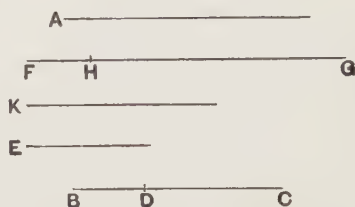
$$x^2 - 2k\rho\sqrt{1+\lambda} \cdot x + \lambda k^2\rho^2 = 0.$$

PROPOSITION 90.

To find the sixth apotome.

Let a rational straight line A be set out, and three numbers E, BC, CD not having to one another the ratio which a square number has to a square number ;

and further let CB also not have to BD the ratio which a square number has to a square number.



Let it be contrived that, as E is to BC , so is the square on A to the square on FG , and, as BC is to CD , so is the square on FG to the square on GH . [x. 6, Por.]

Now since, as E is to BC , so is the square on A to the square on FG , therefore the square on A is commensurable with the square on FG . [x. 6]

But the square on A is rational ;
therefore the square on FG is also rational ;
therefore FG is also rational.

And, since E has not to BC the ratio which a square number has to a square number, therefore neither has the square on A to the square on FG the ratio which a square number has to a square number ;
therefore A is incommensurable in length with FG . [x. 9]

Again, since, as BC is to CD , so is the square on FG to the square on GH , therefore the square on FG is commensurable with the square on GH . [x. 6]

But the square on FG is rational ;
therefore the square on GH is also rational ;
therefore GH is also rational.

And, since BC has not to CD the ratio which a square number has to a square number,

therefore neither has the square on FG to the square on GH the ratio which a square number has to a square number ;
therefore FG is incommensurable in length with GH . [x. 9]

And both are rational ;

therefore FG , GH are rational straight lines commensurable in square only ;

therefore FH is an apotome. [x. 73]

I say next that it is also a sixth apotome.

For since, as E is to BC , so is the square on A to the square on FG ,

and, as BC is to CD , so is the square on FG to the square on GH ,

therefore, *ex aequali*, as E is to CD , so is the square on A to the square on GH . [v. 22]

But E has not to CD the ratio which a square number has to a square number ;

therefore neither has the square on A to the square on GH the ratio which a square number has to a square number ;

therefore A is incommensurable in length with GH ; [x. 9]

therefore neither of the straight lines FG , GH is commensurable in length with the rational straight line A .

Now let the square on K be that by which the square on FG is greater than the square on GH .

Since then, as BC is to CD , so is the square on FG to the square on GH ,

therefore, *convertendo*, as CB is to BD , so is the square on FG to the square on K . [v. 19, Por.]

But CB has not to BD the ratio which a square number has to a square number ;

therefore neither has the square on FG to the square on K the ratio which a square number has to a square number ;

therefore FG is incommensurable in length with K . [x. 9]

And the square on FG is greater than the square on GH by the square on K ;

therefore the square on FG is greater than the square on GH by the square on a straight line incommensurable in length with FG .

And neither of the straight lines FG , GH is commensurable with the rational straight line A set out.

Therefore FH is a sixth apotome. [x. Deff. iii. 6]

Therefore the sixth apotome FH has been found.

Q. E. D.

Let ρ be the given rational straight line.

Take numbers p , $(m+n)$, n which have not to one another the ratio of a square number to a square number, m , n being also chosen such that the ratio $(m+n) : m$ is not that of square to square.

Take x , y such that $p : (m+n) = p^2 : x^2$ (1),

$(m+n) : n = x^2 : y^2$ (2).

Then shall $(x-y)$ be a *sixth apotome*.

For, by (1), x is rational and $\cup \rho$ (3).

By (2), since x is rational,

y is rational and $\cup x$ (4).

Thus [(3), (4)] $(x-y)$ is an apotome.

Again, *ex aequali*, $p : n = p^2 : y^2$,

whence $y \cup \rho$.

Thus x , y are both $\cup \rho$.

Lastly, *convertendo* from (2),

$(m+n) : m = x^2 : (x^2 - y^2)$,

whence $\sqrt{x^2 - y^2} \cup x$.

Therefore $(x-y)$ is a *sixth apotome*.

From (1) and (2) we have

$$x = \rho \sqrt{\frac{m+n}{p}},$$

$$y = \rho \sqrt{\frac{n}{p}},$$

so that the *sixth apotome* may be written

$$\rho \sqrt{\frac{m+n}{p}} - \rho \sqrt{\frac{n}{p}},$$

or, more simply,

$$\sqrt{k} \cdot \rho - \sqrt{\lambda} \cdot \rho.$$

As explained in the note on x. 53, the *sixth apotome* is the lesser root of the equation

$$x^2 - 2\sqrt{k} \cdot \rho x + (k - \lambda) \rho^2 = 0.$$

PROPOSITION 91.

If an area be contained by a rational straight line and a first apotome, the "side" of the area is an apotome.

For let the area AB be contained by the rational straight line AC and the first apotome AD ;

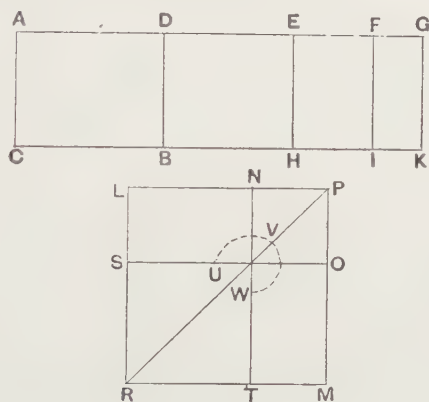
I say that the "side" of the area AB is an apotome.

For, since AD is a first apotome, let DG be its annex ;
therefore AG , GD are rational straight lines commensurable
in square only. [x. 73]

And the whole AG is commensurable with the rational
straight line AC set out,

and the square on AG is greater than the square on GD
by the square on a straight line commensurable in length
with AG ; [x. Def. III. 1]

if therefore there be applied to AG a parallelogram equal to
the fourth part of the square on DG and deficient by a square
figure, it divides it into commensurable parts. [x. 17]



Let DG be bisected at E ,
let there be applied to AG a parallelogram equal to the square
on EG and deficient by a square figure,
and let it be the rectangle AF , FG ;
therefore AF is commensurable with FG .

And through the points E , F , G let EH , FI , GK be drawn
parallel to AC .

Now, since AF is commensurable in length with FG ,
therefore AG is also commensurable in length with each of
the straight lines AF , FG . [x. 15]

But AG is commensurable with AC ;
therefore each of the straight lines AF , FG is commensurable
in length with AC . [x. 12]

And AC is rational ;
 therefore each of the straight lines AF, FG is also rational,
 so that each of the rectangles AI, FK is also rational. [x. 19]

Now, since DE is commensurable in length with EG ,
 therefore DG is also commensurable in length with each of
 the straight lines DE, EG . [x. 15]

But DG is rational and incommensurable in length
 with AC ;

therefore each of the straight lines DE, EG is also rational
 and incommensurable in length with AC ; [x. 13]

therefore each of the rectangles DH, EK is medial. [x. 21]

Now let the square LM be made equal to AI , and let
 there be subtracted the square NO having a common angle
 with it, the angle LPM , and equal to FK ;

therefore the squares LM, NO are about the same diameter.
 [vi. 26]

Let PR be their diameter, and let the figure be drawn.

Since then the rectangle contained by AF, FG is equal to
 the square on EG ,

therefore, as AF is to EG , so is EG to FG . [vi. 17]

But, as AF is to EG , so is AI to EK ,

and, as EG is to FG , so is EK to KF ; [vi. 1]

therefore EK is a mean proportional between AI, KF . [v. 11]

But MN is also a mean proportional between LM, NO ,
 as was before proved, [Lemma after x. 53]

and AI is equal to the square LM , and KF to NO ;

therefore MN is also equal to EK .

But EK is equal to DH , and MN to LO ;

therefore DK is equal to the gnomon UVW and NO .

But AK is also equal to the squares LM, NO ;

therefore the remainder AB is equal to ST .

But ST is the square on LN ;

therefore the square on LN is equal to AB ;

therefore LN is the "side" of AB .

I say next that LN is an apotome.

For, since each of the rectangles AI , FK is rational, and they are equal to LM , NO , therefore each of the squares LM , NO , that is, the squares on LP , PN respectively, is also rational; therefore each of the straight lines LP , PN is also rational.

Again, since DH is medial and is equal to LO , therefore LO is also medial.

Since then LO is medial, while NO is rational, therefore LO is incommensurable with NO .

But, as LO is to NO , so is LP to PN ; [vi. 1]
therefore LP is incommensurable in length with PN . [x. 11]

And both are rational;
therefore LP , PN are rational straight lines commensurable in square only;
therefore LN is an apotome. [x. 73]

And it is the "side" of the area AB ;
therefore the "side" of the area AB is an apotome.

Therefore etc.

This proposition corresponds to x. 54, and the problem solved in it is to find and to classify the *side of a square equal to the rectangle contained by a first apotome and ρ* , or (algebraically) to find

$$\sqrt{\rho(k\rho - k\rho\sqrt{1-\lambda^2})}.$$

First find u , v from the equations

$$\left. \begin{aligned} u + v &= k\rho \\ uv &= \frac{1}{4}k^2\rho^2(1-\lambda^2) \end{aligned} \right\} \dots\dots\dots(1).$$

If u , v represent the values so found, put

$$\left. \begin{aligned} x^2 &= \rho u \\ y^2 &= \rho v \end{aligned} \right\} \dots\dots\dots(2),$$

and $(x-y)$ shall be the square root required.

To prove this Euclid argues thus.

$$\text{By (1),} \quad u : \frac{1}{2}k\rho\sqrt{1-\lambda^2} = \frac{1}{2}k\rho\sqrt{1-\lambda^2} : v,$$

$$\text{whence} \quad \rho u : \frac{1}{2}k\rho^2\sqrt{1-\lambda^2} = \frac{1}{2}k\rho^2\sqrt{1-\lambda^2} : \rho v,$$

$$\text{or} \quad x^2 : \frac{1}{2}k\rho^2\sqrt{1-\lambda^2} = \frac{1}{2}k\rho^2\sqrt{1-\lambda^2} : y^2.$$

But [Lemma after x. 53]

$$\left. \begin{aligned} x^2 : xy &= xy : y^2, \\ xy &= \frac{1}{2}k\rho^2\sqrt{1-\lambda^2} \end{aligned} \right\} \dots\dots\dots(3).$$

so that

Therefore

$$\begin{aligned}(x-y)^2 &= x^2 + y^2 - 2xy \\ &= \rho(u+v) - k\rho^2\sqrt{1-\lambda^2} \\ &= k\rho^2 - k\rho^2\sqrt{1-\lambda^2}.\end{aligned}$$

Thus $(x-y)$ is equal to $\sqrt{\rho(k\rho - k\rho\sqrt{1-\lambda^2})}$.

It has next to be proved that $(x-y)$ is an *apotome*.

From (1) it follows, by X. 17, that

$$u \sim v;$$

thus u, v are both commensurable with $(u+v)$ and therefore with ρ(4).

Hence u, v are both rational,

so that $\rho u, \rho v$ are rational areas;

therefore, by (2), x^2, y^2 are rational and commensurable(5),

whence also x, y are rational straight lines(6).

Next, $k\rho\sqrt{1-\lambda^2}$ is rational and $\sim \rho$;

therefore $\frac{1}{2}k\rho^2\sqrt{1-\lambda^2}$ is a *medial* area.

That is, by (3), xy is a medial area.

But [(5)] y^2 is a *rational* area;

therefore $xy \sim y^2$,

or $x \sim y$.

But [(6)] x, y are both rational.

Therefore x, y are rational and $\sim$;

so that $(x-y)$ is an *apotome*.

To find the form of $(x-y)$ algebraically, we have, by solving (1),

$$u = \frac{1}{2}k\rho(1+\lambda),$$

$$v = \frac{1}{2}k\rho(1-\lambda),$$

whence, from (2),

$$x = \rho\sqrt{\frac{k}{2}(1+\lambda)},$$

$$y = \rho\sqrt{\frac{k}{2}(1-\lambda)},$$

and

$$x-y = \rho\sqrt{\frac{k}{2}(1+\lambda)} - \rho\sqrt{\frac{k}{2}(1-\lambda)}.$$

As explained in the note on X. 54, $(x-y)$ is the lesser positive root of the biquadratic equation

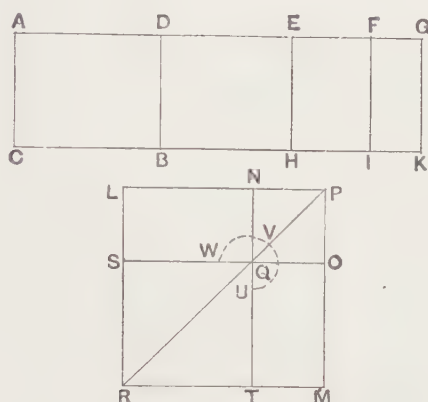
$$x^4 - 2k\rho^2 \cdot x^2 + \lambda^2 k^2 \rho^4 = 0.$$

PROPOSITION 92.

If an area be contained by a rational straight line and a second apotome, the "side" of the area is a first apotome of a medial straight line.

For let the area AB be contained by the rational straight line AC and the second apotome AD ;

I say that the "side" of the area AB is a first apotome of a medial straight line.



For let DG be the annex to AD ;
 therefore AG , GD are rational straight lines commensurable
 in square only, [x. 73]
 and the annex DG is commensurable with the rational straight
 line AC set out,

while the square on the whole AG is greater than the square
 on the annex GD by the square on a straight line commen-
 surable in length with AG . [x. Def. III. 2]

Since then the square on AG is greater than the square
 on GD by the square on a straight line commensurable
 with AG ,

therefore, if there be applied to AG a parallelogram equal to
 the fourth part of the square on GD and deficient by a square
 figure, it divides it into commensurable parts. [x. 17]

Let then DG be bisected at E ,
 let there be applied to AG a parallelogram equal to the square
 on EG and deficient by a square figure,
 and let it be the rectangle AF , FG ;
 therefore AF is commensurable in length with FG .

Therefore AG is also commensurable in length with each
 of the straight lines AF , FG . [x. 15]

But AG is rational and incommensurable in length
 with AC ;

therefore each of the straight lines AF , FG is also rational and incommensurable in length with AC ; [x. 13]

therefore each of the rectangles AI , FK is medial. [x. 21]

Again, since DE is commensurable with EG , therefore DG is also commensurable with each of the straight lines DE , EG . [x. 15]

But DG is commensurable in length with AC .

Therefore each of the rectangles DH , EK is rational. [x. 19]

Let then the square LM be constructed equal to AI , and let there be subtracted NO equal to FK and being about the same angle with LM , namely the angle LPM ;

therefore the squares LM , NO are about the same diameter. [vi. 26]

Let PR be their diameter, and let the figure be drawn.

Since then AI , FK are medial and are equal to the squares on LP , PN ,

the squares on LP , PN are also medial;

therefore LP , PN are also medial straight lines commensurable in square only.

And, since the rectangle AF , FG is equal to the square on EG ,

therefore, as AF is to EG , so is EG to FG , [vi. 17]

while, as AF is to EG , so is AI to EK ,

and, as EG is to FG , so is EK to FK ; [vi. 1]

therefore EK is a mean proportional between AI , FK . [v. 11]

But MN is also a mean proportional between the squares LM , NO ,

and AI is equal to LM , and FK to NO ;

therefore MN is also equal to EK .

But DH is equal to EK , and LO equal to MN ;

therefore the whole DK is equal to the gnomon UVW and NO .

Since then the whole AK is equal to LM , NO ,

and, in these, DK is equal to the gnomon UVW and NO , therefore the remainder AB is equal to TS .

But TS is the square on LN ;
 therefore the square on LN is equal to the area AB ;
 therefore LN is the "side" of the area AB .

I say that LN is a first apotome of a medial straight line.
 For, since EK is rational and is equal to LO ,
 therefore LO , that is, the rectangle LP, PN , is rational.

But NO was proved medial;
 therefore LO is incommensurable with NO .

But, as LO is to NO , so is LP to PN ; [VI. 1]
 therefore LP, PN are incommensurable in length. [X. 11]

Therefore LP, PN are medial straight lines commensurable in square only which contain a rational rectangle;
 therefore LN is a first apotome of a medial straight line. [X. 74]

And it is the "side" of the area AB .

Therefore the "side" of the area AB is a first apotome of a medial straight line.

Q. E. D.

There is an evident flaw in the text in the place (Heiberg, p. 282, ll. 17—20: translation p. 196 above) where it is said that "since then AI, FK are medial and are equal to the squares on LP, PN , the squares on LP, PN are also medial; therefore LP, PN are also medial straight lines commensurable in square only." It is not till the last lines of the proposition (Heiberg, p. 284, ll. 17, 18) that it is proved that LP, PN are incommensurable in length. What should have been proved in the former passage is that the squares on LP, PN are commensurable, so that LP, PN are commensurable in square (not commensurable in square only). I have supplied the step in the note below: "Also $x^2 \cap y^2$, since $u \cap v$." Theon seems to have observed the omission and to have put "and commensurable with one another" after "medial" in the passage quoted, though even this does not show why the squares on LP, PN are commensurable. One ms. (V) also has "only" ($\muόνον$) erased after "commensurable in square."

This proposition amounts to finding and classifying

$$\sqrt{\rho \left(\frac{k\rho}{\sqrt{1-\lambda^2}} - k\rho \right)}.$$

The method is that of the last proposition. Euclid solves, first, the equations

$$\left. \begin{aligned} u + v &= \frac{k\rho}{\sqrt{1-\lambda^2}} \\ uv &= \frac{1}{4} k^2 \rho^2 \end{aligned} \right\} \dots\dots\dots (1).$$

Then, using the values of u, v so found, he puts

$$\left. \begin{aligned} x^2 &= \rho u \\ y^2 &= \rho v \end{aligned} \right\} \dots\dots\dots (2),$$

and $(x - y)$ is the square root required.

That
$$(x - y) = \sqrt{\rho \left(\frac{k\rho}{\sqrt{1 - \lambda^2}} - k\rho \right)}$$

is proved in the same way as is the corresponding fact in x. 91.

From (1)
$$u : \frac{1}{2}k\rho = \frac{1}{2}k\rho : v,$$

so that
$$\rho u : \frac{1}{2}k\rho^2 = \frac{1}{2}k\rho^2 : \rho v.$$

But
$$x^2 : xy = xy : y^2,$$

whence, by (2),
$$xy = \frac{1}{2}k\rho^2 \dots\dots\dots (3).$$

Therefore
$$\begin{aligned} (x - y)^2 &= x^2 + y^2 - 2xy \\ &= \rho(u + v) - k\rho^2 \\ &= \rho \left(\frac{k\rho}{\sqrt{1 - \lambda^2}} - k\rho \right). \end{aligned}$$

Next, we have to prove that $(x - y)$ is a *first apotome of a medial* straight line.

From (1) it follows, by x. 17, that

$$u \sim v \dots\dots\dots (4),$$

therefore u, v are both $\sim (u + v)$.

But [(1)] $(u + v)$ is rational and $\sim \rho$;

therefore u, v are both rational and $\sim \rho \dots\dots\dots (5).$

Therefore $\rho u, \rho v$, or x^2, y^2 , are both *medial* areas, and x, y are medial straight lines $\dots\dots\dots (6).$

Also $x^2 \sim y^2$, since $u \sim v$ [(4)] $\dots\dots\dots (7).$

Now xy , or $\frac{1}{2}k\rho^2$, is a *rational* area;

therefore
$$xy \sim y^2,$$

and
$$x \sim y.$$

Hence [(6), (7), (3)] x, y are medial straight lines commensurable in square only and containing a rational rectangle;

therefore $(x - y)$ is a *first apotome of a medial* straight line.

Algebraical solution of the equations gives

$$u = \frac{1}{2} \frac{1 + \lambda}{\sqrt{1 - \lambda^2}} k\rho,$$

$$v = \frac{1}{2} \frac{1 - \lambda}{\sqrt{1 - \lambda^2}} k\rho,$$

and

$$x - y = \rho \sqrt{\frac{k}{2} \left(\frac{1 + \lambda}{1 - \lambda} \right)} - \rho \sqrt{\frac{k}{2} \left(\frac{1 - \lambda}{1 + \lambda} \right)}^{\frac{1}{2}}$$

As explained in the note on x. 55, this is the lesser positive root of the equation

$$x^4 - \frac{2k\rho^2}{\sqrt{1 - \lambda^2}} x^2 + \frac{\lambda^2}{1 - \lambda^2} k^2 \rho^4 = 0.$$

PROPOSITION 93.

If an area be contained by a rational straight line and a third apotome, the "side" of the area is a second apotome of a medial straight line.

For let the area AB be contained by the rational straight line AC and the third apotome AD ;

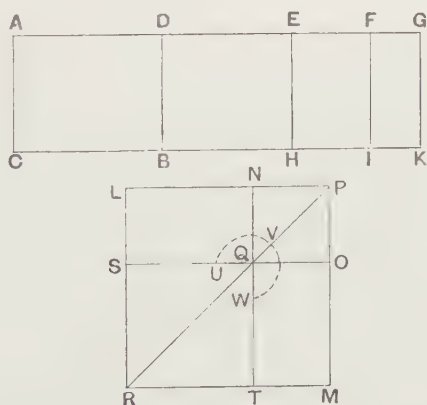
I say that the "side" of the area AB is a second apotome of a medial straight line.

For let DG be the annex to AD ;

therefore AG , GD are rational straight lines commensurable in square only,

and neither of the straight lines AG , GD is commensurable in length with the rational straight line AC set out,

while the square on the whole AG is greater than the square on the annex DG by the square on a straight line commensurable with AG . [x. Def. iii. 3]



Since then the square on AG is greater than the square on GD by the square on a straight line commensurable with AG ,

therefore, if there be applied to AG a parallelogram equal to the fourth part of the square on DG and deficient by a square figure, it will divide it into commensurable parts. [x. 17]

Let then DG be bisected at E ,

let there be applied to AG a parallelogram equal to the square on EG and deficient by a square figure, and let it be the rectangle AF , FG .

Let EH, FI, GK be drawn through the points E, F, G parallel to AC .

Therefore AF, FG are commensurable ;

therefore AI is also commensurable with FK . [vi. 1, x. 11]

And, since AF, FG are commensurable in length,

therefore AG is also commensurable in length with each of the straight lines AF, FG . [x. 15]

But AG is rational and incommensurable in length with AC ;

so that AF, FG are so also. [x. 13]

Therefore each of the rectangles AI, FK is medial. [x. 21]

Again, since DE is commensurable in length with EG , therefore DG is also commensurable in length with each of the straight lines DE, EG . [x. 15]

But GD is rational and incommensurable in length with AC ;

therefore each of the straight lines DE, EG is also rational and incommensurable in length with AC ; [x. 13]

therefore each of the rectangles DH, EK is medial. [x. 21]

And, since AG, GD are commensurable in square only, therefore AG is incommensurable in length with GD .

But AG is commensurable in length with AF , and DG with EG ;

therefore AF is incommensurable in length with EG . [x. 13]

But, as AF is to EG , so is AI to EK ; [vi. 1]

therefore AI is incommensurable with EK . [x. 11]

Now let the square LM be constructed equal to AI , and let there be subtracted NO equal to FK and being about the same angle with LM ;

therefore LM, NO are about the same diameter. [vi. 26]

Let PR be their diameter, and let the figure be drawn.

Now, since the rectangle AF, FG is equal to the square on EG ,

therefore, as AF is to EG , so is EG to FG . [vi. 17]

But, as AF is to EG , so is AI to EK ,
 and, as EG is to FG , so is EK to FK ; [vi. 1]
 therefore also, as AI is to EK , so is EK to FK ; [v. 11]
 therefore EK is a mean proportional between AI , FK .

But MN is also a mean proportional between the squares
 LM , NO ,
 and AI is equal to LM , and FK to NO ;
 therefore EK is also equal to MN .

But MN is equal to LO , and EK equal to DH ;
 therefore the whole DK is also equal to the gnomon UVW
 and NO .

But AK is also equal to LM , NO ;
 therefore the remainder AB is equal to ST , that is, to the
 square on LN ;
 therefore LN is the "side" of the area AB .

I say that LN is a second apotome of a medial straight
 line.

For, since AI , FK were proved medial, and are equal to the
 squares on LP , PN ,
 therefore each of the squares on LP , PN is also medial;
 therefore each of the straight lines LP , PN is medial.

And, since AI is commensurable with FK , [vi. 1, x. 11]
 therefore the square on LP is also commensurable with the
 square on PN .

Again, since AI was proved incommensurable with EK ,
 therefore LM is also incommensurable with MN ,
 that is, the square on LP with the rectangle LP , PN ;
 so that LP is also incommensurable in length with PN ;
 [vi. 1, x. 11]
 therefore LP , PN are medial straight lines commensurable in
 square only.

I say next that they also contain a medial rectangle.

For, since EK was proved medial, and is equal to the
 rectangle LP , PN ,
 therefore the rectangle LP , PN is also medial,
 so that LP , PN are medial straight lines commensurable in
 square only which contain a medial rectangle.

Therefore LN is a second apotome of a medial straight line; [x. 75]

and it is the "side" of the area AB .

Therefore the "side" of the area AB is a second apotome of a medial straight line.

Q. E. D.

Here we are to find and classify the irrational straight line

$$\sqrt{\rho(\sqrt{k} \cdot \rho - \sqrt{k} \cdot \rho \sqrt{1 - \lambda^2})}.$$

Following the same method, we put

$$\left. \begin{aligned} u + v &= \sqrt{k} \cdot \rho \\ uv &= \frac{1}{2} k \rho^2 (1 - \lambda^2) \end{aligned} \right\} \dots\dots\dots (1).$$

Next, u, v being found, let

$$\left. \begin{aligned} x^2 &= \rho u \\ y^2 &= \rho v \end{aligned} \right\} \dots\dots\dots (2);$$

then $(x - y)$ is the square root required and is a *second apotome of a medial straight line*.

That $(x - y)$ is the square root required and that x^2, y^2 are medial areas, so that x, y are medial straight lines, is proved exactly as in the last proposition.

The rectangle xy , being equal to $\frac{1}{2} \sqrt{k} \cdot \rho^2 \sqrt{1 - \lambda^2}$, is also medial.

Now, from (1), by x. 17, $u \cap v$,

whence $u + v \cap u$.

But $(u + v)$, or $\sqrt{k} \cdot \rho \cup \frac{1}{2} \sqrt{k} \cdot \rho \sqrt{1 - \lambda^2}$;

therefore $u \cup \frac{1}{2} \sqrt{k} \cdot \rho \sqrt{1 - \lambda^2}$,

and consequently $\rho u \cup \frac{1}{2} \sqrt{k} \cdot \rho^2 \sqrt{1 - \lambda^2}$,

or $x^2 \cup xy$,

whence $x \cup y$.

And, since $u \cap v$, $\rho u \cap \rho v$,

or $x^2 \cap y^2$

Thus x, y are medial straight lines commensurable in square only.

And xy is a medial area.

Therefore $(x - y)$ is a *second apotome of a medial straight line*.

Its actual form is found by solving equations (1), (2);

thus $u = \frac{1}{2} (\sqrt{k} \cdot \rho + \lambda \sqrt{k} \cdot \rho)$,

$$v = \frac{1}{2} (\sqrt{k} \cdot \rho - \lambda \sqrt{k} \cdot \rho),$$

and

$$x - y = \rho \sqrt{\frac{\sqrt{k}}{2} (1 + \lambda)} - \rho \sqrt{\frac{\sqrt{k}}{2} (1 - \lambda)}.$$

As explained in the note on x. 56, this is the lesser positive root of the equation

$$x^4 - 2 \sqrt{k} \cdot \rho^2 x^2 + \lambda^2 k \rho^4 = 0.$$

PROPOSITION 94.

If an area be contained by a rational straight line and a fourth apotome, the "side" of the area is minor.

For let the area AB be contained by the rational straight line AC and the fourth apotome AD ;

I say that the "side" of the area AB is minor.

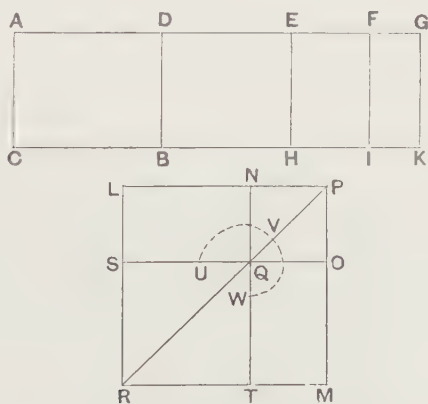
For let DG be the annex to AD ;

therefore AG , GD are rational straight lines commensurable in square only,

AG is commensurable in length with the rational straight line AC set out,

and the square on the whole AG is greater than the square on the annex DG by the square on a straight line incommensurable in length with AG ,

[x. Def. III. 4]



Since then the square on AG is greater than the square on GD by the square on a straight line incommensurable in length with AG ,

therefore, if there be applied to AG a parallelogram equal to the fourth part of the square on DG and deficient by a square figure, it will divide it into incommensurable parts. [x. 18]

Let then DG be bisected at E ,

let there be applied to AG a parallelogram equal to the square on EG and deficient by a square figure,

and let it be the rectangle AF , FG ;

therefore AF is incommensurable in length with FG .

Let EH , FI , GK be drawn through E , F , G parallel to AC , BD .

Since then AG is rational and commensurable in length with AC ,

therefore the whole AK is rational. [x. 19]

Again, since DG is incommensurable in length with AC , and both are rational,

therefore DK is medial. [x. 21]

Again, since AF is incommensurable in length with FG , therefore AI is also incommensurable with FK . [vi. 1, x. 11]

Now let the square LM be constructed equal to AI , and let there be subtracted NO equal to FK and about the same angle, the angle LPM .

Therefore the squares LM , NO are about the same diameter. [vi. 26]

Let PR be their diameter, and let the figure be drawn.

Since then the rectangle AF , FG is equal to the square on EG ,

therefore, proportionally, as AF is to EG , so is EG to FG . [vi. 17]

But, as AF is to EG , so is AI to EK , and, as EG is to FG , so is EK to FK ; [vi. 1]

therefore EK is a mean proportional between AI , FK . [v. 11]

But MN is also a mean proportional between the squares LM , NO ,

and AI is equal to LM , and FK to NO ;

therefore EK is also equal to MN .

But DH is equal to EK , and LO is equal to MN ; therefore the whole DK is equal to the gnomon UVW and NO .

Since, then, the whole AK is equal to the squares LM , NO ,

and, in these, DK is equal to the gnomon UVW and the square NO ,

therefore the remainder AB is equal to ST , that is, to the square on LN ;

therefore LN is the "side" of the area AB .

I say that LN is the irrational straight line called minor.

For, since AK is rational and is equal to the squares on LP, PN ,

therefore the sum of the squares on LP, PN is rational.

Again, since DK is medial,

and DK is equal to twice the rectangle LP, PN ,

therefore twice the rectangle LP, PN is medial.

And, since AI was proved incommensurable with FK , therefore the square on LP is also incommensurable with the square on PN .

Therefore LP, PN are straight lines incommensurable in square which make the sum of the squares on them rational, but twice the rectangle contained by them medial.

Therefore LN is the irrational straight line called minor; [x. 76]

and it is the "side" of the area AB .

Therefore the "side" of the area AB is minor.

Q. E. D.

We have here to find and classify the straight line

$$\sqrt{\rho \left(k\rho - \frac{k\rho}{\sqrt{1+\lambda}} \right)}.$$

As usual, we find u, v from the equations

$$\left. \begin{aligned} u+v &= k\rho \\ uv &= \frac{1}{4} \frac{k^2 \rho^2}{1+\lambda} \end{aligned} \right\} \dots\dots\dots (1),$$

and then, giving u, v their values, we put

$$\left. \begin{aligned} x^2 &= \rho u \\ y^2 &= \rho v \end{aligned} \right\} \dots\dots\dots (2).$$

Then $(x-y)$ is the required square root.

This is proved in the same way as before, and, as before, it is proved that

$$xy = \frac{1}{2} \frac{k\rho^2}{\sqrt{1+\lambda}}.$$

Now, from (1), by x. 18,

$$u \sim v;$$

therefore

$$\rho u \sim \rho v,$$

or

$$x^2 \sim y^2,$$

so that x, y are incommensurable in square.

And $x^2 + y^2$, or $\rho(u+v)$, is a *rational* area ($k\rho^2$).

But $2xy = \frac{k\rho^2}{\sqrt{1+\lambda}}$, which is a *medial* area.

Hence [x. 76] $(x-y)$ is the irrational straight line called *minor*.

Algebraical solution gives

$$u = \frac{1}{2} k \rho \left(1 + \sqrt{\frac{\lambda}{1 + \lambda}} \right),$$

$$v = \frac{1}{2} k \rho \left(1 - \sqrt{\frac{\lambda}{1 + \lambda}} \right),$$

whence $x - y = \rho \sqrt{\frac{k}{2} \left(1 + \sqrt{\frac{\lambda}{1 + \lambda}} \right)} - \rho \sqrt{\frac{k}{2} \left(1 - \sqrt{\frac{\lambda}{1 + \lambda}} \right)}.$

As explained in the note on x. 57, this is the lesser positive root of the equation

$$x^4 - 2k\rho^2 \cdot x^2 + \frac{\lambda}{1 + \lambda} k^2 \rho^4 = 0.$$

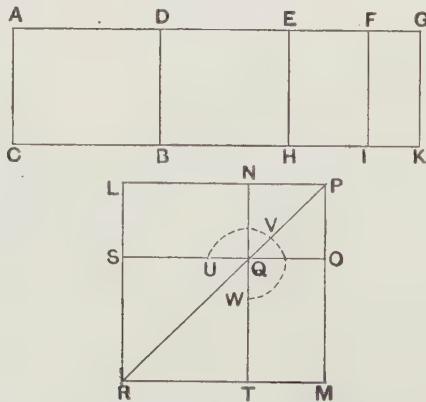
PROPOSITION 95.

If an area be contained by a rational straight line and a fifth apotome, the "side" of the area is a straight line which produces with a rational area a medial whole.

For let the area AB be contained by the rational straight line AC and the fifth apotome AD ;

I say that the "side" of the area AB is a straight line which produces with a rational area a medial whole.

For let DG be the annex to AD ;
therefore AG , GD are rational straight lines commensurable in square only,



the annex GD is commensurable in length with the rational straight line AC set out,
and the square on the whole AG is greater than the square

on the annex DG by the square on a straight line incommensurable with AG . [x. Def. iii. 5]

Therefore, if there be applied to AG a parallelogram equal to the fourth part of the square on DG and deficient by a square figure, it will divide it into incommensurable parts. [x. 18]

Let then DG be bisected at the point E ,
let there be applied to AG a parallelogram equal to the square on EG and deficient by a square figure, and let it be the rectangle AF, FG ;
therefore AF is incommensurable in length with FG .

Now, since AG is incommensurable in length with CA , and both are rational,
therefore AK is medial. [x. 21]

Again, since DG is rational and commensurable in length with AC ,
 DK is rational. [x. 19]

Now let the square LM be constructed equal to AI , and let the square NO equal to FK and about the same angle, the angle LPM , be subtracted;
therefore the squares LM, NO are about the same diameter. [vi. 26]

Let PR be their diameter, and let the figure be drawn.

Similarly then we can prove that LN is the "side" of the area AB .

I say that LN is the straight line which produces with a rational area a medial whole.

For, since AK was proved medial and is equal to the squares on LP, PN ,
therefore the sum of the squares on LP, PN is medial.

Again, since DK is rational and is equal to twice the rectangle LP, PN ,
the latter is itself also rational.

And, since AI is incommensurable with FK ,
therefore the square on LP is also incommensurable with the square on PN ;
therefore LP, PN are straight lines incommensurable in

square which make the sum of the squares on them medial but twice the rectangle contained by them rational.

Therefore the remainder LN is the irrational straight line called that which produces with a rational area a medial whole; [x. 77]

and it is the "side" of the area AB .

Therefore the "side" of the area AB is a straight line which produces with a rational area a medial whole.

Q. E. D.

Here the problem is to find and classify

$$\sqrt{\rho(k\rho\sqrt{1+\lambda}-k\rho)}.$$

As usual, we put

$$\left. \begin{aligned} u+v &= k\rho\sqrt{1+\lambda} \\ uv &= \frac{1}{4}k^2\rho^2 \end{aligned} \right\} \dots\dots\dots (1),$$

and, u, v being found, we take

$$\left. \begin{aligned} x^2 &= \rho u \\ y^2 &= \rho v \end{aligned} \right\} \dots\dots\dots (2).$$

Then $(x-y)$ so found is our required square root.

This fact is proved as before, and, as before, we see that

$$xy = \frac{1}{2}k\rho^2.$$

Now from (1), by x. 78,

$$u \sim v,$$

whence

$$\rho u \sim \rho v,$$

or

$$x^2 \sim y^2,$$

and x, y are incommensurable in square.

Next $(x^2+y^2) = \rho(u+v) = k\rho^2\sqrt{1+\lambda}$, which is a *medial* area.

And $2xy = k\rho^2$, which is a *rational* area.

Hence $(x-y)$ is the "*side*" of a *medial*, minus a *rational*, area. [x. 77]

Algebraical solution gives

$$u = \frac{k\rho}{2}(\sqrt{1+\lambda} + \sqrt{\lambda}),$$

$$v = \frac{k\rho}{2}(\sqrt{1+\lambda} - \sqrt{\lambda}),$$

and therefore

$$x-y = \rho \sqrt{\frac{k}{2}(\sqrt{1+\lambda} + \sqrt{\lambda})} - \rho \sqrt{\frac{k}{2}(\sqrt{1+\lambda} - \sqrt{\lambda})},$$

which is, as explained in the note to x. 58, the lesser positive root of the equation

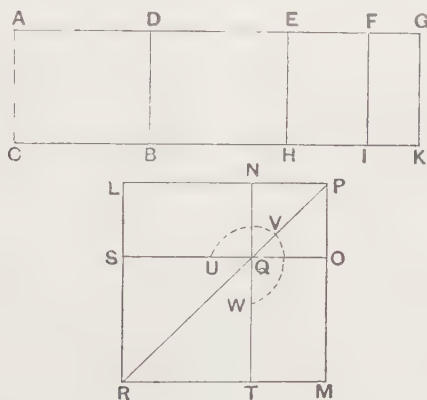
$$x^4 - 2k\rho^2\sqrt{1+\lambda} \cdot x^2 + \lambda k^2\rho^4 = 0.$$

PROPOSITION 96.

If an area be contained by a rational straight line and a sixth apotome, the "side" of the area is a straight line which produces with a medial area a medial whole.

For let the area AB be contained by the rational straight line AC and the sixth apotome AD ;

I say that the "side" of the area AB is a straight line which produces with a medial area a medial whole.



For let DG be the annex to AD ;

therefore AG , GD are rational straight lines commensurable in square only,

neither of them is commensurable in length with the rational straight line AC set out,

and the square on the whole AG is greater than the square on the annex DG by the square on a straight line incommensurable in length with AG . [x. Def. iii. 6]

Since then the square on AG is greater than the square on GD by the square on a straight line incommensurable in length with AG ,

therefore, if there be applied to AG a parallelogram equal to the fourth part of the square on DG and deficient by a square figure, it will divide it into incommensurable parts. [x. 18]

Let then DG be bisected at E ,

let there be applied to AG a parallelogram equal to the square

on EG and deficient by a square figure, and let it be the rectangle AF, FG ;

therefore AF is incommensurable in length with FG .

But, as AF is to FG , so is AI to FK ; [vi. 1]

therefore AI is incommensurable with FK . [x. 11]

And, since AG, AC are rational straight lines commensurable in square only,

AK is medial. [x. 21]

Again, since AC, DG are rational straight lines and incommensurable in length,

DK is also medial. [x. 21]

Now, since AG, GD are commensurable in square only, therefore AG is incommensurable in length with GD .

But, as AG is to GD , so is AK to KD ; [vi. 1]

therefore AK is incommensurable with KD . [x. 11]

Now let the square LM be constructed equal to AI , and let NO equal to FK , and about the same angle, be subtracted;

therefore the squares LM, NO are about the same diameter. [vi. 26]

Let PR be their diameter, and let the figure be drawn.

Then in manner similar to the above we can prove that LN is the "side" of the area AB .

I say that LN is a straight line which produces with a medial area a medial whole.

For, since AK was proved medial and is equal to the squares on LP, PN , therefore the sum of the squares on LP, PN is medial.

Again, since DK was proved medial and is equal to twice the rectangle LP, PN , twice the rectangle LP, PN is also medial.

And, since AK was proved incommensurable with DK , the squares on LP, PN are also incommensurable with twice the rectangle LP, PN .

And, since AI is incommensurable with FK , therefore the square on LP is also incommensurable with the square on PN ;

therefore LP , PN are straight lines incommensurable in square which make the sum of the squares on them medial, twice the rectangle contained by them medial, and further the squares on them incommensurable with twice the rectangle contained by them.

Therefore LN is the irrational straight line called that which produces with a medial area a medial whole; [x. 78] and it is the "side" of the area AB .

Therefore the "side" of the area is a straight line which produces with a medial area a medial whole.

Q. E. D.

We have to find and classify

$$\sqrt{\rho(\sqrt{k} \cdot \rho - \sqrt{\lambda} \cdot \rho)}.$$

Put, as usual,

$$\left. \begin{aligned} u + v &= \sqrt{k} \cdot \rho \\ uv &= \frac{1}{4} \lambda \rho^2 \end{aligned} \right\} \dots\dots\dots (1),$$

and, u , v being thus found, let

$$\left. \begin{aligned} x^2 &= \rho u \\ y^2 &= \rho v \end{aligned} \right\} \dots\dots\dots (2).$$

Then, as before, $(x - y)$ is the square root required.

For, from (1), by x. 18, $u \sim v$,

whence $\rho u \sim \rho v$,

or $x^2 \sim y^2$,

and x , y are incommensurable in square.

Next, $x^2 + y^2 = \rho(u + v) = \sqrt{k} \cdot \rho^2$, which is a *medial* area.

Also $2xy = \sqrt{\lambda} \cdot \rho^2$, which is again a *medial* area.

Lastly, $\sqrt{k} \cdot \rho$, $\sqrt{\lambda} \cdot \rho$ are by hypothesis $\sim$, so that

$$\sqrt{k} \cdot \rho \sim \sqrt{\lambda} \cdot \rho,$$

whence

$$\sqrt{k} \cdot \rho^2 \sim \sqrt{\lambda} \cdot \rho^2,$$

or

$$(x^2 + y^2) \sim 2xy.$$

Thus $(x - y)$ is the "*side*" of a *medial*, minus a *medial*, area [x. 78].

Algebraical solution gives

$$u = \frac{\rho}{2} (\sqrt{k} + \sqrt{k - \lambda}),$$

$$v = \frac{\rho}{2} (\sqrt{k} - \sqrt{k - \lambda}),$$

whence $x - y = \rho \sqrt{\frac{1}{2}(\sqrt{k} + \sqrt{k - \lambda})} - \rho \sqrt{\frac{1}{2}(\sqrt{k} - \sqrt{k - \lambda})}.$

This, as explained in the note on x. 59, is the lesser positive root of the equation

$$x^4 - 2\sqrt{k} \cdot \rho^2 x^2 + (k - \lambda) \rho^4 = 0.$$

And it is applied to the rational straight line CD , producing FM as breadth;

therefore FM is rational and incommensurable in length with CD . [x. 22]

And, since the squares on AG , GB are rational, while twice the rectangle AG , GB is medial, therefore the squares on AG , GB are incommensurable with twice the rectangle AG , GB .

And CL is equal to the squares on AG , GB , and FL to twice the rectangle AG , GB ; therefore DM is incommensurable with FL .

But, as DM is to FL , so is CM to FM ; [vi. 1]
therefore CM is incommensurable in length with FM . [x. 11]

And both are rational;
therefore CM , MF are rational straight lines commensurable in square only;
therefore CF is an apotome. [x. 73]

I say next that it is also a first apotome.

For, since the rectangle AG , GB is a mean proportional between the squares on AG , GB , and CH is equal to the square on AG , KL equal to the square on BG , and NL equal to the rectangle AG , GB , therefore NL is also a mean proportional between CH , KL ; therefore, as CH is to NL , so is NL to KL .

But, as CH is to NL , so is CK to NM , and, as NL is to KL , so is NM to KM ; [vi. 1]
therefore the rectangle CK , KM is equal to the square on NM [vi. 17], that is, to the fourth part of the square on FM .

And, since the square on AG is commensurable with the square on GB , CH is also commensurable with KL .

But, as CH is to KL , so is CK to KM ; [vi. 1]
therefore CK is commensurable with KM . [x. 11]

Since then CM , MF are two unequal straight lines,

and to CM there has been applied the rectangle CK , KM equal to the fourth part of the square on FM and deficient by a square figure,

while CK is commensurable with KM ,

therefore the square on CM is greater than the square on MF by the square on a straight line commensurable in length with CM . [x. 17]

And CM is commensurable in length with the rational straight line CD set out ;

therefore CF is a first apotome. [x. Def. III. 1]

Therefore etc.

Q. E. D.

Here begins the hexad of propositions solving the problems which are the converse of those in the hexad just concluded. Props. 97 to 102 correspond of course to Props. 60 to 65 relating to the binomials etc.

We have in x. 97 to prove that, $(\rho - \sqrt{k} \cdot \rho)$ being an *apotome*,

$$\frac{(\rho \sim \sqrt{k} \cdot \rho)^2}{\sigma}$$

is a *first apotome*, and we have to find it geometrically.

Euclid's procedure may be represented thus.

Take x, y, z such that

$$\left. \begin{aligned} \sigma x &= \rho^2 \\ \sigma y &= k\rho^2 \\ \sigma \cdot 2z &= 2\sqrt{k} \cdot \rho^2 \end{aligned} \right\} \dots\dots\dots (1).$$

Thus $(x + y) - 2z = \frac{(\rho - \sqrt{k} \cdot \rho)^2}{\sigma},$

and we have to prove that $(x + y) - 2z$ is a *first apotome*.

(a) Now $\rho^2 + k\rho^2$, or $\sigma(x + y)$, is rational ;
therefore $(x + y)$ is rational and $\cap \sigma \dots\dots\dots (2).$

And $2\sqrt{k} \cdot \rho^2$, or $\sigma \cdot 2z$, is medial ;
therefore $2z$ is rational and $\cup \sigma \dots\dots\dots (3).$

But, $\sigma(x + y)$ being *rational*, and $\sigma \cdot 2z$ *medial*,

$$\begin{aligned} \sigma(x + y) &\cup \sigma \cdot 2z, \\ (x + y) &\cup 2z. \end{aligned}$$

whence

Therefore, since $(x + y), 2z$ are both rational [(2), (3)],
 $(x + y), 2z$ are rational and $\sim \dots\dots\dots (4).$

Hence $(x + y) - 2z$ is an *apotome*.

(\beta) Since $\sqrt{k} \cdot \rho^2$ is a mean proportional between $\rho^2, k\rho^2$,
 σz is a mean proportional between $\sigma x, \sigma y$ [by (1)].

That is, $\sigma x : \sigma z = \sigma z : \sigma y,$

or $x : z = z : y,$

and $xy = z^2$, or $\frac{1}{4}(2z)^2 \dots\dots\dots (5).$

And, since $\rho^2 \sim k\rho^2$,

$\sigma x \sim \sigma y$,

or

$x \sim y$ (6).

Hence [(5), (6)], by x. 17,

$$\sqrt{(x+y)^2 - (2z)^2} \sim (x+y).$$

And [(4)] $(x+y)$, $2z$ are rational and $\sim$,

while [(2)] $(x+y) \sim \sigma$;

therefore $(x+y) - 2z$ is a *first apotome*.

The actual value of $(x+y) - 2z$ is of course

$$\frac{\rho^2}{\sigma} \{ (1+k) - 2\sqrt{k} \}.$$

PROPOSITION 98.

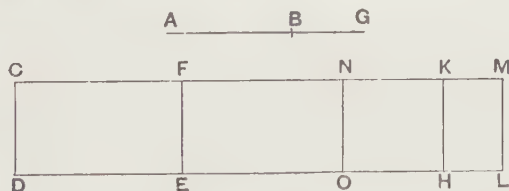
The square on a first apotome of a medial straight line applied to a rational straight line produces as breadth a second apotome.

Let AB be a first apotome of a medial straight line and CD a rational straight line, and to CD let there be applied CE equal to the square on AB , producing CF as breadth;

I say that CF is a second apotome.

For let BG be the annex to AB ;

therefore AG , GB are medial straight lines commensurable in square only which contain a rational rectangle. [x. 74]



To CD let there be applied CH equal to the square on AG , producing CK as breadth, and KL equal to the square on GB , producing KM as breadth;

therefore the whole CL is equal to the squares on AG , GB ;
therefore CL is also medial. [x. 15 and 23, Por.]

And it is applied to the rational straight line CD , producing CM as breadth;

therefore CM is rational and incommensurable in length with CD . [x. 22]

Now, since CL is equal to the squares on AG , GB ,
and, in these, the square on AB is equal to CE ,
therefore the remainder, twice the rectangle AG , GB , is equal
to FL . [ii. 7]

But twice the rectangle AG , GB is rational ;
therefore FL is rational.

And it is applied to the rational straight line FE , producing
 FM as breadth ;
therefore FM is also rational and commensurable in length
with CD . [x. 20]

Now, since the sum of the squares on AG , GB , that is,
 CL , is medial, while twice the rectangle AG , GB , that is, FL ,
is rational,
therefore CL is incommensurable with FL .

But, as CL is to FL , so is CM to FM ; [vi. 1]
therefore CM is incommensurable in length with FM . [x. 11]

And both are rational ;
therefore CM , MF are rational straight lines commensurable
in square only ;
therefore CF is an apotome. [x. 73]

I say next that it is also a second apotome.

For let FM be bisected at N ,
and let NO be drawn through N parallel to CD ;
therefore each of the rectangles FO , NL is equal to the
rectangle AG , GB .

Now, since the rectangle AG , GB is a mean proportional
between the squares on AG , GB ,
and the square on AG is equal to CH ,
the rectangle AG , GB to NL ,
and the square on BG to KL ,
therefore NL is also a mean proportional between CH , KL ;
therefore, as CH is to NL , so is NL to KL .

But, as CH is to NL , so is CK to NM ,
and, as NL is to KL , so is NM to MK ; [vi. 1]
therefore, as CK is to NM , so is NM to KM ; [v. 11]
therefore the rectangle CK , KM is equal to the square on
 NM [vi. 17], that is, to the fourth part of the square on FM .

Since then CM , MF are two unequal straight lines, and the rectangle CK , KM equal to the fourth part of the square on MF and deficient by a square figure has been applied to the greater, CM , and divides it into commensurable parts, therefore the square on CM is greater than the square on MF by the square on a straight line commensurable in length with CM . [x. 17]

And the annex FM is commensurable in length with the rational straight line CD set out ;

therefore CF is a second apotome. [x. Def. III. 2]

Therefore etc.

Q. E. D.

In this case we have to find and classify

$$\frac{(k^{\frac{1}{2}}\rho \sim k^{\frac{3}{2}}\rho)^2}{\sigma}$$

Take x, y, z such that

$$\left. \begin{aligned} \sigma x - k^{\frac{1}{2}}\rho^2 \\ \sigma y = k^{\frac{3}{2}}\rho^2 \\ \sigma \cdot 2z = 2k\rho^2 \end{aligned} \right\} \dots\dots\dots(1).$$

(α) Now $k^{\frac{1}{2}}\rho^2, k^{\frac{3}{2}}\rho^2$ are medial areas ;

therefore $\sigma(x+y)$ is medial,

whence $(x+y)$ is rational and $\smile \sigma \dots\dots\dots(2)$.

But $2k\rho^2$, and therefore $\sigma \cdot 2z$, is rational,

whence $2z$ is rational and $\cap \sigma \dots\dots\dots(3)$.

And, $\sigma(x+y)$ being medial, and $\sigma \cdot 2z$ rational,

$$\sigma(x+y) \smile \sigma \cdot 2z,$$

or $(x+y) \smile 2z$.

Hence $(x+y), 2z$ are rational straight lines commensurable in square only, and therefore $(x+y) - 2z$ is an *apotome*.

(β) We prove, as before, that

$$xy = \frac{1}{4} (2z)^2 \dots\dots\dots(4).$$

Also $k^{\frac{3}{2}}\rho^2 \cap k^{\frac{1}{2}}\rho^2$, or $\sigma x \cap \sigma y$,

so that $x \cap y \dots\dots\dots(5)$.

[This step is omitted in P, and Heiberg accordingly brackets it. The result is, however, assumed.]

Therefore [(4), (5)], by x. 17,

$$\sqrt{(x+y)^2 - (2z)^2} \cap (x+y).$$

And $2z \cap \sigma$.

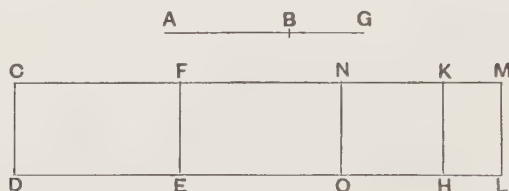
Therefore $(x+y) - 2z$ is a *second apotome*.

Obviously $(x+y) - 2z = \frac{\rho^2}{\sigma} \{ \sqrt{k(1+k)} - 2k \}.$

PROPOSITION 99.

The square on a second apotome of a medial straight line applied to a rational straight line produces as breadth a third apotome.

Let AB be a second apotome of a medial straight line, and CD rational, and to CD let there be applied CE equal to the square on AB , producing CF as breadth ; I say that CF is a third apotome.



For let BG be the annex to AB ; therefore AG , GB are medial straight lines commensurable in square only which contain a medial rectangle. [x. 75]

Let CH equal to the square on AG be applied to CD , producing CK as breadth, and let KL equal to the square on BG be applied to KH , producing KM as breadth ; therefore the whole CL is equal to the squares on AG , GB ; therefore CL is also medial. [x. 15 and 23, Por.]

And it is applied to the rational straight line CD , producing CM as breadth ; therefore CM is rational and incommensurable in length with CD . [x. 22]

Now, since the whole CL is equal to the squares on AG , GB , and, in these, CE is equal to the square on AB , therefore the remainder LF is equal to twice the rectangle AG , GB . [ii. 7]

Let then FM be bisected at the point N , and let NO be drawn parallel to CD ; therefore each of the rectangles FO , NL is equal to the rectangle AG , GB .

But the rectangle AG, GB is medial ;
therefore FL is also medial.

And it is applied to the rational straight line EF , producing FM as breadth ;
therefore FM is also rational and incommensurable in length with CD . [x. 22]

And, since AG, GB are commensurable in square only, therefore AG is incommensurable in length with GB ;
therefore the square on AG is also incommensurable with the rectangle AG, GB . [vi. 1, x. 11]

But the squares on AG, GB are commensurable with the square on AG ,
and twice the rectangle AG, GB with the rectangle AG, GB ;
therefore the squares on AG, GB are incommensurable with twice the rectangle AG, GB . [x. 13]

But CL is equal to the squares on AG, GB ,
and FL is equal to twice the rectangle AG, GB ;
therefore CL is also incommensurable with FL .

But, as CL is to FL , so is CM to FM ; [vi. 1]
therefore CM is incommensurable in length with FM . [x. 11]

And both are rational ;
therefore CM, MF are rational straight lines commensurable in square only ;
therefore CF is an apotome. [x. 73]

I say next that it is also a third apotome.

For, since the square on AG is commensurable with the square on GB ,
therefore CH is also commensurable with KL ,
so that CK is also commensurable with KM . [vi. 1, x. 11]

And, since the rectangle AG, GB is a mean proportional between the squares on AG, GB ,
and CH is equal to the square on AG ,
 KL equal to the square on GB ,
and NL equal to the rectangle AG, GB ,
therefore NL is also a mean proportional between CH, KL ;
therefore, as CH is to NL , so is NL to KL .

But, as CH is to NL , so is CK to NM ,
 and, as NL is to KL , so is NM to KM ; [vi. 1]
 therefore, as CK is to MN , so is MN to KM ; [v. 11]
 therefore the rectangle CK , KM is equal to [the square on
 MN , that is, to] the fourth part of the square on FM .

Since then CM , MF are two unequal straight lines, and
 a parallelogram equal to the fourth part of the square on FM
 and deficient by a square figure has been applied to CM , and
 divides it into commensurable parts,
 therefore the square on CM is greater than the square on
 MF by the square on a straight line commensurable with
 CM . [x. 17]

And neither of the straight lines CM , MF is commensur-
 able in length with the rational straight line CD set out;
 therefore CF is a third apotome. [x. Def. III. 3]

Therefore etc.

Q. E. D.

We have to find and classify

$$\frac{1}{\sigma} \left(k^{\frac{1}{2}} \rho \sim \frac{\sqrt{\lambda \cdot \rho}}{k^{\frac{1}{2}}} \right)^2.$$

Take x, y, z such that

$$\left. \begin{aligned} \sigma x &= \sqrt{k} \cdot \rho^2 \\ \sigma y &= \frac{\lambda}{\sqrt{k}} \cdot \rho^2 \\ \sigma \cdot 2z &= 2 \sqrt{\lambda} \cdot \rho^2 \end{aligned} \right\}.$$

(a) Then $\sigma(x+y)$ is a medial area,

whence $(x+y)$ is rational and $\sim \sigma$ (1).

Also $\sigma \cdot 2z$ is medial,

whence $2z$ is rational and $\sim \sigma$ (2).

Again

$$k^{\frac{1}{2}} \rho \sim \frac{\sqrt{\lambda \cdot \rho}}{k^{\frac{1}{2}}},$$

whence

$$\sqrt{k} \cdot \rho^2 \sim \sqrt{\lambda} \cdot \rho^2.$$

And

$$\sqrt{k} \cdot \rho^2 \sim \left(\sqrt{k} \cdot \rho^2 + \frac{\lambda}{\sqrt{k}} \rho^2 \right),$$

while

$$\sqrt{\lambda} \cdot \rho^2 \sim 2 \sqrt{\lambda} \cdot \rho^2;$$

therefore

$$\left(\sqrt{k} \cdot \rho^2 + \frac{\lambda}{\sqrt{k}} \rho^2 \right) \sim 2 \sqrt{\lambda} \cdot \rho^2,$$

or

$$\sigma(x+y) \sim \sigma \cdot 2z,$$

and

$$(x+y) \sim 2z \text{(3).}$$

Thus [(1), (2), (3)] $(x+y)$, $2z$ are rational and $\sim$, so that $(x+y) - 2z$ is an *apotome*.

(β) $\sigma x \sim \sigma y$, so that $x \sim y$.

And, as before, $xy = \frac{1}{4}(2z)^2$.

Therefore [x. 17] $\sqrt{(x+y)^2 - (2z)^2} \sim (x+y)$.

And neither $(x+y)$ nor $2z$ is $\sim \sigma$.

Therefore $(x+y) - 2z$ is a *third apotome*.

It is of course equal to

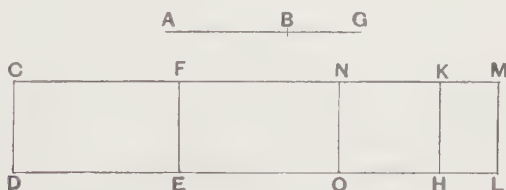
$$\frac{\rho^2}{\sigma} \left\{ \frac{k+\lambda}{\sqrt{k}} - 2\sqrt{\lambda} \right\}.$$

PROPOSITION 100.

The square on a minor straight line applied to a rational straight line produces as breadth a fourth apotome.

Let AB be a minor and CD a rational straight line, and to the rational straight line CD let CE be applied equal to the square on AB and producing CF as breadth;

I say that CF is a fourth apotome.



For let BG be the annex to AB ; therefore AG , GB are straight lines incommensurable in square which make the sum of the squares on AG , GB rational, but twice the rectangle AG , GB medial. [x. 76]

To CD let there be applied CH equal to the square on AG and producing CK as breadth, and KL equal to the square on BG , producing KM as breadth; therefore the whole CL is equal to the squares on AG , GB .

And the sum of the squares on AG , GB is rational; therefore CL is also rational.

And it is applied to the rational straight line CD , producing CM as breadth;

therefore CM is also rational and commensurable in length with CD . [x. 20]

And, since the whole CL is equal to the squares on AG , GB , and, in these, CE is equal to the square on AB , therefore the remainder FL is equal to twice the rectangle AG , GB . [11. 7]

Let then FM be bisected at the point N , and let NO be drawn through N parallel to either of the straight lines CD , ML ; therefore each of the rectangles FO , NL is equal to the rectangle AG , GB .

And, since twice the rectangle AG , GB is medial and is equal to FL , therefore FL is also medial.

And it is applied to the rational straight line FE , producing FM as breadth; therefore FM is rational and incommensurable in length with CD . [x. 22]

And, since the sum of the squares on AG , GB is rational, while twice the rectangle AG , GB is medial, the squares on AG , GB are incommensurable with twice the rectangle AG , GB .

But CL is equal to the squares on AG , GB , and FL equal to twice the rectangle AG , GB ; therefore CL is incommensurable with FL .

But, as CL is to FL , so is CM to MF ; [vi. 1]
therefore CM is incommensurable in length with MF . [x. 11]

And both are rational;
therefore CM , MF are rational straight lines commensurable in square only;
therefore CF is an apotome. [x. 73]

I say that it is also a fourth apotome.

For, since AG , GB are incommensurable in square, therefore the square on AG is also incommensurable with the square on GB .

And CH is equal to the square on AG , and KL equal to the square on GB ;
therefore CH is incommensurable with KL .

But, as CH is to KL , so is CK to KM ; [vi. 1]
therefore CK is incommensurable in length with KM . [x. 11]

And, since the rectangle AG, GB is a mean proportional between the squares on AG, GB ,
and the square on AG is equal to CH ,
the square on GB to KL ,
and the rectangle AG, GB to NL ,
therefore NL is a mean proportional between CH, KL ;
therefore, as CH is to NL , so is NL to KL .

But, as CH is to NL , so is CK to NM ,
and, as NL is to KL , so is NM to KM ; [vi. 1]
therefore, as CK is to MN , so is MN to KM ; [v. 11]
therefore the rectangle CK, KM is equal to the square on MN [vi. 17], that is, to the fourth part of the square on FM .

Since then CM, MF are two unequal straight lines, and the rectangle CK, KM equal to the fourth part of the square on MF and deficient by a square figure has been applied to CM and divides it into incommensurable parts,
therefore the square on CM is greater than the square on MF by the square on a straight line incommensurable with CM . [x. 18]

And the whole CM is commensurable in length with the rational straight line CD set out;
therefore CF is a fourth apotome. [x. Def. III. 4]

Therefore etc.

Q. E. D.

We have to find and classify

$$\frac{1}{\sigma} \left\{ \frac{\rho}{\sqrt{2}} \sqrt{1 + \frac{k}{\sqrt{1+k^2}}} - \frac{\rho}{\sqrt{2}} \sqrt{1 - \frac{k}{\sqrt{1+k^2}}} \right\}^2.$$

We will call this, for brevity,

$$\frac{1}{\sigma} (u - v)^2.$$

Take x, y, z such that

$$\left. \begin{aligned} \sigma x &= u^2 \\ \sigma y &= v^2 \\ \sigma \cdot 2z &= 2uv \end{aligned} \right\},$$

where it has to be remembered that u^2, v^2 are incommensurable, $(u^2 + v^2)$ is rational, and $2uv$ medial.

It follows that $\sigma(x+y)$ is rational and $\sigma \cdot 2z$ medial,
 so that $(x+y)$ is rational and $\cap \sigma$ (1),
 while $2z$ is rational and $\cup \sigma$ (2),
 and $\sigma(x+y) \cup \sigma \cdot 2z$,
 so that $(x+y) \cup 2z$ (3).

Thus [(1), (2), (3)] $(x+y)$, $2z$ are rational and $\sim$,
 so that $(x+y) - 2z$ is an *apotome*.

Next, since $u^2 \cup v^2$,

$$\sigma x \cup \sigma y,$$

or $x \cup y$.

And it is proved, as usual, that

$$xy = z^2 = \frac{1}{4}(2z)^2.$$

Therefore [X. 18] $\sqrt{(x+y)^2 - (2z)^2} \cup (x+y)$.

But $(x+y) \cap \sigma$,

therefore $x+y-2z$ is a *fourth apotome*.

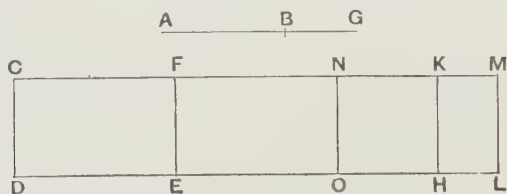
Its value is of course $\frac{\rho^2}{\sigma} \left(1 - \frac{1}{\sqrt{1+k^2}} \right)$.

PROPOSITION 101.

The square on the straight line which produces with a rational area a medial whole, if applied to a rational straight line, produces as breadth a fifth apotome.

Let AB be the straight line which produces with a rational area a medial whole, and CD a rational straight line, and to CD let CE be applied equal to the square on AB and producing CF as breadth;

I say that CF is a fifth apotome.



For let BG be the annex to AB ;

therefore AG , GB are straight lines incommensurable in square which make the sum of the squares on them medial but twice the rectangle contained by them rational. [X. 77]

To CD let there be applied CH equal to the square on AG , and KL equal to the square on GB ;

therefore the whole CL is equal to the squares on AG , GB .

But the sum of the squares on AG , GB together is medial;

therefore CL is medial.

And it is applied to the rational straight line CD , producing CM as breadth;

therefore CM is rational and incommensurable with CD . [x. 22]

And, since the whole CL is equal to the squares on AG , GB , and, in these, CE is equal to the square on AB ,

therefore the remainder FL is equal to twice the rectangle AG , GB . [II. 7]

Let then FM be bisected at N ,

and through N let NO be drawn parallel to either of the straight lines CD , ML ;

therefore each of the rectangles FO , NL is equal to the rectangle AG , GB .

And, since twice the rectangle AG , GB is rational and equal to FL ,

therefore FL is rational.

And it is applied to the rational straight line EF , producing FM as breadth;

therefore FM is rational and commensurable in length with CD . [x. 20]

Now, since CL is medial, and FL rational, therefore CL is incommensurable with FL .

But, as CL is to FL , so is CM to MF ; [VI. 1]

therefore CM is incommensurable in length with MF . [x. 11]

And both are rational;

therefore CM , MF are rational straight lines commensurable in square only;

therefore CF is an apotome. [x. 73]

I say next that it is also a fifth apotome.

For we can prove similarly that the rectangle CK , KM is equal to the square on NM , that is, to the fourth part of the square on FM .

And, since the square on AG is incommensurable with the square on GB ,

while the square on AG is equal to CH ,
and the square on GB to KL ,
therefore CH is incommensurable with KL .

But, as CH is to KL , so is CK to KM ; [vi. 1]
therefore CK is incommensurable in length with KM . [x. 11]

Since then CM , MF are two unequal straight lines,
and a parallelogram equal to the fourth part of the square
on FM and deficient by a square figure has been applied to
 CM , and divides it into incommensurable parts,
therefore the square on CM is greater than the square on
 MF by the square on a straight line incommensurable with
 CM . [x. 18]

And the annex FM is commensurable with the rational
straight line CD set out;
therefore CF is a fifth apotome. [x. Def. III. 5]

Q. E. D.

We have to find and classify

$$\frac{1}{\sigma} \left\{ \frac{\rho}{\sqrt{2(1+k^2)}} \sqrt{\sqrt{1+k^2}+k} - \frac{\rho}{\sqrt{2(1+k^2)}} \sqrt{\sqrt{1+k^2}-k} \right\}^2.$$

Call this $\frac{1}{\sigma} (u-v)^2$, and take x, y, z such that

$$\left. \begin{aligned} \sigma x &= u^2 \\ \sigma y &= v^2 \\ \sigma \cdot 2z &= 2uv \end{aligned} \right\}.$$

In this case u^2, v^2 are incommensurable, $(u^2 + v^2)$ is a medial area and $2uv$ a rational area.

Since $\sigma(x+y)$ is medial and $\sigma \cdot 2z$ rational,
 $(x+y)$ is rational and $\sim \sigma$,
 $2z$ is rational and $\circ \sigma$,

while $(x+y) \circ 2z$.

It follows that $(x+y), 2z$ are rational and $\sim$,
so that $(x+y) - 2z$ is an apotome.

Again, as before, $xy = z^2 = \frac{1}{4}(2z)^2$,
and, since $u^2 \circ v^2$, $\sigma x \circ \sigma y$,
or $x \circ y$.

Hence [x. 18] $\sqrt{(x+y)^2 - (2z)^2} \circ (x+y)$.

And $2z \circ \sigma$.

Therefore $(x+y) - 2z$ is a *fifth apotome*.

It is of course equal to

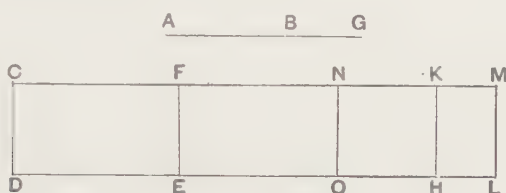
$$\frac{\rho^2}{\sigma} \left(\frac{1}{\sqrt{1+k^2}} - \frac{1}{1+k^2} \right).$$

PROPOSITION 102.

The square on the straight line which produces with a medial area a medial whole, if applied to a rational straight line, produces as breadth a sixth apotome.

Let AB be the straight line which produces with a medial area a medial whole, and CD a rational straight line, and to CD let CE be applied equal to the square on AB and producing CF as breadth;

I say that CF is a sixth apotome.



For let BG be the annex to AB ;

therefore AG , GB are straight lines incommensurable in square which make the sum of the squares on them medial, twice the rectangle AG , GB medial, and the squares on AG , GB incommensurable with twice the rectangle AG , GB . [x. 78]

Now to CD let there be applied CH equal to the square on AG and producing CK as breadth,

and KL equal to the square on BG ;

therefore the whole CL is equal to the squares on AG , GB ;

therefore CL is also medial.

And it is applied to the rational straight line CD , producing CM as breadth;

therefore CM is rational and incommensurable in length with CD . [x. 22]

Since now CL is equal to the squares on AG , GB ,

and, in these, CE is equal to the square on AB ,

therefore the remainder FL is equal to twice the rectangle AG , GB . [II. 7]

And twice the rectangle AG , GB is medial;

therefore FL is also medial.

And it is applied to the rational straight line FE , producing FM as breadth ;
 therefore FM is rational and incommensurable in length with CD . [X. 22]

And, since the squares on AG , GB are incommensurable with twice the rectangle AG , GB ,
 and CL is equal to the squares on AG , GB ,
 and FL equal to twice the rectangle AG , GB ,
 therefore CL is incommensurable with FL .

But, as CL is to FL , so is CM to MF ; [VI. 1]
 therefore CM is incommensurable in length with MF . [X. 11]

And both are rational.

Therefore CM , MF are rational straight lines commensurable in square only ;
 therefore CF is an apotome. [X. 73]

I say next that it is also a sixth apotome.

For, since FL is equal to twice the rectangle AG , GB , let FM be bisected at N ,
 and let NO be drawn through N parallel to CD ;
 therefore each of the rectangles FO , NL is equal to the rectangle AG , GB .

And, since AG , GB are incommensurable in square, therefore the square on AG is incommensurable with the square on GB .

But CH is equal to the square on AG ,
 and KL is equal to the square on GB ;
 therefore CH is incommensurable with KL .

But, as CH is to KL , so is CK to KM ; [VI. 1]
 therefore CK is incommensurable with KM . [X. 11]

And, since the rectangle AG , GB is a mean proportional between the squares on AG , GB ,
 and CH is equal to the square on AG ,
 KL equal to the square on GB ,
 and NL equal to the rectangle AG , GB ,
 therefore NL is also a mean proportional between CH , KL ;
 therefore, as CH is to NL , so is NL to KL .

And for the same reason as before the square on CM is greater than the square on MF by the square on a straight line incommensurable with CM . [x. 18]

And neither of them is commensurable with the rational straight line CD set out ;

therefore CF is a sixth apotome.

[x. Def. iii. 6]

Q. E. D.

We have to find and classify

$$\frac{1}{\sigma} \left\{ \frac{\rho\lambda^{\frac{1}{2}}}{\sqrt{2}} \sqrt{1 + \frac{k}{\sqrt{1+k^2}}} - \frac{\rho\lambda^{\frac{1}{2}}}{\sqrt{2}} \sqrt{1 - \frac{k}{\sqrt{1+k^2}}} \right\}^2.$$

Call this $\frac{1}{\sigma} (u-v)^2$, and put

$$\sigma x = u^2,$$

$$\sigma y = v^2,$$

$$\sigma \cdot 2z = 2uv.$$

Here u^2, v^2 are incommensurable,

$(u^2 + v^2), 2uv$ are both medial areas,

and $(u^2 + v^2) \cup 2uv$.

Since $\sigma(x+y), \sigma \cdot 2z$ are medial and incommensurable,

$(x+y)$ is rational and $\cup \sigma$,

$2z$ is rational and $\cup \sigma$,

and $(x+y) \cup 2z$.

Hence $(x+y), 2z$ are rational and $\sim$,

so that $(x+y) - 2z$ is an apotome.

Again, since u^2, v^2 , or $\sigma x, \sigma y$, are incommensurable,

$$x \cup y.$$

And, as before, $xy = z^2 = \frac{1}{4} (2z)^2$.

Therefore [x. 18] $\sqrt{(x+y)^2 - (2z)^2} \cup (x+y)$.

And neither $(x+y)$ nor $2z$ is $\sim z$;

therefore $(x+y) - 2z$ is a sixth apotome.

It is of course $\frac{\rho^2}{\sigma} \left(\sqrt{\lambda} - \frac{\sqrt{\lambda}}{\sqrt{1+k^2}} \right)^2$.

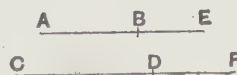
PROPOSITION 103.

A straight line commensurable in length with an apotome is an apotome and the same in order.

Let AB be an apotome,

and let CD be commensurable in length with AB ;

I say that CD is also an apotome and the same in order with AB .



For, since AB is an apotome, let BE be the annex to it ;
therefore AE, EB are rational straight lines commensurable
in square only. [X. 73]

Let it be contrived that the ratio of BE to DF is the same
as the ratio of AB to CD ; [VI. 12]

therefore also, as one is to one, so are all to all ; [V. 12]

therefore also, as the whole AE is to the whole CF , so is AB
to CD .

But AB is commensurable in length with CD .

Therefore AE is also commensurable with CF , and BE
with DF . [X. 11]

And AE, EB are rational straight lines commensurable in
square only ;

therefore CF, FD are also rational straight lines commensur-
able in square only. [X. 13]

Now since, as AE is to CF , so is BE to DF ,
alternately therefore, as AE is to EB , so is CF to FD . [V. 16]

And the square on AE is greater than the square on EB
either by the square on a straight line commensurable with
 AE or by the square on a straight line incommensurable
with it.

If then the square on AE is greater than the square on
 EB by the square on a straight line commensurable with AE ,
the square on CF will also be greater than the square on FD
by the square on a straight line commensurable with CF .
[X. 14]

And, if AE is commensurable in length with the rational
straight line set out,

CF is so also, [X. 12]

if BE , then DF also, [*id.*]

and, if neither of the straight lines AE, EB , then neither of
the straight lines CF, FD . [X. 13]

But, if the square on AE is greater than the square on EB
by the square on a straight line incommensurable with AE ,
the square on CF will also be greater than the square on FD
by the square on a straight line incommensurable with CF .
[X. 14]

And, if AE is commensurable in length with the rational straight line set out,

CF is so also,

if BE , then DF also,

[x. 12]

and, if neither of the straight lines AE , EB , then neither of the straight lines CF , FD .

[x. 13]

Therefore CD is an apotome and the same in order with AB .

Q. E. D.

This and the following propositions to 107 inclusive (like the corresponding theorems x. 66 to 70) are easy and require no explanation. They are equivalent to saying that, if in any of the preceding irrational straight lines $\frac{m}{n}p$ is substituted for p , the resulting irrational is of the same kind and order as that from which it is altered.

PROPOSITION 104.

A straight line commensurable with an apotome of a medial straight line is an apotome of a medial straight line and the same in order.

Let AB be an apotome of a medial straight line, and let CD be commensurable in length with AB ;

I say that CD is also an apotome of a medial straight line and the same in order with AB .



For, since AB is an apotome of a medial straight line, let EB be the annex to it.

Therefore AE , EB are medial straight lines commensurable in square only.

[x. 74, 75]

Let it be contrived that, as AB is to CD , so is BE to DF ;

[vi. 12]

therefore AE is also commensurable with CF , and BE with DF .

[v. 12, x. 11]

But AE , EB are medial straight lines commensurable in square only;

therefore CF , FD are also medial straight lines [x. 23] commensurable in square only;

[x. 13]

therefore CD is an apotome of a medial straight line. [x. 74, 75]

I say next that it is also the same in order with AB .

Since, as AE is to EB , so is CF to FD ,

therefore also, as the square on AE is to the rectangle AE, EB , so is the square on CF to the rectangle CF, FD .

But the square on AE is commensurable with the square on CF ;

therefore the rectangle AE, EB is also commensurable with the rectangle CF, FD . [v. 16, x. 11]

Therefore, if the rectangle AE, EB is rational, the rectangle CF, FD will also be rational, [x. Def. 4]

and if the rectangle AE, EB is medial, the rectangle CF, FD is also medial. [x. 23, Por.]

Therefore CD is an apotome of a medial straight line and the same in order with AB . [x. 74, 75]

Q. E. D.

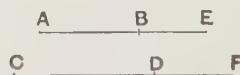
PROPOSITION 105.

A straight line commensurable with a minor straight line is minor.

Let AB be a minor straight line, and CD commensurable with AB ;

I say that CD is also minor.

Let the same construction be made as before;



then, since AE, EB are incommensurable in square, [x. 76]

therefore CF, FD are also incommensurable in square. [x. 13]

Now since, as AE is to EB , so is CF to FD , [v. 12, v. 16]

therefore also, as the square on AE is to the square on EB , so is the square on CF to the square on FD . [vi. 22]

Therefore, *componendo*, as the squares on AE, EB are to the square on EB , so are the squares on CF, FD to the square on FD . [v. 18]

But the square on BE is commensurable with the square on DF ;

therefore the sum of the squares on AE, EB is also commensurable with the sum of the squares on CF, FD . [v. 16, x. 11]

But the sum of the squares on AE, EB is rational; [x. 76] therefore the sum of the squares on CF, FD is also rational.

[x. Def. 4]

Again, since, as the square on AE is to the rectangle AE, EB , so is the square on CF to the rectangle CF, FD , while the square on AE is commensurable with the square on CF ,

therefore the rectangle AE, EB is also commensurable with the rectangle CF, FD .

But the rectangle AE, EB is medial ; [x. 76]

therefore the rectangle CF, FD is also medial ; [x. 23, Por.]

therefore CF, FD are straight lines incommensurable in square which make the sum of the squares on them rational, but the rectangle contained by them medial.

Therefore CD is minor. [x. 76]

Q. E. D.

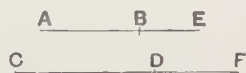
PROPOSITION 106.

A straight line commensurable with that which produces with a rational area a medial whole is a straight line which produces with a rational area a medial whole.

Let AB be a straight line which produces with a rational area a medial whole,

and CD commensurable with AB ;

I say that CD is also a straight line which produces with a rational area a medial whole.



For let BE be the annex to AB ;

therefore AE, EB are straight lines incommensurable in square which make the sum of the squares on AE, EB medial, but the rectangle contained by them rational. [x. 77]

Let the same construction be made.

Then we can prove, in manner similar to the foregoing, that CF, FD are in the same ratio as AE, EB ,

the sum of the squares on AE, EB is commensurable with the sum of the squares on CF, FD ,

and the rectangle AE, EB with the rectangle CF, FD ;

so that CF, FD are also straight lines incommensurable in square which make the sum of the squares on CF, FD medial, but the rectangle contained by them rational.

Therefore CD is a straight line which produces with a rational area a medial whole.

[x. 77]

Q. E. D.

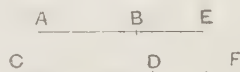
PROPOSITION 107.

A straight line commensurable with that which produces with a medial area a medial whole is itself also a straight line which produces with a medial area a medial whole.

Let AB be a straight line which produces with a medial area a medial whole,

and let CD be commensurable with AB ;

I say that CD is also a straight line which produces with a medial area a medial whole.



For let BE be the annex to AB ,

and let the same construction be made;

therefore AE , EB are straight lines incommensurable in square which make the sum of the squares on them medial, the rectangle contained by them medial, and further the sum of the squares on them incommensurable with the rectangle contained by them.

[x. 78]

Now, as was proved, AE , EB are commensurable with CF , FD ,

the sum of the squares on AE , EB with the sum of the squares on CF , FD ,

and the rectangle AE , EB with the rectangle CF , FD ;

therefore CF , FD are also straight lines incommensurable in square which make the sum of the squares on them medial, the rectangle contained by them medial, and further the sum of the squares on them incommensurable with the rectangle contained by them.

Therefore CD is a straight line which produces with a medial area a medial whole.

[x. 78]

Q. E. D.

PROPOSITION 108.

If from a rational area a medial area be subtracted, the "side" of the remaining area becomes one of two irrational straight lines, either an apotome or a minor straight line.

For from the rational area BC let the medial area BD be subtracted;

I say that the "side" of the remainder EC becomes one of two irrational straight lines, either an apotome or a minor straight line.

For let a rational straight line FG be set out,

to FG let there be applied the rectangular parallelogram GH equal to BC ,

and let GK equal to DB be subtracted;
therefore the remainder EC is equal to LH .

Since then BC is rational, and BD medial, while BC is equal to GH , and BD to GK , therefore GH is rational, and GK medial.

And they are applied to the rational straight line FG ;
therefore FH is rational and commensurable in length with FG , [x. 20]

while FK is rational and incommensurable in length with FG ; [x. 22]

therefore FH is incommensurable in length with FK . [x. 13]

Therefore FH , FK are rational straight lines commensurable in square only;

therefore KH is an apotome [x. 73], and KF the annex to it.

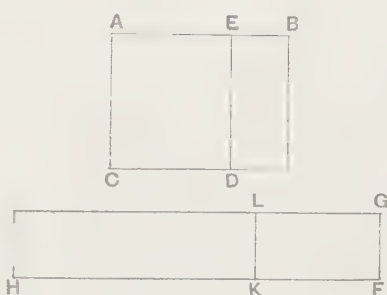
Now the square on HF is greater than the square on FK by the square on a straight line either commensurable with HF or not commensurable.

First, let the square on it be greater by the square on a straight line commensurable with it.

Now the whole HF is commensurable in length with the rational straight line FG set out;

therefore KH is a first apotome.

[x. Def. III. 1]



But the "side" of the rectangle contained by a rational straight line and a first apotome is an apotome. [X. 91]

Therefore the "side" of LH , that is, of EC , is an apotome.

But, if the square on HF is greater than the square on FK by the square on a straight line incommensurable with HF ,

while the whole FH is commensurable in length with the rational straight line FG set out,

KH is a fourth apotome. [X. Def. III. 4]

But the "side" of the rectangle contained by a rational straight line and a fourth apotome is minor. [X. 94]

Q. E. D.

A rational area being of the form $k\rho^2$, and a medial area of the form $\sqrt{\lambda} \cdot \rho^2$, the problem is to classify

$$\sqrt{k\rho^2 - \sqrt{\lambda} \cdot \rho^2}$$

according to the different possible relations between k , λ .

Suppose that

$$\sigma u = k\rho^2,$$

$$\sigma v = \sqrt{\lambda} \cdot \rho^2.$$

Since σu is rational and σv medial,

u is rational and $\cap \sigma$,

while v is rational and $\cup \sigma$.

Therefore

$$u \cup v;$$

thus u , v are rational and $\cap$,

whence $(u - v)$ is an apotome.

The possibilities are now as follows.

$$(1) \quad \sqrt{u^2 - v^2} \cap u,$$

$$(2) \quad \sqrt{u^2 - v^2} \cup u.$$

In both cases $u \cap \sigma$,

so that $(u - v)$ is either (1) a *first apotome*,

or (2) a *fourth apotome*.

In case (1) $\sqrt{\sigma(u - v)}$ is an *apotome* [X. 91],

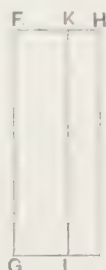
but in case (2) $\sqrt{\sigma(u - v)}$ is a *minor irrational straight line* [X. 94].

PROPOSITION 109.

If from a medial area a rational area be subtracted, there arise two other irrational straight lines, either a first apotome of a medial straight line or a straight line which produces with a rational area a medial whole.

For from the medial area BC let the rational area BD be subtracted.

I say that the "side" of the remainder EC becomes one of two irrational straight lines, either a first apotome of a medial straight line or a straight line which produces with a rational area a medial whole.



For let a rational straight line FG be set out, and let the areas be similarly applied.

It follows then that FH is rational and incommensurable in length with FG ,

while KF is rational and commensurable in length with FG ; therefore FH , FK are rational straight lines commensurable in square only; [x. 13]

therefore KH is an apotome, and FK the annex to it. [x. 73]

Now the square on HF is greater than the square on FK either by the square on a straight line commensurable with HF or by the square on a straight line incommensurable with it.

If then the square on HF is greater than the square on FK by the square on a straight line commensurable with HF , while the annex FK is commensurable in length with the rational straight line FG set out,

KH is a second apotome.

[x. Def. III. 2]

But FG is rational;

so that the "side" of LH , that is, of EC , is a first apotome of a medial straight line.

[x. 92]

But, if the square on HF is greater than the square on FK by the square on a straight line incommensurable with HF , while the annex FK is commensurable in length with the rational straight line FG set out,

KH is a fifth apotome;

[x. Def. III. 5]

so that the "side" of EC is a straight line which produces with a rational area a medial whole.

[x. 95]

In this case we have to classify

$$\sqrt{\sqrt{k} \cdot \rho^2 - \lambda \rho^2}.$$

Suppose that

$$\sigma u = \sqrt{k} \cdot \rho^2,$$

$$\sigma v = \lambda \rho^2.$$

Thus, σu being medial and σv rational,

u is rational and $\cup \sigma$,

while v is rational and $\cap \sigma$.

Thus, as before, u, v are rational and $\sim$,

so that $(u - v)$ is an apotome.

Now either

$$(1) \quad \sqrt{u^2 - v^2} \cap u,$$

$$\text{or } (2) \quad \sqrt{u^2 - v^2} \cup u,$$

while in both cases v is commensurable with σ .

Therefore $(u - v)$ is either (1) a second apotome,

or (2) a fifth apotome,

and hence in case (1) $\sqrt{\sigma(u - v)}$ is the *first apotome of a medial straight line*,

[X. 92]

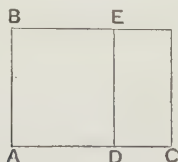
and in case (2) $\sqrt{\sigma(u - v)}$ is the "*side*" of a medial, minus a rational, area.

[X. 95]

PROPOSITION 110.

If from a medial area there be subtracted a medial area incommensurable with the whole, the two remaining irrational straight lines arise, either a second apotome of a medial straight line or a straight line which produces with a medial area a medial whole.

For, as in the foregoing figures, let there be subtracted from the medial area BC the medial area BD incommensurable with the whole;



I say that the "*side*" of EC is one of two irrational straight lines, either a second apotome of a medial straight line or a straight line which produces with a medial area a medial whole.

For, since each of the rectangles BC , BD is medial, and BC is incommensurable with BD , it follows that each of the straight lines FH , FK will be rational and incommensurable in length with FG . [x. 22]

And, since BC is incommensurable with BD , that is, GH with GK , HF is also incommensurable with FK ; [vi. 1, x. 11] therefore FH , FK are rational straight lines commensurable in square only; therefore KH is an apotome. [x. 73]

If then the square on FH is greater than the square on FK by the square on a straight line commensurable with FH , while neither of the straight lines FH , FK is commensurable in length with the rational straight line FG set out, KH is a third apotome. [x. Def. III. 3]

But KL is rational, and the rectangle contained by a rational straight line and a third apotome is irrational, and the "side" of it is irrational, and is called a second apotome of a medial straight line; [x. 93] so that the "side" of LH , that is, of EC , is a second apotome of a medial straight line.

But, if the square on FH is greater than the square on FK by the square on a straight line incommensurable with FH , while neither of the straight lines HF , FK is commensurable in length with FG , KH is a sixth apotome. [x. Def. III. 6]

But the "side" of the rectangle contained by a rational straight line and a sixth apotome is a straight line which produces with a medial area a medial whole. [x. 96]

Therefore the "side" of LH , that is, of EC , is a straight line which produces with a medial area a medial whole.

Q. E. D.

We have to classify $\sqrt{\sqrt{k} \cdot \rho^2 - \sqrt{\lambda} \cdot \rho^2}$,
where $\sqrt{k} \cdot \rho^2$ is incommensurable with $\sqrt{\lambda} \cdot \rho^2$.
Put $\sigma u = \sqrt{k} \cdot \rho^2$,
 $\sigma v = \sqrt{\lambda} \cdot \rho^2$.

Then u is rational and $\cup \sigma$,
 v is rational and $\cup \sigma$,
 and

$$u \cup v.$$

Therefore u, v are rational and $\sim$,
 so that $(u - v)$ is an apotome.

Now either

$$(1) \cdot \sqrt{u^2 - v^2} \cup u,$$

$$\text{or } (2) \cdot \sqrt{u^2 - v^2} \cup u,$$

while in both cases both u and v are $\cup \sigma$.

In case (1) $(u - v)$ is a *third apotome*,
 and in case (2) $(u - v)$ is a *sixth apotome*,

so that $\sqrt{\sigma(u - v)}$ is either (1) a *second apotome of a medial straight line* [x. 93],
 or (2) a "*side*" of the difference between two medial areas [x. 96].

PROPOSITION III.

The apotome is not the same with the binomial straight line.

Let AB be an apotome;

I say that AB is not the same with the binomial straight line.

For, if possible, let it be so;

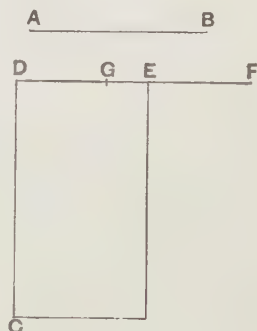
let a rational straight line DC be set out,
 and to CD let there be applied the rectangle CE equal to the square on AB and producing DE as breadth.

Then, since AB is an apotome,
 DE is a first apotome. [x. 97]

Let EF be the annex to it;
 therefore DF, FE are rational straight lines commensurable in square only,
 the square on DF is greater than the square on FE by the square on a straight line commensurable with DF ,
 and DF is commensurable in length with the rational straight line DC set out. [x. Def. III. 1]

Again, since AB is binomial,
 therefore DE is a first binomial straight line. [x. 60]

Let it be divided into its terms at G ,
 and let DG be the greater term;
 therefore DG, GE are rational straight lines commensurable in square only,



the square on DG is greater than the square on GE by the square on a straight line commensurable with DG , and the greater term DG is commensurable in length with the rational straight line DC set out. [x. Def. 11. 1]

Therefore DF is also commensurable in length with DG ; [x. 12]

therefore the remainder GF is also commensurable in length with DF . [x. 15]

But DF is incommensurable in length with EF ;

therefore FG is also incommensurable in length with EF . [x. 13]

Therefore GF , FE are rational straight lines commensurable in square only;

therefore EG is an apotome. [x. 73]

But it is also rational:

which is impossible.

Therefore the apotome is not the same with the binomial straight line.

Q. E. D.

This proposition proves the equivalent of the fact that

$\sqrt{x} + \sqrt{y}$ cannot be equal to $\sqrt{x'} - \sqrt{y'}$, and

$x + \sqrt{y}$ cannot be equal to $x' - \sqrt{y'}$.

We should prove these results by squaring the respective expressions; and Euclid's procedure corresponds to this exactly.

He has to prove that

$\rho + \sqrt{k} \cdot \rho$ cannot be equal to $\rho' - \sqrt{\lambda} \cdot \rho'$.

For, if possible, let this be so.

Take the straight lines $\frac{(\rho + \sqrt{k} \cdot \rho)^2}{\sigma}$, $\frac{(\rho' - \sqrt{\lambda} \cdot \rho')^2}{\sigma}$;

these must be equal, and therefore

$$\frac{\rho^2}{\sigma} (1 + k + 2\sqrt{k}) = \frac{\rho'^2}{\sigma} (1 + \lambda - 2\sqrt{\lambda}) \dots\dots\dots(1).$$

Now $\frac{\rho^2}{\sigma} (1 + k)$, $\frac{\rho'^2}{\sigma} (1 + \lambda)$ are rational and $\wedge$;

therefore $\left\{ \frac{\rho'^2}{\sigma} (1 + \lambda) - \frac{\rho^2}{\sigma} (1 + k) \right\} \wedge \frac{\rho'^2}{\sigma} (1 + \lambda)$
 $\wedge \frac{\rho'^2}{\sigma} \cdot 2\sqrt{\lambda}.$

And, since both sides are rational, it follows that

$$\left\{ \frac{\rho'^2}{\sigma} (1 + \lambda) - \frac{\rho^2}{\sigma} (1 + k) \right\} - \frac{\rho'^2}{\sigma} \cdot 2\sqrt{\lambda} \text{ is an apotome.}$$

But, by (1), this expression is equal to $\frac{\rho^2}{\sigma} \cdot 2\sqrt{k}$, which is *rational*.

Hence an *apotome*, which is *irrational*, is also *rational*:
which is impossible.

This proposition is the connecting link which enables Euclid to prove that *all* the compound irrationals with positive signs above discussed are different from *all* the corresponding compound irrationals with negative signs, while the two sets are all different from one another and from the *medial* straight line. The recapitulation following makes this clear.

The apotome and the irrational straight lines following it are neither the same with the medial straight line nor with one another.

For the square on a medial straight line, if applied to a rational straight line, produces as breadth a straight line rational and incommensurable in length with that to which it is applied, [x. 22]

while the square on an apotome, if applied to a rational straight line, produces as breadth a first apotome, [x. 97]

the square on a first apotome of a medial straight line, if applied to a rational straight line, produces as breadth a second apotome, [x. 98]

the square on a second apotome of a medial straight line, if applied to a rational straight line, produces as breadth a third apotome, [x. 99]

the square on a minor straight line, if applied to a rational straight line, produces as breadth a fourth apotome, [x. 100]

the square on the straight line which produces with a rational area a medial whole, if applied to a rational straight line, produces as breadth a fifth apotome, [x. 101]

and the square on the straight line which produces with a medial area a medial whole, if applied to a rational straight line, produces as breadth a sixth apotome. [x. 102]

Since then the said breadths differ from the first and from one another, from the first because it is rational, and from one another since they are not the same in order, it is clear that the irrational straight lines themselves also differ from one another.

And, since the apotome has been proved not to be the same as the binomial straight line, [x. 111]
but, if applied to a rational straight line, the straight lines

following the apotome produce, as breadths, each according to its own order, apotomes, and those following the binomial straight line themselves also, according to their order, produce the binomials as breadths,

therefore those following the apotome are different, and those following the binomial straight line are different, so that there are, in order, thirteen irrational straight lines in all,

Medial,

Binomial,

First bimedral,

Second bimedral,

Major,

"Side" of a rational plus a medial area,

"Side" of the sum of two medial areas,

Apotome,

First apotome of a medial straight line,

Second apotome of a medial straight line,

Minor,

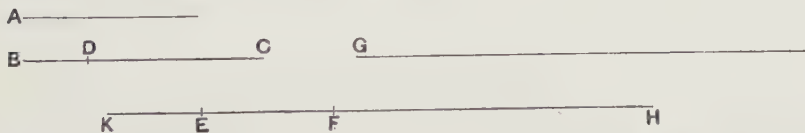
Producing with a rational area a medial whole,

Producing with a medial area a medial whole.

PROPOSITION 112.

The square on a rational straight line applied to the binomial straight line produces as breadth an apotome the terms of which are commensurable with the terms of the binomial and moreover in the same ratio; and further the apotome so arising will have the same order as the binomial straight line.

Let A be a rational straight line,
let BC be a binomial, and let DC be its greater term;
let the rectangle BC , EF be equal to the square on A ;



I say that EF is an apotome the terms of which are commensurable with CD , DB , and in the same ratio, and further EF will have the same order as BC .

For again let the rectangle BD , G be equal to the square on A .

Since then the rectangle BC , EF is equal to the rectangle BD , G ,

therefore, as CB is to BD , so is G to EF . [vi. 16]

But CB is greater than BD ;

therefore G is also greater than EF . [v. 16, v. 14]

Let EH be equal to G ;

therefore, as CB is to BD , so is HE to EF ;

therefore, *separando*, as CD is to BD , so is HF to FE . [v. 17]

Let it be contrived that, as HF is to FE , so is FK to KE ;

therefore also the whole HK is to the whole KF as FK is to KE ;

for, as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents. [v. 12]

But, as FK is to KE , so is CD to DB ; [v. 11]

therefore also, as HK is to KF , so is CD to DB . [*id.*]

But the square on CD is commensurable with the square on DB ; [x. 36]

therefore the square on HK is also commensurable with the square on KF . [vi. 22, x. 11]

And, as the square on HK is to the square on KF , so is HK to KE , since the three straight lines HK , KF , KE are proportional. [v. Def. 9]

Therefore HK is commensurable in length with KE , so that HE is also commensurable in length with EK . [x. 15]

Now, since the square on A is equal to the rectangle EH , BD ,

while the square on A is rational,

therefore the rectangle EH , BD is also rational.

And it is applied to the rational straight line BD ;

therefore EH is rational and commensurable in length with BD ; [x. 20]

so that EK , being commensurable with it, is also rational and commensurable in length with BD .

Since, then, as CD is to DB , so is FK to KE ,
 while CD , DB are straight lines commensurable in square
 only,
 therefore FK , KE are also commensurable in square only.
[x. 11]

But KE is rational ;
 therefore FK is also rational.

Therefore FK , KE are rational straight lines commen-
 surable in square only ;
 therefore EF is an apotome.
[x. 73]

Now the square on CD is greater than the square on DB
 either by the square on a straight line commensurable with
 CD or by the square on a straight line incommensurable
 with it.

If then the square on CD is greater than the square on
 DB by the square on a straight line commensurable with CD ,
 the square on FK is also greater than the square on KE by
 the square on a straight line commensurable with FK . [x. 14]

And, if CD is commensurable in length with the rational
 straight line set out,

so also is FK ; [x. 11, 12]

if BD is so commensurable,

so also is KE ; [x. 12]

but, if neither of the straight lines CD , DB is so commensur-
 able,

neither of the straight lines FK , KE is so.

But, if the square on CD is greater than the square on
 DB by the square on a straight line incommensurable
 with CD ,

the square on FK is also greater than the square on KE by
 the square on a straight line incommensurable with FK . [x. 14]

And, if CD is commensurable with the rational straight
 line set out,

so also is FK ;

if BD is so commensurable,

so also is KE ;

but, if neither of the straight lines CD , DB is so commensurable,

neither of the straight lines FK , KE is so ;

so that FE is an apotome, the terms of which FK , KE are commensurable with the terms CD , DB of the binomial straight line and in the same ratio, and it has the same order as BC .

Q. E. D.

Heiberg considers that this proposition and the succeeding ones are interpolated, though the interpolation must have taken place before Theon's time. His argument is that x. 112—115 are nowhere used, but that x. 111 rounds off the complete discussion of the 13 irrationals (as indicated in the recapitulation), thereby giving what was necessary for use in connexion with the investigation of the five regular solids. For besides x. 73 (used in XIII. 6, 11) x. 94 and 97 are used in XIII. 11, 6 respectively ; and Euclid could not have stopped at x. 97 without leaving the discussion of irrationals imperfect, for x. 98—102 are closely connected with x. 97, and x. 103—111 add, as it were, the coping-stone to the whole doctrine. On the other hand, x. 112—115 are not connected with the rest of the treatise on the 13 irrationals and are not used in the stereometric books. They are rather the germ of a new study and a more abstruse investigation of irrationals *in themselves*. Prop. 115 in particular extends the number of the different kinds of irrationals. As however x. 112—115 are old and serviceable theorems, Heiberg thinks that, though Euclid did not give them, they may have been taken from Apollonius.

I will only point out what seems to me open to doubt in the above, namely that x. 112—114 (excluding 115) are not connected with the rest of the exposition of the 13 irrationals. It seems to me that they *are* so connected. x. 111 has shown us that a *binomial* straight line cannot also be an *apotome*. But x. 112—114 show us *how either of them can be used to rationalise the other*, thus giving what is surely an important relation between them.

x. 112 is the equivalent of rationalising the denominators of the fractions

$$\frac{c^2}{\sqrt{A} + \sqrt{B}}, \quad \frac{c^2}{a + \sqrt{B}},$$

by multiplying numerator and denominator by $\sqrt{A} - \sqrt{B}$ and $a - \sqrt{B}$ respectively.

Euclid proves that $\frac{\sigma^2}{\rho + \sqrt{k} \cdot \rho} = \lambda\rho - \sqrt{k} \cdot \lambda\rho$ ($k < 1$), and his method enables us to see that $\lambda = \sigma^2/(\rho^2 - k\rho^2)$.

The proof is a remarkable instance of the dexterity of the Greeks in using geometry as the equivalent of our algebra. Like so many proofs in Archimedes and Apollonius, it leaves us completely in the dark as to how it was evolved. That the Greeks must have had some analytical method which suggested the steps of such proofs seems certain ; but *what* it was must remain apparently an insoluble mystery.

I will reproduce by means of algebraical symbols the exact course of Euclid's proof.

He has to prove that $\frac{\sigma^2}{\rho + \sqrt{k} \cdot \rho}$ is an apotome related in a certain way to

the binomial straight line $\rho + \sqrt{k} \cdot \rho$. If u be the straight line required, $(u + w) - w$ is shown to be an apotome of the kind described, where w is determined in the following manner.

We have $(\rho + \sqrt{k} \cdot \rho) u = \sigma^2 = \sqrt{k} \cdot \rho \cdot x$, say, $x > u$.
whence $x = u + v$. }(1).

Let $(\rho + \sqrt{k} \cdot \rho) : \sqrt{k} \cdot \rho = (u + v) : u$,
and hence $\rho : \sqrt{k} \cdot \rho = v : u$ (2).

• Let w be taken such that $v : u = (u + w) : w$ (3).

Thus $v : u = (u + v + w) : (u + w)$ (4),
and therefore $\rho : \sqrt{k} \cdot \rho = (u + v + w) : (u + w)$.

From the last proportion,
 $(u + v + w)^2 \sim (u + w)^2$,

and, from the two preceding, $(u + w)$ is a mean proportional between $(u + v + w)$, w , so that

$$(u + v + w)^2 : (u + w)^2 = (u + v + w) : w.$$

Therefore $(u + v + w) \sim w$,
whence $(u + v) \sim w$.

Now $\sqrt{k} \cdot \rho (u + v) = \sigma^2$, which is rational;
therefore $(u + v)$ is rational and $\sim \sqrt{k} \cdot \rho$;
hence w is also rational and $\sim \sqrt{k} \cdot \rho$ (5).

Next, by (2), (3), since ρ , $\sqrt{k} \cdot \rho$ are $\sim$,
 $(u + w) \sim w$,

and w is rational;
therefore $(u + w)$ is rational,
and $(u + w)$, w are rational and $\sim$.

Hence $(u + w) - w$ is an *apotome*.

Now either (I) $\sqrt{\rho^2 - k\rho^2} \sim \rho$,
or (II) $\sqrt{\rho^2 - k\rho^2} \sim \rho$.

In case (I) $\sqrt{(u + w)^2 - w^2} \sim (u + w)$, [(2), (3) and x. 14]
and in case (II) $\sqrt{(u + w)^2 - w^2} \sim (u + w)$. [*id.*]

Then, since [(5)] $w \sim \sqrt{k} \cdot \rho$,
by x. 11 and (2), (3), $(u + w) \sim \rho$ (6).

[This step is omitted in Euclid, but the result is assumed.]

If therefore $\rho \sim \sigma$, $(u + w) \sim \sigma$;
if $\sqrt{k} \cdot \rho \sim \sigma$, $w \sim \sigma$; [(5)]
and, if neither ρ nor $\sqrt{k} \cdot \rho$ is $\sim \sigma$, neither $(u + w)$ nor w will be $\sim \sigma$.

Thus the order of the apotome $(u + w) - w$ is the same as that of the binomial straight line $\rho + \sqrt{k} \cdot \rho$; while [(2), (3)] the terms are proportional and [(5), (6)] commensurable respectively.

We find $(u + w)$, w algebraically thus.

By (1),
$$u = \frac{\sigma^2}{\rho + \sqrt{k} \cdot \rho};$$

and, by (2), (3),
$$\frac{u + w}{w} = \frac{\rho}{\sqrt{k} \cdot \rho},$$

whence
$$w = \frac{u \cdot \sqrt{k} \cdot \rho}{\rho - \sqrt{k} \cdot \rho} = \frac{\sigma^2 \cdot \sqrt{k} \cdot \rho}{\rho^2 - k\rho^2}.$$

Thus
$$u + w = w \cdot \frac{1}{\sqrt{k}} = \frac{\sigma^2 \cdot \rho}{\rho^2 - k\rho^2}.$$

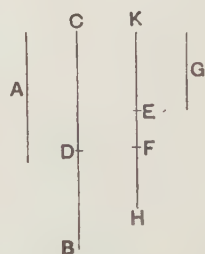
Therefore
$$(u + w) - w = \sigma^2 \cdot \frac{\rho - \sqrt{k} \cdot \rho}{\rho^2 - k\rho^2}.$$

PROPOSITION 113.

The square on a rational straight line, if applied to an apotome, produces as breadth the binomial straight line the terms of which are commensurable with the terms of the apotome and in the same ratio; and further the binomial so arising has the same order as the apotome.

Let A be a rational straight line and BD an apotome, and let the rectangle BD, KH be equal to the square on A , so that the square on the rational straight line A when applied to the apotome BD produces KH as breadth;

I say that KH is a binomial straight line the terms of which are commensurable with the terms of BD and in the same ratio; and further KH has the same order as BD .



For let DC be the annex to BD ; therefore BC, CD are rational straight lines commensurable in square only. [X. 73]

Let the rectangle BC, G be also equal to the square on A .

But the square on A is rational;

therefore the rectangle BC, G is also rational.

And it has been applied to the rational straight line BC ; therefore G is rational and commensurable in length with BC .

[X. 20]

Since now the rectangle BC, G is equal to the rectangle BD, KH ,

therefore, proportionally, as CB is to BD , so is KH to G .
[vi. 16]

But BC is greater than BD ;

therefore KH is also greater than G .
[v. 16, v. 14]

Let KE be made equal to G ;

therefore KE is commensurable in length with BC .

And since, as CB is to BD , so is HK to KE ,

therefore, *convertendo*, as BC is to CD , so is KH to HE .
[v. 19, Por.]

Let it be contrived that, as KH is to HE , so is HF to FE ;

therefore also the remainder KF is to FH as KH is to HE ,
that is, as BC is to CD .
[v. 19]

But BC, CD are commensurable in square only;

therefore KF, FH are also commensurable in square only.
[x. 11]

And since, as KH is to HE , so is KF to FH ,

while, as KH is to HE , so is HF to FE ,

therefore also, as KF is to FH , so is HF to FE ,
[v. 11]

so that also, as the first is to the third, so is the square on the first to the square on the second;
[v. Def. 9]

therefore also, as KF is to FE , so is the square on KF to the square on FH .

But the square on KF is commensurable with the square on FH ,

for KF, FH are commensurable in square;

therefore KF is also commensurable in length with FE , [x. 11]

so that KF is also commensurable in length with KE . [x. 15]

But KE is rational and commensurable in length with BC ;

therefore KF is also rational and commensurable in length with BC .
[x. 12]

And, since, as BC is to CD , so is KF to FH ,

alternately, as BC is to KF , so is DC to FH .
[v. 16]

But BC is commensurable with KF ;

therefore FH is also commensurable in length with CD . [x. 11]

But BC , CD are rational straight lines commensurable in square only;

therefore KF , FH are also rational straight lines [x. Def. 3] commensurable in square only;

therefore KH is binomial. [x. 36]

If now the square on BC is greater than the square on CD by the square on a straight line commensurable with BC , the square on KF will also be greater than the square on FH by the square on a straight line commensurable with KF . [x. 14]

And, if BC is commensurable in length with the rational straight line set out,

so also is KF ;

if CD is commensurable in length with the rational straight line set out,

so also is FH ,

but, if neither of the straight lines BC , CD ,

then neither of the straight lines KF , FH .

But, if the square on BC is greater than the square on CD by the square on a straight line incommensurable with BC , the square on KF is also greater than the square on FH by the square on a straight line incommensurable with KF . [x. 14]

And, if BC is commensurable with the rational straight line set out,

so also is KF ;

if CD is so commensurable,

so also is FH ;

but, if neither of the straight lines BC , CD ,

then neither of the straight lines KF , FH .

Therefore KH is a binomial straight line, the terms of which KF , FH are commensurable with the terms BC , CD of the apotome and in the same ratio, and further KH has the same order as BD .

Q. E. D.

This proposition, which is companion to the preceding, gives us the equivalent of the rationalisation of the denominator of

$$\frac{c^2}{\sqrt{A} - \sqrt{B}} \quad \text{or} \quad \frac{c^2}{a \sim \sqrt{B}}.$$

Euclid (or the writer) proves that

$$\frac{\sigma^2}{\rho - \sqrt{k} \cdot \rho} = \lambda \rho + \lambda \sqrt{k} \cdot \rho, \quad (k < 1)$$

and his method enables us to see that $\lambda = \sigma^2/(\rho^2 - k\rho^2)$.

Let

$$\frac{\sigma^2}{\rho - \sqrt{k} \cdot \rho} = u;$$

and it is proved that u is the binomial straight line $(u - w) + w$, where w is determined as shown below.

$$u(\rho - \sqrt{k} \cdot \rho) = \sigma^2 = \rho x, \text{ say,}$$

whence

$$\rho : (\rho - \sqrt{k} \cdot \rho) = u : x \dots\dots\dots(1),$$

so that

$$x < u.$$

Let then

$$x = u - v.$$

Since

$$(u - v)\rho = \sigma^2, \text{ a rational area,}$$

$$(u - v) \text{ is rational and } \wedge \rho \dots\dots\dots(2).$$

And [(1)]

$$\rho : (\rho - \sqrt{k} \cdot \rho) = u : (u - v),$$

so that, *convertendo*,

$$\rho : \sqrt{k} \cdot \rho = u : v.$$

Suppose that

$$u : v = w : (v - w),$$

so that [v. 19]

$$(u - w) : w = u : v = w : (v - w).$$

Thus, w being a mean proportional between $(u - w)$, $(v - w)$,

$$(u - w)^2 : w^2 = (u - w) : (v - w).$$

But

$$\begin{aligned} (u - w)^2 : w^2 &= u^2 : v^2 \\ &= \rho^2 : k\rho^2 \dots\dots\dots(3), \end{aligned}$$

so that

$$(u - w)^2 \wedge w^2.$$

Therefore

$$\begin{aligned} (u - w) &\wedge (v - w) \\ &\wedge \{(u - w) - (v - w)\} \\ &\wedge (u - v). \end{aligned}$$

Therefore [(2)]

$$(u - w) \text{ is rational and } \wedge \rho \dots\dots\dots(4).$$

And, since

$$\rho : \sqrt{k} \cdot \rho = (u - w) : w,$$

$$w \text{ is rational and } \wedge \sqrt{k} \cdot \rho \dots\dots\dots(5).$$

Hence [(4), (5)] $(u - w)$, w are rational and $\sim$,

so that

$$(u - w) + w \text{ is a binomial straight line.}$$

Now either

$$(I) \quad \sqrt{\rho^2 - k\rho^2} \wedge \rho,$$

or

$$(II) \quad \sqrt{\rho^2 - k\rho^2} \cup \rho.$$

In case (I)

$$\sqrt{(u - w)^2 - w^2} \wedge (u - w),$$

and in case (II)

$$\sqrt{(u - w)^2 - w^2} \cup (u - w). \quad [(3) \text{ and x. 14}]$$

And, if $\rho \oslash \sigma$,

$$(u - w) \oslash \sigma; \quad [(4)]$$

if $\sqrt{k} \cdot \rho \oslash \sigma$,

$$w \oslash \sigma; \quad [5]$$

while, if neither ρ nor $\sqrt{k} \cdot \rho$ is $\oslash \sigma$, neither $(u - w)$ nor w is $\oslash \sigma$.

Hence $(u - w) + w$ is a binomial straight line of the same order as the apotome $\rho - \sqrt{k} \cdot \rho$, its terms are proportional to those of the apotome [(3)], and commensurable with them respectively [(4), (5)].

To find $(u - w)$, w algebraically we have

$$u = \frac{\sigma^2}{\rho - \sqrt{k} \cdot \rho},$$

$$\frac{u - w}{w} = \frac{\rho}{\sqrt{k} \cdot \rho}.$$

From the latter

$$\begin{aligned} w &= \frac{u \cdot \sqrt{k} \cdot \rho}{\rho + \sqrt{k} \cdot \rho} \\ &= \frac{\sigma^2 \cdot \sqrt{k} \cdot \rho}{\rho^2 - k\rho^2}. \end{aligned}$$

Thus

$$u - w = w \cdot \frac{1}{\sqrt{k}} = \frac{\sigma^2 \rho}{\rho^2 - k\rho^2}.$$

Therefore

$$(u - w) + w = \sigma^2 \cdot \frac{\rho + \sqrt{k} \cdot \rho}{\rho^2 - k\rho^2}.$$

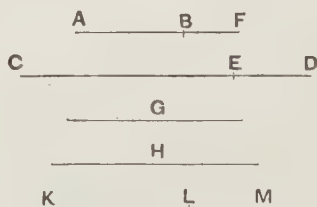
PROPOSITION 114.

If an area be contained by an apotome and the binomial straight line the terms of which are commensurable with the terms of the apotome and in the same ratio, the "side" of the area is rational.

For let an area, the rectangle AB, CD , be contained by the apotome AB and the binomial straight line CD ,

and let CE be the greater term of the latter;

let the terms CE, ED of the binomial straight line be commensurable with the terms AF, FB of the apotome and in the same ratio; and let the "side" of the rectangle AB, CD be G ;



I say that G is rational.

For let a rational straight line H be set out, and to CD let there be applied a rectangle equal to the square on H and producing KL as breadth.

Therefore KL is an apotome.

Let its terms be KM, ML commensurable with the terms CE, ED of the binomial straight line and in the same ratio.

But CE , ED are also commensurable with AF , FB and in the same ratio ;

therefore, as AF is to FB , so is KM to ML .

Therefore, alternately, as AF is to KM , so is BF to LM ; therefore also the remainder AB is to the remainder KL as AF is to KM . [v. 19]

But AF is commensurable with KM ; [x. 12]

therefore AB is also commensurable with KL . [x. 11]

And, as AB is to KL , so is the rectangle CD , AB to the rectangle CD , KL ; [vi. 1]

therefore the rectangle CD , AB is also commensurable with the rectangle CD , KL . [x. 11]

But the rectangle CD , KL is equal to the square on H ; therefore the rectangle CD , AB is commensurable with the square on H .

But the square on G is equal to the rectangle CD , AB ; therefore the square on G is commensurable with the square on H .

But the square on H is rational ;

therefore the square on G is also rational ;

therefore G is rational.

And it is the "side" of the rectangle CD , AB .

Therefore etc.

PORISM. And it is made manifest to us by this also that it is possible for a rational area to be contained by irrational straight lines.

Q. E. D.

This theorem is equivalent to the proof of the fact that

$$\sqrt{(\sqrt{A} - \sqrt{B})(\lambda\sqrt{A} + \lambda\sqrt{B})} = \sqrt{\lambda(A - B)},$$

and

$$\sqrt{(a \sim \sqrt{B})(\lambda a + \lambda\sqrt{B})} = \sqrt{\lambda(a^2 \sim B)}.$$

The result of the theorem x. 112 is used for the purpose thus.

We have to prove that

$$\sqrt{(\rho - \sqrt{k} \cdot \rho)(\lambda\rho + \lambda\sqrt{k} \cdot \rho)}$$

is rational.

By x. 112 we have, if σ is a rational straight line,

$$\frac{\sigma^2}{\lambda\rho + \lambda\sqrt{k} \cdot \rho} = \lambda'\rho - \lambda'\sqrt{k} \cdot \rho \dots\dots\dots(1).$$

Now $\rho : \lambda' \rho = \sqrt{k} . \rho : \lambda' \sqrt{k} . \rho = (\rho - \sqrt{k} . \rho) : (\lambda' \rho - \lambda' \sqrt{k} . \rho)$,
 so that $(\rho - \sqrt{k} . \rho) \cap (\lambda' \rho - \lambda' \sqrt{k} . \rho)$.

Multiplying each by $(\lambda \rho + \lambda \sqrt{k} . \rho)$, we have

$$(\rho - \sqrt{k} . \rho) (\lambda \rho + \lambda \sqrt{k} . \rho) \cap (\lambda \rho + \lambda \sqrt{k} . \rho) (\lambda' \rho - \lambda' \sqrt{k} . \rho) \\ \cap \sigma^2, \quad \text{by (1).}$$

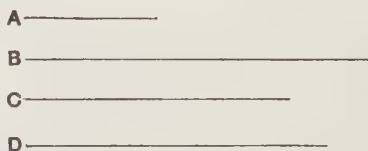
That is, $(\rho - \sqrt{k} . \rho) (\lambda \rho + \lambda \sqrt{k} . \rho)$ is a rational area,
 and therefore $\sqrt{(\rho - \sqrt{k} . \rho) (\lambda \rho + \lambda \sqrt{k} . \rho)}$ is rational.

PROPOSITION 115.

From a medial straight line there arise irrational straight lines infinite in number, and none of them is the same as any of the preceding.

Let A be a medial straight line ;

I say that from A there arise irrational straight lines infinite in number, and none of them is the same as any of the preceding.



Let a rational straight line B be set out,

and let the square on C be equal to the rectangle B, A ;

therefore C is irrational ;

[x. Def. 4]

for that which is contained by an irrational and a rational straight line is irrational.

[deduction from x. 20]

And it is not the same with any of the preceding ;

for the square on none of the preceding, if applied to a rational straight line produces as breadth a medial straight line.

Again, let the square on D be equal to the rectangle B, C ;
 therefore the square on D is irrational.

[deduction from x. 20]

Therefore D is irrational ;

[x. Def. 4]

and it is not the same with any of the preceding, for the square on none of the preceding, if applied to a rational straight line, produces C as breadth.

Similarly, if this arrangement proceeds *ad infinitum*, it is manifest that from the medial straight line there arise irrational straight lines infinite in number, and none is the same with any of the preceding.

Q. E. D.

Heiberg is clearly right in holding that this proposition, at all events, is alien to the general scope of Book x, and is therefore probably an interpolation, made however before Theon's time. It is of the same character as a *scholium* at the end of the Book, which is (along with the interpolated proposition proving, in two ways, the incommensurability of the diagonal of a square with its side) relegated by August as well as Heiberg to an Appendix.

The proposition amounts to this.

The straight line $k^{\frac{1}{2}}\rho$ being medial, if σ be a rational straight line, $\sqrt{k^{\frac{1}{2}}\rho\sigma}$ is a new irrational straight line. So is the mean proportional between this and another rational straight line σ' , and so on indefinitely.

ANCIENT EXTENSIONS OF THE THEORY OF BOOK X.

From the hints given by the author of the commentary found in Arabic by Woepcke (cf. pp. 3—4 above) it would seem probable that Apollonius' extensions of the theory of irrationals took two directions: (1) generalising the *medial* straight line of Euclid, and (2) forming compound irrationals by the addition and subtraction of more than two terms of the sort composing the binomials, apotomes, etc. The commentator writes (Woepcke's article, pp. 694 sqq.):

"It is also necessary that we should know that, not only when we join together two straight lines rational and commensurable in square do we obtain the binomial straight line, but three or four lines produce in an analogous manner the same thing. In the first case, we obtain the trinomial straight line, since the whole line is irrational; and in the second case we obtain the quadrinomial, and so on *ad infinitum*. The proof of the (irrationality of the) line composed of three lines rational and commensurable in square is exactly the same as the proof relating to the combination of two lines.

"But we must start afresh and remark that not only can we take one sole medial line between two lines commensurable in square, but we can take three or four of them and so on *ad infinitum*, since we can take, between any two given straight lines, as many lines as we wish in continued proportion.

"Likewise, in the lines formed by addition not only can we construct the binomial straight line, but we can also construct the trinomial, as well as the first and second trimedial; and, further, the line composed of three straight lines incommensurable in square and such that the one of them gives with each of the two others a sum of squares (which is) rational, while the rectangle contained by the two lines is medial, so that there results a *major* (irrational) composed of three lines.

"And, in an analogous manner, we obtain the straight line which is the 'side' of a rational plus a medial area, composed of three straight lines, and, likewise, that which is the 'side' of (the sum of) two medials."

The generalisation of the medial is apparently after the following manner. Let x, y be two straight lines rational and commensurable in square only and suppose that m means are interposed, so that

$$x : x_1 = x_1 : x_2 = x_2 : x_3 = \dots = x_{m-1} : x_m = x_m : y.$$

We easily derive herefrom

$$\frac{x}{x_r} = \left(\frac{x}{x_1}\right)^r,$$

$$\frac{x}{y} = \left(\frac{x}{x_1}\right)^{m+1},$$

and hence

$$x_1^r = x_r \cdot x^{r-1},$$

$$x_1^{m+1} = y \cdot x^m,$$

so that

$$(x_r \cdot x^{r-1})^{m+1} = (y \cdot x^m)^r,$$

and therefore

$$x_r^{m+1} = x^{m-r+1} \cdot y^r,$$

or

$$x_r = (x^{m-r+1} \cdot y^r)^{\frac{1}{m+1}},$$

which is the generalised medial.

We now pass to the trinomial etc., with the commentator's further remarks about them.

(1) *The trinomial.* "Suppose three rational straight lines commensurable in square only. The line composed of two of these lines, that is, the binomial straight line, is irrational, and, in consequence, the area contained by this line and the remaining line is irrational, and, likewise, the double of the area contained by these two lines will be irrational. Thus the square on the whole line composed of three lines is irrational and consequently the line is irrational, and it is called a trinomial straight line."

It is easy to see that this "proof" is not conclusive as stated. Nor does Woepcke seem to show how the proposition can be proved on Euclidean lines. But I think it would be somewhat as follows.

Suppose x, y, z to be rational and $\sim$.

Then x^2, y^2, z^2 are rational, and $2yz, 2zx, 2xy$ are all medial.

First, $(2yz + 2zx + 2xy)$ cannot be rational.

For suppose this sum equal to a rational area, say σ^2 .

Since

$$2yz + 2zx + 2xy = \sigma^2,$$

$$2zx + 2xy = \sigma^2 - 2yz,$$

or the sum of two medial areas incommensurable with one another is equal to the difference between a rational area and a medial area.

But the "side" of the sum of the two medial areas must [x. 72] be one of two irrationals with a positive sign; and the "side" of the difference between a rational area and a medial area must [x. 108] be one of two irrationals with a negative sign.

And the first "side" cannot be the same as the second [x. 111 and explanation following].

Therefore

$$2zx + 2xy \neq \sigma^2 - 2yz,$$

and

$$2yz + 2zx + 2xy \text{ is consequently irrational.}$$

Therefore

$$(x^2 + y^2 + z^2) \sim (2yz + 2zx + 2xy),$$

whence

$$(x + y + z)^2 \sim (x^2 + y^2 + z^2),$$

so that $(x + y + z)^2$, and therefore also $(x + y + z)$, is irrational.

The commentator goes on :

"And, if we have four lines commensurable in square, as we have said, the procedure will be exactly the same; and we shall treat the succeeding lines in an analogous manner."

Without speculating further as to how the extension was made to the *quadrinomial* etc., we may suppose with Woepcke that Apollonius probably investigated the multinomial

$$\rho + \sqrt{\kappa} \cdot \rho + \sqrt{\lambda} \cdot \rho + \sqrt{\mu} \cdot \rho + \dots$$

(2) The *first trimedial* straight line.

The commentator here says: "Suppose we have three medial lines commensurable in square [only], one of which contains with each of the two others a rational rectangle; then the straight line composed of the two lines is irrational and is called the first bimedial; the remaining line is medial, and the area contained by these two lines is irrational. Consequently the square on the whole line is irrational."

To begin with, the conditions here given are incompatible. If x, y, z be medial straight lines such that xy, xz are both rational,

$$y : z = xy : xz = m : n,$$

and y, z are commensurable *in length* and not *in square only*.

Hence it seems that we must, with Woepcke, understand "three medial straight lines such that *one is commensurable with each of the other two in square only* and makes with it a rational rectangle."

If x, y, z be the three medial straight lines,

$$(x^2 + y^2 + z^2) \cap x^2,$$

so that $(x^2 + y^2 + z^2)$ is medial.

Also we have $2xy, 2xz$ both rational and $2yz$ medial.

Now $(x^2 + y^2 + z^2) + 2yz + 2xy + 2xz$ cannot be rational, for, if it were, the sum of two medial areas, $(x^2 + y^2 + z^2), 2yz$, would be rational: which is impossible. [Cf. x. 72.]

Hence

$$(x + y + z) \text{ is irrational.}$$

(3) The *second trimedial* straight line.

Suppose x, y, z to be medial straight lines commensurable in square only and containing with each other medial rectangles.

Then $(x^2 + y^2 + z^2) \cap x^2$, and is medial.

Also $2yz, 2zx, 2xy$ are all medial areas.

To prove the irrationality in this case I presume that the method would be like that of x. 38 about the *second bimedial*.

Suppose σ to be a rational straight line and let

$$\left. \begin{aligned} (x^2 + y^2 + z^2) &= \sigma t \\ 2yz &= \sigma u \\ 2zx &= \sigma v \\ 2xy &= \sigma w \end{aligned} \right\}$$

Here, since, e.g.,

$$xz : xy = v : w,$$

or

$$z : y = v : w,$$

and similarly

$$x : z = w : u,$$

u, v, w are commensurable in square only.

Also, since

$$\begin{aligned} (x^2 + y^2 + z^2) &\cap x^2 \\ &\cup xy, \end{aligned}$$

t is incommensurable with w .

Similarly t is incommensurable with u, v .

But t, u, v, w are all rational and $\sim \sigma$.

Therefore $(t + u + v + w)$ is a quadrinomial and therefore irrational.

Therefore $\sigma(t + u + v + w)$, or $(x + y + z)^2$, is irrational,

whence $(x + y + z)$ is irrational.

(4) The *major* made up of three straight lines.

The commentator describes this as "the line composed of three straight lines incommensurable in square and such that one of them gives with each of the other two a sum of squares (which is) rational, while the rectangle contained by the two lines is medial."

If x, y, z are the three straight lines, this would indicate

$$(x^2 + y^2) \text{ rational,}$$

$$(x^2 + z^2) \text{ rational,}$$

$$2yz \text{ medial.}$$

Woepcke points out (pp. 696—8, note) the difficulties connected with this supposition or the supposition of

$$(x^2 + y^2) \text{ rational,}$$

$$(x^2 + z^2) \text{ rational,}$$

$$2xy \text{ (or } 2xz) \text{ medial,}$$

and concludes that what is meant is the supposition

$$\left. \begin{array}{l} (x^2 + y^2) \text{ rational} \\ xy \text{ medial} \\ xz \text{ medial} \end{array} \right\}$$

(though the text is against this).

The assumption of $(x^2 + y^2)$ and $(x^2 + z^2)$ being concurrently rational is certainly further removed from Euclid, for x. 33 only enables us to find *one pair* of lines having the property, as x, y .

But we will not pursue these speculations further.

As regards further irrationals formed by *subtraction* the commentator writes as follows.

"Again, it is not necessary that, in the irrational straight lines formed by means of subtraction, we should confine ourselves to making one subtraction only, so as to obtain the apotome, or the first apotome of the medial, or the second apotome of the medial, or the minor, or the straight line which produces with a rational area a medial whole, or that which produces with a medial area a medial whole; but we shall be able here to make two or three or four subtractions.

"When we do that, we show in manner analogous to the foregoing that the lines which remain are irrational and that each of them is one of the lines formed by subtraction. That is to say that, if from a rational line we cut off another rational line commensurable with the whole line in square, we obtain, for remainder, an apotome; and, if we subtract from this line (which is) cut off and rational—that which Euclid calls the *annex* (προσαρμόζοντα)—another rational line which is commensurable with it in square, we obtain, as the remainder, an apotome; likewise, if we cut off from the rational line cut

off from this line (i.e. the annex of the apotome last arrived at) another line which is commensurable with it in square, the remainder is an apotome. The same thing occurs in the subtraction of the other lines."

As Woepcke remarks, the idea is the formation of the successive apotomes $\sqrt{a} - \sqrt{b}$, $\sqrt{b} - \sqrt{c}$, $\sqrt{c} - \sqrt{d}$, etc. We should naturally have expected to see the writer form and discuss the following expressions

$$\begin{aligned} &(\sqrt{a} - \sqrt{b}) - \sqrt{c}, \\ &\{(\sqrt{a} - \sqrt{b}) - \sqrt{c}\} - \sqrt{d}, \text{ etc.} \end{aligned}$$

APPENDIX.

I. THE CONTENTS OF THE SO-CALLED BOOK XIV. BY HYPsicLES.

This supplement to Euclid's Book XIII. is worth reproducing for the sake not only of the additional theorems proved in it but of the historical notices contained in the preface and in one or two later passages. Where I translate literally from the Greek text, I shall use inverted commas; except in such passages I reproduce the contents in briefer form.

I have already quoted from the Preface (Vol. I. pp. 5—6), but I will repeat it here.

"Basilides of Tyre, O Protarchus, when he came to Alexandria and met my father, spent the greater part of his sojourn with him on account of the bond between them due to their common interest in mathematics. And on one occasion, when looking into the tract written by Apollonius about the comparison of the dodecahedron and icosahedron inscribed in one and the same sphere, that is to say, on the question what ratio they bear to one another, they came to the conclusion that Apollonius' treatment of it in this book was not correct; accordingly, as I understood from my father, they proceeded to amend and rewrite it. But I myself afterwards came across another book published by Apollonius, containing a demonstration of the matter in question, and I was greatly attracted by his investigation of the problem. Now the book published by Apollonius is accessible to all; for it has a large circulation in a form which seems to have been the result of later careful elaboration.

"For my part, I determined to dedicate to you what I deem to be necessary by way of commentary, partly because you will be able, by reason of your proficiency in all mathematics and particularly in geometry, to pass an expert judgment upon what I am about to write, and partly because, on account of your intimacy with my father and your friendly feeling towards myself, you will lend a kindly ear to my disquisition. But it is time to have done with the preamble and to begin my treatise itself.

[Prop. I.] "*The perpendicular drawn from the centre of any circle to the side of the pentagon inscribed in the same circle is half the sum of the side of the hexagon and of the side of the decagon inscribed in the same circle.*"

Let ABC be a circle, and BC the side of the inscribed regular pentagon. Take D the centre of the circle, draw DE from D perpendicular to BC , and produce DE both ways to meet the circle in F, A .

I say that DE is half the sum of the side of the hexagon and of the side of the decagon inscribed in the same circle.

Let DC, CF be joined; make GE equal to EF , and join GC .

Since the circumference of the circle is five times the arc BFC ,

and half the circumference of the circle is the arc ACF ,

while the arc FC is half the arc BFC ,

therefore $(\text{arc } ACF) = 5 (\text{arc } FC)$

or $(\text{arc } AC) = 4 (\text{arc } CF)$.

Hence $\angle ADC = 4 \angle CDF$,

and therefore $\angle AFC = 2 \angle CDF$.

Thus $\angle CGF = \angle AFC = 2 \angle CDF$;

therefore [I. 32] $\angle CDG = \angle DCG$,

so that $DG = GC = CF$.

And $GE = EF$;

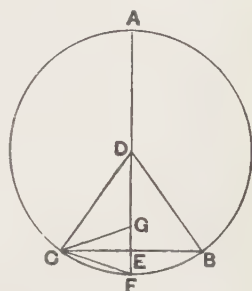
therefore $DE = EF + FC$.

Add DE to each;

therefore $2DE = DF + FC$.

And DF is the side of the regular hexagon, and FC the side of the regular decagon, inscribed in the same circle.

Therefore etc.



"Next it is manifest from the theorem [12] in Book XIII. that the perpendicular drawn from the centre of the circle to the side of the equilateral triangle [inscribed in it] is half of the radius of the circle.

[Prop. 2.] "The same circle circumscribes both the pentagon of the dodecahedron and the triangle of the icosahedron inscribed in the same sphere.

"This is proved by Aristaeus in his work entitled *Comparison of the five figures*. But Apollonius proves in the second edition of his comparison of the dodecahedron with the icosahedron that, as the surface of the dodecahedron is to the surface of the icosahedron, so also is the dodecahedron itself to the icosahedron, because the perpendicular from the centre of the sphere to the pentagon of the dodecahedron and to the triangle of the icosahedron is the same.

"But it is right that I too should prove that

[Prop. 2] *The same circle circumscribes both the pentagon of the dodecahedron and the triangle of the icosahedron inscribed in the same sphere.*

"For this I need the following

Lemma.

"If an equilateral and equiangular pentagon be inscribed in a circle, the sum of the squares on the straight line subtending two sides and on the side of the pentagon is five times the square on the radius."

Let ABC be a circle, AC the side of the pentagon, D the centre ;
draw DF perpendicular to AC and produce it to
 B, E ;
join AB, AE .

I say that

$$BA^2 + AC^2 = 5DE^2.$$

For, since $BE = 2ED$,

$$BE^2 = 4ED^2.$$

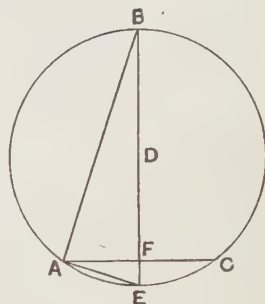
And $BE^2 = BA^2 + AE^2$;

therefore $BA^2 + AE^2 + ED^2 = 5ED^2$.

But $AC^2 = DE^2 + EA^2$;

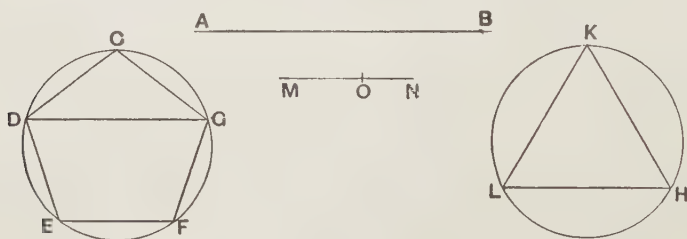
[Eucl. XIII. 10]

therefore $BA^2 + AC^2 = 5DE^2$.



“This being proved, it is required to prove that the same circle circumscribes both the pentagon of the dodecahedron and the triangle of the icosahedron inscribed in the same sphere.”

Let AB be the diameter of the sphere, and let a dodecahedron and an icosahedron be inscribed.



Let $CDEFG$ be one pentagon of the dodecahedron, and KLH one triangle of the icosahedron.

I say that the radii of the circles circumscribing them are equal.

Join DG ; then DG is the side of a cube inscribed in the sphere.

[Eucl. XIII. 17]

Take a straight line MN such that $AB^2 = 5MN^2$.

Now the square on the diameter of the sphere is five times the square on the radius of the circle from which the icosahedron is described.

[XIII. 16, Por.]

Therefore MN is equal to the radius of the circle passing through the five vertices of the icosahedron which form a pentagon.

Cut MN in extreme and mean ratio at O , MO being the greater segment.

Therefore MO is the side of the decagon in the circle with radius MN .

[XIII. 9 and 5, converse]

Now

$$5MN^2 = AB^2 = 3DG^2.$$

[XIII. 15]

But

$$3DG^2 : 3CG^2 = 5MN^2 : 5MO^2$$

(since, if DG is cut in extreme and mean ratio, the greater segment is equal to CG , and, if two straight lines are cut in extreme and mean ratio, their segments are in the same ratio : see lemma later, pp. 518—9).

And

$$5MO^2 + 5MN^2 = 5KL^2.$$

[This follows from XIII. 10, since KL is, by the construction of XIII. 16, the side of the regular pentagon in the circle with radius equal to MN , that is, the circle in which MN is the side of the inscribed hexagon and MO the side of the inscribed decagon.]

Therefore

$$5KL^2 = 3CG^2 + 3DG^2.$$

But

$$5KL^2 = 15 \text{ (radius of circle about } KLFH)^2, \quad [\text{XIII. 12}]$$

and

$$3DG^2 + 3CG^2 = 15 \text{ (radius of circle about } CDEFG)^2.$$

[Lemma above]

Therefore the radii of the two circles are equal.

Q. E. D.

[Prop. 3.] "If there be an equilateral and equiangular pentagon and a circle circumscribed about it, and if a perpendicular be drawn from the centre to one side, then

30 times the rectangle contained by the side and the perpendicular is equal to the surface of the dodecahedron."

Let $ABCDE$ be the pentagon, F the centre of the circle, FG the perpendicular on a side CD .

I say that

$$30CD \cdot FG = 12 \text{ (area of pentagon).}$$

Let CF , FD be joined.

Then, since

$$CD \cdot FG = 2 (\triangle CDF),$$

$$5CD \cdot FG = 10 (\triangle CDF),$$

$$\text{whence } 30CD \cdot FG = 12 \text{ (area of pentagon).}$$

Similarly we can prove that,

[Prop. 4] If ABC be an equilateral triangle in a circle, D the centre, and DE perpendicular to BC ,

$$30BC \cdot DE = \text{(surface of icosahedron).}$$

$$\text{For } DE \cdot BC = 2 (\triangle DBC);$$

$$\text{therefore } 3DE \cdot BC = 6 (\triangle DBC)$$

$$= 2 (\triangle ABC),$$

$$\text{whence } 30DE \cdot BC = 20 (\triangle ABC).$$

It follows that [Prop. 5]

$$\begin{aligned} & \text{(surface of dodecahedron)} : \text{(surface of icosahedron)} \\ &= \text{(side of pentagon)} \cdot \text{(its perpendicular)} : \text{(side of triangle)} \cdot \text{(its perp.)}. \end{aligned}$$

"This being clear, we have next to prove that,

[Prop. 6] As the surface of the dodecahedron is to the surface of the icosahedron, so is the side of the cube to the side of the icosahedron."

Therefore

$$5AF \cdot GH = 10 (\triangle ABD) = 2 (\text{area of pentagon}).$$

And $GH = 2HC$;

therefore

$$5AF \cdot HC = (\text{area of pentagon}),$$

or

$$AF \cdot BH = (\text{area of pentagon}).$$

Proof of theorem.

This being clear, let the circle be set out which circumscribes the pentagon of the dodecahedron and the triangle of the icosahedron inscribed in the same sphere.

Let ABC be the circle, and AB, AC two sides of the pentagon; join BC .

Take E the centre of the circle, join AE and produce it to F .

Let $AE = 2EG$, $KC = 3CH$.

Through G draw DM at right angles to AF meeting the circle at D, M ;

DM is then the side of the inscribed equilateral triangle.

Join AD, AM , which are equal to DM .

Now, since $AG \cdot BH = (\text{area of pentagon}),$

and $AG \cdot GD = (\text{area of triangle}),$

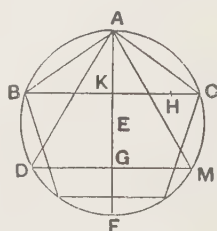
therefore $BH : GD = (\text{area of pentagon}) : (\text{area of triangle}),$

and $12BH : 20GD = (\text{surface of dod.}) : (\text{surface of icos.}).$

But $12BH = 10BC$, since $BH = 5HC$, and $BC = 6HC$;

and $20GD = 10DM$;

therefore $(\text{surface of dodecahedron}) : (\text{surface of icosahedron})$
 $= (\text{side of cube}) : (\text{side of icosahedron}).$



"Next we have to prove that,

[Prop. 7] *If any straight line whatever be cut in extreme and mean ratio, then, as is (1) the straight line the square on which is equal to the sum of the squares on the whole line and on the greater segment to (2) the straight line the square on which is equal to the sum of the squares on the whole and on the lesser segment, so is (3) the side of the cube to (4) the side of the icosahedron."*

Let AHB be the circle circumscribing both the pentagon of the dodecahedron and the triangle of the icosahedron inscribed in the same sphere, C the centre of the circle, and CB any radius divided at D in extreme and mean ratio, CD being the greater segment.

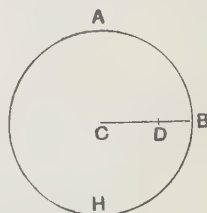
CD is then the side of the decagon inscribed in the circle.

[XIII. 9 and 5, converse]

Let E be the side of the icosahedron, F that of the dodecahedron, and G that of the cube, inscribed in the sphere.

Then E, F are the sides of the equilateral triangle and pentagon inscribed in the circle, and, if G is divided in extreme and mean ratio, the greater segment is equal to F .

[XIII. 17, Por.]



E _____
 F _____
 G _____

$$\begin{array}{ll}
 \text{Thus} & E^2 = 3BC^2, \quad [\text{XIII. 12}] \\
 \text{and} & CB^2 + BD^2 = 3CD^2. \quad [\text{XIII. 4}] \\
 \text{Therefore} & E^2 : CB^2 = (CB^2 + BD^2) : CD^2, \\
 \text{or} & E^2 : (CB^2 + BD^2) = CB^2 : CD^2 \\
 & = G^2 : F^2.
 \end{array}$$

Therefore, alternately and inversely,

$$G^2 : E^2 = F^2 : (CB^2 + BD^2).$$

But $F^2 = BC^2 + CD^2$; for the square on the side of the pentagon is equal to the sum of the squares on the sides of the hexagon and decagon inscribed in the same circle. [XIII. 10]

Therefore $G^2 : E^2 = (BC^2 + CD^2) : (CB^2 + BD^2)$,
which is the result required.

It has now to be proved that

[Prop. 8] (*Side of cube*) : (*side of icosahedron*)
= (*content of dodecahedron*) : (*content of icosahedron*).

Since equal circles circumscribe the pentagon of the dodecahedron and the triangle of the icosahedron inscribed in the same sphere,
and in a sphere equal circular sections are equally distant from the centre,
the perpendiculars from the centre of the sphere to the faces of the two solids are equal;

in other words, the pyramids with the centre as vertex and the pentagons of the dodecahedron and the triangles of the icosahedron respectively as bases are of equal height.

Therefore the pyramids are to one another as their bases.

Thus (12 pentagons) : (20 triangles)
= (12 pyramids on pentagons) : (20 pyramids on triangles),
or (surface of dodecahedron) : (surface of icosahedron)
= (content of dod.) : (content of icos.).

Therefore

(content of dodecahedron) : (content of icosahedron)
= (side of cube) : (side of icosahedron). [Prop. 6]

Lemma.

If two straight lines be cut in extreme and mean ratio, the segments of both are in one and the same ratio.

Let AB be cut in extreme and mean ratio at C , AC being the greater segment;

and let DE be cut in extreme and mean ratio at F , DF being the greater segment.

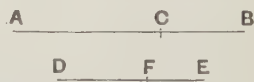
I say that $AB : AC = DE : DF$.

Since $AB \cdot BC = AC^2$,

and $DE \cdot EF = DF^2$,

and $AB \cdot BC : AC^2 = DE \cdot EF : DF^2$,

and $4AB \cdot BC : AC^2 = 4DE \cdot EF : DF^2$.



Componendo,

$$(4AB \cdot BC + AC^2) : AC^2 = (4DE \cdot EF + DF^2) : DF^2,$$

or

$$(AB + BC)^2 : AC^2 = (DE + EF)^2 : DF^2;$$

[II. 8]

therefore

$$(AB + BC) : AC = (DE + EF) : DF.$$

Componendo,

$$(AB + BC + AC) : AC = (DE + EF + DF) : DF,$$

or

$$2AB : AC = 2DE : DF;$$

that is,

$$AB : AC = DE : DF.$$

Summary of results.

If AB be any straight line divided at C in extreme and mean ratio, AC being the greater segment, and if we have a cube, a dodecahedron and an icosahedron inscribed in one and the same sphere, then:

$$(1) \text{ (side of cube) : (side of icosahedron) } = \sqrt{(AB^2 + AC^2)} : \sqrt{(AB^2 + BC^2)};$$

$$(2) \text{ (surface of dod.) : (surface of icos.) } \\ = \text{ (side of cube) : (side of icosahedron);}$$

$$(3) \text{ (content of dod.) : (content of icos.) } \\ = \text{ (surface of dod.) : (surface of icos.);}$$

$$\text{and (4) (content of dodecahedron) : (content of icos.)} \\ = \sqrt{(AB^2 + AC^2)} : \sqrt{(AB^2 + BC^2)}.$$

II. NOTE ON THE SO-CALLED "BOOK XV."

The second of the two Books added to the genuine thirteen is also supplementary to the discussion of the regular solids, but is much inferior to the first, "Book XIV." Its contents are of less interest and the exposition leaves much to be desired, being in some places obscure and in others actually inaccurate. It consists of three portions unequal in length. The first (Heiberg, Vol. v. pp. 40--48) shows how to inscribe certain of the regular solids in certain others, (*a*) a tetrahedron ("pyramid") in a cube, (*b*) an octahedron in a tetrahedron ("pyramid"), (*c*) an octahedron in a cube, (*d*) a cube in an octahedron and (*e*) a dodecahedron in an icosahedron. The second portion (pp. 48--50) explains how to calculate the number of edges and the number of solid angles in the five solids respectively. The third (pp. 50--66) shows how to determine the angle of inclination between faces meeting in an edge of any one of the solids. The method is to construct an isosceles triangle with vertical angle equal to the said angle of inclination; from the middle point of any edge two perpendiculars are drawn to it, one in each of the two faces intersecting in that edge; these perpendiculars (forming an angle which is the inclination of the two faces to one another) are used to determine the two equal sides of an isosceles triangle, and the base of the triangle is easily found from the known properties of the particular solid. The rules for drawing the respective isosceles triangles are first given all together in general terms (pp. 50--52); and the special interest of the passage consists in the fact that the rules are attributed to "Isidorus

our great teacher." This Isidorus is no doubt Isidorus of Miletus, the architect of the Church of St Sophia at Constantinople (about 532 A.D.), whose pupil Eutocius also was; he is often referred to by Eutocius (*Comm. on Archimedes*) as ὁ Μιλήσιος μηχανικὸς Ἰσίδωρος ἡμέτερος διδάσκαλος. Thus the third portion of the Book at all events was written by a pupil of Isidorus in the sixth century. Kluge (*De Euclidis elementorum libris qui feruntur XIV et XV*, Leipzig, 1891) has closely examined the language and style of the three portions and conjectures that they may be the work of different authors; the first portion may, he thinks, date from the end of the third century (the time of Pappus), and the second portion too may be older than the third. Hultsch however (art. "Eukleides" in Pauly-Wissowa's *Real-Encyclopädie der classischen Altertumswissenschaft*, 1907) does not think his arguments convincing.

It may be worth while to set out the particulars of Isidorus' rules for constructing isosceles triangles with vertical angles equal respectively to the angles of inclination between faces meeting in an edge of the several regular solids. A certain base is taken, and then with its extremities as centres and a certain other straight line as radius two circles are drawn; their point of intersection determines the vertex of the particular isosceles triangle. In the case of the cube the triangle is of course right-angled; in the other cases the bases and the equal sides are as shown below.

	Base of isosceles triangle	Equal sides of isosceles triangle
For the tetrahedron	the side of a triangular face	the perpendicular from the vertex of a triangular face to its base
For the octahedron	the diagonal of the square on one side of a triangular face	ditto
For the icosahedron	the chord joining two non-consecutive angular points of the regular pentagon on an edge (the "pentagon of the icosahedron")	ditto
For the dodecahedron	the chord joining two non-consecutive angular points of a pentagonal face [<i>BC</i> in the figure of Eucl. XIII. 17]	the perpendicular from the middle point of the chord joining two non-consecutive angular points of a face to the parallel side of that face [<i>HX</i> in the figure of Eucl. XIII. 17]

NOTES,

CRITICAL AND GEOMETRICAL;

CONTAINING

An Account of those Things in which this Edition differs from the Greek Text; and the Reasons of the Alterations which have been made. As also Observations on some of the Propositions.

By ROBERT SIMSON, M. D.

Emeritus Professor of Mathematics in the University of Glasgow.

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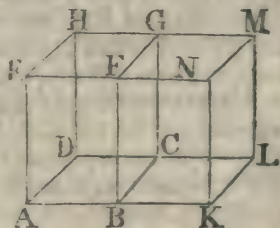
1806.

NOTES, &c.

DEFINITION I. BOOK I.

IT is necessary to consider a solid, that is, a magnitude which has length, breadth, and thickness, in order to understand aright the definitions of a point, line, and superficies; for these all arise from a solid, and exist in it: The boundary, or boundaries which contain a solid are called superficies, or the boundary which is common to two solids which are contiguous, or which divides one solid into two contiguous parts, is called a superficies: Thus, if BCGF be one of the boundaries which contain the solid ABCDEFGH, or which is the common boundary of this solid, and the solid BKLCFNMG, and is therefore in the one as well as in the other solid, is called a superficies, and has no thickness: For, if it have any, this thickness must either be a part of the thickness of the solid AG, or of the solid BM, or a part of the thickness of each of them. It cannot be a part of the thickness of the solid BM; because if this solid be removed from the solid AG, the superficies BCGF, the boundary of the solid AG, remain still the same as it was. Nor can it be a part of the thickness of the solid AG; because if this be removed from the solid BM, the superficies BCGF, the boundary of the solid BM does nevertheless remain, therefore the superficies BCGF has no thickness, but only length and breadth.

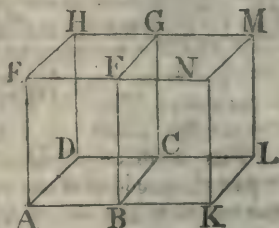
The boundary of a superficies is called a line, or a line is the common boundary of two superficies that are contiguous, or which divides one superficies into two contiguous parts: Thus if BC be one of the boundaries which contain the superficies ABCD, or which is the common boundary of this superficies, and of the superficies KBCL which is contiguous to it, this boundary BC is called a line, and has no breadth: For if it have any, this must be part either of the breadth of the superficies ABCD, or of the superficies KBCL, or part of each of them. It is not part of the breadth of the superficies KBCL; for, if this superficies be removed from the superficies ABCD,



BOOK I. the line BC, which is the boundary of the superficies ABCD, remains the same as it was: Nor can the breadth that BC is supposed to have, be a part of the breadth of the superficies ABCD; because, if this be removed from the superficies KBCL, the line BC, which is the boundary of the superficies KBCL, does nevertheless remain: Therefore the line BC has no breadth: And because the line BC is in a superficies, and that a superficies has no thickness, as was shewn, therefore a line has neither breadth nor thickness, but only length.

The boundary of a line is called a point, or a point is the common boundary or extremity

of two lines that are contiguous: Thus, if B be the extremity of the line AB, or the common extremity of the two lines AB, KB, this extremity is called a point, and has no length: For if it have any, this length must either be part of the length of the line AB, or of the line KB. It is not part



of the length of KB; for if the line KB be removed from AB, the point B which is the extremity of the line AB remains the same as it was: Nor is it part of the length of the line AB; for, if AB be removed from the line KB, the point B, which is the extremity of the line KB, does nevertheless remain: Therefore the point B has no length: And because a point is in a line, and a line has neither breadth nor thickness, therefore a point has no length, breadth, nor thickness. And in this manner the definitions of a point, line, and superficies, are to be understood.

DEF. VII. B. I.

INSTEAD of this definition as it is in the Greek copies, a more distinct one is given from a property of a plane superficies, which is manifestly supposed in the Elements, viz. that a straight line drawn from any point in a plane to any other in it, is wholly in that plane.

DEF. VIII. B. I.

IT seems that he who made this definition designed that it should comprehend not only a plane angle contained by two straight lines, but likewise the angle which some conceive to be made by a straight line and a curve, or by two curve lines which meet one another in a plane: But, though the meaning of the

the words $\epsilon\pi' \epsilon\upsilon\delta\epsilon\iota\chi\epsilon\iota\varsigma$, that is, in a straight line, or in the same direction, be plain, when two straight lines are said to be in a straight line, it does not appear what ought to be understood by these words, when a straight line and a curve, or two curve lines, are said to be in the same direction; at least it cannot be explained in this place; which makes it probable that this definition, and that of the angle of a segment, and what is said of the angle of a semicircle, and the angles of segments, in the 16th and 31st Propositions of Book 3, are the additions of some less skilful editor: On which account, especially since they are quite useless, these definitions are distinguished from the rest by inverted double commas.

BOOK I.

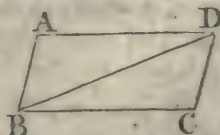
DEF. XVII. B. I.

THE words, "which also divides the circle into two equal parts" are added at the end of this definition in all the copies, but are now left out as not belonging to the definition, being only a corollary from it. Proclus demonstrates it by conceiving one of the parts into which the diameter divides the circle, to be applied to the other; for it is plain they must coincide, else the straight lines from the centre to the circumference would not be all equal: The same thing is easily deduced from the 31st Prop. of Book 3, and the 24th of the same; from the first of which it follows, that semicircles are similar segments of a circle; and from the other, that they are equal to one another.

DEF. XXXIII. B. I.

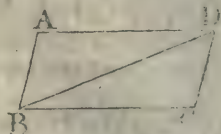
THIS definition has one condition more than is necessary; because every quadrilateral figure which has its opposite sides equal to one another, has likewise its opposite angles equal; and on the contrary.

Let ABCD be a quadrilateral figure, of which the opposite sides AB, CD, are equal to one another; as also AD and BC: Join BD; the two sides AD, DB are equal to the two CB, BD, and the base AB is equal to the base CD; therefore, by Prop. 8. of Book I. the angle ADB is equal to the angle CBD; and, by Prop. 4. B. I. the angle BAD is equal to the angle DCB, and ABD to BDC; and therefore also the angle ADC is equal to the angle ABC.



Book I.

And if the angle BAD be equal to the opposite angle BCD , and the angle ABC to ADC ; the opposite sides are equal: Because, by Prop. 32. B. 1. all the angles of the quadrilateral figure $ABCD$ are together equal to four right angles, and the two angles BAD , ADC are together equal to the two angles BCD , ABC : Wherefore BAD , ADC are the half of all the four angles; that is, BAD and ADC are equal to two right angles: and therefore AB , CD are parallels by Prop. 28. B. 1. In the same manner, AD , BC are parallels: Therefore $ABCD$ is a parallelogram, and its opposite sides are equal, by 34th Prop. B. 1.



PROP. VII. B. I.

THERE are two cases of this proposition, one of which is not in the Greek text, but is as necessary as the other: And that the case left out has been formerly in the text, appears plainly from this, that the second part of Prop. 5. which is necessary to the demonstration of this case, can be of no use at all in the Elements, or any where else, but in this demonstration; because the second part of Prop. 5. clearly follows from the first part, and Prop. 13. B. 1. This part must therefore have been added to Prop. 5. upon account of some proposition betwixt the 5th and 13th, but none of these stand in need of it except the 7th Proposition, on account of which it has been added: Besides, the translation from the Arabic has this case explicitly demonstrated. And Proclus acknowledges, that the second Part of Prop. 5. was added upon account of Prop. 7. but gives a ridiculous reason for it, "that it might afford an answer to objections made against the 7th," as if the case of the 7th, which is left out, were, as he expressly makes it, an objection against the proposition itself. Whoever is curious may read what Proclus says of this in his commentary on the 5th and 7th Propositions; for it is not worth while to relate his trifles at full length.

It was thought proper to change the enunciation of this 7th Prop. so as to preserve the very same meaning; the literal translation from the Greek being extremely harsh, and difficult to be understood by beginners.

PROP. XI. B. I.

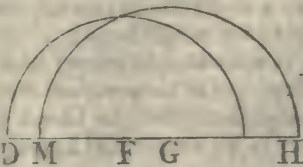
A COROLLARY is added to this proposition, which is necessary to Prop. I. B. II. and otherwise.

PROP. XX. and XXI. B. I.

PROCLUS, in his commentary, relates, that the Epicureans derided this proposition, as being manifest even to asses, and needing no demonstration; and his answer is, that though the truth of it be manifest to our senses, yet it is science which must give the reason why two sides of a triangle are greater than the third: But the right answer to this objection against this and the 21st, and some other plain propositions, is, that the number of axioms ought not to be increased without necessity, as it must be if these propositions be not demonstrated. Mons. Clairault, in the Preface to his Elements of Geometry, published in French at Paris, anno 1741, says, That Euclid has been at the pains to prove, that the two sides of a triangle which is included within another, are together less than the two sides of the triangle which includes it; but he has forgot to add this condition, viz. that the triangles must be upon the same base; because, unless this be added, the sides of the included triangle may be greater than the sides of the triangle which includes it, in any ratio which is less than that of two to one, as Pappus Alexandrinus has demonstrated in Prop. 3. B. 3. of his mathematical collections.

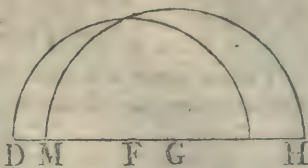
PROP. XXII. B. I.

SOME authors blame Euclid because he does not demonstrate that the two circles made use of in the construction of this problem must cut one another: But this is very plain from the determination he has given, viz. that any two of the straight lines DF, FG, GH must be greater than the third: For who is so dull, though only beginning to learn the Elements, as not to perceive that the circle described from the centre F, at the distance FD, must meet FH betwixt F and H, because FD is less than FH; and that, for the like reason, the circle described from the centre G, at the distance GH or GM, must



Book I.

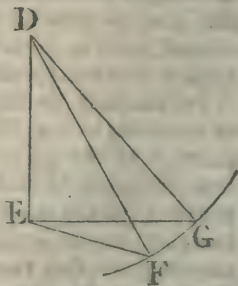
meet DG betwixt D and G; and that these circles must meet one another, because FD and GH are together greater than FG? And this determination is easier to be understood than that which Mr. Thomas Simpson derives from it, and puts instead of Euclid's, in the 49th page of his *Elements of Geometry*, that he may supply the omission he blames Euclid for, which determination is, that any of the three straight lines must be less than the sum, but greater than the difference of the other two: From this he shews the circles must meet one another, in one case; and says, that it may be proved after the same manner in any other case; But the straight line GM, which he bids take from GF may be greater than it, as in the figure here annexed; in which case his demonstration must be changed into another.



PROP. XXIV. B. I.

To this is added, “of the two sides DE, DF, let DE be “that which is not greater than the other;” that is, take that side of the two DE, DF which is not greater than the other, in order to make with it the angle EDG equal to BAC; because without this restriction there might be three different cases of the proposition, as Campanus and others make.

Mr. Thomas Simpson, in p. 262 of the second edition of his *Elements of Geometry*, printed anno 1760, observes in his notes, that it ought to have been shewn, that the points F fall below the line EG. This probably Euclid omitted, as it is very easy to perceive, that DG being equal to DF, the point G is in the circumference of a circle described from the centre D at the distance DF, and must be in that part of it which is above the straight line EF, because DG falls above DF, the angle EDG being greater than the angle EDF.



PROP. XXIX. B. I.

THE proposition which is usually called the 5th postulate, or 11th axiom, by some the 12th, on which this 29th depends, has
given

given a great deal to do, both to ancient and modern geometers: It seems not to be properly placed among the axioms, as indeed it is not self-evident; but it may be demonstrated thus:

DEFINITION I.

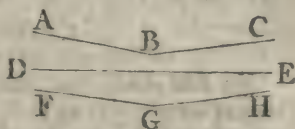
THE distance of a point from a straight line, is the perpendicular drawn to it from the point.

DEF. 2.

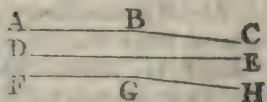
ONE straight line is said to go nearer to, or further from, another straight line, when the distances of the points of the first from the other straight line become less or greater than they were; and two straight lines are said to keep the same distance from one another, when the distance of the points of one of them from the other is always the same.

AXIOM.

A STRAIGHT line cannot first come nearer to another straight line, and then go further from it, before it cuts it; and, in like manner, a straight line cannot go further from another straight line, and then come nearer to it; nor can a straight line keep the same distance from another straight line, and then come nearer to it, or go further from it; for a straight line keeps always the same direction.



For example, the straight line ABC cannot first come nearer to the straight line DE, as from the point A to the point B, and then, from the point B to the point C, go further from the same DE: And, in like manner, the straight line FGH cannot go further from DE, as from F to G, and then, from G to H, come nearer to the same DE: And so in the last case, as in fig. 2.



See the above figure.

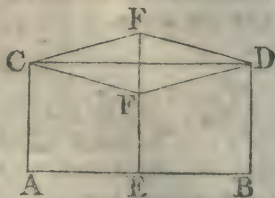
PROP. I.

IF two equal straight lines AC, BD, be each at right angles to the same straight line AB: If the points C, D be joined by the straight line CD, the straight line EF drawn from any point E in AB unto CD, at right angles to AB, shall be equal to AC, or BD.

If EF be not equal to AC, one of them must be greater than the other; let AC be the greater; then, because FE is

Book I.

less than CA, the straight line CFD is nearer to the straight line AB at the point F than at the point C, that is, CF comes nearer to AB from the point C to F: But because DB is greater than FE, the straight line CFD is further from AB at the point D than at F, that is, FD goes further from AB from F to D: Therefore the straight line CFD first comes nearer to the straight line AB, and then goes further from it, before it cuts it; which is impossible. If FE be said to be greater than CA, or DB, the straight line CFD first goes further from the straight line AB, and then comes nearer to it: which is also impossible. Therefore FE is not unequal to AC, that is, it is equal to it.



PROP. II.

If two equal straight lines AC, BD be each at right angles to the same straight line AB; the straight line CD which joins their extremities makes right angles with AC and BD.

Join AD, BC; and because, in the triangles CAB, DBA, CA, AB are equal to DB, BA, and the angle CAB equal to the angle DBA; the base BC is equal^a to the base AD: And in the triangles ACD, BDC, AC, CD are equal to BD, DC, and the base AD is equal to the base BC: Therefore the angle ACD is equal^b to the angle BDC: From any point E in AB draw EF unto CD, at right angles to AB; therefore, by Prop. I. EF is equal to AC, or BD; wherefore, as has been just now shewn, the angle, ACF is equal to the angle EFC: In the same manner, the angle BDF is equal to the angle EFD; but the angles ACD, BDC are equal; therefore the angles EFC and EFD are equal, and right angles^c; wherefore also the angles ACD, BDC are right angles.



COR. Hence, if two straight lines AB, CD be at right angles to the same straight line AC, and if betwixt them a straight line BD be drawn at right angles to either of them, as to AB; then BD is equal to AC, and BDC is a right angle.

If AC be not equal to BD, take BG equal to AC, and join CG: Therefore, by this proposition, the angle ACG is a right angle; but ACD is also a right angle; wherefore the angles ACD,

ACD, ACG are equal to one another, which is impossible. Therefore BD is equal to AC; and by this proposition BDC is a right angle.

Book I.

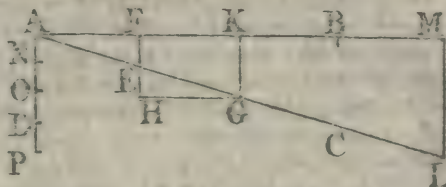
PROP. 3.

If two straight lines which contain an angle be produced, there may be found in either of them a point from which the perpendicular drawn to the other shall be greater than any given straight line.

Let AB, AC be two straight lines which make an angle with one another, and let AD be the given straight line; a point may be found either in AB or AC, as in AC, from which the perpendicular drawn to the other AB shall be greater than AD.

In AC take any point E, and draw EF perpendicular to AB; produce AE to G, so that EG be equal to AE; and produce FE to H, and make EH equal to FE, and join HG. Because, in the triangles AEF, GEH, AE, EF are equal to GE, EH, each to each, and contain equal^a angles, the angle^a $\angle$ 15. 1. GHE is therefore equal^b to the angle AFE which is a right^b angle: Draw GK perpendicular to AB; and because the straight lines FK, HG

are at right angles to FH, and KG at right angles to FK, KG is equal to FH, by Cor. Pr. 2. that is, to the double of FE.



In the same manner if AG be produced to L, so that GL be equal to AG, and LM be drawn perpendicular to AB, then LM is double of GK, and so on. In AD take AN equal to FE, and AO, equal to KG, that is, to the double of FE, or AN; also, take AP, equal to LM, that is, to the double of KG, or AO; and let this be done till the straight line taken be greater than AD: Let this straight line so taken be AP, and because AP is equal to LM, therefore LM is greater than AD. Which was to be done.

PROP. 4.

If two straight lines AB, CD make equal angles EAB, ECD with another straight line EAC towards the same parts of it; AB and CD are at right angles to some straight line.

Bisect

Book I.

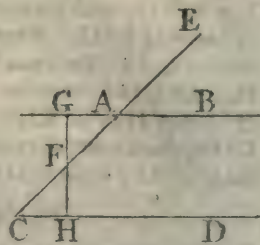
a 15. 1.

b 4. 1.

c 13. 1.

c 14. 1.

Bisect AC in F , and draw FG perpendicular to AB ; take CH in the straight line CD equal to AG , and on the contrary side of AC to that on which AG is, and join FH : Therefore, in the triangles AFG , CFH , the sides FA , AG are equal to FC , CH , each to each, and the angle FAG , that^a is EAB , is equal to the angle FCH ; wherefore^b the angle AGF is equal to CHF , and AFG to the angle CFH : To these last add the common angle AFH ; therefore the two angles AFG , AFH are equal to the two angles CFH , HFA , which two last are equal together to two right angles^c: therefore also AFG , AFH are equal to two right angles, and consequently^d GF and FH are in one straight line. And because AGF is a right angle, CHF which is equal to it is also a right angle: Therefore the straight lines AB , CD are at right angles to GH .



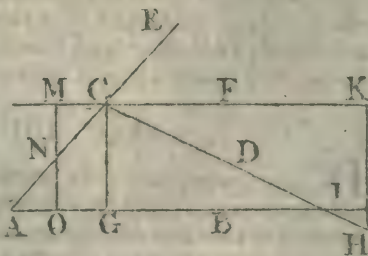
PROP. 5.

If two straight lines BB , CD be cut by a third ACE , so as to make the interior angles BAC , ACD , on the same side of it, together less than two right angles; AB and CD being produced, shall meet one another towards the parts on which are the two angles, which are less than two right angles.

a 23. 1.

At the point C , in the straight line CE , make^a the angle ECF equal to the angle EAB , and draw to AB the straight line CG at right angles to CF : Then, because the angles ECF , EAB are equal to one another, and that the angles ECF , FCA are together equal^b to two right angles, the angles EAB , FCA are equal to two right angles. But by the hypothesis, the angles EAB , ACD are together less than two right angles; therefore the angle

b 13. 1.



FCA is greater than ACD , and CD falls between CF and AB : And because CF and CD make an angle with one another, by Prop. 3. a point may be found in either of them CD , from which the perpendicular drawn to CF shall be greater than

tha_n

than the straight line CG . Let this point be H ; and draw HK perpendicular to CF , meeting AB in L : And because AB , CF contain equal angles with AC on the same side of it, by Prop. 4. AB and CF are at right angles to the straight line MNO , which bisects AC in N , and is perpendicular to CF : Therefore by Cor. Prop. 2. CG and KL , which are at right angles to CF , are equal to one another: And HK is greater than CG , and therefore is greater than KL , and consequently the point H is in KL produced. Wherefore the straight line CDH , drawn betwixt the points C , H , which are on contrary sides of AL , must necessarily cut the straight line AB .

PROP. XXXV. B. I.

THE demonstration of this Proposition is changed, because, if the method which is used in it was followed, there would be three cases to be separately demonstrated, as is done in the translation from the Arabic; for, in the Elements, no case of a Proposition that requires a different demonstration ought to be omitted. On this account, we have chosen the method which Mons. Clairault has given, the first of any, as far as I know, in his Elements, page 21, and which afterwards Mr. Simpson gives in his page 32. But whereas Mr. Simpson makes use of Prop. 26. B. I. from which the equality of the two triangles does not immediately follow, because, to prove that, the 4th of B. I. must likewise be made use of, as may be seen in the very same case in the 34th Prop. B. I. it was thought better to make use only of the 4th of B. I.

PROP. XLV. B. I.

THE straight line KM is proved to be parallel to FL from the 33d Prop. whereas KH is parallel to FG by construction, and KHM , FGL have been demonstrated to be straight lines. A corollary is added from Commandine, as being often used.

PROP. XIII. B. II.

IN this proposition only acute angled triangles are mentioned, whereas it holds true of every triangle: and the demonstrations of the cases omitted are added; Commandine and Clavius have likewise given their demonstrations of these cases.

PROP. XIV. B. II.

IN the demonstration of this, some Greek editor has ignorantly inserted the words, "but if not, one of the two BE ,
" ED ,

BOOK II. "ED, is the greater: Let BE be the greater, and produce it to "F," as if it was of any consequence whether the greater or lesser be produced: Therefore, instead of these words, there ought to be read only, "but if not, produce BE to F."

PROP. I. B. III.

BOOK III. SEVERAL authors, especially among the modern mathematicians and logicians, inveigh too severely against indirect or apogogic demonstrations, and sometimes ignorantly enough; not being aware that there are some things that cannot be demonstrated any other way: Of this the present proposition is a very clear instance, as no direct demonstration can be given of it: Because, besides the definition of a circle, there is no principle or property relating to a circle antecedent to this problem, from which either a direct or indirect demonstration can be deduced: Wherefore it is necessary that the point found by the construction of the problem be proved to be the centre of the circle, by the help of this definition, and some of the preceding propositions: And because, in the demonstration, this proposition must be brought in, viz. straight lines from the centre of a circle to the circumference are equal, and that the point found by the construction cannot be assumed as the centre, for this is the thing to be demonstrated: it is manifest some other point must be assumed as the centre: and if from this assumption an absurdity follows, as Euclid demonstrates there must, then it is not true that the point assumed is the centre; and as any point whatever was assumed, it follows that no point, except that found by the construction, can be the centre, from which the necessity of an indirect demonstration in this case is evident.

PROP. XIII. B. III.

As it is much easier to imagine that two circles may touch one another within in more points than one, upon the same side, than upon opposite sides; the figure of that case ought not to have been omitted; but the construction in the Greek text would not have suited with this figure so well, because the centres of the circles must have been placed near to the circumferences; on which account another construction and demonstration is given, which is the same with the second part of that which Campanus has translated from the Arabic, where,

where, without any reason, the demonstration is divided into Book III.
two parts.

PROP. XV. B. III.

THE converse of the second part of this proposition is wanting, though in the preceding, the converse is added, in a like case, both in the enunciation and demonstration; and it is now added in this. Besides, in the demonstration of the first part of this 15th, the diameter AD (see Commandine's figure), is proved to be greater than the straight line BC by means of another straight line MN; whereas it may be better done without it: on which accounts we have given a different demonstration, like to that which Euclid gives in the preceding 14th, and to that which Theodosius gives in Prop. 6. B. I. of his Spherics, in this very affair.

PROP. XVI. B. III.

IN this we have not followed the Greek nor the Latin translation literally, but have given what is plainly the meaning of the proposition, without mentioning the angle of the semicircle, or that which some call the cornicular angle, which they conceive to be made by the circumference and the straight line which is at right angles to the diameter, at its extremity; which angles have furnished matter of great debate between some of the modern geometrists, and given occasion of deducing strange consequences from them, which are quite avoided by the manner in which we have expressed the proposition. And in like manner, we have given the true meaning of prop. 31. B. 3. without mentioning the angles of the greater or lesser segments. These passages Vieta, with good reason, suspects to be adulterated in the 386th page of his Oper. Math. .

PROP. XX. B. III.

THE first words of the second part of this demonstration, "*κατασθῶ δὴ παλιν*," are wrong translated by Mr. Briggs and Dr. Gregory, "*Rursus inclinetur*;" for the translation ought to be "*Rursus inflectatur*," as Commandine has it: A straight line is said to be inflected either to a straight, or curve line, when a straight line is drawn to this line from a point, and from the point in which it meets it, a straight line making an angle with the former is drawn to another point, as is evident from the 90th prop. of Euclid's Data: For thus the whole line betwixt the first and last points is inflected or broken at
the

Book III. the point of inflection, where the two straight lines meet. And in the like sense two straight lines are said to be inflected from two points to a third point, when they make an angle at this point; as may be seen in the description given by Pappus Alexandrinus of Appollonius's Books de Locis planis, in the preface to his 7th Book: We have made the expression fuller from the 90th Prop. of the Data.

PROP. XXI. B. III.

THERE are two cases of this proposition, the second of which, viz. when the angles are in a segment not greater than a semicircle, is wanting in the Greek: And of this a more simple demonstration is given than that which is in Commandine, as being derived only from the first case, without the help of triangles.

PROP. XXIII. and XXIV. B. III.

In proposition 24. it is demonstrated that the segment AEB must coincide with the segment CFD (see Commandine's figure,) and that it cannot fall otherwise, as CGD, so as to cut the other circle in a third point G, from this, that, if it did, a circle could cut another in more points than two: But this ought to have been proved to be impossible in the 23d prop. as well as that one of the segments cannot fall within the other. This part, then, is left out in the 24th, and put in its proper place, the 23rd proposition.

PROP. XXV. B. III.

THIS proposition is divided into three cases, of which two have the same construction and demonstration; therefore it is now divided only into two cases.

PROP. XXXIII. B. III.

THIS also in the Greek is divided into three cases, of which, two, viz. one, in which the given angle is acute, and the other in which it is obtuse, have exactly the same construction and demonstration: on which account, the demonstration of the last case is left out, as quite superfluous, and the addition of some unskilful editor; besides the demonstration of the case when the angle given is a right angle, is done a round-about way, and is therefore changed to a more simple one, as was done by Clavius.

PROP. XXXV. B. III.

As the 25th and 33rd propositions are divided into more cases, so this 35th is divided into fewer cases than are necessary. Nor can it be supposed that Euclid omitted them because they are easy; as he has given the case, which by far is the easiest of them all, viz. that in which both the straight lines pass through the centre: And in the following proposition he separately demonstrates the case in which the straight line passes through the centre, and that in which it does not pass through the centre: So that it seems Theon, or some other, has thought them too long to insert: But cases that require different demonstrations, should not be left out in the Elements, as was before taken notice of: These cases are in the translation from the Arabic, and are now put into the text.

PROP. XXXVII. B. III.

At the end of this, the words "in the same manner it may be demonstrated, if the centre be in AC," are left out as the addition of some ignorant editor.

DEFINITIONS OF BOOK IV.

WHEN a point is in a straight line, or any other line, this point is by the Greek geometers said *ὑπὲρκειναι*, to be upon, or in that line, and when a straight line or circle meets a circle any way, the one is said *ὑπὲρκειναι* to meet the other: But when a straight line or circle meets a circle so as not to cut it, it is said *ἐφαπτεσθαι*, to touch the circle; and these two terms are never promiscuously used by them: Therefore, in the 5th definition of B. 4. the compound *ἐφαπτηται* must be read, instead of the simple *ὑπὲρκειναι*: And in the 1st, 2d, 3d, and 6th definitions in Commandine's translation, "tangit," must be read instead of "contingit." And in the 2d and 3d definitions of Book 3. the same change must be made: But in the Greek text of propositions 11th, 12th, 13th, 18th, 19th, Book 3. the compound verb is to be put for the simple.

PROP. IV. B. IV.

In this, as also in the 8th and 13th proposition of this book, it is demonstrated indirectly, that the circle touches a straight line; whereas in the 17th, 33d, and 37th propositions of Book 3. the same thing is directly demonstrated: And this way we have

BOOK IV. have chosen to use in the propositions of this book, as it is shorter.

PROP. V. B. IV.

THE demonstration of this has been spoiled by some unskilful hand: For he does not demonstrate, as is necessary, that the two straight lines which bisect the sides of the triangle at right angles must meet one another; and, without any reason, he divides the proposition into three cases; whereas, one and the same construction and demonstration serves for them all, as Campanus has observed; which useless repetitions are now left out: The Greek text also in the corollary is manifestly vitiated, where mention is made of a given angle, though there neither is, nor can be, any thing in the proposition relating to a given angle.

PROP. XV. and XVI. B. IV.

IN the corollary of the first of these, the words equilateral and equiangular are wanting in the Greek: and in prop. 16. instead of the circle ABCD, ought to be read the circumference ABCD: Where mention is made of its containing fifteen equal parts.

DEF. III. B. V.

BOOK V. MANY of the modern mathematicians reject this definition: The very learned Dr. Barrow has explained it at large at the end of his third lecture of the year 1666, in which also he answers the objections made against it as well as the subject would allow: And at the end gives his opinion upon the whole, as follows;

"I shall only add, that the author had, perhaps, no other design in making this definition, than (that he might more fully explain and embellish his subject) to give a general and summary idea of ratio to beginners, by premising this metaphysical definition, to the more accurate definitions of ratios that are the same to one another, or one of which is greater, or less than the other: I call it a metaphysical, for it is not properly a mathematical, definition, since nothing in mathematics depends on it, or is deduced, nor, as I judge, can be deduced from it: And the definition of analogy, which follows, viz. Analogy is the similitude

“ litude of ratios, is of the same kind, and can serve for no purpose in mathematics, but only to give beginners some general, though gross and confused, notion of analogy: But the whole of the doctrine of ratios, and the whole of mathematics, depend upon the accurate mathematical definitions which follow this: To these we ought principally to attend, as the doctrine of ratios is more perfectly explained by them; this third, and others like it, may be entirely spared without any loss to geometry; as we see in the 7th book of the Elements, where the proportion of numbers to one another is defined, and treated of, yet without giving any definition of the ratio of numbers; though such a definition was as necessary and useful to be given in that book as in this: But indeed there is scarce any need of it in either of them: Though I think that a thing of so general and abstracted a nature, and thereby the more difficult to be conceived and explained, cannot be more commodiously defined than as the author has done: Upon which account I thought fit to explain it at large, and defend it against the captious objections of those who attack it.” To this citation from Dr. Barrow I have nothing to add, except that I fully believe the 3d and 8th definitions are not Euclid’s, but added by some unskilful editor.

DEF. XI. B. V.

It was necessary to add the word “ continual” before “ proportionals” in this definition; and thus it is cited in the 33d prop. of Book II.

After this definition ought to have followed the definition of compound ratio, as this was the proper place for it; duplicate and triplicate ratio being species of compound ratio: But Theon has made it the 5th def. of B. 6. where he gives an absurd and entirely useless definition of compound ratio: For this reason we have placed another definition of it betwixt the 11th and 12th of this book, which, no doubt, Euclid gave; for he cites it expressly in Prop. 23. B. 6. and which Clavius, Herigon, and Barrow, have likewise given, but they retain also Theon’s, which they ought to have left out of the Elements.

DEF. XIII. B. V.

THIS, and the rest of the definitions following, contain the explication of some terms which are used in the 5th and following books; which, except a few, are easily enough understood

Book V.

stood from the propositions of this book where they are first mentioned: They seem to have been added by Theon, or some other. However it be, they are explained something more distinctly for the sake of learners.

PROP. IV. B. V.

IN the construction preceding the demonstration of this, the words $\alpha \epsilon \tau \upsilon \chi \epsilon$, any whatever, are twice wanting in the Greek, as also in the Latin translations; and are now added, as being wholly necessary.

Ibid, in the demonstration; in the Greek, and in the Latin translation of Commandine, and in that of Mr. Henry Briggs, which was published at London in 1620, together with the Greek text of the first six books, which translation in this place is followed by Dr. Gregory in his edition of Euclid, there is this sentence following, viz. "and of A and C have been taken "equimultiples K, L; and of B and D, any equimultiples "whatever ($\alpha \epsilon \tau \upsilon \chi \epsilon$) M, N;" which is not true, the words "any whatever," ought to be left out: And it is strange that neither Mr. Briggs, who did right to leave out these words in one place of Prop. 13, of this book, nor Dr. Gregory, who changed them into the word "some" in three places, and left them out in a fourth of that same Prop. 13, did not also leave them out in this place of Prop. 4, and in the second of the two places where they occur in Prop. 17 of this book, in neither of which they can stand consistent with truth: And in none of all these places, even in those which they corrected in their Latin translation, have they cancelled the words $\alpha \epsilon \tau \upsilon \chi \epsilon$ in the Greek text, as they ought to have done.

The same words $\alpha \epsilon \tau \upsilon \chi \epsilon$ are found in four places of Prop. 11 of this book, in the first and last of which they are necessary, but in the second and third, though they are true, they are quite superfluous; as they likewise are in the second of the two places in which they are found in the 12th Prop. and in the like places of Prop. 22, 23, of this book; but are wanting in the last place of Prop. 23, as also in Prop. 25, Book 11.

COR. PROP. IV. B. V.

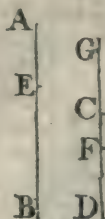
THIS corollary has been unskilfully annexed to this proposition, and has been made instead of the legitimate demonstration, which, without doubt, Theon, or some other editor, has taken away, not from this, but from its proper place in
this

this book: The author of it designed to demonstrate, that if four magnitudes E, G, F, H, be proportionals, they are also proportionals inversely; that is, G is to E, as H to F; which is true; but the demonstration of it does not in the least depend upon this 4th prop. or its demonstration: For, when he says, "because it is demonstrated, that if K be greater than M, L is greater than N," &c. This indeed is shewn in the demonstration of the 4th prop. but not from this, that E, G, F, H, are proportionals; for this last is the conclusion of the proposition. Wherefore these words, "because it is demonstrated," &c. are wholly foreign to his design: And he should have proved, that if K be greater than M, L is greater than N, from this, that E, G, F, H, are proportionals, and from the 5th def. of this book, which he has not; but is done in proposition B, which we have given in its proper place, instead of this corollary; and another corollary is placed after the 4th prop. which is often of use; and is necessary to the demonstration of Prop. 18 of this book.

PROP. V. B. V.

IN the construction which precedes the demonstration of this proposition, it is required that EB may be the same multiple of CG, that AE is of CF; that is, that EB be divided into as many equal parts, as there are parts in AE equal to CF: From which it is evident, that this construction is not Euclid's, for he does not shew the way of dividing straight lines, and far less other magnitudes, into any number of equal parts, until the 9th proposition of B. 6. and he never requires any thing to be done in the construction of which he had not before given the method of doing: For this reason, we have changed the construction to one,

which, without doubt, is Euclid's, in which nothing is required but to add a magnitude to itself a certain number of times; and this is to be found in the translation from the Arabic, though the enunciation of the proposition and the demonstration are there very much spoiled. Jacobus Peletarius, who was the first, as far as I know, who took notice of this error, gives also the right construction in his edition of Euclid, after he had given the other which he blames: He says, he would not leave it out, because it was fine, and might sharpen one's genius to invent others like it;



Book V.

whereas there is not the least difference between the two demonstrations, except a single word in the construction, which very probably has been owing to an unskilful librarian. Clavius likewise gives both the ways ; but neither he nor Peletarius takes notice of the reason why the one is preferable to the other.

PROP. VI. B. V.

THERE are two cases of this proposition, of which only the first and simplest is demonstrated in the Greek : And it is probable Theon thought it was sufficient to give this one, since he was to make use of neither of them in his mutilated edition of the 5th book ; and he might as well have left out the other, as also the fifth proposition, for the same reason. The demonstration of the other case is now added, because both of them, as also the 5th proposition, are necessary to the demonstration of the 18th proposition of this book. The translation from the Arabic gives both cases briefly.

PROP. A. B. V.

THIS proposition is frequently used by geometers, and it is necessary in the 25th prop. of this book, 31st of the 6th, and 34th of the 11th, and 15th of the 12th book : It seems to have been taken out of the Elements by Theon, because it appeared evident enough to him, and others, who substitute the confused and indistinct idea the vulgar have of proportionals, in place of that accurate idea which is to be got from the 5th def. of this book. Nor can there be any doubt that Eudoxus or Euclid gave it a place in the Elements, when we see the 7th and 9th of the same book demonstrated, though they are quite as easy and evident as this. Alphonsus Borellus takes occasion from this proposition to censure the 5th definition of this book very severely, but most unjustly. In p. 126 of his *Euclid Restored*, printed at Pisa in 1658, he says, "Nor can even this least degree of knowledge be obtained from the foresaid property," viz. that which is contained in 5th def. 5. "That, if four magnitudes be proportionals, the third must necessarily be greater than the fourth, when the first is greater than the second : as Clavius acknowledges in the 16th prop. of the 5th book of the elements." But though Clavius makes no such acknowledgment expressly, he has given Borellus a handle to say this of him ; because when Clavius, in the above cited place, censures Commandine, and that very justly, for demonstrating this proposition by help of the 16th of the 5th ; yet he himself gives no demonstration of it, but thinks it plain
from

from the nature of proportionals, as he writes in the end of the 14th and 16th Prop. B. 5. of his edition, and is followed by Herigon in Schol. 1. Prop. 4. B. 5, as if there was any nature of proportionals antecedent to that which is to be derived and understood from the definition of them: And indeed, though it is very easy to give a right demonstration of it, nobody, as far as I know, has given one, except the learned Dr. Barrow, who, in answer to Borellus's objection, demonstrates it indirectly, but very briefly and clearly, from the 5th definition, in the 322d page of his Lect. Mathem. from which definition it may also be easily demonstrated directly: On which account we have placed it next to the propositions concerning equimultiples.

PROP. B. B. V.

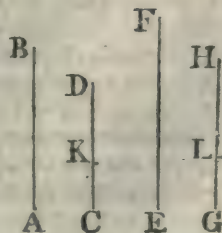
THIS also is easily deduced from the 5th def. B. 5, and therefore is placed next to the other; for it was very ignorantly made a corollary from the fourth Prop. of this Book. See the note on that corollary.

PROP. C. B. V.

THIS is frequently made use of by geometers, and is necessary to the 5th and 6th Propositions of the 10th Book. Clavius, in his notes subjoined to the 8th def. of Book 5. demonstrates it only in numbers, by help of some of the propositions of the 7th Book: in order to demonstrate the property contained in the 5th definition of the 5th Book, when applied to numbers, from the property of proportionals contained in the 20th def. of the 7th Book: And most of the commentators judge it difficult to prove that four magnitudes which are proportionals according to the 20th def. of 7th Book, are also proportionals according to the 5th def. of 5th Book. But this is easily made out as follows:

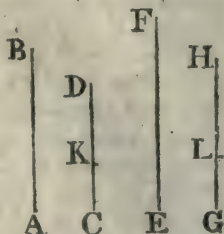
First, if A, B, C, D, be four magnitudes, such that A is the same multiple, or the same part of B, which C is of D: A, B, C, D, are proportionals: This is demonstrated in proposition C.

Secondly, if AB contain the same parts of CD that EF does of GH; in this case likewise AB is to CD, as EF to GH.



BOOK V.

Let CK be a part of CD, and GL the same part of GH ; and let AB be the same multiple of CK, that EF is of GL : Therefore, by Prop. C, of 5th Book, AB is to CK, as EF to GL : And CD, GH are equimultiples, of CK, GL the second and fourth ; wherefore, by Cor. Prop. 4. Book 5. AB is to CD, as EF to GH.



And if four magnitudes be proportionals according to the 5th def. of Book 5, they are also proportionals according to the 20th def. of Book 7.

First, if A be to B, as C to D ; then if A be any multiple or part of B, C is the same multiple or part of D, by Prop. D. of B. 5.

Next, if AB be to CD, as EF to GH ; then if AB contains any parts of CD, EF contains the same parts of GH : For let CK be a part of CD, and GL the same part of GH, and let AB be a multiple of CK : EF is the same multiple of GL : take M the same multiple of GL that AB is of CK ; therefore, by Prop. C. of B. 5. AB is to CK, as M to GL : and CD, GH are equimultiples of CK, GL ; wherefore, by Cor. Prop. 4. B. 5. AB is to CD, as M to GH. And, by the hypothesis, AB is to CD, as EF to GH ; therefore M is equal to EF by Prop. 9. Book 5. and consequently EF is the same multiple of GL that AB is of CK.

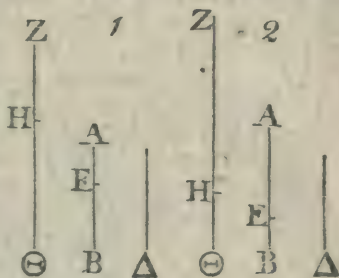
PROP. D. B. V.

THIS is not unfrequently used in the demonstration of other propositions, and is necessary in that of Prop. 9. B. 6. It seems Theon has left it out for the reasons mentioned in the notes at Prop. A.

PROP. VIII. B. V.

In the demonstration of this, as it is now in the Greek, there are two cases (see the demonstration in Hervagius, or Dr. Gregory's edition), of which the first is that in which AE is less than EB ; and in this it necessarily follows, that HΘ the multiple EB is greater than ZH the same multiple of AE, which last multiple, by the construction, is greater than Δ ; whence also HΘ must be greater than Δ : But in the second case, viz. that in which EB is less than AE, though ZH be greater than Δ, yet HΘ may be less than the same Δ ; so that there

there cannot be taken a multiple of Δ which is the first that is greater than K or $H\Theta$, because Δ itself is greater than it: Upon this account, the author of this demonstration found it necessary to change one part of the construction that was made use of in the first case: But he has, without any necessity, changed also another part of it, viz. when he orders to take N that multiple of Δ which is the first that is greater than ZH ; for he might have taken that multiple of Δ which is the first that is greater than $H\Theta$, or K , as was done in the first case: He likewise brings in this K into the demonstration of both cases, without any reason; for it serves to no purpose but to lengthen the demonstration. There is also



a third case which is not mentioned in this demonstration, viz. that in which AE in the first, or EB in the second of the two other cases, is greater than D ; and in this any equimultiples, as the doubles, of AE , EB are to be taken, as is done in this edition, where all the cases are at once demonstrated: And from this it is plain that Theon, or some other unskilful editor, has vitiated this proposition.

PROP. IX. B. V.

Of this there is given a more explicit demonstration than that which is now in the Elements.

PROP. X. B. V.

It was necessary to give another demonstration of this proposition, because that which is in the Greek and Latin, or other editions, is not legitimate: For the words *greater*, *the same*, or *equal*, *lesser*, have a quite different meaning when applied to magnitudes and ratios, as is plain from the 5th and 7th definitions of Book 5. By the help of these let us examine the demonstration of the 10th Prop. which proceeds thus: "Let A have to C a greater ratio than B to C : I say that A is greater than B ; for if it is not greater, it is either equal or less. But A cannot be equal to B , because then each of them would have the same ratio to C ; but they have not. Therefore A is not equal to B ." The force of which reasoning is this: If A had to C the same ratio that B has to C , then if

BOOK V.

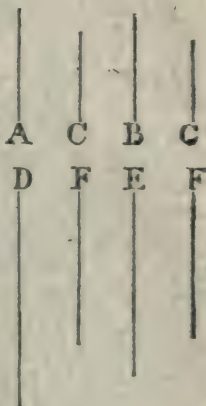
any equimultiples whatever of A and B be taken, and any multiple whatever of C; if the multiple of A be greater than the multiple of C, then, by the 5th def. of Book 5, the multiple of B is also greater than that of C: but, from the hypothesis that A has a greater ratio to C, than B has to C, there must, by the 7th def. of Book 5, be certain equimultiples of A and B, and some multiple of C such, that the multiple of A is greater than the multiple of C, but the multiple of B is not greater than the same multiple of C: And this proposition directly contradicts the preceding; wherefore A is not equal to B. The demonstration of the 10th prop. goes on thus: "But neither is A less than B; because then A would have a less ratio to C than B has to it: But it has not a less ratio, therefore A is not less than B," &c. Here it is said, that "A would have a less ratio to C than B has to C," or, which is the same thing, that B would have a greater ratio to C than A to C; that is, by 7th def. Book 5, there must be some equimultiples of B and A, and some multiple of C, such that the multiple of B is greater than the multiple of C, but the multiple of A is not greater than it: And it ought to have been proved, that this can never happen if the ratio of A to C be greater than the ratio of B to C; that is, it should have been proved, that, in this case, the multiple of A is always greater than the multiple of C, whenever the multiple of B is greater than the multiple of C; for when this is demonstrated, it will be evident that B cannot have a greater ratio to C, than A has to C, or, which is the same thing, that A cannot have a less ratio to C than B has to C. But this is not at all proved in the 10th proposition: But if the 10th were once demonstrated, it would immediately follow from it, but cannot without it be easily demonstrated, as he that tries to do it will find. Wherefore the 10th proposition is not sufficiently demonstrated. And it seems that he who has given the demonstration of the 10th proposition as we now have it, instead of that which Eudoxus or Euclid had given, has been deceived in applying what is manifest, when understood of magnitudes, unto ratios, viz. that a magnitude cannot be both greater and less than another. That those things which are equal to the same are equal to one another, is a most evident axiom when understood of magnitudes; yet Euclid does not make use of it to infer, that those ratios which are the same to the same ratio, are the same to one another; but explicitly demonstrates this in Prop. 11. of Book 5. The demonstration we have given of the 10th prop.

is no doubt the same with that of Eudoxus or Euclid, as it is immediately and directly derived from the definition of a greater ratio, viz. the 7 of the 5.

The above-mentioned proposition, viz. If A have to C a greater ratio than B to C; and if of A and B there be taken certain equimultiples, and some multiple of C; then if the multiple of B be greater than the multiple of C, the multiple of A is also greater than the same, is thus demonstrated:

Let D, E be equimultiples of A, B, and F a multiple of C, such, that E the multiple of B is greater than F; D the multiple of A is also greater than F.

Because A has a greater ratio to C, than B to C, A is greater than B, by the 10th Prop. B. 5, therefore D the multiple of A is greater than E the same multiple of B: And E is greater than F: much more therefore D is greater than F.



PROP. XIII. B. V.

IN Commandine's, Briggs's, and Gregory's translations, at the beginning of this demonstration, it is said, "And the multiple of C is greater than the multiple of D; but the multiple of E is not greater than the multiple of F:" Which words are a literal translation from the Greek: But the sense evidently requires that it be read, "so that the multiple of C be greater than the multiple of D; but the multiple of E be not greater than the multiple of F." And thus this place was restored to the true reading in the first editions of Commandine's Euclid, printed in 8vo. at Oxford: But in the later editions, at least in that of 1747, the error of the Greek text was kept in.

There is a corollary added to prop. 13, as it is necessary to the 20th and 21st prop. of this book, and is as useful as the proposition.

PROP. XIV. B. V.

THE two cases of this, which are not in the Greek, are added; the demonstration of them not being exactly the same with that of the first case.

NOTES.

PROP. XVII. B. V.

THE order of the words in a clause of this is changed to one more natural: As was also done in prop. II.

PROP. XVIII. B. V.

THE demonstration of this is none of Euclid's, nor is it legitimate; for it depends upon this hypothesis, that to any three magnitudes, two of which, at least, are of the same kind, there may be a fourth proportional: which, if not proved, the demonstration now in the text is of no force: But this is assumed without any proof; nor can it, as far as I am able to discern, be demonstrated by the propositions preceding this: so far is it from deserving to be reckoned an axiom, as Clavius, after other commentators, would have it; at the end of the definitions of the 5th book. Euclid does not demonstrate it, nor does he shew how to find the fourth proportional, before the 12th prop. of the 6th book: And he never assumes any thing in the demonstration of a proposition, which he had not before demonstrated; at least, he assumes nothing the existence of which is not evidently possible; for a certain conclusion can never be deduced by the means of an uncertain proposition: Upon this account, we have given a legitimate demonstration of this proposition instead of that in the Greek and other editions, which very probably Theon, at least some other, has put in the place of Euclid's, because he thought it too prolix: And as the 17th prop. of which this 18th is the converse, is demonstrated by help of the 1st and 2nd propositions of this book; so, in the demonstration now given of the 18th, the 5th prop. and both cases of the 6th are necessary, and these two propositions are the converses of the 1st and 2d. Now the 5th and 6th do not enter into the demonstration of any proposition in this book as we now have it: Nor can they be of use in any proposition of the Elements, except in this 18th, and this is a manifest proof, that Euclid made use of them in his demonstration of it, and that the demonstration now given, which is exactly the converse of that of the 17th, as it ought to be, differs nothing from that of Eudoxus or Euclid: For the 5th and 6th have undoubtedly been put into the 5th book for the sake of some propositions in it, as all the other propositions about equimultiples have been.

Hieronymus Saccherius, in his book named "Euclides ab
"omni nævo vindicatus," printed at Milan ann. 1733, in 4to,
acknowledges

acknowledges this blemish in the demonstration of the 18th, and that he may remove it, and render the demonstration we now have of it legitimate, he endeavours to demonstrate the following proposition, which is in page 115 of his book, viz.

"Let A, B, C, D be four magnitudes, of which the two first are of the one kind, and also the two others either of the same kind with the two first, or of some other the same kind with one another. I say the ratio of the third C to the fourth D, is either equal to, or greater, or less than the ratio of the first A to the second B."

And after two propositions premised as lemmas, he proceeds thus :

"Either among all the possible equimultiples of the first A, and of the third C, and, at the same time, among all the possible equimultiples of the second B, and of the fourth D, there can be found some one multiple EF of the first A, and one IK of the second B, that are equal to one another ; and also (in the same case) some one multiple GH of the third C equal to LM the multiple of the fourth D, or such equality is no where to be found. If the first case happen,

"[i. e. if such equality is to be found] it is manifest from what is before demonstrated, that A is to B

A ———	E ——— F
B ———	I ——— K
C ———	G ——— H
D ———	L ——— M

as C to D ; but if such simultaneous equality be not to be found upon both sides, it will be found either upon one side, as upon the side of A [and B] ; or it will be found upon neither side ; if the first happen : therefore (from Euclid's definition of greater and lesser ratio foregoing) A has to B a greater or less ratio than C to D ; according as GH the multiple of the third C is less, or greater than LM the multiple of the fourth D : But if the second case happen ; therefore upon the one side, as upon the side of A the first and B the second, it may happen that the multiple EF [viz. of the first] may be less than IK the multiple of the second, while, on the contrary, upon the other side [viz. of C and D], the multiple GH [of the third C] is greater than the other multiple LM [of the fourth D] : And then (from the same definition of Euclid) the ratio

"of

BOOK V.

"of the first A to the second B, is less than the ratio of the third C to the fourth D; or on the contrary.

"Therefore the axiom [i. e. the proposition before set down] remains demonstrated," &c.

Not in the least; but it still remains undemonstrated; For what he says may happen, may, in innumerable cases, never happen; and therefore his demonstration does not hold: For example, if A be the side, and B the diameter of a square; and C the side, and D the diameter of another square; there can in no case be any multiple of A equal to any of B; nor any one of C equal to one of D, as is well known; and yet it can never happen that when any multiple of A is greater than a multiple of B, the multiple of C can be less than the multiple of D, nor when the multiple of A is less than that of B, the multiple of C can be greater than that of D, viz. taking equimultiples of A and C, and equimultiples of B and D: For A, B, C, D are proportionals; and so if the multiple of A be greater, &c. than that of B, so must that of C be greater, &c. than that of D; by 5th Def. b. 5.

The same objection holds good against the demonstration which some give of the first prop. of the 6th book, which we have made against this of the 18th prop. because it depends upon the same insufficient foundation with the other.

PROP. XIX. B. V.

A COROLLARY is added to this, which is as frequently used as the proposition itself. The corollary which is subjoined to it in the Greek, plainly shews that the 5th book has been vitiated by editors who were not geometers: For the conversion of ratios does not depend upon this 19th, and the demonstration which several of the commentators on Euclid give of conversion is not legitimate, as Clavius has rightly observed, who has given a good demonstration of it which we have put in proposition E; but he makes it a corollary from the 19th, and begins it with the words, "Hence it easily follows," though it does not at all follow from it.

PROP. XX. XXI. XXII. XXIII. XXIV. B. V.

THE demonstration of the 20th and 21st propositions, are shorter than those Euclid gives of easier propositions, either in the preceding or following books: Wherefore it was proper to make them more explicit, and the 22d and 23d propositions are, as they ought to be, extended to any number of magnitudes:

magnitudes: And, in like manner, may the 24th be, as is taken notice of in the corollary; and another corollary is added, as useful as the proposition, and the words, "any whatever" are supplied near the end of prop. 23, which are wanting in the Greek text, and the translations from it.

In a paper writ by Philippus Naudæus, and published after his death, in the history of the Royal Academy of Sciences of Berlin, anno 1745, page 50, the 23d prop. of the 5th book is censured as being obscurely enunciated, and, because of this, prolixly demonstrated: The enunciation there given is not Euclid's, but Tacquet's, as he acknowledges, which, though not so well expressed, is, upon the matter, the same with that which is now in the Elements. Nor is there any thing obscure in it, though the author of the paper has set down the proportionals in a disadvantageous order, by which it appears to be obscure: But no doubt Euclid enunciated this 23d, as well as the 22d, so as to extend it to any number of magnitudes, which, taken two and two, are proportionals, and not of six only; and to this general case, the enunciation which Naudæus gives, cannot be well applied.

The demonstration which is given of this 23d, in that paper, is quite wrong; because, if the proportional magnitudes be plane or solid figures, there can no rectangle, (which he improperly calls a *product*) be conceived to be made by any two of them: And if it should be said, that in this case straight lines are to be taken which are proportional to the figures, the demonstration would this way become much longer than Euclid's: But, even though his demonstration had been right, who does not see that it could not be made use of in the 5th book?

PROP. F, G, H, K. B. V.

THESE propositions are annexed to the 5th book, because they are frequently made use of by both ancient and modern geometers: And in many cases, compound ratios cannot be brought into demonstration, without making use of them.

Whoever desires to see the doctrine of ratios delivered in this 5th book solidly defended, and the arguments brought against it by And. Tacquet, Alph. Borellus and others, fully refuted, may read Dr. Barrow's mathematical lectures, viz. the 7th and 8th of the year 1666.

The 5th book being thus corrected, I most readily agree to what the learned Dr. Barrow says,* "That there is nothing
" in

* Page 336.

“ in the whole body of the Elements of a more subtile invention, nothing more solidly established, and more accurately handled, than the doctrine of proportionals.” And there is some ground to hope, that geometers will think that this could not have been said with as good reason, since Theon’s time till the present.

DEF. II. and V. of B. VI.

Book VI.

THE 2d definition does not seem to be Euclid’s but some unskilful editor’s: For there is no mention made by Euclid, nor, as far as I know, by any other geometer, of reciprocal figures: It is obscurely expressed, which made it proper to render it more distinct: It would be better to put the following definition in place of it, viz.

DEF. II.

Two magnitudes are said to be reciprocally proportional to two others, when one of the first is to one of the other magnitudes, as the remaining one of the last two is to the remaining one of the first.

But the 5th definition, which, since Theon’s time has been kept in the elements, to the great detriment of learners, is now justly thrown out of them, for the reasons given in the notes on the 23d prop. of this book.

PROP. I. and II. B. VI.

To the first of these a corollary is added, which is often used: And the enunciation of the second is made more general.

PROP. III B. VI.

A SECOND case of this, as useful as the first, is given in prop. A; viz. the case in which the exterior angle of a triangle is bisected by a straight line: The demonstration of it is very like to that of the first case, and upon this account may, probably, have been left out, as also the enunciation, by some unskilful editor. At least, it is certain, that Pappus makes use of this case, as an elementary proposition, without a demonstration of it, in Prop. 39, of his 7th Book of Mathematical Collections.

PROP. VII. B. VI.

To this a case is added which occurs not unfrequently in demonstration.

PROP. VIII. B. VI.

It seems plain that some editor has changed the demonstration that Euclid gave of this proposition: For, after he has demonstrated, that the triangles are equiangular to one another, he particularly shews that their sides about the equal angles are proportionals, as if this had not been done in the demonstration of the 4th prop. of this book: This superfluous part is not found in the translation from the Arabic, and is now left out.

PROP. IX. B. VI.

THIS is demonstrated in a particular case, viz. that in which the third part of a straight line is required to be cut off; which is not at all like Euclid's manner: Besides, the author of the demonstration, from four magnitudes being proportionals, concludes that the third of them is the same multiple of the fourth, which the first is of the second; now, this is nowhere demonstrated in the 5th book, as we now have it: But the editor assumes it from the confused notion which the vulgar have of proportionals: On this account it was necessary to give a general and legitimate demonstration of this proposition.

PROP. XVIII. B. VI.

THE demonstration of this seems to be vitiated: For the proposition is demonstrated only in the case of quadrilateral figures, without mentioning how it may be extended to figures of five or more sides: Besides, from two triangles being equiangular, it is inferred, that a side of the one is to the homologous side of the other, as another side of the first is to the side homologous to it of the other, without permutation of the proportionals; which is contrary to Euclid's manner, as is clear from the next proposition: And the same fault occurs again in the conclusion, where the sides about the equal angles are not shewn to be proportionals, by reason of again neglecting permutation. On these accounts, a demonstration is given in Euclid's manner, like to that he makes use of in the 20th prop.

BOOK VI. prop. of this book; and it is extended to five-sided figures, by which it may be seen how to extend it to figures of any number of sides.

PROP. XXIII. B. VI.

NOTHING is usually reckoned more difficult in the elements of geometry by learners, than the doctrine of compound ratio, which Theon has rendered absurd and ungeometrical, by substituting the 5th definition of the 6th book in place of the right definition, which without doubt Eudoxus or Euclid gave, in its proper place, after the definition of triplicate ratio, &c. in the 5th book. Theon's definition is this; a ratio is said to be compounded of ratios *ὅταν αἱ τῶν γολῶν πηλικότητες ἐφ' αὐτάς πολλαπλασιασθεῖσαι ποιῶσι τινὰ*; Which Commandine thus translates: "Quando rationum quantitates inter se multiplicatæ aliquam efficient rationem;" that is, when the quantities of the ratios being multiplied by one another make a certain ratio. Dr. Wallis translates the word *πηλικότητες* "rationem exponentes," the exponents of the ratios: And Dr. Gregory renders the last words of the definition by "illius facit quantitatem," makes the quantity of that ratio: But in whatever sense the "quantities," or "exponents of the ratios," and their "multiplication," be taken, the definition will be ungeometrical and useless: For there can be no multiplication but by a number: Now the quantity or exponent of a ratio (according to Eutochius in his Comment, on Prop. 4. Book 2. of Arch. de Sph. et Cyl. and the moderns explain that term) is the number which multiplied into the consequent term of a ratio produces the antecedent, or, which is the same thing, the number which arises by dividing the antecedent by the consequent; but there are many ratios such, that no number can arise from the division of the antecedent by the consequent; ex. gr. the ratio of which the diameter of a square has to the side of it; and the ratio which the circumference of a circle has to its diameter, and such like. Besides, that there is not the least mention made of this definition in the writings of Euclid, Archimedes, Apollonius, or other ancients, though they frequently make use of compound ratio: And in this 23d prop. of the 6th book, where compound ratio is first mentioned, there is not one word which can relate to this definition, though here, if in any place, it was necessary to be brought in; but the right definition is expressly cited in these words: "But the ratio of K to M is compounded of the ratio of K to L, and

“and of the ratio of L to M.” This definition therefore of Theon is quite useless and absurd: For that Theon brought it into the Elements can scarce be doubted; as it is to be found in his commentary upon Ptolemy’s *Μεγάλη Συνταξις*, page 62. where he also gives a childish explication of it, as agreeing only to such ratios as can be expressed by numbers; and from this place the definition and explication have been exactly copied and prefixed to the definitions of the 6th book, as appears from Hervagius’s edition: But Zambertus and Commandine, in their Latin translations, subjoin the same to these definitions. Neither Campanus, nor, as it seems, the Arabic manuscripts, from which he made his translation, have this definition. Clavius, in his observations upon it, rightly judges, that the definition of compound ratios might have been made after the same manner in which the definitions of duplicate and triplicate ratio are given, viz. “That as in several magnitudes that are continual proportionals, Euclid named the ratio of the first to the third, the duplicate ratio of the first to the second; and the ratio of the first to the fourth, the triplicate ratio of the first to the second, that is, the ratio compounded of two or three intermediate ratios that are equal to one another, and so on; so, in like manner, if there be several magnitudes of the same kind, following one another, which are not continual proportionals, the first is said to have to the last the ratio compounded of all the intermediate ratios,—only for this reason, that these intermediate ratios are interposed betwixt the two extremes, viz. the first and last magnitudes; even as, in the 10th definition of the 5th book, the ratio of the first to the third was called the duplicate ratio, merely upon account of two ratios being interposed betwixt the extremes, that are equal to one another: so that there is no difference betwixt this compounding of ratios, and the duplication or triplication of them which are defined in the 5th book, but that in the duplication, triplication, &c. of ratios, all the interposed ratios are equal to one another; whereas, in the compounding of ratios, it is not necessary that the intermediate ratios should be equal to one another.” Also Mr. Edmund Scarburgh, in his English translation of the first six books, page 238, 266, expressly affirms, that the 5th definition of the 6th book is suppositious, and that the true definition of compound ratio is contained in the 10th definition of the 5th book, viz. the

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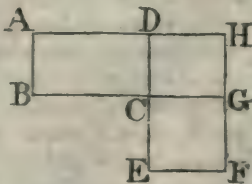
definition

BOOK VI. definition of duplicate ratio, or to be understood from it, to wit, in the same manner as Clavius has explained it in the preceding citation. Yet these, and the rest of the moderns, do notwithstanding retain this 5th def. of the 6th book, and illustrate and explain it by long commentaries, when they ought rather to have taken it quite away from the Elements.

For, by comparing def. 5, book 6, with prop. 5, book 8, it will clearly appear that this definition has been put into the Elements in place of the right one, which has been taken out of them: Because, in prop. 5, book 8, it is demonstrated, that the plane number of which the sides are C, D, has to the plane number of which the sides are E, Z (see Hervagius's or Gregory's edition), the ratio which is compounded of the ratios of their sides; that is, of the ratios of C to E, and D to Z; and by def. 5. book 6, and the explication given of it by all the commentators, the ratio which is compounded of the ratios of C to E, and D to Z, is the ratio of the product made by the multiplication of the antecedents C, D, to the product by the consequents E, Z, that is, the ratio of the plane number of which the sides are C, D, to the plane number of which the sides are E, Z. Wherefore the proposition which is the 5th def. of book 6, is the very same with the 5th prop. of book 8, and therefore it ought necessarily to be cancelled in one of these places; because it is absurd that the same proposition should stand as a definition in one place of the Elements, and be demonstrated in another place of them. Now, there is no doubt that prop. 5, book 8, should have a place in the Elements, as the same thing is demonstrated in it concerning plane numbers, which is demonstrated in prop. 23, book 6, of equiangular parallelograms; wherefore def. 5, book 6, ought not to be in the Elements. And from this it is evident that this definition is not Euclid's, but Theon's, or some other unskillful geometer's.

But nobody, as far as I know, has hitherto shewn the true use of compound ratio, or for what purpose it has been introduced into geometry; for every proposition in which compound ratio is made use of, may without it be both enunciated and demonstrated. Now the use of compound ratio consists wholly in this, that by means of it, circumlocutions may be avoided, and thereby propositions may be more briefly either enunciated or demonstrated, or both may be done; for instance, if this 23d proposition of the sixth book were to be enunciated, without mentioning compound ratio, it might be
done

done as follows : If two parallelograms be equiangular, and if as a side of the first to a side of the second, so any assumed straight line be made to a second straight line ; and as the other side of the first to the other side of the second, so the second straight line be made to a third. The first parallelogram is to the second, as the first straight line to the third. And the demonstration would be exactly the same as we now have it. But the ancient geometers, when they observed this enunciation could be made shorter, by giving a name to the ratio which the first straight line has to the last, by which name the intermediate ratios might likewise be signified, of the first to the second, and of the second to the third, and so on, if there were more of them, they called this ratios of the first to the last, the ratio compounded of the ratio of the first to the second, and of the second to the third straight line ; that is, in the present example, of the ratios which are the same with the ratios of the sides, and by this they expressed the proposition more briefly thus : If there be two equiangular parallelograms, they have to one another the ratio which is the same with that which is compounded of ratios that are the same with the ratios of the sides ; which is shorter than the preceding enunciation, but has precisely the same meaning. Or yet shorter thus : Equiangular parallelograms have to one another the ratio which is the same with that which is compounded of the ratios of their sides. And these two enunciations, the first especially, agree to the demonstration which is now in the Greek. The proposition may be more briefly demonstrated, as Candalla does, thus : Let ABCD, CEFG be two equiangular parallelograms, and complete the parallelogram CDHG ; then, because there are three parallelograms, AC, CH, CF, the first AC (by the definition of compound ratio) has to the third CF, the ratio which is compounded of the ratio of the first AC to the second CH, and of the ratio of CH to the third CF ; but the parallelogram AC is to the parallelogram CH, as the straight line BC to CG ; and the parallelogram CH is to CF, as the straight line CD is to CE ; therefore the parallelogram AC has to CF the ratio which is compounded of ratios that are the same with the ratios of the sides. And to this demonstration agrees the enunciation which is at present in the text, viz. equiangular



BOOK VI.

parallelograms have to one another the ratio which is compounded of the ratios of the sides: for the vulgar reading, "which is compounded of their sides," is absurd. But, in this edition, we have kept the demonstration which is in the Greek text, though not so short as Candalla's; because the way of finding the ratio which is compounded of the ratio of the sides, that is, of finding the ratio of the parallelograms, is shewn in that, but not in Candalla's demonstration; whereby beginners may learn, in like cases, how to find the ratio which is compounded of two or more given ratios.

From what has been said, it may be observed, that in any magnitudes whatever of the same kind A, B, C, D, &c. the ratio compounded of the ratios of the first to the second, of the second to the third, and so on to the last, is only a name or expression by which the ratio which the first A has to the last D is signified, and by which at the same time the ratios of all the magnitudes A to B, B to C, C to D, from the first to the last, to one another, whether they be the same, or be not the same, are indicated; as in magnitudes which are continual proportionals A, B, C, D, &c. the duplicate ratio of the first to the second is only a name, or expression by which the ratio of the first A to the third C is signified, and by which, at the same time, is shewn, that there are two ratios of the magnitudes from the first to the last, viz. of the first A to the second B, and of the second B to the third or last C, which are the same with one another; and the triplicate ratio of the first to the second is a name or expression by which the ratio of the first A to the fourth D is signified, and by which, at the same time, is shewn, that there are three ratios of the magnitudes from the first to the last, viz. of the first A to the second B, and of B to the third C, and of C to the fourth or last D, which are all the same with one another; and so in the case of any other multiply ratios. And that this is the right explication of the meaning of these ratios is plain from the definitions of duplicate and triplicate ratio, in which Euclid makes use of the word *ἀπὸ τῶν*, is said to be, or is called; which word he, no doubt, made use of also in the definition of compound ratio, which Theon, or some other, has expunged from the Elements; for the very same word is still retained in the wrong definition of compound ratio, which is now the 5th of the 6th book: But in the citation of these definitions it is sometimes retained, as in the demonstration of prop. 19, book

6, "the first is said to have, ἔχει λεγεται, to the third the duplicate ratio," &c. which is wrong translated by Commandine and others, "has," instead of "is said to have:" and sometimes it is left out, as in the demonstration of prop. 33, of the 11th book, in which we find, "the first has, ἔχει, to the third the triplicate ratio;" but without doubt ἔχει, "has," in this place, signifies the same as ἔχει λεγεται, is said to have: so likewise in Prop. 23, B. 6, we find this citation, "but the ratio of K to M is compounded, συγκεῖται of the ratio of K to L, and the ratio L to M," which is a shorter way of expressing the same thing, which, according to the definition, ought to have been expressed by συγκεῖσθαι λεγεται, is said to be compounded.

From these remarks, together with the propositions subjoined to the 5th book, all that is found concerning compound ratio, either in the ancient or modern geometers, may be understood and explained.

PROP. XXIV. B. VI.

IT seems that some unskilful editor has made up this demonstration as we now have it, out of two others; one of which may be made from the 2d prop. and the other from the 4th of this book. For after he has, from the 2d of this book, and composition and permutation, demonstrated, that the sides about the angle common to the two parallelograms are proportionals, he might have immediately concluded, that the sides about the other equal angles were proportionals, viz. from Prop. 34, B. I. and Prop. 7, B. 5. This he does not, but proceeds to shew, that the triangles and parallelograms are equiangular: and in a tedious way, by help of Prop. 4. of this book, and the 22d of book 5, deduces the same conclusion: From which it is plain, that this ill-composed demonstration is not Euclid's: These superfluous things are now left out, and a more simple demonstration is given from the 4th prop. of this book, the same which is in the translation from the Arabic, by help of the 2d prop. and composition; but in this the author neglects permutation, and does not shew the parallelograms to be equiangular, as is proper to do for the sake of beginners.

PROP. XXV. B. VI.

IT is very evident that the demonstration which Euclid had given of this proposition has been vitiated by some unskilful hand: For, after this editor had demonstrated, that, "as the rectilineal figure ABC is to the rectilineal KGH, so is the

BOOK VI. "parallelogram BE to the parallelogram EF;" nothing more should have been added but this, "and the rectilinear figure ABC is equal to the parallelogram BE; therefore the rectilinear KGH is equal to the parallelogram EF," viz. from prop. 14, book 5. But betwixt these two sentences he has inserted this; "wherefore, by permutation, as the rectilinear figure ABC to the parallelogram BE, so is the rectilinear KGH to the parallelogram EF; by which, it is plain, he thought it was not so evident to conclude, that the second of four proportions is equal to the fourth from the equality of the first and third, which is a thing demonstrated in the 14th prop. of B. 5, as to conclude that the third is equal to the fourth, from the equality of the first and second, which is no where demonstrated in the Elements as we now have them: But though this proposition, viz. the third of four proportionals is equal to the fourth, if the first be equal to the second, had been given in the Elements by Euclid, as very probably it was, yet he would not have made use of it in this place; because, as was said, the conclusion could have been immediately deduced without this superfluous step by permutation: This we have shewn at the greater length, both because it affords a certain proof of the vitiation of the text of Euclid; for the very same blunder is found twice in the Greek text of prop. 23, book 11, and twice in prop. 2, book 12, and in the 5, 11, 12, and 18th of that book; in which places of book 12, except the last of them, it is rightly left out in the Oxford edition of Commandine's translation; And also that geometers may beware of making use of permutation in the like cases: for the moderns not unfrequently commit this mistake, and among others Commandine himself in his commentary on prop. 5, book 3, p. 6, b. of Pappus Alexandrinus, and in other places: The vulgar notion of proportionals has, it seems, pre-occupied many so much, that they do not sufficiently understand the true nature of them.

Besides, though the rectilinear figure ABC, to which another is to be made similar, may be of any kind whatever; yet in the demonstration the Greek text has "triangle" instead of "rectilinear figure," which error is corrected in the above-named Oxford edition.

PROP. XXVII. B. VI.

THE second case of this has *ἀλλως*, otherwise, prefixed to it, as if it was a different demonstration, which probably has been done by some unskilful librarian. Dr. Gregory has
rightly

rightly left it out: The scheme of this second case ought to be marked with the same letters of the alphabet which are in the scheme of the first, as is now done.

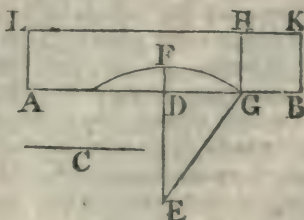
PROP. XXVIII, and XXIX. B. VI.

THESE two problems, to the first of which the 27th prop. is necessary, are the most general and useful of all in the Elements, and are most frequently made use of by the ancient geometers in the solution of other problems; and therefore are very ignorantly left out by Tacquet and Dechaes in their editions of the Elements, who pretend that they are scarce of any use: The cases of these problems, wherein it is required to apply a rectangle which shall be equal to a given square; to a given straight line, either deficient or exceeding by a square; are very often made use of by geometers: And, on this account, it is thought proper, for the sake of beginners, to give their constructions as follows:

1. To apply a rectangle which shall be equal to a given square, to a given straight line, deficient by a square: But the given square must not be greater than that upon the half of the given line.

Let AB be the given straight line, and let the square upon the given straight line C be that to which the rectangle to be applied must be equal, and this square by the determination is not greater than that upon half of the straight line AB.

Bisect AB in D, and if the square upon AD be equal to the square upon C, the thing required is done: But if it be not equal to it, AD must be greater than C, according to the determination: Draw DE at right angles to AB and make it equal to C; produce ED to F, so that EF be equal to AD or DB, and from the centre E, at the distance EF, describe a circle meeting AB in G, and upon GB describe the square GBKH, and complete the rectangle AGHL; also join EG: And because AB is bisected in D, the rectangle AG, GB together with the square of DG is equal^a to (the square of DB, that is, of EF or EG, that is



Book VI.

to) the squares of ED, DG: Take away the square of DG from each of these equals; therefore the remaining rectangle AG, GB is equal to the square of ED, that is, of C: But the rectangle AG, GP is the rectangle AH, because GH is equal to GB; therefore the rectangle AH is equal to the given square upon the straight line C. Wherefore the rectangle AH, equal to the given square upon C, has been applied to the given straight line AB, deficient by the square GK. Which was to be done.

2. To apply a rectangle which shall be equal to a given square, to a given straight line, exceeding by a square.

Let AB be the given straight line, and let the square upon the given straight line C be that to which the rectangle to be applied must be equal.

Bisect AB in D, and draw BE at right angles to it, so that BE be equal to C; and having joined DE, from the centre D at the distance DE describe a circle meeting AB produced in G; upon BG describe the square

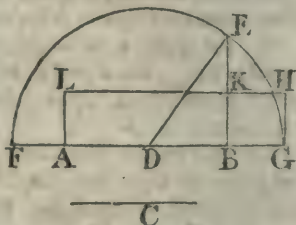
BGHL, and complete the rectangle AGHL. And because AB is bisected in D, and produced to G, the rectangle AG, GB together with the square of DB is equal^a to (the square of DG, or DE, that is, to) the squares

of EB, BD. From each of these

equals take the square of DB; therefore the remaining rectangle AG, GB is equal to the square of BE, that is, to the square upon C. But the rectangle AG, GB is the rectangle AH, because GH is equal to GB. Therefore the rectangle AH is equal to the square upon C. Wherefore the rectangle AH, equal to the given square upon C, has been applied to the given straight line AB, exceeding by the square GK. Which was to be done.

3. To apply a rectangle to a given straight line which shall be equal to a given rectangle, and be deficient by a square. But the given rectangle must not be greater than the square upon the half of the given straight line.

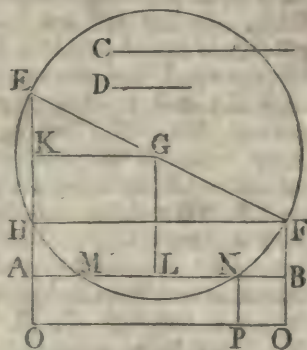
Let AB be the given straight line, and let the given rectangle be that which is contained by the straight lines C, D, which is not greater than the square upon the half of AB; it is required to apply to AB a rectangle equal to the rectangle C, D, deficient by a square.



Draw

Draw AE, BF at right angles to AB, upon the same side of ^{Book VI.} it, and make AE equal to C, and BF to D: join EF and bisect it in G; and from the centre G, at the distance GE, describe a circle meeting AE again in H: Join HF, and draw GK parallel to it, and GL parallel to AE, meeting AB in L.

Because the angle EHF in a semicircle is equal to the right angle EAB, AB and HF are parallels, and AH and BF are parallels; wherefore AH is equal to BF, and the rectangle EA, AH equal to the rectangle EA, BF, that is, to the rectangle C, D: And because EG, GF are equal to one another, and AE, LG, BF parallels; therefore AL and LB are equal, also EK is equal to KH and the rectangle C, D, from ^{3.3.} the determination, is not greater than the square of AL, the half of AB; wherefore the rectangle EA, AH is not greater than the square of AL, that is, of KG: Add to each the square of KE; therefore the square^b of AK is not greater than the ^{6.2.} squares of EK, KG, that is, than the square of EG; and consequently the straight line AK or GL is not greater than GE. Now, if GE be equal to GL, the circle EHF touches AB in L, and therefore the square of AL is ^c equal to the rectangle EA, AH, that is, to the given rectangle C, D; and that which was required is done: But if EG, GL be unequal, EG must be the greater: and



therefore the circle EHF cuts the straight line AB: let it cut it in the points M, N, and upon NB describe the square NBOP, and complete the rectangle ANPQ: Because LM is equal to ^d ^{3.3.} LN, and it has been proved that AL is equal to LB; therefore AM is equal to NB, and the rectangle AN, NB equal to the rectangle NA, AM, that is, to the rectangle ^e EA, AH, ^e ^{Cor. 36.3.} or the rectangle C, D: But the rectangle AN, NB is the rectangle AP, because PN is equal to NB: Therefore the rectangle AP is equal to the rectangle C, D; and the rectangle AP equal to the given rectangle C, D, has been applied to the given straight line AB, deficient by the square BP. Which was to be done.

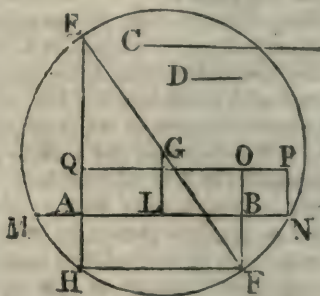
BOOK VI.

4. To apply a rectangle to a given straight line that shall be equal to a given rectangle, exceeding by a square.

Let AB be the given straight line, and the rectangle C, D the given rectangle, it is required to apply a rectangle to AB equal to C, D , exceeding by a square.

Draw AE, BF at right angles to AB , on the contrary sides of it, and make AE equal to C , and BF equal to D : Join EF , and bisect it in G ; and from the centre G , at the distance GE , describe a circle meeting AE again in H ; join HF , and draw GL parallel to AE ;

let the circle meet AB produced in M, N , and upon BN describe the square $BNOP$, and complete the rectangle $ANPQ$; because the angle EHF in a semicircle is equal to the right angle EAB , AB , and HF are parallels, and therefore AH and BF are equal, and the rectangle EA, AH equal to the rectangle EA, BF , that is,



to the rectangle C, D : And because ML is equal to LN , and AL to LB , therefore MA is equal to BN , and the rectangle AN, NB to MA, AN , that is, ^a to the rectangle EA, AH , or the rectangle C, D : Therefore the rectangle AN, NB , that is, AP , is equal to the rectangle C, D ; and to the given straight line AB the rectangle AP has been applied equal to the given rectangle C, D , exceeding by the square BP . Which was to be done.

Willebrordus Snellius was the first, as far as I know, who gave these constructions of the 3d and 4th Problems in his *Apollonius Batavus*: And afterwards the learned Dr. Halley gave them in the Scholium of the 18th Prop. of the 8th Book of *Apollonius's Conics* restored by him.

The 3d Problem is otherwise enunciated thus: To cut a given straight line AB in the point N , so as to make the rectangle AN, NB , equal to a given space: Or, which is the same thing, having given AB the sum of the sides of a rectangle, and the magnitude of it being likewise given, to find its sides.

And the 4th Problem is the same with this, To find a point N in the given straight line AB produced, so as to make the

BOOK VI.

PROP. XXXI. B. VI.

IN the demonstration of this, the inversion of proportionals is twice neglected, and is now added, that the conclusion may be legitimately made by help of the 24th Prop. of B. 5, as Clavius had done.

PROP. XXXII. B. VI.

THE enunciation of the preceding 26th Prop. is not general enough ; because not only two similar parallelograms that have an angle common to both, are about the same diameter ; but likewise two similar parallelograms that have vertically opposite angles, have their diameters in the same straight lines : But there seems to have been another, and that a direct demonstration of these cases, to which this 32d Proposition was needful ; And the 32d may be otherwise, and something more briefly demonstrated, as follows :

PROP. XXXII. B. VI.

If two triangles which have two sides of the one, &c.

Let GAF, HFC be two triangles which have two sides AG, GF, proportional to the two sides FH, HC, viz. AG to GF, as FH to HC; and let AG be parallel to FH, and GF to HC; AF and FC are in a straight line.

A rectangle ABCD is divided into four quadrants by a horizontal line EH and a vertical line GK. A diagonal line AC is drawn. The top-left quadrant is labeled AGFE, the top-right is GHD, the bottom-left is EBF, and the bottom-right is FHC. The area of the top-left quadrant is given as 31.1, and the area of the bottom-right quadrant is given as 30.1.

The 26th Prop. is demonstrated from the 32d, as follows:

If two similar and similarly placed parallelograms have an angle common to both, or vertically opposite angles; their diameters are in the same straight line.

First,

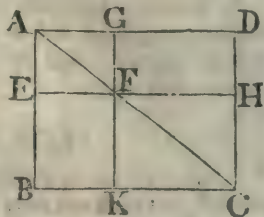
BOOK VI.

First, let the parallelograms ABCD, AEFG have the angle BAD common to both, and be similar, and similarly placed: ABCDG, AEFG are about the same diameter.

Produce EF, GF, to H, K, and join FA, FC; then because the parallelograms ABCD, AEFG are similar, DA is to AB,

* Cor. 19.5.

as GA to AE: Wherefore the remainder DG is^a to the remainder EB, as GA to AE: But DG is equal to FH, EB to HC, and AE to CF: Therefore as FH to HC, so is AG to GF; and FH, HC are parallel to AG, GF; and the triangles AGF, FHC are joined at one angle in the point F; wherefore AF, FC are in the same straight line^b.



* 32. 6.

Next, Let the parallelograms KFHC, GFEA which are similar and similarly placed, have their angles KFH, GFE vertically opposite; their diameters AF, FC are in the same straight line.

Because AG, GF are parallel to FH, HC; and that AG is to GF, as FH to HC; therefore AF, FC are in the same straight line^b.

PROP. XXXIII. B. VI.

The words "because they are at the centre," are left out as the addition of some unskilful hand.

In the Greek, as also in the Latin translation, the words *ἅ εἴ τι*, "any whatever," are left out in the demonstration of both parts of the proposition, and are now added as quite necessary; and, in the demonstration of the second part, where the triangle BGC, is proved to be equal to CGK, the illative particle *αρα*, in the Greek text ought to be omitted.

The second part of the proposition is an addition of Theon's, as he tells in his commentary on Ptolemy's *Μεγαλη Συνταξις*, p. 50.

PROP. B. C. D. B. VI.

THESE three propositions are added, because they are frequently made use of by geometers.

DEF.

DEF. IX and XI. B. XI.

THE similitude of plane figures is defined from the equality of their angles, and the proportionality of the sides about the equal angles; for from the proportionality of the sides only, or only from the equality of the angles, the similitude of the figures does not follow, except in the case when the figures are triangles: The similar position of the sides which contain the figures, to one another, depending partly upon each of these: And for the same reason, those are similar solid figures which have all their solid angles equal, each to each, and are contained by the same number of similar plane figures: For there are some solid figures contained by similar plane figures, of the same number, and even of the same magnitude, that are neither similar nor equal, as shall be demonstrated after the notes on the 10th definition: Upon this account it was necessary to amend the definition of similar solid figures, and so place the definition of a solid angle before it: And from this and the 10th definition, it is sufficiently plain how much the Elements have been spoiled by unskilful editors.

DEF. X. B. XI.

SINCE the meaning of the word "equal" is known and established before it comes to be used in this definition: therefore the proposition which is the 10th definition of this book, is a theorem, the truth or falsehood of which ought to be demonstrated, not assumed; so that Theon, or some other editor, has ignorantly turned a theorem, which ought to be demonstrated in this 10th definition: That figures are similar, ought to be proved from the definition of similar figures; that they are equal, ought to be demonstrated from the axiom, "Magnitudes that wholly coincide are equal to one another;" or from Prop. A. of Book 5, or the 9th Prop. or the 14th of the same Book, from one of which the equality of all kinds of figures must ultimately be deduced. In the preceding books, Euclid has given no definition of equal figures, and it is certain he did not give this: For what is called the first def. of the third book, is really a theorem in which these circles are said to be equal, that have the straight lines from the centres to the circumferences equal, which is plain, from the definition of a circle; and therefore has by
some

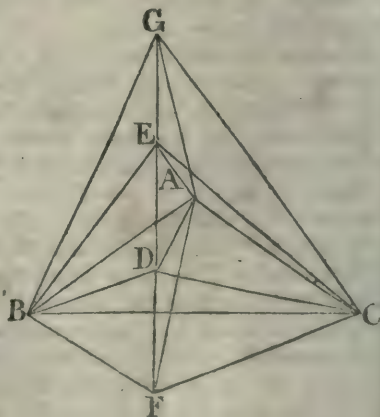
BOOK XI. some editor been improperly placed among the definitions. The equality of figures ought not to be defined, but demonstrated; Therefore, though it were true, that solid figures contained by the same number of similar and equal plane figures are equal to one another, yet he would justly deserve to be blamed who would make a definition of this proposition, which ought to be demonstrated. But if this proposition be not true, must it not be confessed, that geometers have, for these thirteen hundred years, been mistaken in this elementary matter? And this should teach us modesty, and to acknowledge how little, through the weakness of our minds, we are able to prevent mistakes, even in the principles of sciences which are justly reckoned amongst the most certain; for that the proposition is not universally true, can be shewn by many examples: The following is sufficient:

12. 11. Let there be any plane rectilineal figure, as the triangle ABC, and from a point D within it draw^a the straight line DE at right angles to the plane ABC; in DE take DE, DF equal to one another, upon the opposite sides of the plane, and let G be any point in EF; join DA, DB, DC; EA, EB, EC; FA, FB, FC; GA, GB, GC: because the straight line EDF is at right angles to the plane ABC, it makes right angles with DA, DB, DC which it meets in that plane; and in the triangles EDB, FDB, ED and DB are equal to FD and DB, each to each, and they contain right angles; therefore the base EB is equal^b

4. 1. to the base FB; in the same manner EA is equal to FA, and EC to FC: And in the triangles EBA, FBA, EB, BA, are equal to FB, BA, and the base EA is equal to the base FA; wherefore the angle EBA is equal^c to the angle FBA, and the triangle EBA equal^b to the triangle FBA, and the other angles equal to the other angles; therefore these triangles are

24. 6.
§ 1. def.

similar^d: In the same manner the triangle EBC is similar to the



the triangle FBC , and the triangle EAC to FAC ; therefore there are two solid figures, each of which is contained by six triangles, one of them by three triangles, the common vertex of which is the point G , and their bases the straight lines AB , BC , CA , and by three other triangles, the common vertex of which is the point E , and their bases the same lines AB , BC , CA : The other solid is contained by the same three triangles the common vertex of which is G , and their bases AB , BC , CA : and by three other triangles of which the common vertex is the point F , and their bases the same straight lines AB , BC , CA : Now the three triangles GAB , GBC , GCA are common to both solids, and the three others EAB , EBC , ECA of the first solid, have been shewn equal and similar to the three others FAB , FBC , FCA of the other solid, each to each: therefore these two solids are contained by the same number of equal and similar planes: But that they are not equal is manifest, because the first of them is contained in the other: Therefore it is not universally true that solids are equal which are contained by the same number of equal and similar planes.

COR. From this it appears that two unequal solid angles may be contained by the same number of equal plane angles.

For the solid angle at B , which is contained by the four plane angles EBA , EBC , GBA , GBC is not equal to the solid angle at the same point B which is contained by the four plane angles FBA , FBC , GBA , GBC ; for this last contains the other: And each of them is contained by four plane angles, which are equal to one another, each to each, or are the self same, as has been proved: And indeed there may be innumerable solid angles all unequal to one another which are each of them contained by plane angles that are equal to one another, each to each: It is likewise manifest, that the before-mentioned solids are not similar, since their solid angles are not all equal.

And that there may be innumerable solid angles all unequal to one another, which are each of them contained by the same plane angles disposed in the same order, will be plain from the three following propositions.

PROP. I. PROBLEM.

THREE magnitudes, A , B , C being given, to find a fourth such, that every three shall be greater than the remaining one.
Let

BOOK XI.

Let D be the fourth: therefore D must be less than A, B, C together: Of the three A, B, C, let A be that which is not less than either of the two B and C: And first, let B and C together be not less than A; therefore B, C, D together are greater than A: and because A is not less than B; A, C, D together are greater than B: In the like manner A, B, D together are greater than C; Wherefore in the case in which B and C together are not less than A, any magnitude D which is less than A, B, C together, will answer the problem.

But if B and C together be less than A; then, because it is required that B, C, D together be greater than A, from each of these taking away B, C, the remaining one D must be greater than the excess of A above B and C: Take therefore any magnitude D which is less than A, B, C together, but greater than the excess of A above B and C: Then, B, C, D together are greater than A; and because A is greater than either B or C, much more will A and D, together with either of the two B, C be greater than the other: And, by the construction, A, B, C are together greater than D.

COR. If besides it be required, that A and B together shall not be less than C and D together; the excess of A and B together above C must not be less than D, that is, D must not be greater than that excess.

PROP. II. PROBLEM.

FOUR magnitudes A, B, C, D being given, of which A and B together are not less than C and D together, and such that any three of them whatever are greater than the fourth; it is required to find a fifth magnitude E such, that any two of the three A, B, E shall be greater than the third, and also that any two of the three C, D, E shall be greater than the third. Let A be not less than B, and C not less than D.

First, Let the excess of C above D be not less than the excess of A above B: It is plain that a magnitude E can be taken which is less than the sum of C and D, but greater than the excess of C above D; let it be taken; then E is greater likewise than the excess of A above B; wherefore E and B together are greater than A; and A is not less than B; therefore A and E together are greater than B: And, by the hypothesis, A and B together are not less than C and D together, and C and D together are greater than E; therefore likewise A and B are greater than E.

But

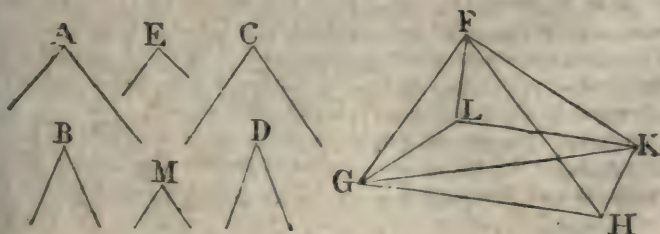
But let the excess of A above B be greater than the excess of C above D: And because, by the hypothesis, the three B, C, D are together greater than the fourth A; C and D together are greater than the excess of A above B: Therefore a magnitude may be taken which is less than C and D together, but greater than the excess of A above B. Let this magnitude be E: and because E is greater than the excess of A above B, B together with E is greater than A: And, as in the preceding case, it may be shewn that A together with E is greater than B, and that A together with B is greater than E: Therefore, in each of the cases, it has been shewn, that any two of the three A, B, E are greater than the third.

And because in each of the cases E is greater than the excess of C above D, E together with D is greater than C; and by the hypothesis, C is not less than D; therefore E together with C is greater than D; and, by the construction, C and D together are greater than E; Therefore any two of the three C, D, E are greater than the third.

PROP. III. THEOREM.

THERE may be innumerable solid angles all unequal to one another, each of which is contained by the same four plane angles, placed in the same order.

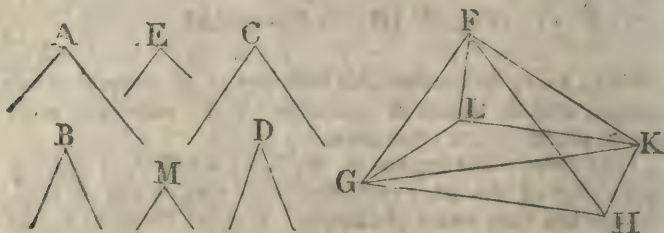
Take three plane angles, A, B, C, of which A is not less than either of the other two, and such, that A and B together are less than two right angles: and, by Problem 1, and its corollary, find a fourth angle D such, that any three whatever of the angles A, B, C, D be greater than the remaining angle, and such, that A and B together be not less than C and D together: And, by Problem 2, find a fifth angle E such, that any two of the angles A, B, E, be greater than the third,



and also that any two of the angles C, D, E be greater than the

Book XI.

the third: And because A and B together are less than two right angles, the double of A and B together is less than four right angles: But A and B together are greater than the angle E; wherefore the double of A, B together is greater than the three angles A, B, E together, which three are consequently less than four right angles; and every two of the same angles, A, B, E are greater than the third; therefore, by prop. 23, 11, a solid angle may be made contained by three plane angles, equal to the angles A, B, E, each to each. Let this be the angle F, contained by the three plane angles GFH, HFK, GFK, which are equal to the angles A, B, E, each to each: And because the angles C, D together are not greater than the angles A, B together, therefore the angles C, D, E are not greater than the angles A, B, E: But these last three are less than four right angles, as has been demonstrated: wherefore also the angles C, D, E are together less than four right angles, and every two of them are greater than the third; therefore a solid angle may be made, which shall be contained by three plane angles equal to the angles C, D, E, each to



23. 11.

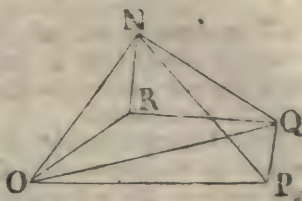
each: And, by prop. 26, 11, at the point F, in the straight line FG, a solid angle may be made equal to that which is contained by the three plane angles that are equal to the angles C, D, E: Let this be made, and let the angle GFK, which is equal to E, be one of the three; and let KFL, GFL be the other two which are equal to the angles C, D, each to each. Thus there is a solid angle constituted at the point F, contained by the four plane angles GFH, HFK, KFL, GFL, which are equal to the angles A, B, C, D, each to each.

Again, Find another angle M such, that every two of the three angles A, B, M be greater than the third, and also every two of the three C, D, M be greater than the third:

And,

And, as in the preceding part, it may be demonstrated, that Book XI.

the three A, B, M are less than four right angles, as also that the three C, D, M are less than four right angles. Make therefore^a a solid angle at N contained by the three plane angles ONP, PNQ, ONQ , which are equal to A, B, M , each to each: And by prop. 26, 11, make at the



^a 23. 11.

point N , in the straight line ON , a solid angle contained by three plane angles, of which one is the angle ONQ equal to M , and the other two are the angles QNR, ONR , which are equal to the angles C, D , each to each. Thus, at the point N , there is a solid angle contained by the four plane angles ONP, PNQ, QNR, ONR which are equal to the angles A, B, C, D , each to each. And that the two solid angles at the points P, N , each of which is contained by the above-named four plane angles, are not equal to one another, or that they cannot coincide, will be plain by considering that the angles GFK, ONQ : that is, the angles E, M , are unequal by the construction; and therefore the straight lines GF, FK cannot coincide with ON, NQ , nor consequently can the solid angles, which therefore are unequal.

And because from the four plane angles A, B, C, D , there can be found innumerable other angles that will serve the same purpose with the angles E and M : it is plain that innumerable other solid angles may be constituted which are each contained by the same four plane angles, and all of them unequal to one another. Q. E. D.

And from this it appears, that Clavius and other authors are mistaken, who assert that those solid angles are equal which are contained by the same number of plane angles that are equal to one another, each to each. Also it is plain, that the 26th prop. of Book 11, is by no means sufficiently demonstrated, because the equality of two solid angles, whereof each is contained by three plane angles which are equal to one another, each to each, is only assumed, and not demonstrated.

NOTES.

PROP. I. B. XI.

THE words at the end of this, "for a straight line cannot meet a straight line in more than one point," are left out, as an addition by some unskilful hand; for this is to be demonstrated, not assumed.

Mr. Thomas Simpson, in his notes at the end of the second edition of his *Elements of Geometry*, p. 262, after repeating the words of this note, adds "Now, can it possibly shew any want of skill in an editor (he means Euclid or Theon) to refer to an axiom which Euclid himself hath laid down, Book I, "N^o 14," he means Barrow's Euclid, for it is the 10th in the Greek, "and not to have demonstrated, what no man can demonstrate?" But all that in this case can follow from that axiom is, that, if two straight lines could meet each other in two points, the parts of them betwixt these points must coincide, and so they would have a segment betwixt these points common to both. Now, as it has not been shewn in Euclid, that they cannot have a common segment, this does not prove that they cannot meet in two points, from which their not having a common segment is deduced in the Greek edition: But, on the contrary, because they cannot have a common segment, as is shewn in Cor. of 11th Prop. Book I, of 4to edition, it follows plainly, that they cannot meet in two points, which the remarker says no man can demonstrate.

Mr. Simpson, in the same notes, p. 265, justly observes, that in the corollary of Prop. 11, Book I, 4to edit. the straight lines AB, BD, BC, are supposed to be all in the same plane, which cannot be assumed in 1st Prop. Book 11. This, soon after the 4to edition was published, I observed and corrected as it is now in this edition: He is mistaken in thinking the 10th axiom he mentions here to be Euclid's; it is none of Euclid's, but is the 10th in Dr. Barrow's edition, who had it from Herigon's *Cursus*, vol. I. and in place of it the corollary of 11th Prop. Book 1, was added.

PROP. II. B. XI.

THIS proposition seems to have been changed and vitiated by some editor; for all the figures defined in the 1st Book of the *Elements*, and among them triangles, are, by the hypothesis, plane figures; that is, such as are described in a plane; wherefore the second part of the enunciation needs no demonstration. Besides, a convex superficies may be terminated by
three

three straight lines meeting one another: The thing that should have been demonstrated is, that two or three straight lines, that meet one another, are in one plane. And as this is not sufficiently done, the enunciation and demonstration are changed into those now put into the text. Book XI.

PROP. III. B. XI.

IN this proposition the following words near to the end of it are left out, viz. "therefore DEB, DFB are not straight lines; "in the like manner it may be demonstrated, that there can be "no other straight line between the points D, B:" Because from this, that two lines include a space, it only follows that one of them is not a straight line: And the force of the argument lies in this, viz. if the common section of the planes be not a straight line, then two straight lines could include a space, which is absurd; therefore the common section is a straight line.

PROP. IV. B. XI.

THE words "and the triangle AED to the triangle BEC" are omitted, because the whole conclusion of the 4th Prop. B. I, has been so often repeated in the preceding books, it was needless to repeat it here.

PROP. V. B. XI.

IN this, near to the end, *επιπλάτω* ought to be left out in the Greek text: And the word "plane" is rightly left out in the Oxford edition of Commandine's translation.

PROP. VII. B. XI.

THIS proposition has been put into this book by some unskilful editor, as is evident from this, that straight lines which are drawn from one point to another in a plane, are, in the preceding books, supposed to be in that plane: And if they were not, some demonstrations in which one straight line is supposed to meet another would not be conclusive, because these lines would not meet one another: For instance, in Prop. 30, B. I, the straight line GK would not meet EF, if GK were not in the plane in which are the parallels AB, CD, and in which, by hypothesis, the straight line EF is: Besides, this 7th Proposition is demonstrated by the preceding 3d; in which the very thing which is proposed to be demonstrated in the 7th is twice assumed, viz. that the straight line drawn from one point to another in a plane, is in that plane; and the same thing is assumed in the preceding 6th Prop. in which the straight line

Book XI.

which joins the points B, D that are in the plane to which AB and CD are at right angles, is supposed to be in that plane: And the 7th, of which another demonstration is given, is kept in the book merely to preserve the number of the propositions; for it is evident, from the 7th and 35th definitions of the 1st book, though it had not been in the Elements.

PROP. VIII. B. XI.

IN the Greek, and in Commandine's and Dr. Gregory's translations, near to the end of this Proposition, are the following words: "But DC is in the plane through BA, AD," instead of which, in the Oxford edition of Commandine's translation is rightly put, "but DC is in the plane through BD, DA." But all the editions have the following words, viz. "because A, B, BD are in the plane through BD, DA, and "DC is in the plane in which are AB, BD," which are manifestly corrupted, or have been added to the text; for there was not the least necessity to go so far about to shew that DC is in the same plane in which are BD, DA, because it immediately follows, from Prop. 7, preceding, that BD, DA, are in the plane in which are the parallels AB, CD: Therefore, instead of these words, there ought only to be, "because all "three are in the plane in which are the parallels AB, CD."

PROP. XV. B. XI.

AFTER the words, "and because BA is parallel to GH," the following are added, "for each of them is parallel to DE, "and are not both in the same plane with it," as being manifestly forgotten to be put into the text.

PROP. XVI. B. XI.

IN this, near to the end, instead of the words, "but straight "lines which meet neither way," ought to be read, "but "straight lines in the same plane which produced meet neither "way." Because, though in citing this definition in Prop. 27, Book I, it was not necessary to mention the words, "in the "same plane," all the straight lines in the books preceding this being in the same plane; yet here it was quite necessary.

PROP. XX. B. XI.

IN this, near the beginning, are the words, "But if not, let "BAC be the greater:" But the angle BAC may happen to be equal to one of the other two: Wherefore this place should be

be read thus, "But if not, let the angle BAC be not less than either of the other two, but greater than DAB."

At the end of this proposition it is said, "in the same manner it may be demonstrated," though there is no need of any demonstration; because the angle BAC being not less than either of the other two, it is evident that BAC together with one of them is greater than the other.

PROP. XXII. B. XI.

AND likewise in this, near the beginning, it is said, "But if not, let the angles at B, E, H be unequal, and let the angle at B be greater than either of those at E, H:" Which words manifestly shew this place to be vitiated, because the angle at B may be equal to one of the other two. They ought therefore to be read thus, "But if not, let the angles at B, E, H be unequal, and let the angle at B be not less than either of the other two at E, H: Therefore the straight line AC is not less than either of the two DF, GK."

PROP. XXIII. B. XI.

The demonstration of this is made something shorter, by not repeating in the third case the things which were demonstrated in the first; and by making use of the construction which Campanus has given; but he does not demonstrate the second and third cases: The construction and demonstration of the third case are made a little more simple than in the Greek text.

PROP. XXIV. B. XI.

The word "similar" is added to the enunciation of this proposition, because the planes containing the solids which are to be demonstrated to be equal to one another, in the 25th proposition, ought to be similar and equal, that the equality of the solids may be inferred from Prop. C. of this Book: And in the Oxford edition of Commandine's translation, a corollary is added to Prop. 24, to shew that the parallelograms mentioned in this proposition are similar, that the equality of the solids in Prop. 25, may be deduced from the 10th def. of Book II.

PROP. XXV and XXVI. B. XI.

IN the 25th Prop. solid figures which are contained by the same number of similar and equal plane figures, are supposed to be equal to one another. And it seems that Theon, or some

Book XI. other editor, that he might save himself the trouble of demonstrating the solid figures mentioned in this proposition to be equal to one another, has inserted the 10th def. of this Book, to serve instead of a demonstration: which was very ignorantly done.

Likewise in the 26th Prop. two solid angles are supposed to be equal: If each of them be contained by three plane angles which are equal to one another, each to each. And it is strange enough, that none of the commentators on Euclid have, as far as I know, perceived, that something is wanting in the demonstrations of these two propositions. Clavius, indeed, in a note upon the 11th def. of this Book, affirms, that it is evident that those solid angles are equal which are contained by the same number of plane angles, equal to one another, each to each, because they will coincide, if they be conceived to be placed within one another; but this is said without any proof, nor is it always true, except when the solid angles are contained by three plane angles only, which are equal to one another, each to each: And in this case the proposition is the same with this, that two spherical triangles that are equilateral to one another, are also equiangular to one another, and can coincide: which ought not to be granted without a demonstration. Euclid does not assume this in the case of rectilineal triangles, but demonstrates in Prop. 8, Book 1, that triangles which are equilateral to one another, are also equiangular to one another; and from this their total equality appears by Prop. 4. Book 1. And Menelaus, in the 4th Prop. of his first Book of Spherics, explicitly demonstrates, that spherical triangles which are mutually equilateral, are also equiangular to one another; from which it is easy to shew that they must coincide, providing they have their sides disposed in the same order and situation.

To supply these defects, it was necessary to add the three Propositions marked A, B, C to this Book. For the 25th, 26th, and 28th Propositions of it, and consequently eight others, viz. the 27th, 31st, 32d, 33d, 34th, 36th, 37th, and 40th of the same, which depend upon them, have hitherto stood upon an infirm foundation; as also, the 8th, 12th Cor. of 17th and 18th of 12th Book, which depend upon the 9th definition. For it has been shewn in the notes on def. 10th of this book, that solid figures which are contained by the same number of similar and equal plane figures, as also solid angles that are contained by the same number of equal plane angles, are not always equal to one another.

It

It is to be observed that Tacquet, in his Euclid, defines equal solid angles to be such, "as being put within one another do coincide;" but this is an axiom, not a definition; for it is true of all magnitudes whatever. He made this useless definition, that by it he might demonstrate the 36th Prop. of this Book, without the help of the 35th of the same: Concerning which demonstration, see the note upon Prop. 36.

PROP. XXVIII. B. XI.

IN this it ought to have been demonstrated, not assumed, that the diagonals are in one plane. Clavius has supplied this defect.

PROP. XXIX. B. XI.

THERE are three cases of this proposition; the first is, when the two parallelograms opposite to the base AB have a side common to both; the second is, when these parallelograms are separated from one another; and the third, when there is a part of them common to both; and to this last only, the demonstration that has hitherto been in the Elements does agree. The first case is immediately deduced from the preceding 28th Prop. which seems for this purpose to have been premised to this 29th, for it is necessary to none but to it, and to the 40th of this book, as we now have it, to which last, it would, without doubt, have been premised, if Euclid had not made use of it in the 29th; but some unskilful editor has taken it away from the Elements, and has mutilated Euclid's demonstration of the other two cases, which is now restored, and serves for both at once.

PROP. XXX. B. XI.

IN the demonstration of this, the opposite planes of the solid CP, in the figure in this edition, that is, of the solid CO in Commandine's figure, are not proved to be parallel; which it is proper to do for the sake of learners.

PROP. XXXI. B. XI.

THERE are two cases of this proposition; the first is, when the insisting straight lines are at right angles to the bases; the other, when they are not; the first case is divided again into two others, one of which is, when the bases are equiangular parallelograms; the other, when they are not equiangular:

The

Book, XI.

The Greek editor makes no mention of the first of these two last cases, but has inserted the demonstration of it as a part of that of the other : And therefore should have taken notice of it in a corollary ; but we thought it better to give these two cases separately : The demonstration also is made something shorter by following the way Euclid has made use of in Prop. 14, Book 6. Besides, in the demonstration of the case in which the insisting straight lines are not at right angles to the bases, the editor does not prove that the solids described in the construction, are parallelopipeds, which it is not to be thought that Euclid neglected : also the words, “ of which the insisting straight lines are not in the same straight lines,” have been added by some unskilful hand ; for they may be in the same straight lines.

PROP. XXXII. B. XI.

THE editor has forgot to order the parallelogram FH to be applied in the angle FGH equal to the angle LCG, which is necessary. Clavius has supplied this.

Also, in the construction, it is required to complete the solid of which the base is FH, and altitude the same with that of the solid CD : But this does not determine the solid to be completed, since there may be innumerable solids upon the same base, and of the same altitude : It ought therefore to be said, “ complete the solid of which the base is FH, and one of its insisting straight lines is FD :” The same correction must be made in the following Proposition 33.

PROP. D. B. XI.

IT is very probable that Euclid gave this proposition a place in the Elements, since he gave the like proposition concerning equiangular parallelograms in the 23d B. 6.

PROP. XXXIV. B. XI.

IN this the words, ὧν αἱ ἐφεστῶσαι ἐκείνη ἐπὶ τῶν αὐτῶν εὐθειῶν. “ of which the insisting straight lines are not in the same straight lines,” are thrice repeated : but these words ought either to be left out, as they are by Clavius, or, in place of them, ought to be put, “ whether the insisting straight lines be, “ or be not in the same straight lines :” For the other case is, without any reason, excluded ; also the words ὧν καὶ ὁ ψ, of “ which

“which the altitudes,” are twice put for $\epsilon\iota\ \alpha\iota\ \epsilon\pi\iota\sigma\tau\omega\sigma\iota$, “of Book XI.
“which the insisting straight lines;” which is a plain mistake: ~~~~~
For the altitude is always at right angles to the base.

PROP. XXXV. B. XI.

THE angles ABH, DEM are demonstrated to be right angles in a shorter way than in the Greek; and in the same way ACH, DFM may be demonstrated to be right angles: Also the repetition of the same demonstration, which begins with “in the same manner,” is left out, as it was probably added to the text by some editor: for the words “in like manner “we may demonstrate,” are not inserted except when the demonstration is not given, or when it is something different from the other if it be given, as in Prop. 26, of this Book. Campanus has not this repetition.

We have given another demonstration of the corollary, besides the one in the original, by help of which the following 36th Prop. may be demonstrated without the 35th.

PROP. XXXVI. B. XI.

TACQUET in his Euclid demonstrates this proposition without the help of the 35th; but it is plain, that the solids mentioned in the Greek text in the enunciation of the proposition as equiangular, are such that their solid angles are contained by three plane angles equal to one another, each to each; as is evident from the construction. Now Tacquet does not demonstrate, but assumes these solid angles to be equal to one another; for he supposes the solids to be already made, and does not give the construction by which they are made: But, by the second demonstration of the preceding corollary, his demonstration is rendered legitimate likewise in the case where the solids are constructed as in the text.

PROP. XXXVII. B. XI.

IN this it is assumed, that the ratios which are triplicate of those ratios which are the same with one another, are likewise the same with one another; and that those ratios are the same with one another, of which the triplicate ratios are the same with one another; but this ought not to be granted without a demonstration; nor did Euclid assume the first and easiest of these two propositions, but demonstrated it in the case of duplicate ratios, in the 22nd Prop. Book 6. On this account, another demonstration is given of this Proposition like to that which Euclid gives in Prop. 22, Book 6, as Clavius has done.

Book XI.

PROP. XXXVIII. B. XI.

* 17. 12.
in other
editions.

WHEN it is required to draw a perpendicular from a point in one plane which is at right angles to another plane, unto this last plane, it is done by drawing a perpendicular from the point to the common section of the planes; for this perpendicular will be perpendicular to the plane, by def. 4, of this Book: And it would be foolish in this case to do it by the 11th Prop. of the same: But Euclid^a, Apollonius, and other geometers, when they have occasion for this problem, direct a perpendicular to be drawn from the point to the plane, and conclude that it will fall upon the common section of the planes, because this is the very same thing as if they had made use of the construction above mentioned, and then concluded that the straight line must be perpendicular to the plane: but is expressed in fewer words: Some editor, not perceiving this, thought it was necessary to add this proposition, which can never be of any use to the 11th book, and its being near to the end among propositions with which it has no connection, is a mark of its having been added to the text.

PROP. XXXIX. B. XI.

IN this it is supposed, that the straight lines which bisect the sides of the opposite planes, are in one plane, which ought to have been demonstrated; as is now done.

B. XII.

Book XII.

THE learned Mr. Moore, Professor of Greek in the University of Glasgow, observed to me, that it plainly appears from Archimedes's Epistle to Dositheus, prefixed to his books of the Sphere and Cylinder, which epistle he has restored from ancient manuscripts, that Eudoxus was the author of the chief propositions in this 12th book.

PROP. II. B. XII.

At the beginning of this it is said, "if it be not so, the square of BD shall be to the square of FH, as the circle ABCD is to some space either less than the circle EFGH, or greater than it." And the like is to be found near to the end of this proposition, as also in Prop. 5, 11, 12, 18, of this Book: Con-

cerning which it is to be observed, that in the demonstration Book XII.
of theorems, it is sufficient, in this and the like cases, that a thing made use of in the reasoning can possibly exist, providing this be evident, though it cannot be exhibited or found by a geometrical construction: So, in this place, it is assumed, that there may be a fourth proportional to these three magnitudes, viz. the squares of BD, FH, and the circle ABCD; because it is evident that there is some square equal to the circle ABCD though it cannot be found geometrically; and to the three rectilineal figures, viz. the squares of BD, FH, and the square which is equal to the circle ABCD, there is a fourth square proportional; because to the three straight lines which are their sides, there is a fourth straight line proportional^a, • 12. 6. and this fourth square, or a space equal to it, is the space which in this proposition is denoted by the letter S: And the like is to be understood in the other places above cited: And it is probable that this has been shewn by Euclid, but left out by some editor; for the lemma which some unskilful hand has added to this proposition explains nothing of it.

PROP. III. B. XII.

IN the Greek text and the translations, it is said, “and because the two straight lines BA, AC which meet one another,” &c. here the angles BAC, KHL, are demonstrated to be equal to one another, by 10th Prop. B. 11, which had been done before: Because the triangle EAG was proved to be similar to the triangle KHL: This repetition is left out, and the triangles BAC, KHL are proved to be similar in a shorter way by Prop. 21. B. 6.

PROP. IV. B. XII.

A FEW things in this are more fully explained, than in the Greek text.

PROP. V. B. XII.

IN this, near to the end, are the words, *ὡς ἔμπροσθεν ἐδείχθη*, “as was before shewn;” and the same are found again in the end of Prop. 18, of this Book; but the demonstration referred to, except it be in the useless lemma annexed to the 2d Prop. is nowhere in these Elements, and has been perhaps left out by some editor, who has forgot to cancel those words also.

PROP. VI. B. XII.

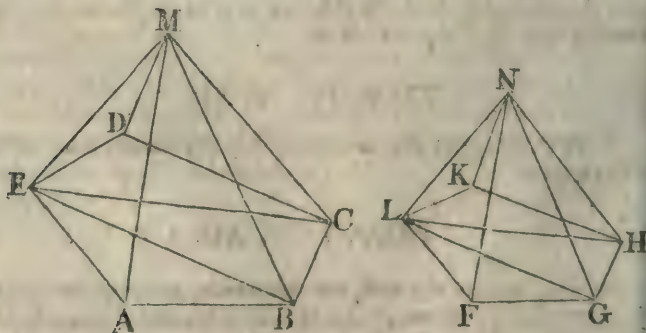
A SHORTER demonstration is given of this; and that which is in the Greek text may be made shorter by a step than it is: For the author of it makes use of the 22d Prop. of B. 5, twice: Whereas once would have served his purpose; because that proposition extends to any number of magnitudes which are proportionals taken two and two, as well as to three which are proportional to other three.

COR. PROP. VIII. B. XII.

THE demonstration of this is imperfect, because it is not shewn, that the triangular pyramids into which those upon multangular bases are divided, are similar to one another, as ought necessarily to have been done, and is done in the like case in Prop. 12 of this Book: The full demonstration of the corollary is as follows:

Upon the polygonal bases $ABCDE$, $FGHKL$, let there be similar and similarly situated pyramids which have the points M , N for their vertices: The pyramid $ABCDEM$ has to the pyramid $FGHKLN$ the triplicate ratio of that which the side AB has to the homologous side FG .

Let the polygons be divided into the triangles ABE , EBC , ECD ; FGL , LGH , LHK , which are similar^a, each to each: And because the pyramids are similar, therefore^b the triangle EAM is similar to the triangle LFN , and the triangle ABM to FGN : Wherefore^c ME is to EA , as NL to LF ; and as



AE to EB , so is FL to LG , because the triangles EAB , LFG are similar; therefore, ex æquali, as ME to EB , so is NL to LG .

LG: In like manner it may be shewn, that EB is to BM, as ^{Book XII.} LG to GN; therefore, again, ex æquali, as EM to MB, so is LN to NG: Wherefore the triangles EMB, LNG having their sides proportionals, are^d equiangular, and similar to one another^{d 5. 6.} Therefore the pyramids which have the triangles EAB, LFG for their bases, and the points M, N for their vertices, are similar^b to one another, for their solid angles are^c, ^{b 11. def. 11.} equal, and the solids themselves are contained by the same number of similar planes: In the same manner the pyramid EBCM may be shewn to be similar to the pyramid LGHN, and the pyramid ECDM to LHKM: And because the pyramids EABM, LFGN are similar, and have triangular bases, the pyramid EABM has^f to LFGN the triplicate ratio of that which EB has to the homologous side LG. And, in the same manner, the pyramid EBCM has to the pyramid LGHN the triplicate ratio of that which EB has to LG: Therefore as the pyramid EABM is to the pyramid LFGN, so is the pyramid EBCM to the pyramid LGHN: In like manner, as the pyramid EBCM is to LGHN, so is the pyramid ECDM to the pyramid LHKM: And as one of the antecedents is to one of the consequents, so are all the antecedents to all the consequents: Therefore as the pyramid EABM to the pyramid LFGN, so is the whole pyramid ABCDEM to the whole pyramid FGHLKN: And the pyramid EABM has to the pyramid LFGN the triplicate ratio of that which AB has to FG; therefore the whole pyramid has to the whole pyramid the triplicate ratio of that which AB has to the homologous side FG. Q. E. D.

PROP. XI and XII. B. XII.

THE order of the letters of the alphabet is not observed in these two propositions, according to Euclid's manner, and is now restored: By which means, the first part of prop. 12, may be demonstrated in the same words with the first part of Prop. 11.; on this account the demonstration of that first part is left out and assumed from Prop. 11.

PROP. XIII. B. XII.

In this proposition, the common section of a plane parallel to the bases of a cylinder, with the cylinder itself, is supposed to be a circle, and it was thought proper briefly to demonstrate it; from whence it is sufficiently manifest, that this plane divides the cylinder into two others: And the same thing is understood to be supplied in Prop. 14.

PROP. XV. B. XII.

“AND complete the cylinders AX, EO,” both the enunciation and exposition of the proposition represent the cylinders as well as the cones, as already described: Wherefore the reading ought rather to be, “and let the cones be ALC, “ENG; and the cylinders AX, EO.”

The first case in the second part of the demonstration is wanting; and something also in the second case of that part, before the repetition of the construction is mentioned; which are now added.

PROP. XVII. B. XII.

IN the enunciation of this proposition, the Greek words *εις την μειζονα σφαηραν στερεον πολυεδρον ιγχαψαι με ψαυον της ελασσονος σφαηρας κατα την επιφανειαν* are thus translated by Commandine and others, “in majori solidum polyhedrum describere quod minoris sphaeræ superficiem non tangat;” that is, “to describe in the greater sphere a solid polyhedron which shall not meet the superficies of the lesser sphere:” Whereby they refer the words *κατα την επιφανειαν* to these next to them *της ελασσονος σφαηρας*: But they ought by no means to be thus translated; for the solid polyhedron doth not only meet the superficies of the lesser sphere, but pervades the whole of that sphere: Therefore the aforesaid words are to be referred to *το στερεον πολυεδρον*, and ought thus to be translated, viz. to describe in the greater sphere a solid polyhedron whose superficies shall not meet the lesser sphere; as the meaning of the proposition necessarily requires.

The demonstration of the proposition is spoiled and mutilated: For some easy things are very explicitly demonstrated, while others not so obvious are not sufficiently explained; for example, when it is affirmed, that the square of KB is greater than the double of the square of BZ, in the first demonstration; and that the angle BZK is obtuse, in the second: Both which ought to have been demonstrated: Besides, in the first demonstration, it is said, “draw KΩ from the point K, perpendicular to BD;” whereas it ought to have been said, “join “KV,” and it should have been demonstrated, that KV is perpendicular to BD: For it is evident from the figure in Her-vagius’s and Gregory’s editions, and from the words of the demon-

demonstration, that the Greek editor did not perceive that the perpendicular drawn from the point K to the straight line BD must necessarily fall upon the point V, for in the figure it is made to fall upon the point Ω , a different point from V, which is likewise supposed in the demonstration. Commandine seems to have been aware of this; for in this figure he marks one and the same point with two letters V, Ω ; and before Commandine, the learned John Dee, in the commentary he annexes to this proposition in Henry Billingsley's translation of the Elements, printed at London, ann. 1570, expressly takes notice of this error, and gives a demonstration suited to the construction in the Greek text, by which he shews that the perpendicular drawn from the point K to BD, must necessarily fall upon the point V.

Likewise it is not demonstrated, that the quadrilateral figures SOPT, TPRY, and the triangle YRX, do not meet the lesser sphere, as was necessary to have been done: only Clavius, as far as I know, has observed this, and demonstrated it by a lemma, which is now premised to this proposition, something altered, and more briefly demonstrated.

In a corollary of this proposition, it is supposed that a solid polyhedron is described in the other sphere similar to that which is described in the sphere BCDE; but, as the construction by which this may be done is not given, it was thought proper to give it, and to demonstrate, that the pyramids in it are similar to those of the same order in the solid polyhedron described in the sphere BCDE.

FROM the preceding notes, it is sufficiently evident how much the Elements of Euclid, who was a most accurate geometer, have been vitiated and mutilated by ignorant editors. The opinion which the greatest part of learned men have entertained concerning the present Greek edition, viz. that it is very little or nothing different from the genuine work of Euclid, has without doubt deceived them, and made them less attentive and accurate in examining that edition; whereby several errors, some of them gross enough, have escaped their notice from the age in which Theon lived to this time. Upon which account there is some ground to hope that the pains we have taken in correcting those errors, and freeing the Elements as far as we could from blemishes, will not be unacceptable to good judges, who can discern when demonstrations are legitimate, and when they are not.

Book XII.

The objections which, since the first edition, have been made against some things in the notes, especially against the doctrine of proportionals, have either been fully answered in Dr. Barrow's *Lect. Mathemat.* and in these notes; or are such, except one which has been taken notice of in the note on Prop. 1. Book II, as shew that the person who made them has not sufficiently considered the things against which they are brought; so that it is not necessary to make any further answer to these objections and others like them against Euclid's definition of proportionals, of which definition Dr. Barrow justly says in page 297 of the above named book, that "*Nisi machinis impulsa validioribus æternum persistet inconcussa.*"

END OF THE NOTES.

NOTES ON EUCLID'S ELEMENTS.

THE article *Eucleides* in Dr Smith's *Dictionary of Greek and Roman Biography* was written by Professor De Morgan; it contains an account of the works of Euclid, and of the various editions of them which have been published. To that article we refer the student who desires full information on these subjects. Perhaps the only work of importance relating to Euclid which has been published since the date of that article is a work on the *Porisms of Euclid* by Chasles; Paris, 1860.

Euclid appears to have lived in the time of the first Ptolemy, B.C. 323—283, and to have been the founder of the Alexandrian mathematical school. The work on Geometry known as *The Elements of Euclid* consists of thirteen books; two other books have sometimes been added, of which it is supposed that Hypsicles was the author. Besides the *Elements*, Euclid was the author of other works, some of which have been preserved and some lost.

We will now mention the three editions which are the most valuable for those who wish to read the *Elements of Euclid* in the original Greek.

(1) The Oxford edition in folio, published in 1703 by David Gregory, under the title *Εὐκλείδου τὰ σωζόμενα*. "As an edition of the whole of Euclid's works, this stands alone, there being no other in Greek." *De Morgan*.

(2) *Euclidis Elementorum Libri sex priores...edidit Joannes Gulielmus Camerer*. This edition was published at Berlin in two volumes octavo, the first volume in 1824 and the second in 1825. It contains the first six books of the *Elements* in Greek with a Latin Translation, and very good notes which form a mathematical commentary on the subject.

(3) *Euclidis Elementa ex optimis libris in usum tironum Græce edita ab Ernesto Ferdinando August*. This edition was published at Berlin in two volumes octavo, the first volume in 1826 and the second in 1829. It contains the thirteen books of the *Elements* in Greek, with a collection of various readings.

A third volume, which was to have contained the remaining works of Euclid, never appeared. "To the scholar who wants one edition of the Elements we should decidedly recommend this, as bringing together all that has been done for the text of Euclid's greatest work." *De Morgan*.

An edition of the whole of Euclid's works in the original has long been promised by Teubner the well-known German publisher, as one of his series of compact editions of Greek and Latin authors; but we believe there is no hope of its early appearance.

Robert Simson's edition of the *Elements of Euclid*, which we have in substance adopted in the present work, differs considerably from the original. The English reader may ascertain the contents of the original by consulting the work entitled *The Elements of Euclid with dissertations...by James Williamson*. This work consists of two volumes quarto; the first volume was published at Oxford in 1781, and the second at London in 1788. Williamson gives a close translation of the thirteen books of the *Elements* into English, and he indicates by the use of Italics the words which are not in the original but which are required by our language.

Among the numerous works which contain notes on the *Elements of Euclid* we will mention four by which we have been aided in drawing up the selection given in this volume.

An Examination of the first six Books of Euclid's Elements by William Austin...Oxford, 1781.

*Euclid's Elements of Plane Geometry with copious notes...*by John Walker. London, 1827.

*The first six books of the Elements of Euclid with a Commentary...*by Dionysius Lardner, fourth edition. London, 1834.

Short supplementary remarks on the first six Books of Euclid's Elements, by Professor De Morgan, in the *Companion to the Almanac* for 1849.

We may also notice the following works:

*Geometry, Plane, Solid, and Spherical...*London 1830; this forms part of the Library of Useful Knowledge.

Théorèmes et Problèmes de Géométrie Élémentaire par Eugène Catalan...Troisième édition. Paris, 1858.

For the History of Geometry the student is referred to Montucla's *Histoire des Mathématiques*, and to Chasles's *Aperçu historique sur l'origine et le développement des Méthodes en Géométrie...*

THE FIRST BOOK.

Definitions. The first seven definitions have given rise to considerable discussion, on which however we do not propose to enter. Such a discussion would consist mainly of two subjects, both of which are unsuitable to an elementary work, namely, an examination of the origin and nature of some of our elementary ideas, and a comparison of the original text of Euclid with the substitutions for it proposed by Simson and other editors. For the former subject the student may hereafter consult Whewell's *History of Scientific Ideas* and Mill's *Logic*, and for the latter the notes in Camerer's edition of the *Elements of Euclid*.

We will only observe that the ideas which correspond to the words *point*, *line*, and *surface*, do not admit of such definitions as will really supply the ideas to a person who is destitute of them. The so-called definitions may be regarded as cautions or restrictions. Thus a *point* is not to be supposed to have any *size*, but only *position*; a *line* is not to be supposed to have any *breadth* or *thickness*, but only *length*; a *surface* is not to be supposed to have any *thickness*, but only *length* and *breadth*.

The eighth definition seems intended to include the cases in which an angle is formed by the meeting of two *curved* lines, or of a *straight* line and a *curved* line; this definition however is of no importance, as the only angles ever considered are such as are formed by straight lines. The definition of a plane rectilinear angle is important; the beginner must carefully observe that no change is made in an angle by prolonging the lines which form it, away from the angular point.

Some writers object to such definitions as those of an equilateral triangle, or of a square, in which the existence of the object defined is *assumed* when it ought to be *demonstrated*. They would present them in such a form as the following: if there be a triangle having three equal sides, let it be called an equilateral triangle.

Moreover, some of the definitions are introduced prematurely. Thus, for example, take the definitions of a right-angled triangle and an obtuse-angled triangle; it is not shewn until I. 17, that a triangle cannot have both a right angle and an obtuse angle, and so cannot be at the same time right-angled and obtuse-angled. And before Axiom 11 has been given, it is conceivable

that the same angle may be greater than one right angle, and less than another right angle, that is, obtuse and acute at the same time.

The definition of a square assumes more than is necessary. For if a four-sided figure have all its sides equal and *one* angle a right angle, it may be shewn that *all* its angles are right angles; or if a four-sided figure have all its angles *equal*, it may be shewn that they are all *right angles*.

Postulates. The postulates state what processes we assume that we can effect, namely, that we can draw a straight line between two given points, that we can produce a straight line to any length, and that we can describe a circle from a given centre with a given distance as radius. It is sometimes stated that the postulates amount to requiring the use of a *ruler* and *compasses*. It must however be observed that the ruler is not supposed to be a *graduated* ruler, so that we cannot use it to measure off assigned lengths. And we do not require the compasses for any other process than describing a circle from a given point with a given distance as radius; in other words, the compasses may be supposed to close of themselves, as soon as one of their points is removed from the paper.

Axioms. The axioms are called in the original *Common Notions*. It is supposed by some writers that Euclid intended his postulates to include all demands which are peculiarly geometrical, and his common notions to include only such notions as are applicable to all kinds of magnitude as well as to space magnitudes. Accordingly, these writers remove the last three axioms from their place and put them among the postulates; and this transposition is supported by some manuscripts and some versions of the *Elements*.

The fourth axiom is sometimes referred to in editions of Euclid when in reality more is required than this axiom expresses. Euclid says, that if A and B be unequal, and C and D equal, the sum of A and C is *unequal* to the sum of B and D . What Euclid often requires is something more, namely, that if A be greater than B , and C and D be equal, the sum of A and C is *greater* than the sum of B and D . Such an axiom as this is required, for example, in I. 17. A similar remark applies to the fifth axiom.

In the eighth axiom the words "that is, which exactly fill the same space," have been introduced without the authority of

the original Greek. They are objectionable, because *lines* and *angles* are magnitudes to which the axiom may be applied, but they cannot be said to *fill space*.

On the *method of superposition* we may refer to papers by Professor Kelland in the *Transactions of the Royal Society of Edinburgh*, Vols. XXI. and XXIII.

The eleventh axiom is not required before I. 14, and the twelfth axiom is not required before I. 29; we shall not consider these axioms until we arrive at the propositions in which they are respectively required for the first time.

The first book is chiefly devoted to the properties of triangles and parallelograms.

We may observe that Euclid himself does not distinguish between problems and theorems except by using at the end of the investigation phrases which correspond to Q.E.F. and Q.E.D. respectively.

I. 2. This problem admits of *eight* cases in its figure. For it will be found that the given point may be joined with *either* end of the given straight line, then the equilateral triangle may be described on *either* side of the straight line which is drawn, and the side of the equilateral triangle which is produced may be produced through *either* extremity. These various cases may be left for the exercise of the student, as they present no difficulty.

There will not however always be eight different straight lines obtained which solve the problem. For example, if the point *A* falls on *BC* produced, some of the solutions obtained coincide; this depends on the fact which follows from I. 32, that the angles of all equilateral triangles are equal.

I. 5. "Join *FC*." Custom seems to allow this singular expression as an abbreviation for "draw the straight line *FC*," or for "join *F* to *C* by the straight line *FC*."

In comparing the triangles *BFC*, *CGB*, the words "and the base *BC* is common to the two triangles *BFC*, *CGB*" are usually inserted, with the authority of the original. As however these words are of no use, and tend to perplex a beginner, we have followed the example of some editors and omitted them.

A *corollary* to a proposition is an inference which may be deduced immediately from that proposition. Many of the corollaries in the *Elements* are not in the original text, but introduced by the editors.

It has been suggested to demonstrate I. 5 by *superposition*. Conceive the isosceles triangle ABC to be taken up, and then replaced so that AB falls on the old position of AC , and AC falls on the old position of AB . Thus, in the manner of I. 4, we can shew that the angle ABC is equal to the angle ACB .

I. 6 is the *converse* of part of I. 5. One proposition is said to be the converse of another when the conclusion of each is the hypothesis of the other. Thus in I. 5 the hypothesis is the equality of the sides, and one conclusion is the equality of the angles; in I. 6 the hypothesis is the equality of the angles and the conclusion is the equality of the sides. When there is more than one hypothesis or more than one conclusion to a proposition, we can form more than one converse proposition. For example, as another converse of I. 5 we have the following: if the angles formed by the base of a triangle and the sides produced be equal, the sides of the triangle are equal; this proposition is true and will serve as an exercise for the student.

The converse of a true proposition is not necessarily true; the student however will see, as he proceeds, that Euclid shews that the converses of many geometrical propositions are true.

I. 6 is an example of the *indirect* mode of demonstration, in which a result is established by shewing that some absurdity follows from supposing the required result to be untrue. Hence this mode of demonstration is called the *reductio ad absurdum*. Indirect demonstrations are often less esteemed than direct demonstrations; they are said to shew that a theorem is true rather than to shew *why* it is true. Euclid uses the *reductio ad absurdum* chiefly when he is demonstrating the converse of some former theorem; see I. 14, 19, 25, 40.

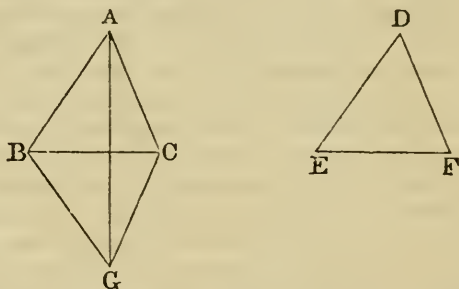
Some remarks on *indirect demonstration* by Professor Sylvester, Professor De Morgan, and Dr Adamson will be found in the volumes of the *Philosophical Magazine* for 1852 and 1853.

I. 6 is not required by Euclid before he reaches II. 4; so that I. 6 might be removed from its present place and demonstrated hereafter in other ways if we please. For example, I. 6 might be placed after I. 18 and demonstrated thus. Let the angle ABC be equal to the angle ACB : then the side AB shall be equal to the side AC . For if not, one of them must be greater than the other; suppose AB greater than AC . Then the angle ACB is greater than the angle ABC , by I. 18. But this is impossible, because

the angle ACB is equal to the angle ABC , by hypothesis. Or I. 6 might be placed after I. 26 and demonstrated thus. Bisect the angle BAC by a straight line meeting the base at D . Then the triangles ABD and ACD are equal in all respects, by I. 26.

I. 7 is only required in order to lead to I. 8. The two might be superseded by another demonstration of I. 8, which has been recommended by many writers.

Let ABC , DEF be two triangles, having the sides AB , AC equal to the sides DE , DF , each to each, and the base BC equal to the base EF : the angle BAC shall be equal to the angle EDF .



For, let the triangle DEF be applied to the triangle ABC , so that the bases may coincide, the equal sides be conterminous, and the vertices fall on opposite sides of the base. Let GBC represent the triangle DEF thus applied, so that G corresponds to D . Join AG . Since, by hypothesis, BA is equal to BG , the angle BAG is equal to the angle BGA , by I. 5. In the same manner the angle CAG is equal to the angle CGA . Therefore the whole angle BAC is equal to the whole angle BGC , that is, to the angle EDF .

There are two other cases; for the straight line AG may pass through B or C , or it may fall outside BC : these cases may be treated in the same manner as that which we have considered.

I. 8. It may be observed that the two triangles in I. 8 are equal in *all respects*; Euclid however does not assert more than the equality of the angles opposite to the bases, and when he requires more than this result he obtains it by using I. 4.

I. 9. Here the equilateral triangle DEF is to be described on the side *remote* from A , because if it were described on the *same* side, its vertex, F , might coincide with A , and then the construction would fail.

I. 11. The corollary was added by Simson. It is liable to serious objection. For we do not know how the perpendicular BE is to be drawn. If we are to use I. 11 we must produce AB , and then we must assume that there is only *one* way of producing AB , for otherwise we shall not know that there is only *one* perpendicular; and thus we assume what we have to demonstrate.

Simson's corollary might come after I. 13 and be demonstrated thus. If possible let the two straight lines ABC , ABD have the segment AB common to both. From the point B draw any straight line BE . Then the angles ABE and EBC are equal to two right angles, by I. 13, and the angles ABE and EBD are also equal to two right angles, by I. 13. Therefore the angles ABE and EBC are equal to the angles ABE and EBD . Therefore the angle EBC is equal to the angle EBD ; which is absurd.

But if the question whether two straight lines can have a common segment is to be considered at all in the Elements, it might occur at an earlier place than Simson has assigned to it. For example, in the figure to I. 5, if two straight lines could have a common segment AB , and then separate at B , we should obtain two different angles formed on the other side of BC by these produced parts, and each of them would be equal to the angle BCG . The opinion has been maintained that even in I. 1, it is tacitly assumed that the straight lines AC and BC cannot have a common segment at C where they meet; see Camerer's *Euclid*, pages 30 and 36.

Simson never formally refers to his corollary until XI. 1. The corollary should be omitted, and the tenth axiom should be extended so as to amount to the following; if two straight lines coincide in two points they must coincide both beyond and between those points.

I. 12. Here the straight line is said to be of *unlimited* length, in order that we may ensure that it shall meet the circle.

Euclid distinguishes between the terms *at right angles* and *perpendicular*. He uses the term *at right angles* when the straight line is drawn from a point *in* another, as in I. 11; and he uses the term *perpendicular* when the straight line is drawn from a point *without* another, as in I. 12. This distinction however is often disregarded by modern writers.

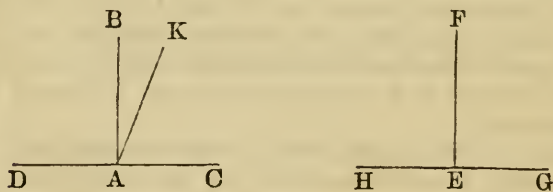
I. 14. Here Euclid first requires his eleventh axiom. For

in the demonstration we have the angles ABC and ABE equal to two right angles, and also the angles ABC and ABD equal to two right angles; and then the former two right angles are equal to the latter two right angles by the aid of the eleventh axiom. Many modern editions of Euclid however refer *only* to the first axiom, as if that alone were sufficient; a similar remark applies to the demonstrations of I. 15, and I. 24. In these cases we have omitted the reference purposely, in order to avoid perplexing a beginner; but when his attention is thus drawn to the circumstance he will see that the first and eleventh axioms are both used.

We may observe that errors, in the references with respect to the eleventh axiom, occur in other places in many modern editions of Euclid. Thus for example in III. 1, at the step "therefore the angle FDB is equal to the angle GDB ," a reference is given to the first axiom *instead* of to the eleventh.

There seems no objection on Euclid's principles to the following *demonstration* of his eleventh axiom.

Let AB be at right angles to DAC at the point A , and EF at right angles to HEG at the point E : then shall the angles BAC and FEG be equal.



Take any length AC , and make AD, EH, EG all equal to AC . Now apply HEG to DAC , so that H may be on D , and HG on DC , and B and F on the same side of DC ; then G will coincide with C , and E with A . Also EF shall coincide with AB ; for if not, suppose, if possible, that it takes a different position as AK . Then the angle DAK is equal to the angle HEF , and the angle CAK to the angle GEF ; but the angles HEF and GEF are equal, by hypothesis; therefore the angles DAK and CAK are equal. But the angles DAB and CAB are also equal, by hypothesis; and the angle CAB is greater than the angle CAK ; there-

fore the angle DAB is greater than the angle CAK . Much more then is the angle DAK greater than the angle CAK . But the angle DAK was shewn to be equal to the angle CAK ; which is absurd. Therefore EF must coincide with AB ; and therefore the angle FEG coincides with the angle BAC , and is equal to it.

I. 18, I. 19. In order to assist the student in remembering which of these two propositions is demonstrated directly and which indirectly, it may be observed that the order is similar to that in I. 5 and I. 6.

I. 20. "Proclus, in his commentary, relates, that the Epicureans derided Prop. 20, as being manifest even to asses, and needing no demonstration; and his answer is, that though the truth of it be manifest to our senses, yet it is science which must give the reason why two sides of a triangle are greater than the third: but the right answer to this objection against this and the 21st, and some other plain propositions, is, that the number of axioms ought not to be increased without necessity, as it must be if these propositions be not demonstrated." *Simson*.

I. 21. Here it must be carefully observed that the two straight lines are to be drawn *from the ends of the side* of the triangle. If this condition be omitted the two straight lines will not necessarily be less than two sides of the triangle.

I. 22. "Some authors blame Euclid because he does not demonstrate that the two circles made use of in the construction of this problem must cut one another: but this is very plain from the determination he has given, namely, that any two of the straight lines DF , FG , GH , must be greater than the third. For who is so dull, though only beginning to learn the Elements, as not to perceive that the circle described from the centre F , at the distance FD , must meet FH betwixt F and H , because FD is less than FH ; and that for the like reason, the circle described from the centre G , at the distance GH ...must meet DG betwixt D and G ; and that these circles must meet one another, because FD and GH are together greater than FG ?" *Simson*.

The condition that B and C are greater than A , ensures that the circle described from the centre G shall not fall entirely within the circle described from the centre F ; the condition that A and B are greater than C , ensures that the circle described

from the centre F shall not fall entirely within the circle described from the centre G ; the condition that A and C are greater than B , ensures that one of these circles shall not fall entirely without the other. Hence the circles must meet. It is easy to see this as Simson says, but there is something arbitrary in Euclid's selection of what is to be *demonstrated* and what is to be *seen*, and Simson's language suggests that he was really conscious of this.

I. 24. In the construction, the condition that DE is to be the side which is not greater than the other, was added by Simson; unless this condition be added there will be *three* cases to consider, for F may fall *on* EG , or *above* EG , or *below* EG . It may be objected that even if Simson's condition be added, it ought to be shewn that F will fall *below* EG . Simson accordingly says "...it is very easy to perceive, that DG being equal to DF , the point G is in the circumference of a circle described from the centre D at the distance DF , and must be in that part of it which is above the straight line EF , because DG falls above DF , the angle EDG being greater than the angle EDF ." Or we may shew it in the following manner. Let H denote the point of intersection of DF and EG . Then, the angle DHG is greater than the angle DEG , by I. 16; the angle DEG is not less than the angle DGE , by I. 19; therefore the angle DHG is greater than the angle DGH . Therefore DH is less than DG , by I. 20. Therefore DH is less than DF .

If Simson's condition be omitted, we shall have two other cases to consider besides that in Euclid. If F falls *on* EG , it is obvious that EF is less than EG . If F falls *above* EG , the sum of DF and EF is less than the sum of DG and EG , by I. 21; and therefore EF is less than EG .

I. 26. It will appear after I. 32 that two triangles which have two angles of the one equal to two angles of the other, each to each, have also their third angles equal. Hence we are able to include the two cases of I. 26 in one enunciation thus; *if two triangles have all the angles of the one respectively equal to all the angles of the other, each to each, and have also a side of the one, opposite to any angle, equal to the side opposite to the equal angle in the other, the triangles shall be equal in all respects.*

The first twenty-six propositions constitute a distinct section

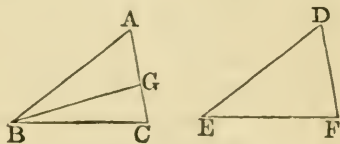
of the first Book of the *Elements*. The principal results are those contained in Propositions 4, 8, and 26; in each of these Propositions it is shewn that two triangles which agree in three respects agree entirely. There are two other cases which will naturally occur to a student to consider besides those in Euclid; namely, (1) when two triangles have the three angles of the one respectively equal to the three angles of the other, (2) when two triangles have two sides of the one equal to two sides of the other, each to each, and an angle opposite to one side of one triangle equal to the angle opposite to the equal side of the other triangle. In the first of these two cases the student will easily see, after reading I. 29, that the two triangles are not necessarily equal. In the second case also the triangles are not necessarily equal, as may be shewn by an example; in the figure of I. 11, suppose the straight line FB drawn; then in the two triangles FBE , FBD , the side FB and the angle FBC are common, and the side FE is equal to the side FD , but the triangles are not equal in all respects. In certain cases, however, the triangles will be equal in all respects, as will be seen from a proposition which we shall now demonstrate.

If two triangles have two sides of the one equal to two sides of the other, each to each, and the angles opposite to a pair of equal sides equal; then if the angles opposite to the other pair of equal sides be both acute, or both obtuse, or if one of them be a right angle, the two triangles are equal in all respects.

Let ABC and DEF be two triangles; let AB be equal to DE , and BC equal to EF , and the angle A equal to the angle D .

First, suppose the angles C and F acute angles.

If the angle B be equal to the angle E , the triangles ABC , DEF are equal in all respects, by I. 4. If the angle B be not equal to the angle E , one of them must be greater than the other; suppose the angle B greater than the angle E , and make the angle ABG equal to the angle E . Then the triangles ABG , DEF are equal in all respects, by I. 26; therefore BG is equal to EF , and the angle BGA is equal to the angle EFD . But the angle EFD is acute, by hypothesis; therefore the angle BGA is acute. Therefore the angle BGC is obtuse, by I. 13. But it has

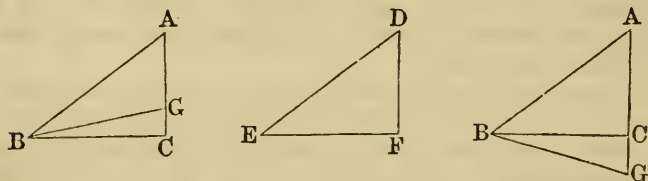
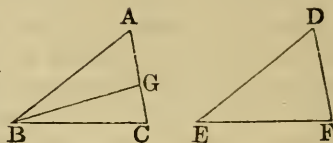


been shewn that BG is equal to EF ; and EF is equal to BC , by hypothesis; therefore BG is equal to BC . Therefore the angle BGC is equal to the angle BCG , by I. 5; and the angle BCG is acute, by hypothesis; therefore the angle BGC is acute. But BGC was shewn to be obtuse; which is absurd.

Therefore the angles ABC , DEF are not unequal; that is, they are equal. Therefore the triangles ABC , DEF are equal in all respects, by I. 4.

Next, suppose the angles at C and F obtuse angles. The demonstration is similar to the above.

Lastly, suppose one of the angles a right angle, namely, the angle C . If the angle B be not equal to the angle E , make the



angle ABG equal to the angle E . Then it may be shewn, as before, that BG is equal to BC , and therefore the angle BGC is equal to the angle BCG , that is, equal to a right angle. Therefore two angles of the triangle BGC are equal to two right angles; which is impossible, by I. 17. Therefore the angles ABC and DEF are not unequal; that is, they are equal. Therefore the triangles ABC , DEF are equal in all respects, by I. 4.

If the angles A and D are both right angles, or both obtuse, the angles C and F must be both acute, by I. 17. If AB is less than BC , and DE less than EF , the angles at C and F must be both acute, by I. 18 and I. 17.

The propositions from I. 27 to I. 34 inclusive may be said to constitute the second section of the first Book of the *Elements*. They relate to the theory of parallel straight lines. In I. 29 Euclid uses for the first time his twelfth axiom. The theory of parallel straight lines has always been considered the great difficulty of elementary geometry, and many attempts have been made

to overcome this difficulty in a better way than Euclid has done. We shall not give an account of these attempts. The student who wishes to examine them may consult Camerer's *Euclid*, Gergonne's *Annales de Mathématiques*, Volumes xv and xvi, the work by Colonel Perronet Thompson entitled *Geometry without Axioms*, the article *Parallels* in the *English Cyclopædia*, a memoir by Professor Baden Powell in the second volume of the *Memoirs of the Ashmolean Society*, an article by M. Bouniakofsky in the *Bulletin de l'Académie Impériale*, Volume v, St Pétersbourg, 1863, articles in the volumes of the *Philosophical Magazine* for 1856 and 1857, and a dissertation entitled *Sur un point de l'histoire de la Géométrie chez les Grecs.....* par A. J. H. Vincent. Paris, 1857.

Speaking generally it may be said that the methods which differ substantially from Euclid's involve, in the first place an axiom as difficult as his, and then an intricate series of propositions; while in Euclid's method after the axiom is once admitted the remaining process is simple and clear.

One modification of Euclid's axiom has been proposed, which appears to diminish the difficulty of the subject. This consists in assuming instead of Euclid's axiom the following; *two intersecting straight lines cannot be both parallel to a third straight line*. The propositions in the *Elements* are then demonstrated as in Euclid up to I. 28, inclusive. Then, in I. 29, we proceed with Euclid up to the words, "therefore the angles BGH , GHD are less than two right angles." We then infer that BGH and GHD must meet: because if a straight line be drawn through G so as to make the interior angles together equal to two right angles this straight line will be parallel to CD , by I. 28; and, by our axiom, there cannot be two parallels to CD , both passing through G .

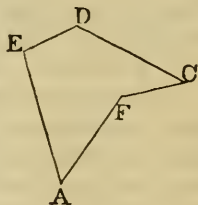
This form of making the necessary assumption has been recommended by various eminent mathematicians, among whom may be mentioned Playfair and De Morgan. By postponing the consideration of the axiom until it is wanted, that is, until after I. 28, and then presenting it in the form here given, the theory of parallel straight lines appears to be treated in the easiest manner that has hitherto been proposed.

I. 30. Here we may in the same way shew that if AB and EF are each of them parallel to CD , they are parallel to each other. It has been said that the case considered in the text is so obvious as to need no demonstration; for if AB and CD can

never meet EF , which lies between them, they cannot meet one another.

I. 32. The corollaries to I. 32 were added by Simson. In the second corollary it ought to be stated what is meant by an *exterior* angle of a rectilinear figure. At each angular point let *one* of the sides meeting at that point be produced; then the exterior angle at that point is the angle contained between this produced part and the side which is not produced. *Either* of the sides may be produced, for the two angles which can thus be obtained are equal, by I. 15.

The rectilinear figures to which Euclid confines himself are those in which the angles all face inwards; we may here however notice another class of figures. In the accompanying diagram the angle AFC faces outwards, and it is an angle less than two right angles; this angle however is not one of the interior angles of the figure $AEDCF$. We may consider the corresponding interior angle to be the excess of four right angles above the angle AFC ; such an angle, greater than two right angles, is called a *re-entrant* angle.



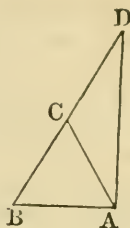
The *first* of the corollaries to I. 32 is true for a figure which has a re-entrant angle or re-entrant angles; but the *second* is not.

I. 32. If two triangles have two angles of the one equal to two angles of the other each to each they shall also have their third angles equal. This is a very important result, which is often required in the *Elements*. The student should notice how this result is established on Euclid's principles. By Axioms 11 and 2 one pair of right angles is equal to any other pair of right angles. Then, by I. 32, the three angles of one triangle are together equal to the three angles of any other triangle. Then, by Axiom 2, the sum of the two angles of one triangle is equal to the sum of the two equal angles of the other; and then, by Axiom 3, the third angles are equal.

After I. 32 we can draw a straight line at right angles to a given straight line from its extremity, without producing the given straight line.

Let AB be the given straight line. It is required to draw from A a straight line at right angles to AB .

On AB describe the equilateral triangle ABC . Produce BC to D , so that CD may be equal to CB . Join AD . Then AD shall be at right angles to AB . For, the angle CAD is equal to the angle CDA , and the angle CAB is equal to the angle CBA , by I. 5. Therefore the angle BAD is equal to the two angles ABD , BDA , by Axiom 2. Therefore the angle BAD is a right angle, by I. 32.



The propositions from I. 35 to I. 48 inclusive may be said to constitute the third section of the first Book of the *Elements*. They relate to equality of area in figures which are not necessarily identical in form.

I. 35. Here Simson has altered the demonstration given by Euclid, because, as he says, there would be three cases to consider in following Euclid's method. Simson however uses the third Axiom in a peculiar manner, when he first takes a triangle from a trapezium, and then another triangle from the *same* trapezium, and infers that the remainders are equal. If the demonstration is to be conducted strictly after Euclid's manner, three cases must be made, by dividing the latter part of the demonstration into two. In the left-hand figure we may suppose the point of intersection of BE and DC to be denoted by G . Then, the triangle ABE is equal to the triangle DCF ; take away the triangle DGE from each; then the figure $ABGD$ is equal to the figure $EGCF$; add the triangle GBC to each; then the parallelogram $ABCD$ is equal to the parallelogram $EBCF$. In the right-hand figure we have the triangle AEB equal to the triangle DFC ; add the figure $BEDC$ to each; then the parallelogram $ABCD$ is equal to the parallelogram $EBCF$.

The equality of the parallelograms in I. 35 is an equality of area, and not an identity of figure. Legendre proposed to use the word *equivalent* to express the equality of area, and to restrict the word *equal* to the case in which magnitudes admit of superposition and coincidence. This distinction, however, has not been generally adopted, probably because there are few cases in which any ambiguity can arise; in such cases we may say especially, *equal in area*, to prevent misconception.

Cresswell, in his *Treatise of Geometry*, has given a demonstration of I. 35 which shews that the parallelograms may be

divided into pairs of pieces admitting of superposition and coincidence; see also his Preface, page x.

I. 38. An important case of I. 38 is that in which the triangles are on equal bases and have a *common* vertex.

I. 40. We may demonstrate I. 40 without adopting the indirect method. Join BD , CD . The triangles DBC and DEF are equal, by I. 38; the triangles ABC and DEF are equal, by hypothesis; therefore the triangles DBC and ABC are equal, by the first Axiom. Therefore AD is parallel to BC , by I. 39. *Philosophical Magazine*, October 1850.

I. 44. In I. 44, Euclid does not shew that AH and FG will meet. "I cannot help being of opinion that the construction would have been more in Euclid's manner if he had made GH equal to BA and then joining HA had proved that HA was parallel to GB by the thirty-third proposition." *Williamson*.

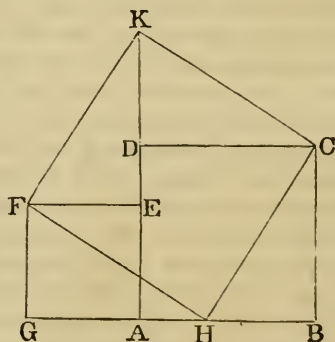
I. 47. Tradition ascribed the discovery of I. 47 to Pythagoras. Many demonstrations have been given of this celebrated proposition; the following is one of the most interesting.

Let $ABCD$, $AEFG$ be any two squares, placed so that their bases may join and form one straight line. Take GH and EK each equal to AB , and join HC , CK , KF , FH .

Then it may be shewn that the triangle HBC is equal in all respects to the triangle FEK , and the triangle KDC to the triangle FGH . Therefore the two squares are together equivalent to the figure $CKFH$. It

may then be shewn, with the aid of I. 32, that the figure $CKFH$ is a square. And the side CH is the hypotenuse of a right-angled triangle of which the sides CB , BH are equal to the sides of the two given squares. This demonstration requires no proposition of Euclid after I. 32, and it shews how two given squares may be cut into pieces which will fit together so as to form a third square. *Quarterly Journal of Mathematics*, Vol. I.

A large number of demonstrations of this proposition are collected in a dissertation by Joh. Jos. Ign. Hoffmann, entitled *Der Pythagorische Lehrsatz....Zweyte...Ausgabe*. Mainz. 1821.



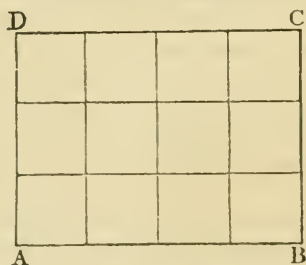
THE SECOND BOOK.

THE second book is devoted to the investigation of relations between the rectangles contained by straight lines divided into segments in various ways.

When a straight line is divided into two parts, each part is called a segment by Euclid. It is found convenient to extend the meaning of the word *segment*, and to lay down the following definition. When a point is taken in a straight line, or in the straight line produced, the distances of the point from the ends of the straight line are called segments of the straight line. When it is necessary to distinguish them, such segments are called *internal* or *external*, according as the point is in the straight line, or in the straight line produced.

The student cannot fail to notice that there is an analogy between the first ten propositions of this book and some elementary facts in Arithmetic and Algebra.

Let $ABCD$ represent a rectangle which is 4 inches long and 3 inches broad. Then, by drawing straight lines parallel to the sides, the figure may be divided into 12 squares, each square being described on a side which represents an inch in length. A square described on a side measuring an inch is called, for shortness, a *square inch*. Thus if a rectangle is 4 inches long and 3 inches



broad it may be divided into 12 square inches; this is expressed by saying, that its area is equal to 12 square inches, or, more briefly, that it contains 12 square inches. And a similar result is easily seen to hold in all similar cases. Suppose, for example, that a rectangle is 12 feet long and 7 feet broad; then its area is equal to 12 times 7 square feet, that is to 84 square feet; this may be expressed briefly in common language thus; if a rectangle measures 12 feet by 7 it contains 84 square feet. It must be carefully observed that the sides of the rectangle are supposed to be measured by the same unit of length. Thus if a rectangle is a yard in length, and a foot and a half in breadth, we

must express each of these dimensions in terms of the same unit; we may say that the rectangle measures 36 inches by 18 inches, and contains 36 times 18 square inches, that is, 648 square inches.

Thus universally, if one side of a rectangle contain a unit of length an exact number of times, and if an adjacent side of the rectangle also contain the same unit of length an exact number of times, the product of these numbers will be the number of square units contained in the rectangle.

Next suppose we have a *square*, and let its side be 5 inches in length. Then, by our rule, the area of the square is 5 times 5 square inches, that is 25 square inches. Now the number 25 is called in Arithmetic the square of the number 5. And universally, if a straight line contain a unit of length an exact number of times, the area of the square described *on* the straight line is denoted by the square *of* the number which denotes the length of the straight line.

Thus we see that there is in general a connexion between the product of two numbers and the rectangle contained by two straight lines, and in particular a connexion between the square *of* a number and the square *on* a straight line; and in consequence of this connexion the first ten propositions in Euclid's Second Book correspond to propositions in Arithmetic and Algebra.

The student will perceive that we speak of the square described *on* a straight line, when we refer to the geometrical figure, and of the square *of* a number when we refer to Arithmetic. The editors of Euclid generally use the words "square described *upon*" in I. 47 and I. 48, and afterwards speak of the square *of* a straight line. Euclid himself retains throughout the same form of expression, and we have imitated him.

Some editors of Euclid have added Arithmetical or Algebraical demonstrations of the propositions in the second book, founded on the connexion we have explained. We have thought it unnecessary to do this, because the student who is acquainted with the elements of Arithmetic and Algebra will find no difficulty in supplying such demonstrations himself, so far as they are usually given. We say *so far as they are usually given*, because these demonstrations usually imply that the sides of rectangles can always be expressed *exactly* in terms of some unit of length; whereas the student will find hereafter that this is not the case, owing to the existence of what are technically called *incommensurable* magnitudes. We do not enter on this subject,

as it would lead us too far from Euclid's *Elements of Geometry*, with which we are here occupied.

The first ten propositions in the second book of Euclid may be arranged and enunciated in various ways; we will briefly indicate this, but we do not consider it of any importance to distract the attention of a beginner with these diversities.

II. 2 and II. 3 are particular cases of II. 1.

II. 4 is very important; the following particular case of it should be noticed; *the square described on a straight line made up of two equal straight lines is equal to four times the square described on one of the two equal straight lines.*

II. 5 and II. 6 may be included in one enunciation thus; *the rectangle under the sum and difference of two straight lines is equal to the difference of the squares described on those straight lines; or thus, the rectangle contained by two straight lines together with the square described on half their difference, is equal to the square described on half their sum.*

II. 7 may be enunciated thus; *the square described on a straight line which is the difference of two other straight lines is less than the sum of the squares described on those straight lines by twice the rectangle contained by those straight lines.* Then from this and II. 4, and the second Axiom, we infer that *the square described on the sum of two straight lines, and the square described on their difference, are together double of the sum of the squares described on the straight lines; and this enunciation includes both II. 9 and II. 10, so that the demonstrations given of these propositions by Euclid might be superseded.*

II. 8 coincides with the second form of enunciation which we have given to II. 5 and II. 6, bearing in mind the particular case of II. 4 which we have noticed.

II. 11. When the student is acquainted with the elements of Algebra he should notice that II. 11 gives a geometrical construction for the solution of a particular quadratic equation.

II. 12, II. 13. These are interesting in connexion with I. 47; and, as the student may see hereafter, they are of great importance in Trigonometry; they are however not required in any of the parts of Euclid's *Elements* which are usually read. The converse of I. 47 is proved in I. 48; and we can easily shew that converses of II. 12 and II. 13 are true.

Take the following, which is the converse of II. 12; *if the square described on one side of a triangle be greater than the sum*

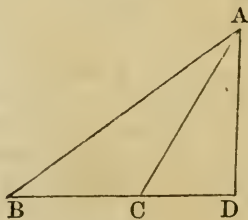
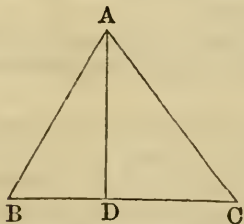
of the squares described on the other two sides, the angle opposite to the first side is obtuse.

For the angle cannot be a right angle, since the square described on the first side would then be equal to the sum of the squares described on the other two sides, by I. 47; and the angle cannot be acute, since the square described on the first side would then be less than the sum of the squares described on the other two sides, by II. 13; therefore the angle must be obtuse.

Similarly we may demonstrate the following, which is the converse of II. 13; *if the square described on one side of a triangle be less than the sum of the squares described on the other two sides, the angle opposite to the first side is acute.*

II. 13. Euclid enunciates II. 13 thus; *in acute-angled triangles, &c.*; and he gives only the first case in the demonstration. But, as Simson observes, the proposition holds for any triangle; and accordingly Simson supplies the second and third cases. It has, however, been often noticed that the same demonstration is applicable to the first and second cases; and it would be a great improvement as to brevity and clearness to take these two cases together. Then the whole demonstration will be as follows.

Let ABC be any triangle, and the angle at B one of its acute angles; and, if AC be not perpendicular to BC , let fall on BC , produced if necessary, the perpendicular AD from the opposite angle: the square on AC opposite to the angle B , shall be less than the squares on CB , BA , by twice the rectangle CB , BD .



First, suppose AC not perpendicular to BC .

The squares on CB , BD are equal to twice the rectangle CB , BD , together with the square on CD . [II. 7.]

To each of these equals add the square on DA .

Therefore the squares on CB , BD , DA are equal to twice the rectangle CB , BD , together with the squares on CD , DA .

But the square on AB is equal to the squares on BD , DA ,

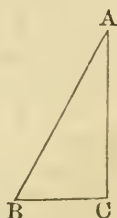
and the square on AC is equal to the squares on CD , DA , because the angle BDA is a right angle. [I. 47.]

Therefore the squares on CB , BA are equal to the square on AC , together with twice the rectangle CB , BD ; that is, the square on AC alone is less than the squares on CB , BA , by twice the rectangle CB , BD .

Next, suppose AC perpendicular to BC . Then BC is the straight line intercepted between the perpendicular and the acute angle at B .

And the square on AB is equal to the squares on AC , CB . [I. 47.]

Therefore the square on AC is less than the squares on AB , BC , by twice the square on BC .



II. 14. This is not required in any of the parts of Euclid's *Elements* which are usually read; it is included in VI. 22.

THE THIRD BOOK.

THE third book of the *Elements* is devoted to properties of circles.

Different opinions have been held as to what is, or should be, included in the third definition of the third book. One opinion is that the definition only means that the circles do not cut in the *neighbourhood of the point of contact*, and that it must be shewn that they do not cut elsewhere. Another opinion is that the definition means that the circles do not cut at all; and this seems the correct opinion. The definition may therefore be presented more distinctly thus. Two circles are said to touch internally when their circumferences have one or more common points, and when every point in one circle is within the other circle, except the common point or points. Two circles are said to touch externally when their circumferences have one or more common points, and when every point in each circle is without the other circle, except the common point or points. It is then shewn in the third Book that the circumferences of two circles which touch can have only *one* common point.

A straight line which touches a circle is often called a *tangent* to the circle, or briefly, a *tangent*.

It is very convenient to have a word to denote a portion of

the boundary of a circle, and accordingly we use the word *arc*. Euclid himself uses *circumference* both for the whole boundary and for a portion of it.

III. 1. In the construction, *DC* is said to be *produced* to *E*; this assumes that *D* is within the circle, which Euclid demonstrates in III. 2.

III. 3. This consists of two parts, each of which is the converse of the other; and the whole proposition is the converse of the corollary in III. 1.

III. 5 and III. 6 should have been taken together. They amount to this, *if the circumferences of two circles meet at a point they cannot have the same centre*, so that circles which have the same centre and one point in their circumferences common, must coincide altogether. It would seem as if Euclid had made three cases, one in which the circles cut, one in which they touch internally, and one in which they touch externally, and had then omitted the last case as evident.

III. 7, III. 8. It is observed by Professor De Morgan that in III. 7 it is *assumed* that the angle *FEB* is greater than the angle *FEC*, the hypothesis being only that the angle *DFB* is greater than the angle *DFC*; and that in III. 8 it is *assumed* that *K* falls within the triangle *DLM*, and *E* without the triangle *DMF*. He intimates that these assumptions may be established by means of the following two propositions which may be given in order after I. 21.

The perpendicular is the shortest straight line which can be drawn from a given point to a given straight line; and of others that which is nearer to the perpendicular is less than the more remote, and the converse; and not more than two equal straight lines can be drawn from the given point to the given straight line, one on each side of the perpendicular.

Every straight line drawn from the vertex of a triangle to the base is less than the greater of the two sides, or than either of them if they be equal.

The following proposition is analogous to III. 7 and III. 8.

If any point be taken on the circumference of a circle, of all the straight lines which can be drawn from it to the circumference, the greatest is that in which the centre is; and of any others, that which is nearer to the straight line which passes through the centre is always greater than one more remote; and from the same point there can be drawn to the circumference two straight lines, and

only two, which are equal to one another, one on each side of the greatest line.

The first two parts of this proposition are contained in III. 15; all three parts might be demonstrated in the manner of III. 7, and they should be demonstrated, for the third part is really required, as we shall see in the note on III. 10.

III. 9. The point E might be supposed to fall *within* the angle ADC . It cannot then be shewn that DC is greater than DB , and DB greater than DA , but only that either DC or DA is less than DB ; this however is sufficient for establishing the proposition.

Euclid has given two demonstrations of III. 9, of which Simson has chosen the second. Euclid's other demonstration is as follows. Join D with the middle point of the straight line AB ; then it may be shewn that this straight line is at right angles to AB ; and therefore the centre of the circle must lie in this straight line, by III. 1, Corollary. In the same manner it may be shewn that the centre of the circle must lie in the straight line which joins D with the middle point of the straight line BC . The centre of the circle must therefore be at D , because two straight lines cannot have more than one common point.

III. 10. Euclid has given two demonstrations of III. 10, of which Simson has chosen the second. Euclid's first demonstration resembles his first demonstration of III. 9. He shews that the centre of each circle is on the straight line which joins K with the middle point of the straight line BG , and also on the straight line which joins K with the middle point of the straight line BH ; therefore K must be the centre of each circle.

The demonstration which Simson has chosen requires some additions to make it complete. For the point K might be supposed to fall *without* the circle DEF , or *on* its circumference, or *within* it; and of these three suppositions Euclid only considers the last. If the point K be supposed to fall *without* the circle DEF we obtain a contradiction of III. 8; which is absurd. If the point K be supposed to fall *on* the circumference of the circle DEF we obtain a contradiction of the proposition which we have enunciated at the end of the note on III. 7 and III. 8; which is absurd.

What is demonstrated in III. 10 is that the circumferences of two circles cannot have more than *two* common points; there is

nothing in the demonstration which assumes that the circles *cut* one another, but the enunciation refers to this case only because it is shewn in III. 13 that if two circles *touch* one another, their circumferences cannot have more than *one* common point.

III. 11, III. 12. The enunciations as given by Simson and others speak of *the* point of contact; it is however not shewn until III. 13 that there is only *one* point of contact. It should be observed that the demonstration in III. 11 will hold even if D and H be supposed to coincide, and that the demonstration in III. 12 will hold even if C and D be supposed to coincide. We may combine III. 11 and III. 12 in one enunciation thus.

If two circles touch one another their circumferences cannot have a common point out of the direction of the straight line which joins the centres.

III. 11 may be deduced from III. 7. For GH is the least line that can be drawn from G to the circumference of the circle whose centre is F , by III. 7. Therefore GH is less than GA , that is, less than GD ; which is absurd. Similarly III. 12 may be deduced from III. 8.

III. 13. Simson observes, "As it is much easier to imagine that two circles may touch one another within in more points than one, upon the same side, than upon opposite sides, the figure of that case ought not to have been omitted; but the construction in the Greek text would not have suited with this figure so well, because the centres of the circles must have been placed near to the circumferences; on which account another construction and demonstration is given, which is the same with the second part of that which Campanus has translated from the Arabic, where, without any reason, the demonstration is divided into two parts."

It would not be obvious from this note which figure Simson himself supplied, because it is uncertain what he means by the "same side" and "opposite sides." It is the left-hand figure in the first part of the demonstration. Euclid, however, seems to be quite correct in omitting this figure, because he has shewn in III. 11 that if two circles touch internally there cannot be a point of contact out of the direction of the straight line which joins the centres. Thus, in order to shew that there is only one point of contact, it is sufficient to put the second supposed point of contact on the direction of the straight line which joins the

centres. Accordingly in his own demonstration Euclid confines himself to the right-hand figure; and he shews that this case cannot exist, because the straight line BD would be a diameter of both circles, and would therefore be bisected at two different points; which is absurd.

Euclid might have used a similar method for the second part of the proposition; for as there cannot be a point of contact out of the straight line joining the centres, *it is obviously impossible* that there can be a second point of contact when the circles touch externally. It is easy to see this; but Euclid preferred a method in which there is more formal reasoning.

We may observe that Euclid's mode of dealing with the contact of circles has often been censured by commentators, but apparently not always with good reason. For example, Walker gives another demonstration of III. 13; and says that Euclid's is worth nothing, and that Simson fails; for it is not proved that two circles which touch cannot have any arc common to both circumferences. But it is shewn in III. 10 that this is impossible; Walker appears to have supposed that III. 10 is limited to the case of circles which *cut*. See the note on III. 10.

III. 17. It is obvious from the construction in III. 17 that *two* straight lines can be drawn from a given external point to touch a given circle; and these two straight lines are equal in length and equally inclined to the straight line which joins the given external point with the centre of the given circle.

After reading III. 31 the student will see that the problem in III. 17 may be solved in another way, as follows: describe a circle on AE as diameter; then the points of intersection of this circle with the given circle will be the points of contact of the two straight lines which can be drawn from A to touch the given circle.

III. 18. It does not appear that III. 18 adds anything to what we have already obtained in III. 16. For in III. 16 it is shewn, that there is only one straight line which touches a given circle at a given point, and that the angle between this straight line and the radius drawn to the point of contact is a right angle.

III. 20. There are two assumptions in the demonstration of III. 20. Suppose that A is double of B and C double of D ; then in the first part it is assumed that the sum of A and C is double of the sum of B and D , and in the second part it is as-

sumed that the difference of A and C is double of the difference of B and D . The former assumption is a particular case of V. 1, and the latter is a particular case of V. 5.

An important extension may be given to III. 20 by introducing angles greater than two right angles. For, in the first figure, suppose we draw the straight lines BF and CF . Then, the angle BEA is double of the angle BFA , and the angle CEA is double of the angle CFA ; therefore the sum of the angles BEA and CEA is double of the angle BFC . The sum of the angles BEA and CEA is greater than two right angles; we will call the sum, the *re-entrant* angle BEC . Thus the re-entrant angle BEC is double of the angle BFC . (See note on I. 32). If this extension be used some of the demonstrations in the third book may be abbreviated. Thus III. 21 may be demonstrated without making two cases; III. 22 will follow immediately from the fact that the sum of the angles at the centre is equal to four right angles; and III. 31 will follow immediately from III. 20.

III. 21. In III. 21 Euclid himself has given only the first case; the second case has been added by Simson and others. In either of the figures of III. 21 if a point be taken on the same side of BD as A , the angle contained by the straight lines which join this point to the extremities of BD is *greater* or *less* than the angle BAD , according as the point is *within* or *without* the angle BAD ; this follows from I. 21.

We shall have occasion to refer to IV. 5 in some of the remaining notes to the third Book; and the student is accordingly recommended to read that proposition at the present stage.

The following proposition is very important. *If any number of triangles be constructed on the same base and on the same side of it, with equal vertical angles, the vertices will all lie on the circumference of a segment of a circle.*

For take any one of these triangles, and describe a circle round it, by IV. 5; then the vertex of any other of the triangles must be on the circumference of the segment containing the assumed vertex, since, by the former part of this note, the vertex cannot be without the circle or within the circle.

III. 22. The converse of III. 22 is true and very important; namely, *if two opposite angles of a quadrilateral be together equal to two right angles, a circle may be circumscribed about the quadrilateral.* For, let $ABCD$ denote the quadrila-

teral. Describe a circle round the triangle ABC , by IV. 5. Take any point E , on the circumference of the segment cut off by AC , and on the same side of AC as D is. Then, the angles at B and E are together equal to two right angles, by III. 22; and the angles at B and D are together equal to two right angles, by hypothesis. Therefore the angle at E is equal to the angle at D . Therefore, by the preceding note D is on the circumference of the same segment as E .

III. 32. The converse of III. 32 is true and important; namely, *if a straight line meet a circle, and from the point of meeting a straight line be drawn cutting the circle, and the angle between the two straight lines be equal to the angle in the alternate segment of the circle, the straight line which meets the circle shall touch the circle.*

This may be demonstrated indirectly. For, if possible, suppose that the straight line which meets the circle does not touch it. Draw through the point of meeting a straight line to touch the circle. Then, by III. 32 and the hypothesis, it will follow that two different straight lines pass through the same point, and make the same angle, on the same side, with a third straight line which also passes through that point; but this is impossible.

III. 35, III. 36. The following proposition constitutes a large part of the demonstrations of III. 35 and III. 36. *If any point be taken in the base, or the base produced, of an isosceles triangle, the rectangle contained by the segments of the base is equal to the difference of the square on the straight line joining this point to the vertex and the square on the side of the triangle.*

This proposition is in fact demonstrated by Euclid, without using any property of the circle; if it were enunciated and demonstrated before III. 35 and III. 36 the demonstrations of these two propositions might be shortened and simplified.

The following converse of III. 35 and the Corollary of III. 36 may be noticed. *If two straight lines AB , CD intersect at O , and the rectangle AO , OB be equal to the rectangle CO , OD , the circumference of a circle will pass through the four points A , B , C , D .*

For a circle may be described round the triangle ABC , by IV. 5; and then it may be shewn indirectly, by the aid of III. 35 or the Corollary of III. 36 that the circumference of this circle will also pass through D .

THE FOURTH BOOK.

THE fourth Book of the Elements consists entirely of problems. The first five propositions relate to triangles of any kind; the remaining propositions relate to polygons which have all their sides equal and all their angles equal. A polygon which has all its sides equal and all its angles equal is called a *regular* polygon.

IV. 4. By a process similar to that in IV. 4 we can describe a circle which shall touch one side of a triangle and the other two sides produced. Suppose, for example, that we wish to describe a circle which shall touch the side BC , and the sides AB and AC produced: bisect the angle between AB produced and BC , and bisect the angle between AC produced and BC ; then the point at which the bisecting straight lines meet will be the centre of the required circle. The demonstration will be similar to that in IV. 4.

A circle which touches one side of a triangle and the other two sides produced, is called an *escribed* circle of the triangle.

We can also describe a triangle equiangular to a given triangle, and such that one of its sides and the other two sides produced shall touch a given circle. For, in the figure of IV. 3 suppose AK produced to meet the circle again; and at the point of intersection draw a straight line touching the circle; this straight line with parts of NB and NC , will form a triangle, which will be equiangular to the triangle MLN , and therefore equiangular to the triangle EDF ; and one of the sides of this triangle, and the other two sides produced, will touch the given circle.

IV. 5. Simson introduced into the demonstration of IV. 5 the part which shews that DF and EF will meet. It has also been proposed to shew this in the following way: join DE ; then the angles EDF and DEF are together less than the angles ADF and AEF , that is, they are together less than two right angles; and therefore DF and EF will meet, by Axiom 12. This assumes that ADE and AED are *acute* angles; it may however be easily shewn that DE is parallel to BC , so that the triangle ADE is equiangular to the triangle ABC ; and we must therefore select the two sides AB and AC such that ABC and ACB may be acute angles.

IV. 10. The vertical angle of the triangle in IV. 10 is easily seen to be the fifth part of two right angles; and as it

may be bisected, we can thus divide a right angle geometrically into five equal parts.

It follows from what is given in the fourth Book of the Elements that the circumference of a circle can be divided into 3, 6, 12, 24, ... equal parts; and also into 4, 8, 16, 32, ... equal parts; and also into 5, 10, 20, 40, ... equal parts; and also into 15, 30, 60, 120, ... equal parts. Hence also regular polygons having as many sides as any of these numbers may be inscribed in a circle, or described about a circle. This however does not enable us to describe a regular polygon of any assigned number of sides; for example, we do not know how to describe geometrically a regular polygon of 7 sides.

It was first demonstrated by Gauss in 1801, in his *Disquisitiones Arithmeticae*, that it is possible to describe geometrically a regular polygon of $2^n + 1$ sides, provided $2^n + 1$ be a prime number; the demonstration is not of an elementary character. As an example, it follows that a regular polygon of 17 sides can be described geometrically; this example is discussed in Catalan's *Théorèmes et Problèmes de Géométrie Élémentaire*.

For an approximate construction of a regular heptagon see the *Philosophical Magazine* for February and for April, 1864.

THE FIFTH BOOK.

THE fifth Book of the Elements is on *Proportion*. Much has been written respecting Euclid's treatment of this subject; besides the Commentaries on the Elements to which we have already referred, the student may consult the articles *Ratio* and *Proportion* in the *English Cyclopædia*, and the tract on the *Connexion of Number and Magnitude* by Professor De Morgan.

The fifth Book relates not merely to length and space, but to any kind of magnitude of which we can form multiples.

V. Def. 1. The word *part* is used in two senses in Geometry. Sometimes the word denotes any magnitude which is less than another of the same kind, as in the axiom, *the whole is greater than its part*. In this sense the word has been used up to the present point, but in the fifth Book Euclid confines the word to a more restricted sense. This restricted sense agrees with that which is given in Arithmetic and Algebra to the term *aliquot part*, or to the term *submultiple*.

V. Def. 3. Simson considers that the definitions 3 and 8 are "not Euclid's, but added by some unskilful editor." Other commentators also have rejected these definitions as useless. The last word of the third definition should be *quantuplicity*, not *quantity*; so that the definition indicates that ratio refers to the *number of times* which one magnitude contains another. See De Morgan's *Differential and Integral Calculus*, page 18.

V. Def. 4. This definition amounts to saying that the quantities must be of the *same kind*.

V. Def. 5. The fifth definition is the foundation of Euclid's doctrine of proportion. The student will find in works on Algebra a comparison of Euclid's definition of proportion with the simpler definitions which are employed in Arithmetic and Algebra. Euclid's definition is applicable to *incommensurable* quantities, as well as to *commensurable* quantities.

We should recommend the student to read the first proposition of the sixth Book immediately after the fifth definition of the fifth Book; he will there see how Euclid applies his definition, and will thus obtain a better notion of its meaning and importance.

Compound Ratio. The definition of compound ratio was supplied by Simson. The Greek text does not give any definition of compound ratio here, but gives one as the fifth definition of the sixth Book, which Simson rejects as absurd and useless.

V. Defs. 18, 19, 20. The definitions 18, 19, 20 are not presented by Simson precisely as they stand in the original. The last sentence in definition 18 was supplied by Simson. Euclid does not connect definitions 19 and 20 with definition 18. In 19 he defines *ordinate proportion*, and in 20 he defines *perturbate proportion*. Nothing would be lost if Euclid's definition 18 were entirely omitted, and the term *ex æquali* never employed. Euclid employs such a term in the enunciations of V. 20, 21, 22, 23; but it seems quite useless, and is accordingly neglected by Simson and others in their translations.

The axioms given after the definitions of the fifth Book are not in Euclid; they were supplied by Simson.

The propositions of the fifth Book might be divided into four sections. Propositions 1 to 6 relate to the properties of equimultiples. Propositions 7 to 10 and 13 and 14 connect the notion of the *ratio* of magnitudes with the ordinary notions of

greater, equal, and less. Propositions 11, 12, 15 and 16 may be considered as introduced to shew that, *if four quantities of the same kind be proportionals they will also be proportionals when taken alternately*. The remaining propositions shew that magnitudes are proportional by *composition*, by *division*, and *ex æquo*.

In this division of the fifth Book propositions 13 and 14 are supposed to be placed immediately after proposition 10; and they might be taken in this order without any change in Euclid's demonstrations.

The propositions headed *A, B, C, D, E* were supplied by Simson.

V. 1, 2, 3, 5, 6. These are simple propositions of Arithmetic, though they are here expressed in terms which make them appear less familiar than they really are. For example, V. 1 "states no more than that *ten acres and ten roods make ten times as much as one acre and one rood*." *De Morgan*.

In V. 5 Simson has substituted another construction for that given by Euclid, because Euclid's construction assumes that we can divide a given straight line into any assigned number of equal parts, and this problem is not solved until VI. 9.

V. 18. This demonstration is Simson's. We will give here Euclid's demonstration.

Let AE be to EB as CF is to FD : AB shall be to BE as CD is to DF .

For, if not, AB will be to BE as CD is to some magnitude less than DF , or greater than DF .

First, suppose that AB is to BE as CD is to DG , which is less than DF .

Then, because AB is to BE as CD is to DG , therefore AE is to EB as CG is to GD . [V. 17.]

But AE is to EB as CF is to FD , [Hypothesis.] therefore CG is to GD as CF is to FD . [V. 11.]

But CG is greater than CF ;

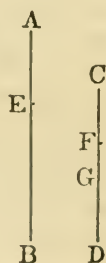
therefore GD is greater than FD .

But GD is less than FD ; which is impossible.

In the same manner it may be shewn that AB is not to BE as CD is to a magnitude greater than DF .

Therefore AB is to BE as CD is to DF .

The objection urged by Simson against Euclid's demonstration is that "it depends upon this hypothesis, that to any three magnitudes, two of which, at least, are of the same kind, there



[Hypothesis.
[V. 14.]

may be a fourth proportional: Euclid does not demonstrate it, nor does he shew how to find the fourth proportional, before the 12th Proposition of the 6th Book."

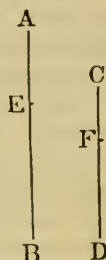
The following demonstration is given by Austin in his *Examination of the first six books of Euclid's Elements*.

Let AE be to EB as CF is to FD : AB shall be to BE as CD is to DF .

For, because AE is to EB as CF is to FD , therefore, alternately, AE is to CF as EB is to FD . [V. 16.]

And as one of the antecedents is to its consequent so is the sum of the antecedents to the sum of the consequents; [V. 12.]

therefore as EB is to FD so are AE and EB together to CF and FD together, that is, AB is to CD as EB is to FD .



Therefore, alternately, AB is to EB as CD is to FD . [V. 16.]

V. 25. The first step in the demonstration of this proposition is "take AG equal to E and CH equal to F "; and here a reference is sometimes given to I. 3. But the magnitudes in the proposition are not necessarily *straight lines*, so that this reference to I. 3 should not be given; it must however be assumed that we can perform on the magnitudes considered, an operation similar to that which is performed on straight lines in I. 3. Since the fifth Book of the Elements treats of magnitudes generally, and not merely of lengths, areas, and angles, there is no reference made in it to any proposition of the first four Books.

Simson adds four propositions relating to compound ratio, which he distinguishes by the letters F, G, H, K ; it seems however unnecessary to reproduce them as they are now rarely read and never required.

THE SIXTH BOOK.

THE sixth Book of the Elements consists of the application of the theory of proportion to establish properties of geometrical figures.

VI. Def. 1. For an important remark bearing on the first definition, see the note on VI. 5.

VI. Def. 2. The second definition is useless, for Euclid makes no mention of reciprocal figures.

VI. *Def.* 4. The fourth definition is strictly only applicable to a triangle, because no other figure has a point which can be exclusively called its *vertex*. The altitude of a parallelogram is the perpendicular drawn to the base from any point in the opposite side.

VI. 2. The enunciation of this important proposition is open to objection, for the manner in which the sides may be cut is not sufficiently limited. Suppose, for example, that AD is double of DB , and CE double of EA ; the sides are then cut proportionally, for each side is divided into two parts, one of which is double of the other; but DE is not parallel to BC . It should therefore be stated in the enunciation that *the segments terminated at the vertex of the triangle are to be homologous terms in the ratios, that is, are to be the antecedents or the consequents of the ratios.*

It will be observed that there are three figures corresponding to three cases which may exist; for the straight line drawn parallel to one side may cut the other sides, or may cut the other sides when they are produced through the extremities of the base, or may cut the other sides when they are produced through the vertex. In all these cases the triangles which are shewn to be equal have their vertices at the extremities of the base of the given triangle, and have for their common base the straight line which is, either by hypothesis or by demonstration, parallel to the base of the triangle. The triangle with which these two triangles are compared has the same base as they have, and has its vertex coinciding with the vertex of the given triangle.

VI. A. This proposition was supplied by Simson.

VI. 4. We have preferred to adopt the term "triangles which are equiangular to one another," instead of "equiangular triangles," when the words are used in the sense they bear in this proposition. Euclid himself does not use the term *equiangular triangle* in the sense in which the modern editors use it in the Corollary to I. 5, so that he is not prevented from using the term in the sense it bears in the enunciation of VI. 4 and elsewhere; but modern editors, having already employed the term in one sense ought to keep to that sense. In the demonstrations, where Euclid uses such language as "the triangle ABC is equiangular to the triangle DEF ," the modern editors sometimes adopt it, and sometimes change it to "the triangles ABC and DEF are equiangular."

In VI. 4 the manner in which the two triangles are to be

placed is very imperfectly described; their bases are to be in the same straight line and contiguous, their vertices are to be on the same side of the base, and each of the two angles which have a common vertex is to be equal to the remote angle of the other triangle.

By superposition we might deduce VI. 4 immediately from VI. 2.

VI. 5. The hypothesis in VI. 5 involves more than is directly asserted; the enunciation should be, "if the sides of two triangles, *taken in order*, about each of their angles ;" that is, some restriction equivalent to the words *taken in order* should be introduced. It is quite possible that there should be two triangles ABC , DEF , such that AB is to BC as DE is to EF , and BC to CA as DF is to ED , and therefore, by V. 23, AB to AC as DF is to EF ; in this case the sides of the triangles about each of their angles are proportionals, but not in the same order, and the triangles are not necessarily equiangular to one another. For a numerical illustration we may suppose the sides of one triangle to be 3, 4 and 5 feet respectively, and those of another to be 12, 15 and 20 feet respectively. *Walker*.

Each of the two propositions VI. 4 and VI. 5 is the converse of the other. They shew that if two triangles have either of the two properties involved in the definition of similar figures they will have the other also. This is a special property of triangles. In other figures either of the properties may exist alone. For example, any rectangle and a square have their angles equal, but not their sides proportional; while a square and any rhombus have their sides proportional, but not their angles equal.

VI. 7. In VI. 7 the enunciation is imperfect; it should be, "if two triangles have one angle of the one equal to one angle of the other, and the sides about two other angles proportionals, so that the sides subtending the equal angles are homologous; then if each" The imperfection is of the same nature as that which is pointed out in the note on VI. 5. *Walker*.

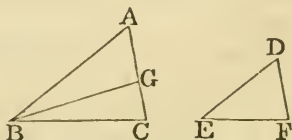
The proposition might be conveniently broken up and the essential part of it presented thus: *if two triangles have two sides of the one proportional to two sides of the other, and the angles opposite to one pair of homologous sides equal, the angles which are opposite to the other pair of homologous sides shall either be equal, or be together equal to two right angles.*

For, the angles included by the proportional sides must be

either equal or unequal. If they are equal, then since the triangles have two angles of the one equal to two angles of the other, each to each, they are equiangular to one another. We have therefore only to consider the case in which the angles included by the proportional sides are unequal.

Let the triangles ABC , DEF have the angle at A equal to the angle at D , and AB to BC as DE is to EF , but the angle ABC not equal to the angle DEF : the angles ACB and DFE shall be together equal to two right angles.

For, one of the angles ABC , DEF must be greater than the other; suppose ABC the greater; and make the angle ABG equal to the angle DEF . Then it may be shewn, as in VI. 7, that BG is equal to



BC , and the angle BGA equal to the angle EFD .

Therefore the angles ACB and DFE are together equal to the angles BGC and AGB , that is, to two right angles.

Then the results enunciated in VI. 7 will readily follow. For if the angles ACB and DFE are both greater than a right angle, or both less than a right angle, or if one of them be a right angle, they must be equal.

VI. 8. In the demonstration of VI. 8, as given by Simson, it is inferred that two triangles which are similar to a third triangle are similar to each other; this is a particular case of VI. 21, which the student should consult, in order to see the validity of the inference.

VI. 9. The word *part* is here used in the restricted sense of the first definition of the fifth Book. VI. 9 is a particular case of VI. 10.

VI. 10. The most important case of this proposition is that in which a straight line is to be divided either *internally* or *externally* into two parts which shall be in a given ratio.

The case in which the straight line is to be divided *internally* is given in the text; suppose, for example, that the given ratio is that of AE to EC ; then AB is divided at G in the given ratio.

Suppose, however, that AB is to be divided *externally* in a given ratio; that is, suppose that AB is to be produced so that the whole straight line made up of AB and the part produced may be to the part produced in a given ratio. Let the given ratio

be that of AC to CE . Join EB ; through C draw a straight line parallel to EB ; then this straight line will meet AB , produced through B , at the required point.

VI. 11. This is a particular case of VI. 12.

VI. 14. The following is a full exhibition of the steps which lead to the result that FB and BG are in one straight line.

The angle DBF is equal to the angle GBE ; [*Hypothesis.*
add to each the angle FBE ;

therefore the angles DBF , FBE are together equal to the angles GBE , FBE . [*Axiom 2.*

But the angles DBF , FBE are together equal to two right angles; [I. 13.

therefore the angles GBE , FBE are together equal to two right angles; [*Axiom 1.*

therefore FB and BG are in one straight line. [I. 14.

VI. 15. This may be inferred from VI. 14, since a triangle is half of a parallelogram with the same base and altitude.

It is not difficult to establish a third proposition conversely connected with the two involved in VI. 14, and a third proposition similarly conversely connected with the two involved in VI. 15. These propositions are the following.

Equal parallelograms which have their sides reciprocally proportional, have their angles equal, each to each.

Equal triangles which have the sides about a pair of angles reciprocally proportional, have those angles equal or together equal to two right angles.

We will take the latter proposition.

Let ABC , ADE be equal triangles; and let CA be to AD as AE is to AB : either the angle BAC shall be equal to the angle DAE , or the angles BAC and DAE shall be together equal to two right angles.

[The student can construct the figure for himself.]

Place the triangles so that CA and AD may be in one straight line; then if EA and AB are in one straight line the angle BAC is equal to the angle DAE . [I. 15.

If EA and AB are not in one straight line, produce BA through A to F , so that AF may be equal to AE ; join DF and EF .

Then because CA is to AD as AE is to AB , [*Hypothesis.*
and AF is equal to AE , [*Construction.*

therefore CA is to AD as AF is to AB . [V. 9, V. 11.

Therefore the triangle DAF is equal to the triangle BAC . [VI. 15.

But the triangle DAE is equal to the triangle BAC . [*Hypothesis*.
Therefore the triangle DAE is equal to the triangle DAF . [*Ax. 1*.
Therefore EF is parallel to AD . [*I. 39*.

Suppose now that the angle DAE is greater than the angle DAF .

Then the angle CAE is equal to the angle AEF , [*I. 29*.
and therefore the angle CAE is equal to the angle AFE , [*I. 5*.
and therefore the angle CAE is equal to the angle BAC . [*I. 29*.
Therefore the angles BAC and DAE are together equal to two right angles.

Similarly the proposition may be demonstrated if the angle DAE is less than the angle DAF .

VI. 16. This is a particular case of VI. 14.

VI. 17. This is a particular case of VI. 16.

VI. 22. There is a step in the second part of VI. 22 which requires examination. After it has been shewn that the figure SR is equal to the similar and similarly situated figure NH , it is added "therefore PR is equal to GH ." In the Greek text reference is here made to a *lemma* which follows the proposition. The word *lemma* is occasionally used in mathematics to denote an auxiliary proposition. From the unusual circumstance of a reference to something following, Simson probably concluded that the lemma could not be Euclid's, and accordingly he takes no notice of it.

The following is the substance of the lemma.

If PR be not equal to GH , one of them must be greater than the other; suppose PR greater than GH .

Then, because SR and NH are similar figures, PR is to PS as GH is to GN . [*VI. Definition 1*.

But PR is greater than GH , [*Hypothesis*.
therefore PS is greater than GN . [*V. 14*.

Therefore the triangle RPS is greater than the triangle HGN . [*I. 4, Axiom 9*.

But, because SR and NH are similar figures, the triangle RPS is equal to the triangle HGN ; [*VI. 20*.
which is impossible.

Therefore PR is equal to GH .

VI. 23. In the figure of VI. 23 suppose BD and GE drawn. Then the triangle BCD is to the triangle GCE as the parallelogram AC is to the parallelogram CF . Hence the result may be extended to triangles, and we have the following theorem,

triangles which have one angle of the one equal to one angle of the other, have to one another the ratio which is compounded of the ratios of their sides.

Then VI. 19 is an immediate consequence of this theorem. For let ABC and DEF be similar triangles, so that AB is to BC as DE is to EF ; and therefore, alternately, AB is to DE as BC is to EF . Then, by the theorem, the triangle ABC has to the triangle DEF the ratio which is compounded of the ratios of AB to DE and of BC to EF , that is, the ratio which is compounded of the ratios of BC to EF and of BC to EF . And, from the definitions of duplicate ratio and of compound ratio, it follows that the ratio compounded of the ratios of BC to EF and of BC to EF is the duplicate ratio of BC to EF .

VI. 25. It will be easy for the student to exhibit in detail the process of shewing that BC and CF are in one straight line, and also LE and EM ; the process is exactly the same as that in I. 45, by which it is shewn that KH and HM are in one straight line, and also FG and GL .

It seems that VI. 25 is out of place, since it separates propositions so closely connected as VI. 24 and VI. 26. We may enunciate VI. 25 in familiar language thus: *to make a figure which shall have the form of one figure and the size of another.*

VI. 26. This proposition is the converse of VI. 24; it might be extended to the case of two similar and similarly situated parallelograms which have a pair of angles *vertically opposite*.

We have omitted in the sixth Book Propositions 27, 28, 29, and the first solution which Euclid gives of Proposition 30, as they appear now to be never required, and have been condemned as useless by various modern commentators; see Austin, Walker, and Lardner. Some idea of the nature of these propositions may be obtained from the following statement of the problem proposed by Euclid in VI. 29. AB is a given straight line; it has to be produced through B to a point O , and a parallelogram described on AO subject to the following conditions; the parallelogram is to be equal to a given rectilineal figure, and the parallelogram on the base BO which can be cut off by a straight line through B is to be similar to a given parallelogram.

VI. 32. This proposition seems of no use. Moreover the enunciation is imperfect. For suppose ED to be produced

through D to a point F , such that DF is equal to DE ; and join CF . Then the triangle CDF will satisfy all the conditions in Euclid's enunciation, as well as the triangle CDE ; but CF and CB are not in one straight line. It should be stated that the bases must lie on corresponding sides of both the parallels; the bases CF and BC do not lie on corresponding sides of the parallels AB and DC , and so the triangle CDF would not fulfil all the conditions, and would therefore be excluded.

VI. 33. In VI. 33 Euclid implicitly gives up the restriction, which he seems to have adopted hitherto, that no angle is to be considered greater than two right angles. For in the demonstration the angle BGL may be any multiple whatever of the angle BGC , and so may be greater than any number of right angles.

VI. B, C, D . These propositions were introduced by Simson. The important proposition VI. D occurs in the *Μεγάλη Σύνταξις* of Ptolemy.

THE ELEVENTH BOOK.

IN addition to the first six Books of the Elements it is usual to read part of the eleventh Book. For an account of the contents of the other Books of the Elements the student is referred to the article *Euclides* in Dr Smith's *Dictionary of Greek and Roman Biography*, and to the article *Irrational Quantities* in the *English Cyclopædia*. We may state briefly that Books VII, VIII, IX treat on Arithmetic, Book X on Irrational Quantities, and Books XI, XII on Solid Geometry.

XI. *Def. 10*. This definition is omitted by Simson, and justly, because, as he shews, it is not true that solid figures contained by the same number of similar and equal plane figures are equal to one another. For, conceive two pyramids, which have their bases similar and equal, but have different altitudes. Suppose one of these bases applied exactly on the other; then if the vertices be put on opposite sides of the base a certain solid is formed, and if the vertices be put on the same side of the base another solid is formed. The two solids thus formed are contained by the same number of similar and equal plane figures, but they are not equal.

It will be observed that in this example one of the solids has a *re-entrant* solid angle; see page 264. It is however true that

two *convex* solid figures are equal if they are contained by equal plane figures similarly arranged; see Catalan's *Théorèmes et Problèmes de Géométrie Élémentaire*. This result was first demonstrated by Cauchy, who turned his attention to the point at the request of Legendre and Malus; see the *Journal de l'École Polytechnique*, Cahier 16.

XI. Def. 26. The word tetrahedron is now often used to denote a solid bounded by any four triangular faces, that is, a pyramid on a triangular base; and when the tetrahedron is to be such as Euclid defines, it is called a regular tetrahedron.

Two other definitions may conveniently be added.

A straight line is said to be parallel to a plane when they do not meet if produced.

The angle made by two straight lines which do not meet is the angle contained by two straight lines parallel to them, drawn through any point.

XI. 21. In XI. 21 the first case only is given in the original. In the second case a certain condition must be introduced, or the proposition will not be true; the polygon *BCDEF* must have no *re-entrant* angle. See note on I. 32.

The propositions in Euclid on Solid Geometry which are now not read, contain some very important results respecting the volumes of solids. We will state these results, as they are often of use; the demonstrations of them are now usually given as examples of the *Integral Calculus*.

We have already explained in the notes to the second Book how the area of a figure is measured by the number of square inches or square feet which it contains. In a similar manner the volume of a solid is measured by the number of *cubic inches* or *cubic feet* which it contains; a *cubic inch* is a cube in which each of the faces is a square inch, and a *cubic foot* is similarly defined.

The volume of a prism is found by multiplying the number of square inches in its base by the number of inches in its altitude; the volume is thus expressed in cubic inches. Or we may multiply the number of square feet in the base by the number of feet in the altitude; the volume is thus expressed in cubic feet. By the *base* of a prism is meant either of the *two equal, similar, and parallel figures* of XI. Definition 13; and the *altitude* of the prism is the perpendicular distance between these two planes.

The rule for the volume of a prism involves the fact that *prisms on equal bases and between the same parallels are equal in volume.*

A parallelepiped is a particular case of a prism. The volume of a pyramid is one third of the volume of a prism on the same base and having the same altitude.

For an account of what are called the *five regular solids* the student is referred to the chapter on *Polyhedrons* in the Treatise on *Spherical Trigonometry*.

THE TWELFTH BOOK.

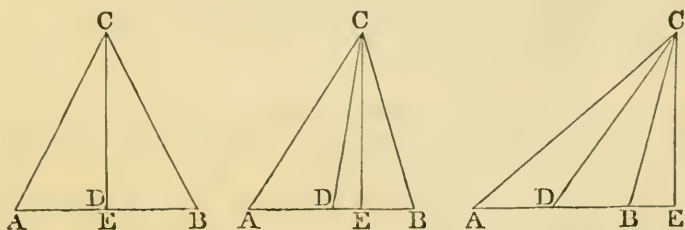
Two propositions are given from the twelfth Book, as they are very important, and are required in the University Examinations. The Lemma is the first proposition of the tenth Book, and is required in the demonstration of the second proposition of the twelfth Book.

A P P E N D I X.

THIS Appendix consists of a collection of important propositions which will be found useful, both as affording geometrical exercises, and as exhibiting results which are often required in mathematical investigations. The student will have no difficulty in drawing for himself the requisite figures in the cases where they are not given.

1. *The sum of the squares on the sides of a triangle is equal to twice the square on half the base, together with twice the square on the straight line which joins the vertex to the middle point of the base.*

Let ABC be a triangle; and let D be the middle point of the base AB . Draw CE perpendicular to the base



meeting it at E ; then E may be either in AB or in AB produced.

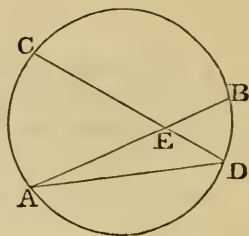
First, let E coincide with D ; then the proposition follows immediately from I. 47.

Next, let E not coincide with D ; then of the two angles ADC and BDC , one must be obtuse and one acute. Suppose the angle ADC obtuse. Then, by II. 12, the square on AC is equal to the squares on AD , DC , together with twice the rectangle AD , DE ; and, by II. 13, the square on BC together with twice the rectangle BD , DE is equal to the squares on BD , DC . Therefore, by Axiom 2, the squares on AC , BC , together with twice the rectangle BD , DE are equal to the squares on AD , DB , and twice the square on DC , together with twice the rectangle AD , DE . But AD is equal to DB . Therefore the squares on AC , BC are equal to twice the squares on AD , DC .

2. *If two chords intersect within a circle, the angle which they include is measured by half the sum of the intercepted arcs.*

Let the chords AB and CD of a circle intersect at E ; join AD .

The angle AEC is equal to the angles ADE , and DAE , by I. 32; that is, to the angles standing on the arcs AC and BD . Thus the angle AEC is equal to an angle at the *circumference* of the circle standing on the sum of the arcs AC and BD ; and is therefore equal to an angle at the *centre* of the circle standing on half the sum of these arcs.

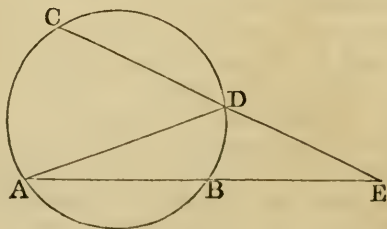


Similarly the angle CEB is measured by half the sum of the arcs CB and AD .

3. *If two chords produced intersect without a circle, the angle which they include is measured by half the difference of the intercepted arcs.*

Let the chords AB and CD of a circle, produced, intersect at E ; join AD .

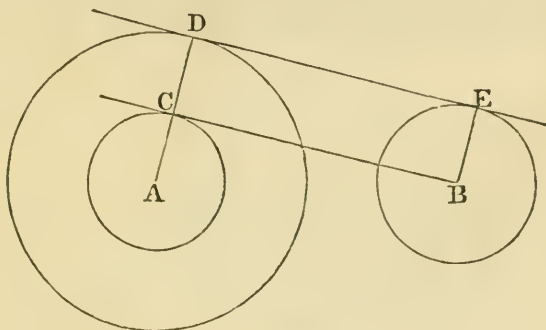
The angle ADC is equal to the angles EAD and AED , by I. 32. Thus the angle AEC is equal to the difference of the angles ADC and BAD ; that is, to an angle at the *circumference* of the circle standing on an arc which is the



difference of AC and BD ; and is therefore equal to an angle at the *centre* of the circle standing on half the difference of these arcs.

4. *To draw a straight line which shall touch two given circles.*

Let A be the centre of the greater circle, and B the centre of the less circle. With centre A , and radius equal to the difference of the radii of the given circles, describe a circle; from B draw a straight line touching the circle



so described at C . Join AC and produce it to meet the circumference at D . Draw the radius BE parallel to AD , and on the same side of AB ; and join DE . Then DE shall touch both circles.

See I. 33, I. 29, and III. 16 Corollary.

Since two straight lines can be drawn from B to touch the described circle, two solutions can be obtained; and the two straight lines which are thus drawn to touch the two given circles can be shewn to meet AB , produced through B , at the same point. The construction is applicable when each of the given circles is without the other, and also when they intersect.

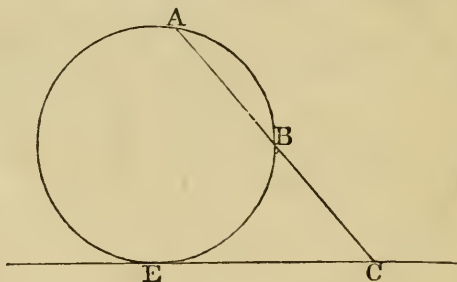
When each of the given circles is without the other we can obtain two other solutions. For, describe a circle with A as a centre and radius equal to the sum of the radii of the given circles; and continue as before, except that BE and AD will now be on *opposite* sides of AB . The two straight lines which are thus drawn to touch the two given circles can be shewn to intersect AB at the same point.

5. *To describe a circle which shall pass through three given points not in the same straight line.*

This is solved in Euclid IV. 5.

6. *To describe a circle which shall pass through two given points on the same side of a given straight line, and touch that straight line.*

Let A and B be the given points; join AB and produce it to meet the given straight line at C . Make a square equal to the rectangle CA, CB (II. 14), and on the



given straight line take CE equal to a side of this square. Describe a circle through A, B, E (5); this will be the circle required (III. 37).

Since E can be taken on either side of C , there are two solutions.

The construction fails if AB is parallel to the given straight line. In this case bisect AB at D , and draw DC at right angles to AB , meeting the given straight line at C . Then describe a circle through A, B, C .

7. *To describe a circle which shall pass through a given point and touch two given straight lines.*

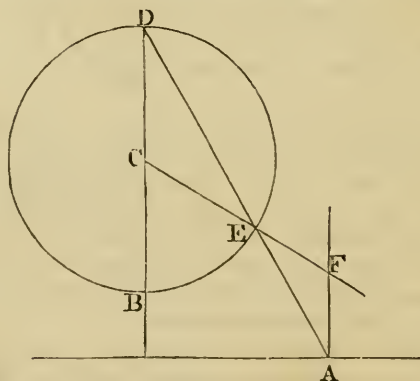
Let A be the given point; produce the given straight lines to meet at B , and join AB . Through B draw a straight line, bisecting that angle included by the given straight lines within which A lies; and in this bisecting straight line take any point C . From C draw a perpendicular on one of the given straight lines, meeting it at D ; with centre C , and radius CD , describe a circle, meeting AB , produced if necessary, at E . Join CE ; and through A draw a straight line parallel to CE , meeting BC , produced if

8. *To describe a circle which shall touch three given straight lines, not more than two of which are parallel.*

Proceed as in Euclid IV. 4. If the given straight lines form a triangle, four circles can be described, namely, one as in Euclid, and three others each touching one side of the triangle and the other two sides produced. If two of the given straight lines are parallel, two circles can be described, namely, one on each side of the third given straight line.

9. *To describe a circle which shall touch a given circle, and touch a given straight line at a given point.*

Let A be the given point in the given straight line, and C be the centre of the given circle. Through C draw a straight line perpendicular to the given straight line,



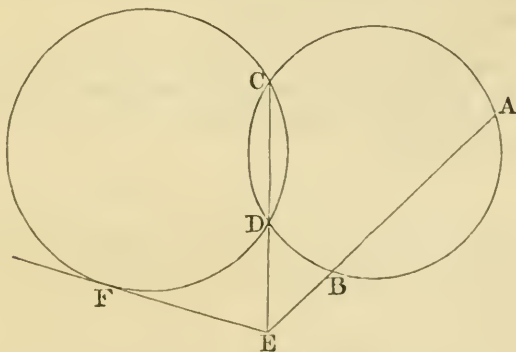
and meeting the circumference of the circle at B and D , of which D is the more remote from the given straight line. Join AD , meeting the circumference of the circle at E . From A draw a straight line at right angles to the given straight line, meeting CE produced at F . Then F shall be the centre of the required circle, and FA its radius.

For the angle AEF is equal to the angle CED (I. 15); and the angle EAF is equal to the angle CDE (I. 29); therefore the angle AEF is equal to the angle EAF ; therefore AF is equal to EF (I. 6).

In a similar manner another solution may be obtained by joining AB . If the given straight line falls without the given circle, the circle obtained by the first solution touches the given circle externally, and the circle obtained by the second solution touches the given circle internally. If the given straight line cuts the given circle, both the circles obtained touch the given circle externally.

10. *To describe a circle which shall pass through two given points and touch a given circle.*

Let A and B be the given points. Take any point C on the circumference of the given circle, and describe a circle through A, B, C . If this described circle touches the given circle, it is the required circle. But if not, let D



be the other point of intersection of the two circles. Let AB and CD be produced to meet at E ; from E draw a straight line touching the given circle at F . Then a circle described through A, B, F shall be the required circle. See III. 35 and III. 37.

There are two solutions, because two straight lines can be drawn from E to touch the given circle.

If the straight line which bisects AB at right angles passes through the centre of the given circle, the construction fails, for AB and CD are parallel. In this case F must be determined by drawing a straight line parallel to AB so as to touch the given circle.

11. *To describe a circle which shall touch two given straight lines and a given circle.*

Draw two straight lines parallel to the given straight lines, at a distance from them equal to the radius of the given circle, and on the sides of them remote from the centre of the given circle. Describe a circle touching the straight lines thus drawn, and passing through the centre of the given circle (7). A circle having the same centre as the circle thus described, and a radius equal to the excess of its radius over that of the given circle, will be the required circle.

Two solutions will be obtained, because there are two solutions of the problem in 7; the circles thus obtained touch the given circle externally.

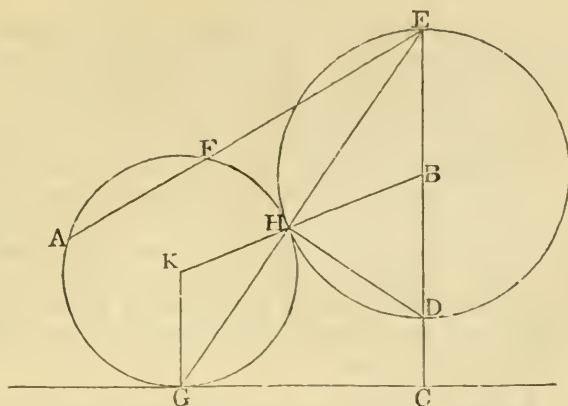
We may obtain two circles which touch the given circle internally, by drawing the straight lines parallel to the given straight lines on the sides of them adjacent to the centre of the given circle.

12. *To describe a circle which shall pass through a given point and touch a given straight line and a given circle.*

We will suppose the given point and the given straight line without the circle; other cases of the problem may be treated in a similar manner.

Let A be the given point, and B the centre of the given circle. From B draw a perpendicular to the given straight line, meeting it at C , and meeting the circumference of the given circle at D and E , so that D is between B and C . Join EA and determine a point F in EA , produced if necessary, such that the rectangle EA, EF may be equal to the rectangle EC, ED ; this can be done by describing a circle through A, C, D , which will meet EA at the required point (III. 36, *Corollary*). Describe a circle to pass through A and F and touch the given straight line (6); this shall be the required circle.

For, let the circle thus described touch the given straight line at G ; join EG meeting the given circle at H ,



and join DH . Then the triangles EHD and ECG are similar; and therefore the rectangle EC, ED is equal to the rectangle EG, EH (III. 31, VI. 4, VI. 16). Thus the rectangle EA, EF is equal to the rectangle EH, EG ; and therefore H is on the circumference of the described circle (III. 36, *Corollary*). Take K the centre of the described circle; join KG, KH , and BH . Then it may be shewn that the angles KHG and EHB are equal (I. 29, I. 5). Therefore KHB is a straight line; and therefore the described circle touches the given circle.

Two solutions will be obtained, because there are two solutions of the problem in 6; the circles thus described touch the given circle externally.

By joining DA instead of EA we can obtain two solutions in which the circles described touch the given circle internally.

13. *To describe a circle which shall touch a given straight line and two given circles.*

Let A be the centre of the larger circle and B the centre of the smaller circle. Draw a straight line parallel to the given straight line, at a distance from it equal to the radius of the smaller circle, and on the side of it remote from A . Describe a circle with A as centre, and radius equal to the difference of the radii of the given circles. Describe a circle which shall pass through B , touch externally the circle just described, and also touch the straight line which has been drawn parallel to the given straight line (12). Then a circle having the same centre as the second described circle, and a radius equal to the excess of its radius over the radius of the smaller given circle, will be the required circle.

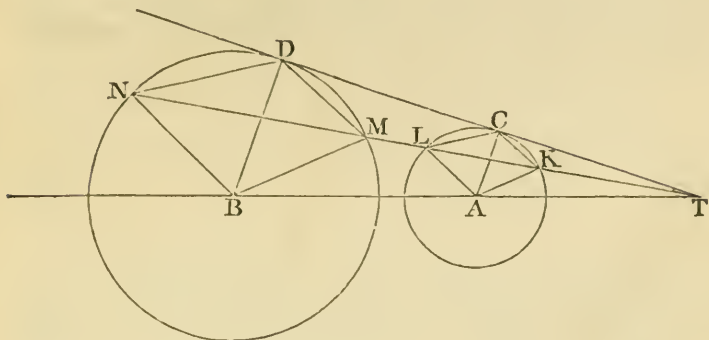
Two solutions will be obtained, because there are two solutions of the problem in 12; the circles thus described touch the given circles externally.

We may obtain in a similar manner circles which touch the given circles internally, and also circles which touch one of the given circles internally and the other externally.

14. *Let A be the centre of a circle, and B the centre of a larger circle; let a straight line be drawn touching the former circle at C and the latter circle at D , and meeting AB produced through A at T . From T draw any straight line meeting the smaller circle at K and L , and the larger circle at M and N ; so that the five letters T, K, L, M, N are in this order. Then the straight lines AK, KC, CL, LA shall be respectively parallel to the straight lines BM, MD, DN, NB ; and the rectangle TK, TN shall be equal to the rectangle TL, TM , and equal to the rectangle TC, TD .*

Join AC, BD . Then the triangles TAC and TBD are

equiangular; and therefore TA is to TB as AC is to BD (VI. 4, V. 16), that is, as AK is to BM .



Therefore the triangles TAK and TBM are similar (VI. 7); therefore the angle TAK is equal to the angle TBM ; and therefore AK is parallel to BM . Similarly AL is parallel to BN . And because AK is parallel to BM and AC parallel to BD , the angle CAK is equal to the angle DBM ; and therefore the angle CLK is equal to the angle DNM (III. 20); and therefore CL is parallel to DN . Similarly CK is parallel to DM .

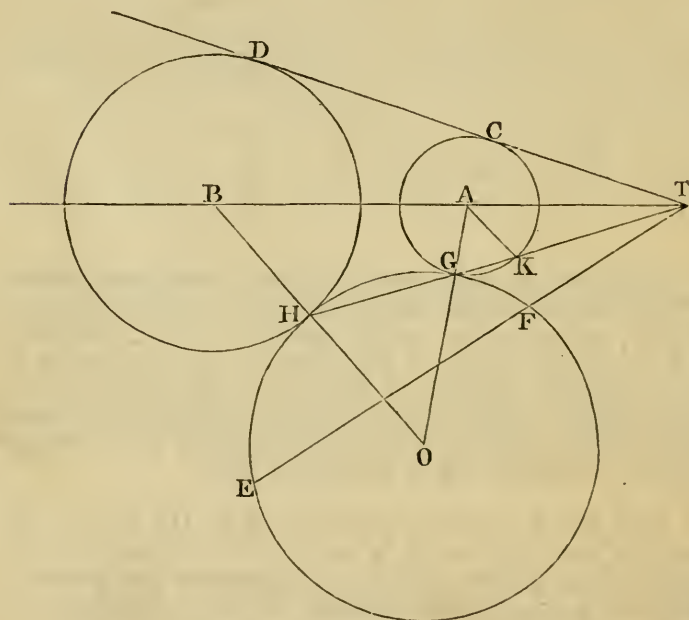
Now TM is to TD as TD is to TN (III. 37, VI. 16); and TM is to TD as TK is to TC (VI. 4); therefore TK is to TC as TD is to TN ; and therefore the rectangle TK , TN is equal to the rectangle TC , TD . Similarly the rectangle TL , TM is equal to the rectangle TC , TD .

If each of the given circles is without the other we may suppose the straight line which touches both circles to meet AB at T between A and B , and the above results will all hold, provided we interchange the letters K and L ; so that the five letters are now to be in the following order, L , K , T , M , N .

The point T is called a *centre of similitude* of the two circles.

15. *To describe a circle which shall pass through a given point and touch two given circles.*

Let A be the centre of the smaller circle and B the centre of the larger circle; and let E be the given point.



Draw a straight line touching the former circle at C and the latter at D , and meeting the straight line AB , produced through A , at T . Join TE and divide it at F so that the rectangle TE, TF may be equal to the rectangle TC, TD . Then describe a circle to pass through E and F , and touch either of the given circles (10); this shall be the required circle.

For suppose that the circle is described so as to touch the smaller given circle; let G be the point of contact; we have then to shew that the described circle will also touch the larger given circle. Join TG , and produce it to meet the larger given circle at H . Then the rectangle TG, TH is equal to the rectangle TC, TD (14); therefore the rectangle TG, TH is equal to the rectangle TE, TF ; and therefore the described circle passes through H .

Let O be the centre of this circle, so that OGA is a straight line; we have to shew that OHB is a straight line.

Let TG intersect the smaller circle again at K ; then AK is parallel to BH (14); therefore the angle AKT is equal to the angle BHG ; and the angle AKG is equal to the angle AGK , which is equal to the angle OGH , which is equal to the angle OHG . Therefore the angles BHG and OHG together are equal to AKT and AKG together; that is, to two right angles. Therefore OHB is a straight line.

Two solutions will be obtained, because there are two solutions of the problem in 10. Also, if each of the given circles is without the other, two other solutions can be obtained by taking for T the point between A and B where a straight line touching the two given circles meets AB . The various solutions correspond to the circumstance that the contact of circles may be external or internal.

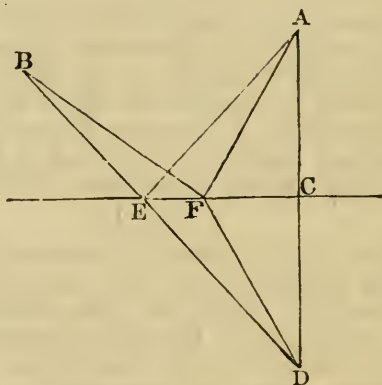
16. *To describe a circle which shall touch three given circles.*

Let A be the centre of that circle which is not greater than either of the other circles; let B and C be the centres of the other circles. With centre B , and radius equal to the excess of the radius of the circle with centre B over the radius of the circle with centre A , describe a circle. Also with centre C , and radius equal to the excess of the radius of the circle with centre C over the radius of the circle with centre A , describe a circle. Describe a circle to touch externally these two described circles and to pass through A (15). Then a circle having the same centre as the last described circle, and having a radius equal to

the excess of its radius over the radius of the circle with centre A , will touch externally the three given circles.

In a similar way we may describe a circle touching internally the three given circles, or touching one of them externally and the two others internally, or touching one of them internally and the two others externally.

17. *In a given indefinite straight line it is required to find a point such that the sum of its distances from two given points on the same side of the straight line shall be the least possible.*



Let A and B be the two given points. From A draw a perpendicular to the given straight line meeting it at C ; and produce AC to D so that CD may be equal to AC . Join DB meeting the given straight line at E . Then E shall be the required point.

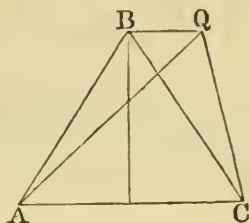
For, let F be any other point in the given straight line. Then, because AC is equal to DC , and EC is common to the two triangles ACE , DCE ; and that the right angle ACE is equal to the right angle DCE ; therefore AE is equal to DE . Similarly, AF is equal to DF . And the sum of DF and FB is greater than BD (I. 20): therefore the sum of AF and FB is greater than BD ; that is, the sum of AF and FB is greater than the sum of DE and EB ; therefore the sum of AF and FB is greater than the sum of AE and EB .

18. *The perimeter of an isosceles triangle is less than that of any other triangle of equal area standing on the same base.*

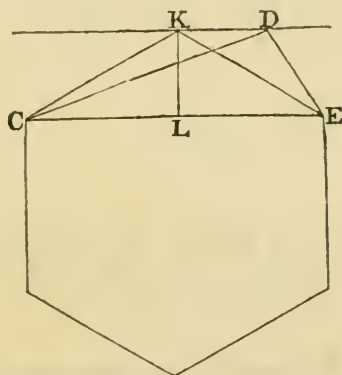
Let ABC be an isosceles triangle; AQC any other triangle equal in area and standing on the same base AC .

Join BQ ; then BQ is parallel to AC (I. 39).

And it will follow from 17 that the sum of AQ and QC is greater than the sum of AB and BC .



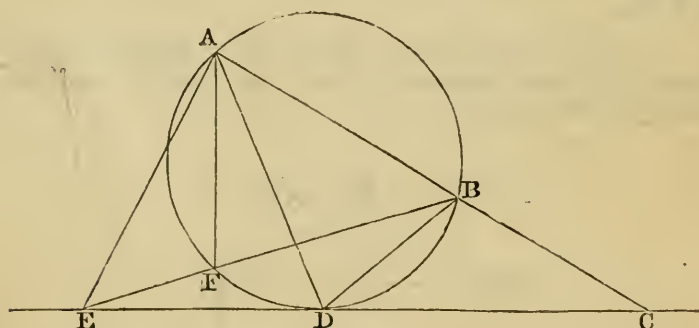
19. *If a polygon be not equilateral a polygon may be found of the same number of sides, and equal in area, but having a less perimeter.*



For, let CD , DE be two adjacent unequal sides of the polygon. Join CE . Through D draw a straight line parallel to CE . Bisect CE at L ; from L draw a straight line at right angles to CE meeting the straight line drawn through D at K . Then by removing from the given polygon the triangle CDE and applying the triangle CKE , we obtain a polygon having the same number of sides as the given polygon, and equal to it in area, but having a less perimeter (18).

20. *A and B are two given points on the same side of a given straight line, and AB produced meets the given straight line at C; of all points in the given straight line on each side of C, it is required to determine that at which AB subtends the greatest angle.*

Describe a circle to pass through *A* and *B*, and to touch the given straight line on that side of *C* which is to be considered (6). Let *D* be the point of contact: *D* shall be the required point.



For, take any other point *E* in the given straight line, on the same side of *C* as *D* is; draw *EA*, *EB*; then one at least of these straight lines will cut the circumference *ADB*.

Suppose that *BE* cuts the circumference at *F*; join *AF*. Then the angle *AFB* is equal to the angle *ADB* (III. 21); and the angle *AFB* is greater than the angle *AEB* (I. 16); therefore the angle *ADB* is greater than the angle *AEB*.

21. *A and B are two given points within a circle; and AB is drawn and produced both ways so as to divide the whole circumference into two arcs; it is required to determine the point in each of these arcs at which AB subtends the greatest angle.*

Describe a circle to pass through A and B and to touch the circumference considered (10): the point of contact will be the required point. The demonstration is similar to that in the preceding proposition.

22. *A and B are two given points without a given circle; it is required to determine the points on the circumference of the given circle at which AB subtends the greatest and least angles.*

Suppose that neither AB nor AB produced cuts the given circle.

Describe two circles to pass through A and B , and to touch the given circle (10): the point of contact of the circle which touches the given circle externally will be the point where the angle is greatest, and the point of contact of the circle which touches the given circle internally will be the point where the angle is least. The demonstration is similar to that in 20.

If AB cuts the given circle, both the circles obtained by 10 touch the given circle internally; in this case the angle subtended by AB at a point of contact is less than the angle subtended at any other point of the circumference of the given circle which is on the same side of AB . Here the angle is greatest at the points where AB cuts the circle, and is there equal to two right angles.

If AB produced cuts the given circle, both the circles obtained by 10 touch the given circle externally; in this case the angle subtended by AB at a point of contact is greater than the angle subtended at any other point of the circumference of the given circle which is on the same side of AB . Here the angle is least at the points where AB produced cuts the circle, and is there zero.

23. *If there be four magnitudes such that the first is to the second as the third is to the fourth; then shall the first together with the second be to the excess of the first above the second as the third together with the fourth is to the excess of the third above the fourth.*

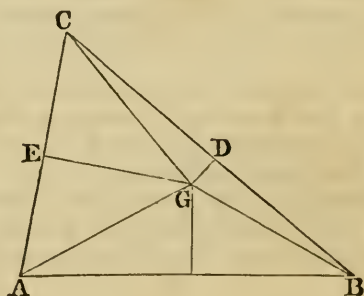
For, the first together with the second is to the second as the third together with the fourth is to the fourth (V. 18). Therefore, alternately, the first together with the second is to the third together with the fourth as the second is to the fourth (V. 16).

Similarly, by V. 17 and V. 16, the excess of the first above the second is to the excess of the third above the fourth as the second is to the fourth.

Therefore, by V. 11, the first together with the second is to the excess of the first above the second as the third together with the fourth is to the excess of the third above the fourth.

24. *The straight lines drawn at right angles to the sides of a triangle from the points of bisection of the sides meet at the same point.*

Let ABC be a triangle; bisect BC at D , and bisect CA at E ; from D draw a straight line at right angles to BC , and from E draw a straight line at right angles to CA ;

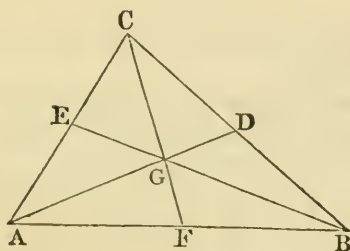


let these straight lines meet at G : we have then to shew that the straight line which bisects AB at right angles also passes through G . From the triangles BDG and

CDG we can shew that BG is equal to CG ; and from the triangles CEG and AEG we can shew that CG is equal to AG ; therefore BG is equal to AG . Then if we draw a straight line from G to the middle point of AB we can shew that this straight line is at right angles to AB : that is, the line which bisects AB at right angles passes through G .

25. *The straight lines drawn from the angles of a triangle to the points of bisection of the opposite sides meet at the same point.*

Let ABC be a triangle; bisect BC at D , bisect CA at E , and bisect AB at F ; join BE and CF meeting at G ;



join AG and GD : then AG and GD shall lie in a straight line.

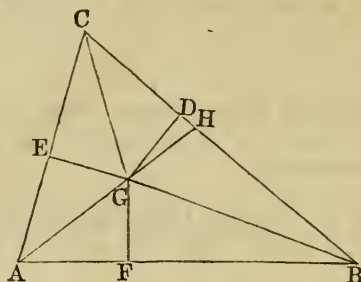
The triangle BEA is equal to the triangle BEC , and the triangle GEA is equal to the triangle GEC (I. 38); therefore, by the third Axiom, the triangle BGA is equal to the triangle BGC .

Similarly, the triangle CGA is equal to the triangle CGB .

Therefore the triangle BGA is equal to the triangle CGA . And the triangle BGD is equal to the triangle CGD (I. 38); therefore the triangles BGA and BGD together are equal to the triangles CGA and CGD together. Therefore the triangles BGA and BGD together are equal to half the triangle ABC . Therefore G must fall on the straight line AD ; that is, AG and GD lie in a straight line.

26. *The straight lines which bisect the angles of a triangle meet at the same point.*

Let ABC be a triangle; bisect the angles at B and C



by straight lines meeting at G ; join AG : then AG shall bisect the angle at A .

From G draw GD perpendicular to BC , GE perpendicular to CA , and GF perpendicular to AB .

From the triangles BGF and BGD we can shew that GF is equal to GD ; and from the triangles CGE and CGD we can shew that GE is equal to GD ; therefore GF is equal to GE . Then from the triangles AFG and AEG we can shew that the angle FAG is equal to the angle EAG .

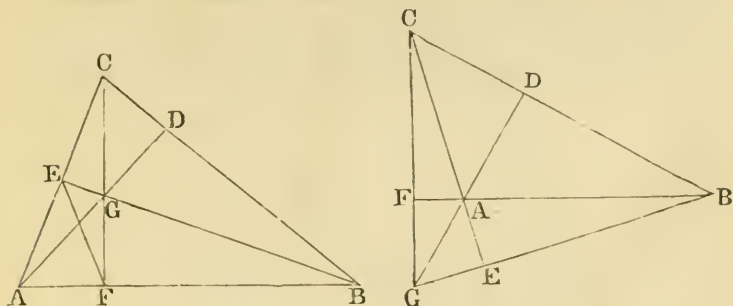
The theorem may also be demonstrated thus. Produce AG to meet BC at H . Then AB is to BH as AG is to GH , and AC is to CH as AG is to GH (VI. 3); therefore AB is to BH as AC is to CH (V. 11); therefore AB is to AC as BH is to CH (V. 16); therefore the straight line AH bisects the angle at A (VI. 3).

27. *Let two sides of a triangle be produced through the base; then the straight lines which bisect the two exterior angles thus formed, and the straight line which bisects the vertical angle of the triangle, meet at the same point.*

This may be shewn like 26: if we adopt the second method we shall have to use VI. A.

28. *The perpendiculars drawn from the angles of a triangle on the opposite sides meet at the same point.*

Let ABC be a triangle; and first suppose that it is not obtuse angled. From B draw BE perpendicular to CA ;



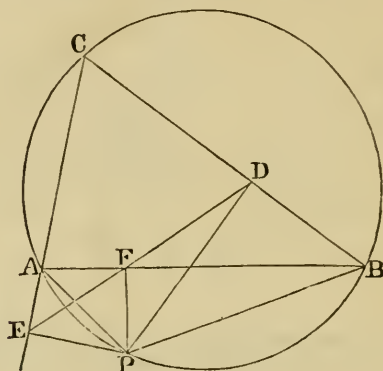
from C draw CF perpendicular to AB ; let these perpendiculars meet at G ; join AG , and produce it to meet BC at D : then AD shall be perpendicular to BC .

For a circle will go round $AEGF$ (*Note on III. 22*); therefore the angle FAG is equal to the angle FEG (III. 21). And a circle will go round $BCEF$ (III. 31, *Note on III. 21*); therefore the angle FEB is equal to the angle FCB . Therefore the angle BAD is equal to the angle BCF . And the angle at B is common to the two triangles BAD and BCF . Therefore the third angle BDA is equal to the third angle BFC (*Note on I. 32*). But the angle BFC is a right angle, by construction; therefore the angle BDA is a right angle.

In the same way the theorem may be demonstrated when the triangle is obtuse angled. Or this case may be deduced from what has been already shewn. For suppose the angle at A obtuse, and let the perpendicular from B on the opposite side meet that side *produced* at E , and let the perpendicular from C on the opposite side meet that side *produced* at F ; and let BE and CF be produced to meet at G . Then in the triangle BCG the perpendiculars BF and CE meet at A ; therefore by the former case the straight line GA produced will be perpendicular to BC .

29. *If from any point in the circumference of the circle described round a triangle perpendiculars be drawn to the sides of the triangle, the three points of intersection are in the same straight line.*

Let ABC be a triangle, P any point on the circumference of the circumscribing circle; from P draw PD ,



PE , PF perpendiculars to the sides BC , CA , AB respectively: D , E , F shall be in the same straight line.

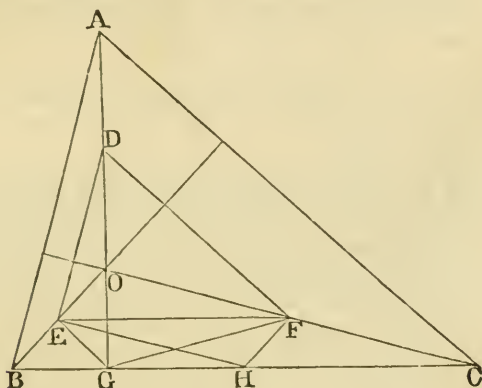
[We will suppose that P is on the arc cut off by AB , on the opposite side from C , and that E is on CA produced through A ; the demonstration will only have to be slightly modified for any other figure.]

A circle will go round $PEAF$ (*Note on III. 22*); therefore the angle PFE is equal to the angle PAE (*III. 21*). But the angles PAE and PAC are together equal to two right angles (*I. 13*); and the angles PAC and PBC are together equal to two right angles (*III. 22*). Therefore the angle PAE is equal to the angle PBC ; therefore the angle PFE is equal to the angle PBC .

Again, a circle will go round $PFDB$ (*Note on III. 21*); therefore the angles PFD and PBD are together equal to two right angles (*III. 22*). But the angle PBD has been shewn equal to the angle PFE . Therefore the angles PFD and PFE are together equal to two right angles. Therefore EF and FD are in the same straight line.

30. *ABC is a triangle, and O is the point of intersection of the perpendiculars from A, B, C on the opposite sides of the triangle: the circle which passes through the middle points of OA, OB, OC will pass through the feet of the perpendiculars and through the middle points of the sides of the triangle.*

Let *D, E, F* be the middle points of *OA, OB, OC* respectively; let *G* be the foot of the perpendicular from *A* on *BC*, and *H* the middle point of *BC*.



Then *OBG* is a right-angled triangle and *E* is the middle point of the hypotenuse *OB*; therefore *EG* is equal to *EO*; therefore the angle *EGO* is equal to the angle *EOG*. Similarly, the angle *FGO* is equal to the angle *FOG*. Therefore the angle *FGE* is equal to the angle *FOE*. But the angles *FOE* and *BAC* are together equal to two right angles; therefore the angles *FGE* and *BAC* are together equal to two right angles. And the angle *BAC* is equal to the angle *EDF*, because *ED, DF* are parallel to *BA, AC* (VI. 2). Therefore the angles *FGE* and *EDF* are together equal to two right angles. Hence *G* is on the circumference of the circle which passes through *D, E, F* (Note on III. 22).

Again, *FH* is parallel to *OB*, and *EH* parallel to *OC*; therefore the angle *EHF* is equal to the angle *EGF*. Therefore *H* is also on the circumference of the circle.

Similarly, the two points in each of the other sides of the triangle ABC may be shewn to be on the circumference of the circle.

The circle which is thus shewn to pass through these nine points may be called the *Nine points circle*: it has some curious properties, of which we will now give two.

The radius of the Nine points circle is half of the radius of the circle described round the original triangle.

For the triangle DEF has its sides respectively halves of the sides of the triangle ABC , so that the triangles are similar. Hence the radius of the circle described round DEF is half of the radius of the circle described round ABC .

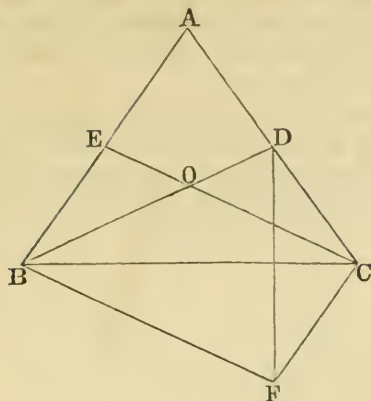
If S be the centre of the circle described round the triangle ABC , the centre of the Nine points circle is the middle point of SO .

For HS is at right angles to BC , and therefore parallel to GO . Hence the straight line which bisects HG at right angles must bisect SO . And H and G are on the circumference of the Nine points circle, so that the straight line which bisects HG at right angles must pass through the centre of the Nine points circle. Similarly, from the other sides of the triangle ABC two other straight lines can be obtained, which pass through the centre of the Nine points circle and also bisect SO . Hence the centre of the Nine points circle must coincide with the middle point of SO .

We may state that the *Nine points circle* of any triangle touches the inscribed circle and the escribed circles of the triangle: a demonstration of this theorem will be found in the *Nouvelles Annales de Mathématiques* for 1842, page 196. For the history of this theorem see the volume of the same Journal for 1863, page 562.

31. *If two straight lines bisecting two angles of a triangle and terminated at the opposite sides be equal, the bisected angles shall be equal.*

Let ABC be a triangle; let the straight line BD bisect the angle at B , and be terminated at the side AC ; and let the straight line CE bisect the angle at C , and be terminated at the side AB ; and let the straight line BD be equal to the straight line CE : then the angle at B shall be equal to the angle at C .



For, let BD and CE meet at O ; then if the angle OBC be not equal to the angle OCB , one of them must be greater than the other; let the angle OBC be the greater. Then, because CB and BD are equal to BC and CE , each to each; but the angle CBD is greater than the angle BCE ; therefore CD is greater than BE (I. 24).

On the other side of the base BC make the triangle BCF equal to the triangle CBE , so that BF may be equal to CE , and CF equal to BE (I. 22); and join DF .

Then because BF is equal to BD , the angle BFD is equal to the angle BDF . And the angle OCD is, by hypothesis, less than the angle OBE ; and the angle COD is equal to the angle BOE ; therefore the angle ODC is greater than the angle OEB (I. 32), and therefore the angle ODC is greater than the angle BFC .

Hence, by taking away the equal angles BDF and BFD , the angle FDC is greater than the angle DFC ; and therefore CF is greater than CD (I. 19); therefore BE is greater than CD .

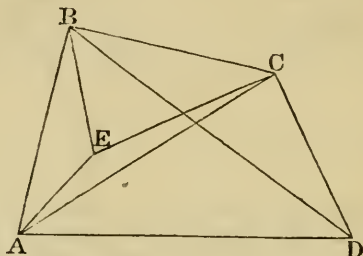
But it was shewn that CD is greater than BE ; which is absurd.

Therefore the angles OBC and OCB are not unequal, that is, they are equal; and therefore the angle ABC is equal to the angle ACB .

[For the history of this theorem see *Lady's and Gentleman's Diary* for 1859, page 88.]

32. *If a quadrilateral figure does not admit of having a circle described round it, the sum of the rectangles contained by the opposite sides is greater than the rectangle contained by the diagonals.*

Let $ABCD$ be a quadrilateral figure which does not admit of having a circle described round it; then the rectangle AB, DC , together with the rectangle BC, AD , shall be greater than the rectangle AC, BD .



For, make the angle ABE equal to the angle DBC , and the angle BAE equal to the angle BDC ; then the triangle ABE is similar to the triangle BDC (VI. 4); therefore AB is to AE as DB is to DC ; and therefore the rectangle AB, DC is equal to the rectangle AE, DB .

Join EC . Then, since the angle ABE is equal to the angle DBC , the angle CBE is equal to the angle DBA . And because the triangles ABE and BDC are similar, AB is to DB as BE is to BC ; therefore the triangles ABD and EBC are similar (VI. 6); therefore CB is to CE as DB is to DA ; and therefore the rectangle CB, DA is equal to the rectangle CE, DB .

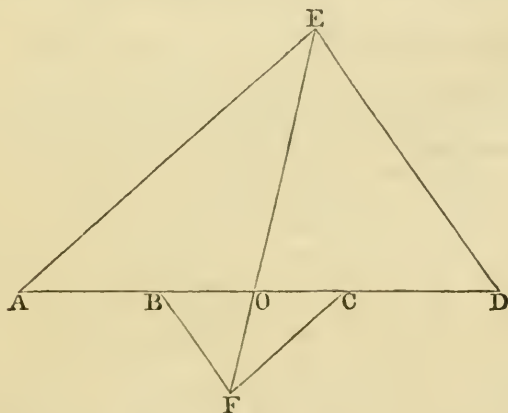
Therefore the rectangle AB, DC , together with the rectangle BC, AD is equal to the rectangle AE, BD together with the rectangle CE, BD ; that is, equal to the rectangle contained by BD and the sum of AE and EC . But the sum of AE and EC is greater than AC (I. 20); therefore the rectangle AB, DC , together with the rectangle BC, AD is greater than the rectangle AC, BD .

33. *If the rectangle contained by the diagonals of a quadrilateral be equal to the sum of the rectangles contained by the opposite sides, a circle can be described round the quadrilateral.*

This is the converse of VI. *D*; it can be demonstrated indirectly with the aid of 32.

34. *It is required to find a point in a given straight line, such that the rectangle contained by its distances from two given points in the straight line may be equal to the rectangle contained by its distances from two other given points in the straight line.*

Let *A, B, C, D* be four given points in the same straight line: it is required to find a point in the straight



line, such that the rectangle contained by its distances from *A* and *B* may be equal to the rectangle contained by its distances from *C* and *D*.

On *AD* describe any triangle *AED*; and on *CB* describe a similar triangle *CFB*, so that *CF* is parallel to *AE*, and *BF* to *DE*; join *EF*, and let it meet the given straight line at *O*. Then *O* shall be the required point.

For, *OE* is to *OA* as *OF* is to *OC* (VI. 4); therefore *OE* is to *OF* as *OA* is to *OC* (V. 16). Similarly *OE* is to *OF* as *OD* is to *OB*. Therefore *OA* is to *OC* as *OD* is to *OB* (V. 11). Therefore the rectangle *OA, OB* is equal to the rectangle *OC, OD*.

The figure will vary slightly according to the situation of the four given points, but corresponding to an assigned situation there will be only *one* point such as is required. For suppose there could be such a point P , besides the point O which is determined by the construction given above; and that the points are in the order A, C, D, B, O, P . Join PE , and let it meet CF , produced at G ; join BG . Then the rectangle PA, PB is, by hypothesis, equal to the rectangle PC, PD ; and therefore PA is to PC as PD is to PB . But PA is to PC as PE is to PG (VI. 2); therefore PD is to PB as PE is to PG (V. 11); therefore BG is parallel to DE .

But, by the construction, BF is parallel to ED ; therefore BG and BF are themselves parallel (I. 30); which is absurd. Therefore P is not such a point as is required.

ON GEOMETRICAL ANALYSIS.

35. The substantives *analysis* and *synthesis*, and the corresponding adjectives *analytical* and *synthetical*, are of frequent occurrence in mathematics. In general *analysis* means decomposition, or the separating a whole into its parts, and *synthesis* means composition, or making a whole out of its parts. In Geometry however these words are used in a more special sense. In synthesis we begin with results already established, and end with some new result; thus, by the aid of theorems already demonstrated, and problems already solved, we demonstrate some new theorem, or solve some new problem. In analysis we begin with assuming the truth of some theorem or the solution of some problem, and we deduce from the assumption consequences which we can compare with results already established, and thus test the validity of our assumption.

36. The propositions in Euclid's Elements are all exhibited synthetically; the student is only employed in examining the soundness of the reasoning by which each successive addition is made to the collection of geometrical truths already obtained; and there is no hint given as to the manner in which the propositions were originally discovered. Some of the constructions and demonstrations appear rather artificial, and we are thus naturally induced to enquire whether any rules can be discovered by which we may be guided easily and naturally to the investigation of new propositions.

37. Geometrical analysis has sometimes been described in language which might lead to the expectation that directions could be given which would enable a student to proceed to the demonstration of any proposed theorem, or the solution of any proposed problem, with confidence of success; but no such directions can be given. We will state the exact extent of these directions. Suppose that a new theorem is proposed for investigation, or a new problem for trial. Assume the truth of the theorem or the solution of the problem, and deduce consequences from this assumption combined with results which have been already established. If a consequence can be deduced which contradicts some result already established, this amounts to a demonstration that our assumption is inadmissible; that is, the theorem is not true, or the problem cannot be solved. If a consequence can be deduced which coincides with some result already established, we cannot say that the assumption is inadmissible; and it *may happen* that by starting from the consequence which we deduced, and retracing our steps, we can succeed in giving a synthetical demonstration of the theorem, or solution of the problem. These directions however are very vague, because no certain rule can be prescribed by which we are to combine our assumption with results already established; and moreover no test exists by which we can ascertain whether a valid consequence which we have drawn from an assumption will enable us to establish the assumption itself. That a proposition may be false and yet furnish consequences which are true, can be seen from a simple example. Suppose a theorem were proposed for investigation in the following words; *one angle of a triangle is to another as the side opposite to the first angle is to the side opposite to the other*. If this be assumed to be true we can immediately deduce Euclid's result in I. 19; but from Euclid's result in I. 19 we cannot retrace our steps and establish the proposed theorem, and in fact the proposed theorem is false.

Thus the only definite statement in the directions respecting Geometrical analysis is, that if a consequence can be deduced from an assumed proposition which contradicts a result already established, that assumed proposition must be false.

38. We may mention, in particular, that a consequence would contradict results already established, if we could shew that it would lead to the solution of a problem already given up as impossible. There are three famous problems which are now admitted to be beyond the power of Geometry; namely, to find a straight line equal in length to the circumference of a given circle, to trisect any given angle, and to find two mean proportionals between two given straight lines. The grounds on which the geometrical solution of these problems is admitted to be impossible cannot be explained without a knowledge of the higher parts of mathematics; the student of the Elements may however be content with the fact that innumerable attempts have been made to obtain solutions, and that these attempts have been made in vain.

The first of these problems is usually referred to as the *Quadrature of the Circle*. For the history of it the student should consult the article in the *English Cyclopædia* under that head, and also a series of papers in the *Athenæum* for 1863 and subsequent years, entitled a *Budget of Paradoxes*, by Professor De Morgan.

For *approximate* solutions of the problem we may refer to Davies's edition of Hutton's *Course of Mathematics*, Vol. I. page 400, the *Lady's and Gentleman's Diary* for 1855, page 86, and the *Philosophical Magazine* for April, 1862.

The third of the three problems is often referred to as the *Duplication of the Cube*. See the note on VI. 13 in *Lardner's Euclid*, and a dissertation by C. H. Biering entitled *Historia Problematis Cubi Duplicandi*...Havniæ, 1844.

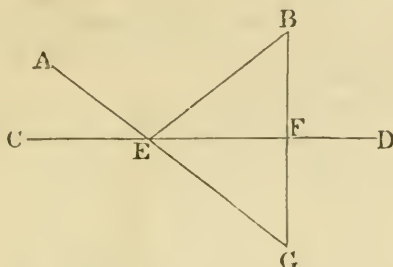
We will now give some examples of Geometrical analysis.

39. *From two given points it is required to draw to the same point in a given straight line, two straight lines equally inclined to the given straight line.*

Let A and B be the given points, and CD the given straight line.

Suppose AE and EB to be the two straight lines equally inclined to CD . Draw BF perpendicular to CD , and produce AE and BF to meet at G . Then the angle

BED is equal to the angle AEC , by hypothesis; and the angle AEC is equal to the angle DEG (I. 15). Hence the

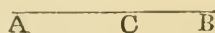


triangles BEF and GEF are equal in all respects (I. 26); therefore FG is equal to FB .

This result shews how we may synthetically solve the problem. Draw BF perpendicular to CD , and produce it to G , so that FG may be equal to FB ; then join AG , and AG will intersect CD at the required point.

40. To divide a given straight line into two parts such that the difference of the squares on the parts may be equal to a given square.

Let AB be the given straight line, and suppose C the required point.



Then the difference of the squares on AC and BC is to be equal to a given square. But the difference of the squares on AC and BC is equal to the rectangle contained by their sum and difference; therefore this rectangle must be equal to the given square. Hence we have the following synthetical solution. On AB describe a rectangle equal to the given square (I. 45); then the difference of AC and CB will be equal to the side of the rectangle adjacent to AB , and is therefore known. And the sum of AC and CB is known. Thus AC and CB are known.

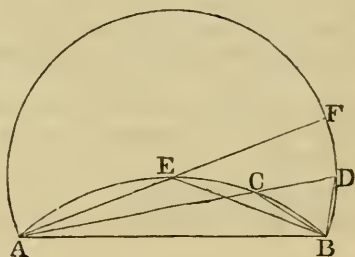
It is obvious that the given square must not exceed the square on AB , in order that the problem may be possible.

There are two positions of C , if it is not specified which of the two segments AC and CB is to be greater than the other; but only one position, if it is specified.

In like manner we may solve the problem, to produce a given straight line so that the square on the whole straight line made up of the given straight line and the part produced, may exceed the square on the part produced by a given square, which is not less than the square on the given straight line.

The two problems may be combined in one enunciation thus, to divide a given straight line internally or externally so that the difference of the squares on the segments may be equal to a given square.

41. To find a point in the circumference of a given segment of a circle, so that the straight lines which join the point to the extremities of the straight line on which the segment stands may be together equal to a given straight line.



Let ACB be the circumference of the given segment, and suppose C the required point, so that the sum of AC and CB is equal to a given straight line.

Produce AC to D so that CD may be equal to CB ; and join DB .

Then AD is equal to the given straight line. And the angle ACB is equal to the sum of the angles CDB and CBD (I. 32), that is, to twice the angle CDB (I. 5). Therefore the angle ADB is half of the angle in the given segment. Hence we have the following synthetical solution. Describe on AB a segment of a circle containing an angle equal to half the angle in the given segment. With A as centre, and a radius equal to the given straight line, describe a circle. Join A with a point of intersection of this circle and the segment which has been described; this

joining straight line will cut the circumference of the given segment at a point which solves the problem.

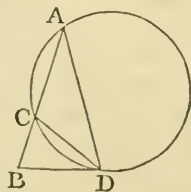
The given straight line must exceed AB and it must not exceed a certain straight line which we will now determine. Suppose the circumference of the given segment bisected at E : join AE , and produce it to meet the circumference of the described segment at F . Then AE is equal to EB (III. 28), and EB is equal to EF for the same reason that CB is equal to CD . Thus EA , EB , EF are all equal; and therefore E is the centre of the circle of which ADB is a segment (III. 9). Hence AF is the longest straight line which can be drawn from A to the circumference of the described segment; so that the given straight line must not exceed twice AE .

42. *To describe an isosceles triangle having each of the angles at the base double of the third angle.*

This problem is solved in IV. 10; we may suppose the solution to have been discovered by such an analysis as the following.

Suppose the triangle ABD such a triangle as is required, so that each of the angles at B and D is double of the angle at A .

Bisect the angle at D by the straight line DC . Then the angle ADC is equal to the angle at A ; therefore CA is equal to CD . The angle CBD is equal to the angle ADB , by hypothesis; the angle CDB is equal to the angle at A ; therefore the third angle BCD is equal to the third angle ABD (I. 32). Therefore BD is equal to CD (I. 6); and therefore BD is equal to AC .



Since the angle BDC is equal to the angle at A , the straight line BD will touch at D the circle described round the triangle ACD (Note on III. 32). Therefore the rectangle AB , BC is equal to the square on BD (III. 36). Therefore the rectangle AB , BC is equal to the square on AC .

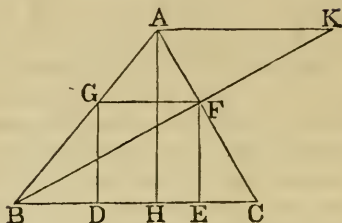
Therefore AB is divided at C in the manner required in II. 11.

Hence the synthetical solution of the problem is evident.

43. *To inscribe a square in a given triangle.*

Let ABC be the given triangle, and suppose $DEFG$ the required square.

Draw AH perpendicular to BC , and AK parallel to BC ; and let BF produced meet AK at K .



Then BG is to GF

as BA is to AK , and BG is to GD as BA is to AH (VI. 4).

But GF is equal to GD , by hypothesis.

Therefore BA is to AK as BA is to AH (V. 7, V. 11).

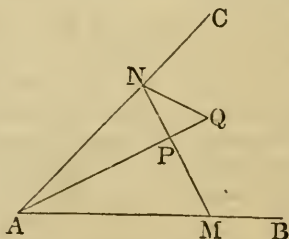
Therefore AH is equal to AK (V. 7).

Hence we have the following synthetical solution. Draw AK parallel to BC , and equal to AH ; and join BK . Then BK meets AC at one of the corners of the required square, and the solution can be completed.

44. *Through a given point between two given straight lines, it is required to draw a straight line, such that the rectangle contained by the parts between the given point and the given straight lines may be equal to a given rectangle.*

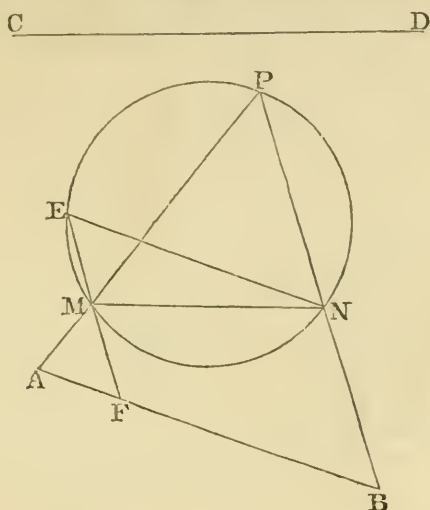
Let P be the given point, and AB and AC the given straight lines; suppose MPN the required straight line, so that the rectangle MP , PN is equal to a given rectangle.

Produce AP to Q , so that the rectangle AP , PQ may be equal to the given rectangle. Then the rectangle MP , PN is equal to the rectangle AP , PQ . Therefore a circle will go round $AMQN$ (Note on III. 35). Therefore the angle PNQ is equal to the angle PAM (III. 21).



Hence we have the following synthetical solution. Produce AP to Q , so that the rectangle AP , PQ may be equal to the given rectangle; describe on PQ a segment of a circle containing an angle equal to the angle PAM ; join P with a point of intersection of this circle and AC ; the straight line thus drawn solves the problem.

45. *In a given circle it is required to inscribe a triangle so that two sides may pass through two given points, and the third side be parallel to a given straight line.*



Let A and B be the given points, and CD the given straight line. Suppose PMN to be the required triangle inscribed in the given circle.

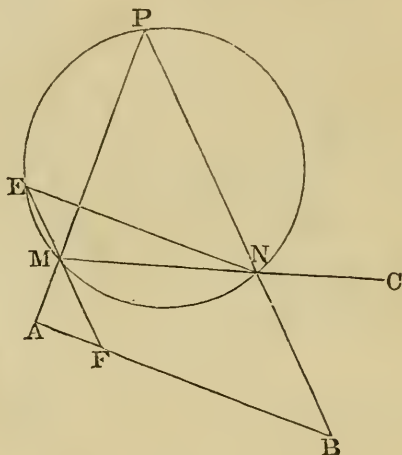
Draw NE parallel to AB ; join EM , and produce it if necessary to meet AB at F .

If the point F were known the problem might be considered solved. For ENM is a known angle, and therefore the chord EM is known in magnitude. And then, since F is a known point, and EM is a known magnitude, the position of M becomes known.

We have then only to shew how F is to be determined. The angle MEN is equal to the angle MFA (I. 29). The angle MEN is equal to the angle MPN (III. 21). Hence MAF and BAP are similar triangles (VI. 4). Therefore MA is to AF as BA is to AP . Therefore the rectangle MA, AP is equal to the rectangle BA, AF (VI. 16). But since A is a *given* point the rectangle MA, AP is known; and AB is known; thus AF is determined.

46. In a given circle it is required to inscribe a triangle so that the sides may pass through three given points.

Let A, B, C be the three given points. Suppose PMN to be the required triangle inscribed in the given circle.



Draw NE parallel to AB , and determine the point F as in the preceding problem. We shall then have to describe in the given circle a triangle EMN so that two of its sides may pass through given points, F and C , and the third side be parallel to a given straight line AB . This can be done by the preceding problem.

This example and the preceding are taken from the work of Catalan already cited. The present problem is sometimes called *Castillon's* and sometimes *Cramer's*; the history of the general researches to which it has given rise will be found in a series of papers in the *Mathematician*, Vol. III. by the late T. S. Davies.

ON LOCI.

47. A *locus* consists of all the points which satisfy certain conditions and of those points alone. Thus, for example, the locus of the points which are at a given distance

from a given point is the surface of the sphere described from the given point as centre, with the given distance as radius; for all the points on this surface, and no other points, are at the given distance from the given point. If we restrict ourselves to all the points in a fixed plane which are at a given distance from a given point, the locus is the circumference of the circle described from the given point as centre, with the given distance as radius. In future we shall restrict ourselves to loci which are situated in a fixed plane, and which are properly called *plane loci*.

Several of the propositions in Euclid furnish good examples of loci. Thus the locus of the vertices of all triangles which are on the same base and on the same side of it, and which have the same area, is a straight line parallel to the base; this is shewn in I. 37 and I. 39.

Again, the locus of the vertices of all triangles which are on the same base and on the same side of it, and which have the same vertical angle, is a segment of a circle described on the base; for it is shewn in III. 21, that all the points thus determined satisfy the assigned conditions, and it is easily shewn that no other points do.

We will now give some examples. In each example we ought to shew not only that all the points which we indicate as the locus do fulfil the assigned conditions, but that no other points do. This second part however we leave to the student in all the examples except the last two; in these, which are more difficult, we have given the complete investigation.

48. *Required the locus of points which are equidistant from two given points.*

Let A and B be the two given points; join AB ; and draw a straight line through the middle point of AB at right angles to AB ; then it may be easily shewn that this straight line is the required locus.

49. *Required the locus of the vertices of all triangles on a given base AB , such that the square on the side terminated at A may exceed the square on the side terminated at B , by a given square.*

Suppose C to denote a point on the required locus; from C draw a perpendicular on the given base, meeting it, pro-

duced if necessary, at D . Then the square on AC is equal to the squares on AD and CD , and the square on BC is equal to the squares on BD and CD (I. 47); therefore the square on AC exceeds the square on BC by as much as the square on AD exceeds the square on BD . Hence D is a fixed point either in AB or in AB produced through B (40). And the required locus is the straight line drawn through D , at right angles to AB .

50. *Required the locus of a point such that the straight lines drawn from it to touch two given circles may be equal.*

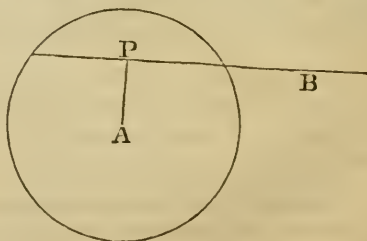
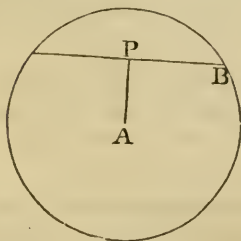
Let A be the centre of the greater circle, B the centre of a smaller circle; and let P denote any point on the required locus. Since the straight lines drawn from P to touch the given circles are equal, the squares on these straight lines are equal. But the squares on PA and PB exceed these equal squares by the squares on the radii of the respective circles. Hence the square on PA exceeds the square on PB , by a known square, namely a square equal to the excess of the square on the radius of the circle of which A is the centre over the square on the radius of the circle of which B is the centre. Hence, the required locus is a certain straight line which is at right angles to AB (49).

This straight line is called the *radical axis* of the two circles.

If the given circles intersect, it follows from III. 36, that the straight line which is the locus coincides with the produced parts of the common chord of the two circles.

51. *Required the locus of the middle points of all the chords of a circle which pass through a fixed point.*

Let A be the centre of the given circle; B the fixed



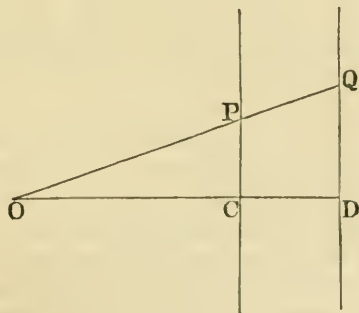
point; let any chord of the circle be drawn so that, produced if necessary, it may pass through B . Let P be the middle point of this chord, so that P is a point on the required locus.

The straight line AP is at right angles to the chord of which P is the middle point (III. 3); therefore P is on the circumference of a circle of which AB is a diameter. Hence if B be within the given circle the locus is the circumference of the circle described on AB as diameter; if B be without the given circle the locus is that part of the circumference of the circle described on AB as diameter, which is within the given circle.

52. O is a fixed point from which any straight line is drawn meeting a fixed straight line at P ; in OP a point Q is taken such that OQ is to OP in a fixed ratio: determine the locus of Q .

We shall shew that the locus of Q is a straight line.

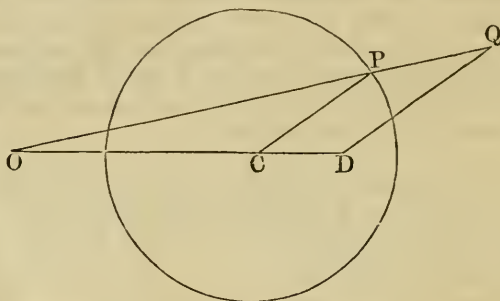
For draw a perpendicular from O on the fixed straight line, meeting it at C ; in OC take a point D such that OD is to OC in the fixed ratio; draw from O any straight line OP meeting the fixed straight line at P , and in OP take a point Q such that OQ is to OP in the fixed ratio; join



QD . The triangles ODQ and OCP are similar (VI. 6); therefore the angle ODQ is equal to the angle OCP , and is therefore a right angle. Hence Q lies in the straight line drawn through D at right angles to OD .

53. *O is a fixed point from which any straight line is drawn meeting the circumference of a fixed circle at P; in OP a point Q is taken such that OQ is to OP in a fixed ratio: determine the locus of Q.*

We shall shew that the locus is the circumference of a circle.



For let C be the centre of the fixed circle; in OC take a point D such that OD is to OC in the fixed ratio, and draw any radius CP of the fixed circle; draw DQ parallel to CP meeting OP , produced if necessary, at Q . Then the triangles OCP and ODQ are similar (VI. 4), and therefore OQ is to OP as OD is to OC , that is, in the fixed ratio. Therefore Q is a point on the locus. And DQ is to CP in the fixed ratio, so that DQ is of constant length. Hence the locus is the circumference of a circle of which D is the centre.

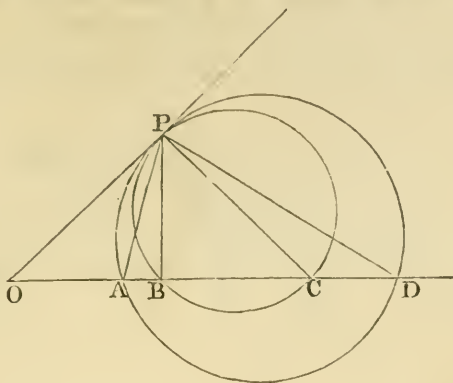
54. *There are four given points A, B, C, D in a straight line; required the locus of a point at which AB and CD subtend equal angles.*

Find a point O in the straight line, such that the rectangle OA, OD may be equal to the rectangle OB, OC (34), and take OK such that the square on OK may be equal to either of these rectangles (II. 14): the circumference of the circle described from O as centre, with radius OK , shall be the required locus.

[We will take the case in which the points are in the following order, O, A, B, C, D .]

For let P be any point on the circumference of this

circle. Describe a circle round PAD , and also a circle



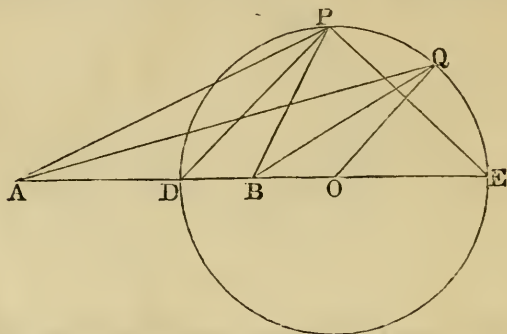
round PBC ; then OP touches each of these circles (III. 37); therefore the angle OPA is equal to the angle PDA , and the angle OPB is equal to the angle PCB (III. 32). But the angle OPB is equal to the angles OPA and APB together, and the angle PCB is equal to the angles CPD and PDA together (I. 32). Therefore the angles OPA and APB together are equal to the angles CPD and PDA together; and the angle OPA has been shewn equal to the angle PDA ; therefore the angle APB is equal to the angle CPD .

We have thus shewn that any point on the circumference of the circle satisfies the assigned conditions; we shall now shew that any point which satisfies the assigned conditions is on the circumference of the circle.

For take any point Q which satisfies the required conditions. Describe a circle round QAD , and also a circle round QBC . These circles will touch the same straight line at Q ; for the angles AQB and CQD are equal, and the converse of III. 32 is true. Let this straight line which touches both circles at Q be drawn; and let it meet the straight line containing the four given points at R . Then the rectangle RA, RD is equal to the rectangle RB, RC ; for each is equal to the square on RQ (III. 36). Therefore R must coincide with O (34); and therefore RQ must be equal to OK . Thus Q must be on the circumference of the circle of which O is the centre, and OK the radius.

55. *Required the locus of the vertices of all the triangles ABC which stand on a given base AB , and have the side AC to the side BC in a constant ratio.*

If the sides AC and BC are to be equal, the locus is



the straight line which bisects AB at right angles. We will suppose that the ratio is greater than a ratio of equality; so that AC is to be the greater side.

Divide AB at D so that AD is to DB in the given ratio (VI. 10); and produce AB to E , so that AE is to EB in the given ratio. Let P be any point in the required locus; join PD and PE . Then PD bisects the angle APB , and PE bisects the angle between BP and AP produced. Therefore the angle DPE is a right angle. Therefore P is on the circumference of a circle described on DE as diameter.

We have thus shewn that any point which satisfies the assigned conditions is on the circumference of the circle described on DE as diameter; we shall now shew that any point on the circumference of this circle satisfies the assigned conditions.

Let Q be any point on the circumference of this circle, QA shall be to QB in the assigned ratio. For, take O the centre of the circle; and join QO . Then, by construction, AE is to EB as AD is to DB , and therefore, alternately, AE is to AD as EB is to DB ; therefore the sum of AE and AD is to their difference as the sum of EB and DB is to their difference (23); that is, twice AO is to twice DO as twice DO is to twice BO ; therefore AO is to DO as DO is

to BO ; that is, AO is to OQ as QO is to OB . Therefore the triangles AOQ and QOB are similar triangles (VI. 6); and therefore AQ is to QB as QO is to BO . This shews that the ratio of AQ to BQ is *constant*; we have still to shew that this ratio is the same as the assigned ratio.

We have already shewn that AO is to DO as DO is to BO ; therefore, the difference of AO and DO is to DO as the difference of DO and BO is to BO (V. 17); that is, AD is to DO as BD is to BO ; therefore AD is to BD as DO is to BO ; that is, AD is to DB as QO is to BO . This shews that the ratio of QO to BO is the same as the assigned ratio.

ON MODERN GEOMETRY.

56. We have hitherto restricted ourselves to Euclid's Elements, and propositions which can be demonstrated by strict adherence to Euclid's methods. In modern times various other methods have been introduced, and have led to numerous and important results. These methods may be called semi-geometrical, as they are not confined within the limits of the ancient pure geometry; in fact the power of the modern methods is obtained chiefly by combining arithmetic and algebra with geometry. The student who desires to cultivate this part of mathematics may consult Townsend's *Chapters on the Modern Geometry of the Point, Line, and Circle*.

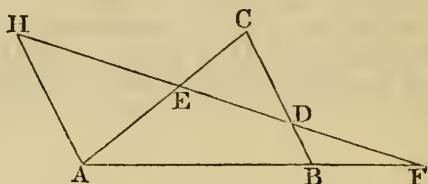
We will give as specimens some important theorems, taken from what is called the theory of transversals.

Any line, straight or curved, which cuts a system of other lines is called a *transversal*; in the examples which we shall give, the lines will be straight lines, and the system will consist of three straight lines forming a triangle.

We will give a brief enunciation of the theorem which we are about to prove, for the sake of assisting the memory in retaining the result; but the enunciation will not be fully comprehended until the demonstration is completed.

57. *If a straight line cut the sides, or the sides produced, of a triangle, the product of three segments in order is equal to the product of the other three segments.*

Let ABC be a triangle, and let a straight line be drawn cutting the side BC at D , the side CA at E , and the side AB produced through B at F . Then BD and DC are



called *segments* of the side BC , and CE and EA are called *segments* of the side CA , and also AF and FB are called *segments* of the side AB .

Through A draw a straight line parallel to BC , meeting DF produced at H .

Then the triangles CED and EAH are equiangular to one another; therefore AH is to CD as AE is to EC (VI. 4).

Therefore the rectangle AH, EC is equal to the rectangle CD, AE (VI. 16).

Again, the triangles FAH and FBD are equiangular to one another; therefore AH is to BD as FA is to FB (VI. 4). Therefore the rectangle AH, FB is equal to the rectangle BD, FA (VI. 16).

Now suppose the straight lines represented by numbers in the manner explained in the notes to the second Book of the Elements. We have then two results which we can express arithmetically: namely, the *product* $AH.EC$ is equal to the *product* $CD.AE$; and the *product* $AH.FB$ is equal to the *product* $BD.FA$.

Therefore, by the principles of arithmetic, the product $AH.EC.BD.FA$ is equal to the product $AH.FB.CD.AE$, and therefore, by the principles of arithmetic, the product $BD.CE.AF$ is equal to the product $DC.EA.FB$.

This is the result intended by the enunciation given above. Each product is made by three segments, one from

every side of the triangle : and the two segments which terminated at any angular point of the triangle are never in the same product. Thus if we begin one product with the segment BD , the other segment of the side BC , namely DC , occurs in the other product ; then the segment CE occurs in the first product, so that the two segments CD and CE , which terminate at C , do not occur in the same product ; and so on.

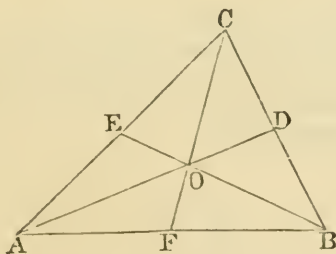
The student should for exercise draw another figure for the case in which the transversal meets *all* the sides *produced*, and obtain the same result.

58. Conversely, it may be shewn by an indirect proof that if the product $BD.CE.AF$ be equal to the product $DC.EA.FB$, the three points D, E, F lie in the same straight line.

59. *If three straight lines be drawn through the angular points of a triangle to the opposite sides, and meet at the same point, the product of three segments in order is equal to the product of the other three segments.*

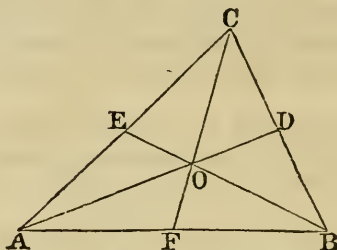
Let ABC be a triangle. From the angular points to the opposite sides let the straight lines AOD, BOE, COF be drawn, which meet at the point O : the product $AF.BD.CE$ shall be equal to the product $FB.DC.EA$.

For the triangle ABD is cut by the transversal FOC , and therefore by the theorem in 57 the following products are equal, $AF.BC.DO$, and $FB.CD.OA$.



Again, the triangle ACD is cut by the transversal EOB , and therefore by the theorem in 57 the following products are equal, $AO.DB.CE$ and $OD.BC.EA$.

Therefore, by the principles of arithmetic, the following products are equal, $AF \cdot BC \cdot DO \cdot AO \cdot DB \cdot CE$ and $FB \cdot CD \cdot OA \cdot OD \cdot BC \cdot EA$. Therefore the following



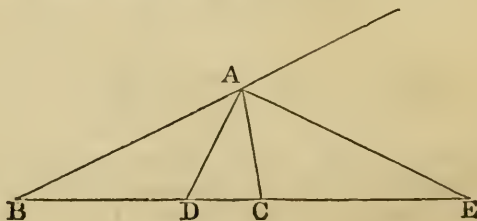
products are equal, $AF \cdot BD \cdot CE$ and $FB \cdot DC \cdot EA$.

We have supposed the point O to be within the triangle; if O be without the triangle two of the points D, E, F will fall on the sides produced.

60. Conversely, it may be shewn by an indirect proof that if the product $AF \cdot BD \cdot CE$ be equal to the product $FB \cdot DC \cdot EA$, the three straight lines AD, BE, CF meet at the same point.

61. We may remark that in geometrical problems the following terms sometimes occur, used in the same sense as in arithmetic; namely *arithmetical progression*, *geometrical progression*, and *harmonical progression*. A proposition respecting harmonical progression, which deserves notice, will now be given.

62. Let ABC be a triangle; let the angle A be bisected by a straight line which meets BC at D , and let the exterior angle at A be bisected by a straight line which meets BC , produced through C , at E : then BD, BC, BE shall be in harmonical progression.



For BD is to DC as BA is to AC (VI. 3); and BE is to EC as BA is to AC (VI. A). Therefore BD is to DC as BE is to EC (V. 11). Therefore BD is to BE as DC is to EC (V. 16). Thus of the three straight lines BD, BC, BE , the first is to the third as the excess of the second over the first is to the excess of the third over the second. Therefore BD, BC, BE are in harmonical progression.

This result is sometimes expressed by saying that BE is divided harmonically at D and C .



EXERCISES IN EUCLID.

I. 1 to 15.

1. On a given straight line describe an isosceles triangle having each of the sides equal to a given straight line.

2. In the figure of I. 2 if the diameter of the smaller circle is the radius of the larger, shew where the given point and the vertex of the constructed triangle will be situated.

3. If two straight lines bisect each other at right angles, any point in either of them is equidistant from the extremities of the other.

4. If the angles ABC and ACB at the base of an isosceles triangle be bisected by the straight lines BD , CD , shew that DBC will be an isosceles triangle.

5. BAC is a triangle having the angle B double of the angle A . If BD bisects the angle B and meets AC at D , shew that BD is equal to AD .

6. In the figure of I. 5 if FC and BG meet at H shew that FH and GH are equal.

7. In the figure of I. 5 if FC and BG meet at H , shew that AH bisects the angle BAC .

8. The sides AB , AD of a quadrilateral $ABCD$ are equal, and the diagonal AC bisects the angle BAD : shew that the sides CB and CD are equal, and that the diagonal AC bisects the angle BCD .

9. ACB , ADB are two triangles on the same side of AB , such that AC is equal to BD , and AD is equal to BC , and AD and BC intersect at O : shew that the triangle AOB is isosceles.

10. The opposite angles of a rhombus are equal.

11. A diagonal of a rhombus bisects each of the angles through which it passes.

12. If two isosceles triangles are on the same base the straight line joining their vertices, or that straight line produced, will bisect the base at right angles.

13. Find a point in a given straight line such that its distances from two given points may be equal.

14. Through two given points on opposite sides of a given straight line draw two straight lines which shall meet in that given straight line, and include an angle bisected by that given straight line.

15. A given angle BAC is bisected; if CA is produced to G and the angle BAG bisected, the two bisecting lines are at right angles.

16. If four straight lines meet at a point so that the opposite angles are equal, these straight lines are two and two in the same straight line.

I. 16 to 26.

17. ABC is a triangle and the angle A is bisected by a straight line which meets BC at D ; shew that BA is greater than BD , and CA greater than CD .

18. In the figure of I. 17 shew that ABC and ACB are together less than two right angles, by joining A to any point in BC .

19. $ABCD$ is a quadrilateral of which AD is the longest side and BC the shortest; shew that the angle ABC is greater than the angle ADC , and the angle BCD greater than the angle BAD .

20. If a straight line be drawn through A one of the angular points of a square, cutting one of the opposite sides, and meeting the other produced at F , shew that AF is greater than the diagonal of the square.

21. The perpendicular is the shortest straight line that can be drawn from a given point to a given straight line; and of others, that which is nearer to the perpendicular is less than the more remote; and two, and only two, equal straight lines can be drawn from the given point to the given straight line, one on each side of the perpendicular.

22. The sum of the distances of any point from the three angles of a triangle is greater than half the sum of the sides of the triangle.

23. The four sides of any quadrilateral are together greater than the two diagonals together.

24. The two sides of a triangle are together greater than twice the straight line drawn from the vertex to the middle point of the base.

25. If one angle of a triangle is equal to the sum of the other two, the triangle can be divided into two isosceles triangles.

26. If the angle C of a triangle is equal to the sum of the angles A and B , the side AB is equal to twice the straight line joining C to the middle point of AB .

27. Construct a triangle, having given the base, one of the angles at the base, and the sum of the sides.

28. The perpendiculars let fall on two sides of a triangle from any point in the straight line bisecting the angle between them are equal to each other.

29. In a given straight line find a point such that the perpendiculars drawn from it to two given straight lines shall be equal.

30. Through a given point draw a straight line such that the perpendiculars on it from two given points may be on opposite sides of it and equal to each other.

31. A straight line bisects the angle A of a triangle ABC ; from B a perpendicular is drawn to this bisecting straight line, meeting it at D , and BD is produced to meet AC or AC produced at E ; shew that BD is equal to DE .

32. AB , AC are any two straight lines meeting at A : through any point P draw a straight line meeting them at E and F , such that AE may be equal to AF .

33. Two right-angled triangles have their hypotenuses equal, and a side of one equal to a side of the other: shew that they are equal in all respects.

I. 27 to 31.

34. Any straight line parallel to the base of an isosceles triangle makes equal angles with the sides.

35. If two straight lines A and B are respectively parallel to two others C and D , shew that the inclination of A to B is equal to that of C to D .

36. A straight line is drawn terminated by two parallel straight lines; through its middle point any straight line is

drawn and terminated by the parallel straight lines. Shew that the second straight line is bisected at the middle point of the first.

37. If through any point equidistant from two parallel straight lines, two straight lines be drawn cutting the parallel straight lines, they will intercept equal portions of these parallel straight lines.

38. If the straight line bisecting the exterior angle of a triangle be parallel to the base, shew that the triangle is isosceles. ✕

39. Find a point B in a given straight line CD , such that if AB be drawn to B from a given point A , the angle ABC will be equal to a given angle.

40. If a straight line be drawn bisecting one of the angles of a triangle to meet the opposite side, the straight lines drawn from the point of section parallel to the other sides, and terminated by these sides, will be equal.

41. The side BC of a triangle ABC is produced to a point D ; the angle ACB is bisected by the straight line CE which meets AB at E . A straight line is drawn through E parallel to BC , meeting AC at F , and the straight line bisecting the exterior angle ACD at G . Shew that EF is equal to FG .

42. AB is the hypotenuse of a right-angled triangle ABC : find a point D in AB such that DB may be equal to the perpendicular from D on AC .

43. ABC is an isosceles triangle: find points D, E in the equal sides AB, AC such that BD, DE, EC may all be equal.

44. A straight line drawn at right angles to BC the base of an isosceles triangle ABC cuts the side AB at D and CA produced at E : shew that AED is an isosceles triangle.

I. 32.

45. From the extremities of the base of an isosceles triangle straight lines are drawn perpendicular to the sides; shew that the angles made by them with the base are each equal to half the vertical angle.

46. On the sides of any triangle ABC equilateral triangles BCD, CAE, ABF are described, all external: shew that the straight lines AD, BE, CF are all equal.

47. What is the magnitude of an angle of a regular octagon?

48. Through two given points draw two straight lines forming with a straight line given in position an equilateral triangle.

49. If the straight lines bisecting the angles at the base of an isosceles triangle be produced to meet, they will contain an angle equal to an exterior angle of the triangle.

50. A is the vertex of an isosceles triangle ABC , and BA is produced to D , so that AD is equal to BA ; and DC is drawn: shew that BCD is a right angle.

51. ABC is a triangle, and the exterior angles at B and C are bisected by the straight lines BD , CD respectively, meeting at D : shew that the angle BDC together with half the angle BAC make up a right angle.

52. Shew that any angle of a triangle is obtuse, right, or acute, according as it is greater than, equal to, or less than the other two angles of the triangle taken together.

✶ 53. Construct an isosceles triangle having the vertical angle four times each of the angles at the base.

54. In the triangle ABC the side BC is bisected at E and AB at G ; AE is produced to F so that EF is equal to AE , and CG is produced to H so that GH is equal to CG : shew that FB and HB are in one straight line.

55. Construct an isosceles triangle which shall have one-third of each angle at the base equal to half the vertical angle.

56. AB , AC are two straight lines given in position: it is required to find in them two points P and Q , such that, PQ being joined, AP and PQ may together be equal to a given straight line, and may contain an angle equal to a given angle.

57. Straight lines are drawn through the extremities of the base of an isosceles triangle, making angles with it on the side remote from the vertex, each equal to one-third of one of the equal angles of the triangle and meeting the sides produced: shew that three of the triangles thus formed are isosceles.

58. AEB , CED are two straight lines intersecting at E ; straight lines AC , DB are drawn forming two triangles ACE , BED ; the angles ACE , DBE are bisected by the straight lines CF , BF , meeting at F . Shew that the angle CFB is equal to half the sum of the angles EAC , EDB .

59. The straight line joining the middle point of the hypotenuse of a right-angled triangle to the right angle is equal to half the hypotenuse.

60. From the angle A of a triangle ABC a perpendicular is drawn to the opposite side, meeting it, produced if necessary, at D ; from the angle B a perpendicular is drawn to the opposite side, meeting it, produced if necessary, at E : shew that the straight lines which join D and E to the middle point of AB are equal.

61. From the angles at the base of a triangle perpendiculars are drawn to the opposite sides, produced if necessary: shew that the straight line joining the points of intersection will be bisected by a perpendicular drawn to it from the middle point of the base.

62. In the figure of I. 1, if C and H be the points of intersection of the circles, and AB be produced to meet one of the circles at K , shew that CHK is an equilateral triangle.

63. The straight lines bisecting the angles at the base of an isosceles triangle meet the sides at D and E : shew that DE is parallel to the base.

64. AB, AC are two given straight lines, and P is a given point in the former: it is required to draw through P a straight line to meet AC at Q , so that the angle APQ may be three times the angle AQP .

65. Construct a right-angled triangle, having given the hypotenuse and the sum of the sides.

66. Construct a right-angled triangle, having given the hypotenuse and the difference of the sides.

67. Construct a right-angled triangle, having given the hypotenuse and the perpendicular from the right angle on it.

68. Construct a right-angled triangle, having given the perimeter and an angle.

69. Trisect a right angle.

70. Trisect a given finite straight line.

71. From a given point it is required to draw to two parallel straight lines, two equal straight lines at right angles to each other.

72. Describe a triangle of given perimeter, having its angles equal to those of a given triangle.

I. 33, 34.

73. If a quadrilateral have two of its opposite sides parallel, and the two others equal but not parallel, any two of its opposite angles are together equal to two right angles.

74. If a straight line which joins the extremities of two equal straight lines, not parallel, make the angles on the same side of it equal to each other, the straight line which joins the other extremities will be parallel to the first.

75. No two straight lines drawn from the extremities of the base of a triangle to the opposite sides can possibly bisect each other.

76. If the opposite sides of a quadrilateral are equal it is a parallelogram.

77. If the opposite angles of a quadrilateral are equal it is a parallelogram.

78. The diagonals of a parallelogram bisect each other.

79. If the diagonals of a quadrilateral bisect each other it is a parallelogram.

80. If the straight line joining two opposite angles of a parallelogram bisect the angles the four sides of the parallelogram are equal.

81. Draw a straight line through a given point such that the part of it intercepted between two given parallel straight lines may be of given length.

82. Straight lines bisecting two adjacent angles of a parallelogram intersect at right angles.

83. Straight lines bisecting two opposite angles of a parallelogram are either parallel or coincident.

84. If the diagonals of a parallelogram are equal all its angles are equal.

85. Find a point such that the perpendiculars let fall from it on two given straight lines shall be respectively equal to two given straight lines. How many such points are there?

86. It is required to draw a straight line which shall be equal to one straight line and parallel to another, and be terminated by two given straight lines.

87. On the sides AB , BC , and CD of a parallelogram $ABCD$ three equilateral triangles are described, that on BC towards the same parts as the parallelogram, and those on AB , CD towards the opposite parts: shew that the

distances of the vertices of the triangles on AB , CD from that on BC are respectively equal to the two diagonals of the parallelogram.

88. If the angle between two adjacent sides of a parallelogram be increased, while their lengths do not alter, the diagonal through their point of intersection will diminish.

89. A , B , C are three points in a straight line, such that AB is equal to BC : shew that the sum of the perpendiculars from A and C on any straight line which does not pass between A and C is double the perpendicular from B on the same straight line.

90. If straight lines be drawn from the angles of any parallelogram perpendicular to any straight line which is outside the parallelogram, the sum of those from one pair of opposite angles is equal to the sum of those from the other pair of opposite angles.

91. If a six-sided plane rectilineal figure have its opposite sides equal and parallel, the three straight lines joining the opposite angles will meet at a point.

92. AB , AC are two given straight lines; through a given point E between them it is required to draw a straight line GEH such that the intercepted portion GH shall be bisected at the point E .

93. Inscribe a rhombus within a given parallelogram, so that one of the angular points of the rhombus may be at a given point in a side of the parallelogram.

94. $ABCD$ is a parallelogram, and E , F , the middle points of AD and BC respectively; shew that BE and DF will trisect the diagonal AC .

Exercise 2 10th

I. 35 to 45.

95. $ABCD$ is a quadrilateral having BC parallel to AD ; shew that its area is the same as that of the parallelogram which can be formed by drawing through the middle point of DC a straight line parallel to AB .

96. $ABCD$ is a quadrilateral having BC parallel to AD , E is the middle point of DC ; shew that the triangle AEB is half the quadrilateral.

97. Shew that any straight line passing through the middle point of the diameter of a parallelogram and terminated by two opposite sides, bisects the parallelogram.

98. Bisect a parallelogram by a straight line drawn through a given point within it.

99. Construct a rhombus equal to a given parallelogram.

100. If two triangles have two sides of the one equal to two sides of the other, each to each, and the sum of the two angles contained by these sides equal to two right angles, the triangles are equal in area.

101. A straight line is drawn bisecting a parallelogram $ABCD$ and meeting AD at E and BC at F : shew that the triangles EBF and CED are equal.

102. Shew that the four triangles into which a parallelogram is divided by its diagonals are equal in area.

103. Two straight lines AB and CD intersect at E , and the triangle AEC is equal to the triangle BED : shew that BC is parallel to AD .

104. $ABCD$ is a parallelogram; from any point P in the diagonal BD the straight lines PA , PC are drawn. Shew that the triangles PAB and PCB are equal.

105. If a triangle is described having two of its sides equal to the diagonals of any quadrilateral, and the included angle equal to either of the angles between these diagonals, then the area of the triangle is equal to the area of the quadrilateral.

106. The straight line which joins the middle points of two sides of any triangle is parallel to the base.

107. Straight lines joining the middle points of adjacent sides of a quadrilateral form a parallelogram.

108. D , E are the middle points of the sides AB , AC of a triangle, and CD , BE intersect at F : shew that the triangle BFC is equal to the quadrilateral $ADFE$.

109. The straight line which bisects two sides of any triangle is half the base.

110. In the base AC of a triangle take any point D ; bisect AD , DC , AB , BC at the points E , F , G , H respectively: shew that EG is equal and parallel to FH .

111. Given the middle points of the sides of a triangle, construct the triangle.

112. If the middle points of any two sides of a triangle be joined, the triangle so cut off is one quarter of the whole.

113. The sides AB , AC of a given triangle ABC are bisected at the points E , F ; a perpendicular is drawn from A to the opposite side, meeting it at D . Shew that the

angle FDE is equal to the angle BAC . Shew also that $AFDE$ is half the triangle ABC .

114. Two triangles of equal area stand on the same base and on opposite sides: shew that the straight line joining their vertices is bisected by the base or the base produced.

115. Three parallelograms which are equal in all respects are placed with their equal bases in the same straight line and contiguous; the extremities of the base of the first are joined with the extremities of the side opposite to the base of the third, towards the same parts: shew that the portion of the new parallelogram cut off by the second is one half the area of any one of them.

116. $ABCD$ is a parallelogram; from D draw any straight line DFG meeting BC at F and AB produced at G ; draw AF and CG : shew that the triangles ABF , CFG are equal.

117. ABC is a given triangle: construct a triangle of equal area, having for its base a given straight line AD , coinciding in position with AB .

118. ABC is a given triangle: construct a triangle of equal area, having its vertex at a given point in BC and its base in the same straight line as AB .

119. $ABCD$ is a given quadrilateral: construct another quadrilateral of equal area having AB for one side, and for another a straight line drawn through a given point in CD parallel to AB .

120. $ABCD$ is a quadrilateral: construct a triangle whose base shall be in the same straight line as AB , vertex at a given point P in CD , and area equal to that of the given quadrilateral.

121. ABC is a given triangle: construct a triangle of equal area, having its base in the same straight line as AB , and its vertex in a given straight line parallel to AB .

122. Bisect a given triangle by a straight line drawn through a given point in a side.

123. Bisect a given quadrilateral by a straight line drawn through a given angular point.

124. If through the point O within a parallelogram $ABCD$ two straight lines are drawn parallel to the sides, and the parallelograms OB and OD are equal, the point O is in the diagonal AC .

I. 46 to 48.

125. On the sides AC , BC of a triangle ABC , squares $ACDE$, $BCFH$ are described: shew that the straight lines AF and BD are equal.

126. The square on the side subtending an acute angle of a triangle is less than the squares on the sides containing the acute angle.

127. The square on the side subtending an obtuse angle of a triangle is greater than the squares on the sides containing the obtuse angle.

128. If the square on one side of a triangle be less than the squares on the other two sides, the angle contained by these sides is an acute angle; if greater, an obtuse angle.

129. A straight line is drawn parallel to the hypotenuse of a right-angled triangle, and each of the acute angles is joined with the points where this straight line intersects the sides respectively opposite to them: shew that the squares on the joining straight lines are together equal to the square on the hypotenuse and the square on the straight line drawn parallel to it.

130. If any point P be joined to A , B , C , D , the angular points of a rectangle, the squares on PA and PC are together equal to the squares on PB and PD .

131. In a right-angled triangle if the square on one of the sides containing the right angle be three times the square on the other, and from the right angle two straight lines be drawn, one to bisect the opposite side, and the other perpendicular to that side, these straight lines divide the right angle into three equal parts.

132. If ABC be a triangle whose angle A is a right angle, and BE , CF be drawn bisecting the opposite sides respectively, shew that four times the sum of the squares on BE and CF is equal to five times the square on BC .

133. On the hypotenuse BC , and the sides CA , AB of a right-angled triangle ABC , squares $BDEC$, AF , and AG are described: shew that the squares on DG and EF are together equal to five times the square on BC .

II. 1 to 11.

134. A straight line is divided into two parts; shew that if twice the rectangle of the parts is equal to the sum of the squares described on the parts, the straight line is bisected.

135. Divide a given straight line into two parts such that the rectangle contained by them shall be the greatest possible.

136. Construct a rectangle equal to the difference of two given squares.

137. Divide a given straight line into two parts such that the sum of the squares on the two parts may be the least possible.

138. Shew that the square on the sum of two straight lines together with the square on their difference is double the squares on the two straight lines.

139. Divide a given straight line into two parts such that the sum of their squares shall be equal to a given square.

140. Divide a given straight line into two parts such that the square on one of them may be double the square on the other.

141. In the figure of II. 11 if CH be produced to meet BF at L , shew that CL is at right angles to BF .

142. In the figure of II. 11 if BE and CH meet at O , shew that AO is at right angles to CH .

143. Shew that in a straight line divided as in II. 11 the rectangle contained by the sum and difference of the parts is equal to the rectangle contained by the parts.

II. 12 to 14.

144. The square on the base of an isosceles triangle is equal to twice the rectangle contained by either side and by the straight line intercepted between the perpendicular let fall on it from the opposite angle and the extremity of the base.

145. In any triangle the sum of the squares on the sides is equal to twice the square on half the base together with twice the square on the straight line drawn from the vertex to the middle point of the base.

146. ABC is a triangle having the sides AB and AC equal; if AB is produced beyond the base to D so that BD is equal to AB , shew that the square on CD is equal to the square on AB , together with twice the square on BC .

147. The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals.

148. The base of a triangle is given and is bisected by the centre of a given circle: if the vertex be at any point of the circumference, shew that the sum of the squares on the two sides of the triangle is invariable.

149. In any quadrilateral the squares on the diagonals are together equal to twice the sum of the squares on the straight lines joining the middle points of opposite sides.

150. If a circle be described round the point of intersection of the diameters of a parallelogram as a centre, shew that the sum of the squares on the straight lines drawn from any point in its circumference to the four angular points of the parallelogram is constant.

151. The squares on the sides of a quadrilateral are together greater than the squares on its diagonals by four times the square on the straight line joining the middle points of its diagonals.

152. In AB the diameter of a circle take two points C and D equally distant from the centre, and from any point E in the circumference draw EC , ED : shew that the squares on EC and ED are together equal to the squares on AC and AD .

153. In BC the base of a triangle take D such that the squares on AB and BD are together equal to the squares on AC and CD , then the middle point of AD will be equally distant from B and C .

154. The square on any straight line drawn from the vertex of an isosceles triangle to the base is less than the square on a side of the triangle by the rectangle contained by the segments of the base.

155. A square $BDEC$ is described on the hypotenuse BC of a right-angled triangle ABC : shew that the squares on DA and AC are together equal to the squares on EA and AB .

156. ABC is a triangle in which C is a right angle, and DE is drawn from a point D in AC perpendicular to

AB : shew that the rectangle AB, AE is equal to the rectangle AC, AD .

157. If a straight line be drawn through one of the angles of an equilateral triangle to meet the opposite side produced, so that the rectangle contained by the whole straight line thus produced and the part of it produced is equal to the square on the side of the triangle, shew that the square on the straight line so drawn will be double the square on a side of the triangle.

158. In a triangle whose vertical angle is a right angle a straight line is drawn from the vertex perpendicular to the base: shew that the square on this perpendicular is equal to the rectangle contained by the segments of the base.

159. In a triangle whose vertical angle is a right angle a straight line is drawn from the vertex perpendicular to the base: shew that the square on either of the sides adjacent to the right angle is equal to the rectangle contained by the base and the segment of it adjacent to that side.

160. In a triangle ABC the angles B and C are acute: if E and F be the points where perpendiculars from the opposite angles meet the sides AC, AB , shew that the square on BC is equal to the rectangle AB, BF , together with the rectangle AC, CE .

161. Divide a given straight line into two parts so that the rectangle contained by them may be equal to the square described on a given straight line which is less than half the straight line to be divided.

III. 1 to 15.

162. Describe a circle with a given centre cutting a given circle at the extremities of a diameter.

163. Shew that the straight lines drawn at right angles to the sides of a quadrilateral inscribed in a circle from their middle points intersect at a fixed point.

164. If two circles cut each other, any two parallel straight lines drawn through the points of section to cut the circles are equal.

165. Two circles whose centres are A and B intersect at C ; through C two chords DCE and FCG are drawn equally inclined to AB and terminated by the circles: shew that DE and FG are equal.

166. Through either of the points of intersection of two given circles draw the greatest possible straight line terminated both ways by the two circumferences.

167. If from any point in the diameter of a circle straight lines are drawn to the extremities of a parallel chord, the squares on these straight lines are together equal to the squares on the segments into which the diameter is divided.

168. A and B are two fixed points without a circle PQR ; it is required to find a point P in the circumference, so that the sum of the squares described on AP and BP may be the least possible.

169. If in any two given circles which touch one another, there be drawn two parallel diameters, an extremity of each diameter, and the point of contact, shall lie in the same straight line.

170. A circle is described on the radius of another circle as diameter, and two chords of the larger circle are drawn, one through the centre of the less at right angles to the common diameter, and the other at right angles to the first through the point where it cuts the less circle. Shew that these two chords have the segments of the one equal to the segments of the other, each to each.

171. Through a given point within a circle draw the shortest chord.

172. O is the centre of a circle, P is any point in its circumference, PN a perpendicular on a fixed diameter: shew that the straight line which bisects the angle OPN always passes through one or the other of two fixed points.

173. Three circles touch one another externally at the points A, B, C ; from A , the straight lines AB, AC are produced to cut the circle BC at D and E : shew that DE is a diameter of BC , and is parallel to the straight line joining the centres of the other circles.

174. Circles are described on the sides of a quadrilateral as diameters: shew that the common chord of any adjacent two is parallel to the common chord of the other two.

175. Describe a circle which shall touch a given circle, have its centre in a given straight line, and pass through a given point in the given straight line.

III. 16 to 19.

176. Shew that two tangents can be drawn to a circle from a given external point, and that they are of equal length. ✓

177. Draw parallel to a given straight line a straight line to touch a given circle. ✓

178. Draw perpendicular to a given straight line a straight line to touch a given circle. ✓

179. In the diameter of a circle produced, determine a point so that the tangent drawn from it to the circumference shall be of given length. ✓

180. Two circles have the same centre: shew that all chords of the outer circle which touch the inner circle are equal. ✓

181. Through a given point draw a straight line so that the part intercepted by the circumference of a given circle shall be equal to a given straight line not greater than the diameter. ✓

182. Two tangents are drawn to a circle at the opposite extremities of a diameter, and cut off from a third tangent a portion AB : if C be the centre of the circle shew that ACB is a right angle. ✓

183. Describe a circle that shall have a given radius and touch a given circle and a given straight line. ✓

184. A circle is drawn to touch a given circle and a given straight line. Shew that the points of contact are always in the same straight line with a fixed point in the circumference of the given circle. ✓

185. Draw a straight line to touch each of two given circles. ✓

186. Draw a straight line to touch one given circle so that the part of it contained by another given circle shall be equal to a given straight line not greater than the diameter of the latter circle. ✓

187. Draw a straight line cutting two given circles so that the chords intercepted within the circles shall have given lengths. ✓

188. A quadrilateral is described so that its sides touch a circle: shew that two of its sides are together equal to the other two sides. ✓

189. Shew that no parallelogram can be described about a circle except a rhombus. ✓

190. ABD , ACE are two straight lines touching a circle at B and C , and if DE be joined DE is equal to BD and CE together: shew that DE touches the circle.

191. If a quadrilateral be described about a circle the angles subtended at the centre of the circle by any two opposite sides of the figure are together equal to two right angles.

192. Two radii of a circle at right angles to each other when produced are cut by a straight line which touches the circle: shew that the tangents drawn from the points of section are parallel to each other.

193. A straight line is drawn touching two circles: shew that the chords are parallel which join the points of contact and the points where the straight line through the centres meets the circumferences.

194. If two circles can be described so that each touches the other and three of the sides of a quadrilateral figure, then the difference between the sums of the opposite sides is double the common tangent drawn across the quadrilateral.

195. AB is the diameter and C the centre of a semicircle: shew that O the centre of any circle inscribed in the semicircle is equidistant from C and from the tangent to the semicircle parallel to AB .

196. If from any point without a circle straight lines be drawn touching it, the angle contained by the tangents is double the angle contained by the straight line joining the points of contact and the diameter drawn through one of them.

197. A quadrilateral is bounded by the diameter of a circle, the tangents at its extremities, and a third tangent: shew that its area is equal to half that of the rectangle contained by the diameter and the side opposite to it.

198. If a quadrilateral, having two of its sides parallel, be described about a circle, a straight line drawn through the centre of the circle, parallel to either of the two parallel sides, and terminated by the other two sides, shall be equal to a fourth part of the perimeter of the figure.

199. A series of circles touch a fixed straight line at a fixed point: shew that the tangents at the points where they cut a parallel fixed straight line all touch a fixed circle.

200. Of all straight lines which can be drawn from two given points to meet in the convex circumference of a

given circle, the sum of the two is least which make equal angles with the tangent at the point of concurrence.

201. C is the centre of a given circle, CA a radius, B a point on a radius at right angles to CA ; join AB and produce it to meet the circle again at D , and let the tangent at D meet CB produced at E ; shew that BDE is an isosceles triangle.

202. Let the diameter BA of a circle be produced to P , so that AP equals the radius; through A draw the tangent AED , and from P draw PEC touching the circle at C and meeting the former tangent at E ; join BC and produce it to meet AED at D : then will the triangle DEC be equilateral.

III. 20 to 22.

203. Two tangents AB , AC are drawn to a circle; D is any point on the circumference outside of the triangle ABC : shew that the sum of the angles ABD and ACD is constant.

204. P , Q are any points in the circumferences of two segments described on the same straight line AB , and on the same side of it; the angles PAQ , PBQ are bisected by the straight lines AR , BR meeting at R : shew that the angle ARB is constant.

205. Two segments of a circle are on the same base AB , and P is any point in the circumference of one of the segments; the straight lines APD , BPC are drawn meeting the circumference of the other segment at D and C ; AC and BD are drawn intersecting at Q . Shew that the angle AQB is constant.

206. APB is a fixed chord passing through P a point of intersection of two circles AQP , PBR ; and QPR is any other chord of the circles passing through P : shew that AQ and RB when produced meet at a constant angle.

207. AOB is a triangle; C and D are points in BO and AO respectively, such that the angle ODC is equal to the angle OBA : shew that a circle may be described round the quadrilateral $ABCD$.

208. $ABCD$ is a quadrilateral inscribed in a circle, and the sides AB , CD when produced meet at O : shew that the triangles AOC , BOD are equiangular.

209. Shew that no parallelogram except a rectangle can be inscribed in a circle.

210. A triangle is inscribed in a circle: shew that the sum of the angles in the three segments exterior to the triangle is equal to four right angles.

211. A quadrilateral is inscribed in a circle: shew that the sum of the angles in the four segments of the circle exterior to the quadrilateral is equal to six right angles.

212. Divide a circle into two parts so that the angle contained in one segment shall be equal to twice the angle contained in the other.

213. Divide a circle into two parts so that the angle contained in one segment shall be equal to five times the angle contained in the other.

214. If the angle contained by any side of a quadrilateral and the adjacent side produced, be equal to the opposite angle of the quadrilateral, shew that any side of the quadrilateral will subtend equal angles at the opposite angles of the quadrilateral.

215. If any two consecutive sides of a hexagon inscribed in a circle be respectively parallel to their opposite sides, the remaining sides are parallel to each other.

216. A , B , C , D are four points taken in order on the circumference of a circle; the straight lines AB , CD produced intersect at P , and AD , BC at Q : shew that the straight lines which respectively bisect the angles APC , AQC are perpendicular to each other.

217. If a quadrilateral be inscribed in a circle, and a straight line be drawn making equal angles with one pair of opposite sides, it will make equal angles with the other pair.

218. A quadrilateral can have one circle inscribed in it and another circumscribed about it: shew that the straight lines joining the opposite points of contact of the inscribed circle are perpendicular to each other.

III. 23 to 30.

219. The straight lines joining the extremities of the chords of two equal arcs of a circle, towards the same parts are parallel to each other.

- 220. The straight lines in a circle which join the extremities of two parallel chords are equal to each other.
- 221. AB is a common chord of two circles; through C any point of one circumference straight lines CAD , CBE are drawn terminated by the other circumference: shew that the arc DE is invariable.
- 222. Through a point C in the circumference of a circle two straight lines ACB , DCE are drawn cutting the circle at B and E : shew that the straight line which bisects the angles ACE , DOB meets the circle at a point equidistant from B and E .
- 223. The straight lines bisecting any angle of a quadrilateral inscribed in a circle and the opposite exterior angle, meet in the circumference of the circle.
- 224. AB is a diameter of a circle, and D is a given point on the circumference: draw a chord DE on one side of AB so that one arc between the chord and diameter may be three times the other.
- 225. From A and B two of the angular points of a triangle ABC , straight lines are drawn so as to meet the opposite sides at P and Q in given equal angles: shew that the straight line joining P and Q will be of the same length in all triangles on the same base AB , and having vertical angles equal to C .
- 226. If two equal circles cut each other, and if through one of the points of intersection a straight line be drawn terminated by the circles, the straight lines joining its extremities with the other point of intersection are equal.
- 227. OA , OB , OC are three chords of a circle; the angle AOB is equal to the angle BOC , and OA is nearer to the centre than OB . From B a perpendicular is drawn on OA , meeting it at P , and a perpendicular on OC produced, meeting it at Q : shew that AP is equal to CQ .
- 228. AB is a given finite straight line; through A two indefinite straight lines are drawn equally inclined to AB ; any circle passing through A and B meets these straight lines at L and M . Shew that if AB be between AL and AM the sum of AL and AM is constant; if AB be not between AL and AM the difference of AL and AM is constant.
- 229. AOB and COD are diameters of a circle at right angles to each other; E is a point in the arc AC , and EFG is a chord meeting COD at F , and drawn in such a

direction that EF is equal to the radius. Shew that the arc BG is equal to three times the arc AE .

230. The straight lines which bisect the vertical angles of all triangles on the same base and on the same side of it, and having equal vertical angles, all intersect at the same point.

231. If two circles touch each other internally, any chord of the greater circle which touches the less shall be divided at the point of its contact into segments which subtend equal angles at the point of contact of the two circles.

III. 31.

232. Right-angled triangles are described on the same hypotenuse: shew that the angular points opposite the hypotenuse all lie on a circle described on the hypotenuse as diameter.

233. The circles described on the equal sides of an isosceles triangle as diameters, will intersect at the middle point of the base.

234. The greatest rectangle which can be inscribed in a circle is a square.

235. The hypotenuse AB of a right-angled triangle ABC is bisected at D , and EDF is drawn at right angles to AB , and DE and DF are cut off each equal to DA ; CE and CF are joined: shew that the last two straight lines will bisect the angle C and its supplement respectively.

236. On the side AB of any triangle ABC as diameter a circle is described; EF is a diameter parallel to BC : shew that the straight lines EB and FB bisect the interior and exterior angles at B .

237. If AD , CE be drawn perpendicular to the sides BC , AB of a triangle ABC , and DE be joined, shew that the angles ADE and ACE are equal to each other.

238. If two circles ABC , ABD intersect at A and B , and AC , AD be two diameters, shew that the straight line CD will pass through B .

239. If O be the centre of a circle and OA a radius and a circle be described on OA as diameter, the circum-

ference of this circle will bisect any chord drawn through it from A to meet the exterior circle.

240. Describe a circle touching a given straight line at a given point, such that the tangents drawn to it from two given points in the straight line may be parallel.

241. Describe a circle with a given radius touching a given straight line, such that the tangents drawn to it from two given points in the straight line may be parallel.

242. If from the angles at the base of any triangle perpendiculars are drawn to the opposite sides, produced if necessary, the straight line joining the points of intersection will be bisected by a perpendicular drawn to it from the centre of the base.

243. AD is a diameter of a circle; B and C are points on the circumference on the same side of AD ; a perpendicular from D on BC produced through C , meets it at E : shew that the square on AD is greater than the sum of the squares on AB , BC , CD , by twice the rectangle BC , CE .

244. AB is the diameter of a semicircle, P is a point on the circumference, PM is perpendicular to AB ; on AM , BM as diameters two semicircles are described, and AP , BP meet these latter circumferences at Q , R : shew that QR will be a common tangent to them.

245. AB , AC are two straight lines, B and C are given points in the same; BD is drawn perpendicular to AC , and DE perpendicular to AB ; in like manner CF is drawn perpendicular to AB , and FG to AC . Shew that EG is parallel to BC .

246. Two circles intersect at the points A and B , from which are drawn chords to a point C in one of the circumferences, and these chords, produced if necessary, cut the other circumference at D and E : shew that the straight line DE cuts at right angles that diameter of the circle ABC which passes through C .

247. If squares be described on the sides and hypotenuse of a right-angled triangle, the straight line joining the intersection of the diagonals of the latter square with the right angle is perpendicular to the straight line joining the intersections of the diagonals of the two former.

248. C is the centre of a given circle, CA a straight line less than the radius; find the point of the circumference at which CA subtends the greatest angle.

249. AB is the diameter of a semicircle, D and E are any two points in its circumference. Shew that if the chords joining A and B with D and E each way intersect at F and G , then FG produced is at right angles to AB .

250. Two equal circles touch one another externally, and through the point of contact chords are drawn, one to each circle, at right angles to each other: shew that the straight line joining the other extremities of these chords is equal and parallel to the straight line joining the centres of the circles.

251. A circle is described on the shorter diagonal of a rhombus as a diameter, and cuts the sides; and the points of intersection are joined crosswise with the extremities of that diagonal: shew that the parallelogram thus formed is a rhombus with angles equal to those of the first.

252. If two chords of a circle meet at a right angle within or without a circle, the squares on their segments are together equal to the squares on the diameter.

III. 32 to 34.

253. B is a point in the circumference of a circle, whose centre is C ; PA , a tangent at any point P , meets CB produced at A , and PD is drawn perpendicular to CB : shew that the straight line PB bisects the angle APD .

254. If two circles touch each other, any straight line drawn through the point of contact will cut off similar segments.

255. AB is any chord, and AD is a tangent to a circle at A . DPQ is any straight line parallel to AB , meeting the circumference at P and Q . Shew that the triangle PAD is equiangular to the triangle QAB .

256. Two circles $ABDH$, ABG , intersect each other at the points A , B ; from B a straight line BD is drawn in the one to touch the other; and from A any chord whatever is drawn cutting the circles at G and H : shew that BG is parallel to DH .

257. Two circles intersect at A and B . At A the tangents AC , AD are drawn to each circle and terminated

by the circumference of the other. If CB , BD be joined, shew that AB or AB produced, if necessary, bisects the angle CBD .

258. Two circles intersect at A and B , and through P any point in the circumference of one of them the chords PA and PB are drawn to cut the other circle at C and D : shew that CD is parallel to the tangent at P .

259. If from any point in the circumference of a circle a chord and tangent be drawn, the perpendiculars dropped on them from the middle point of the subtended arc are equal to one another.

260. AB is any chord of a circle, P any point on the circumference of the circle; PM is a perpendicular on AB and is produced to meet the circle at Q ; and AN is drawn perpendicular to the tangent at P : shew that the triangle NAM is equiangular to the triangle PAQ .

261. Two diameters AOB , COD of a circle are at right angles to each other; P is a point in the circumference; the tangent at P meets COD produced at Q , and AP , BP meet the same line at R , S respectively: shew that RQ is equal to SQ .

262. Construct a triangle, having given the base, the vertical angle, and the point in the base on which the perpendicular falls.

263. Construct a triangle, having given the base, the vertical angle, and the altitude.

264. Construct a triangle, having given the base, the vertical angle, and the length of the straight line drawn from the vertex to the middle point of the base.

265. Having given the base and the vertical angle of a triangle, shew that the triangle will be greatest when it is isosceles.

266. From a given point A without a circle whose centre is O draw a straight line cutting the circle at the points B and C , so that the area BOC may be the greatest possible.

267. Two straight lines containing a constant angle always pass through two fixed points, their position being otherwise unrestricted: shew that the straight line bisecting the angle always passes through one or other of two fixed points.

268. Given one angle of a triangle, the side opposite

it, and the sum of the other two sides, construct the triangle.

III. 35 to 37.

✓ 269. If two circles cut one another, the tangents drawn to the two circles from any point in the common chord produced are equal.

270. Two circles intersect at A and B : shew that AB produced bisects their common tangent.

✓ 271. If AD , CE are drawn perpendicular to the sides BC , AB of a triangle ABC , shew that the rectangle contained by BC and BD is equal to the rectangle contained by BA and BE .

272. If through any point in the common chord of two circles which intersect one another, there be drawn any two other chords, one in each circle, their four extremities shall all lie in the circumference of a circle.

273. From a given point as centre describe a circle cutting a given straight line in two points, so that the rectangle contained by their distances from a fixed point in the straight line may be equal to a given square.

274. Two circles $ABCD$, $EBCF$, having the common tangents AE and DF , cut one another at B and C , and the chord BC is produced to cut the tangents at G and H : shew that the square on GH exceeds the square on AE or DF by the square on BC .

275. A series of circles intersect each other, and are such that the tangents to them from a fixed point are equal: shew that the straight lines joining the two points of intersection of each pair will pass through this point.

276. ABC is a right-angled triangle; from any point D in the hypotenuse BC a straight line is drawn at right angles to BC , meeting CA at E and BA produced at F : shew that the square on DE is equal to the difference of the rectangles BD , DC and AE , EC ; and that the square on DF is equal to the sum of the rectangles BD , DC and AF , FB .

277. It is required to find a point in the straight line which touches a circle at the end of a given diameter, such that when a straight line is drawn from this point to the other extremity of the diameter, the rectangle contained

by the part of it without the circle and the part within the circle may be equal to a given square not greater than that on the diameter.

IV. 1 to 4.

278. In IV. 3 shew that the straight lines drawn through A and B to touch the circle will meet.

279. In IV. 4 shew that the straight lines which bisect the angles B and C will meet.

280. In IV. 4 shew that the straight line DA will bisect the angle at A .

281. If the circle inscribed in a triangle ABC touch the sides AB, AC at the points D, E , and a straight line be drawn from A to the centre of the circle meeting the circumference at G , shew that the point G is the centre of the circle inscribed in the triangle ADE .

282. Shew that the straight lines joining the centres of the circles touching one side of a triangle and the others produced, pass through the angular points of the triangle.

283. A circle touches the side BC of a triangle ABC and the other two sides produced: shew that the distance between the points of contact of the side BC with this circle and with the inscribed circle, is equal to the difference between the sides AB and AC .

284. A circle is inscribed in a triangle ABC , and a triangle is cut off at each angle by a tangent to the circle. Shew that the sides of the three triangles so cut off are together equal to the sides of ABC .

285. D is the centre of the circle inscribed in a triangle BAC , and AD is produced to meet the straight line drawn through B at right angles to BD at O : shew that O is the centre of the circle which touches the side BC and the sides AB, AC produced.

286. Three circles are described, each of which touches one side of a triangle ABC , and the other two sides produced. If D be the point of contact of the side BC , E that of AC , and F that of AB , shew that AE is equal to BD , BF to CE , and CD to AF .

287. Describe a circle which shall touch a given circle and two given straight lines which themselves touch the given circle.

288. If the three points be joined in which the circle inscribed in a triangle meets the sides, shew that the resulting triangle is acute angled.

289. Two opposite sides of a quadrilateral are together equal to the other two, and each of the angles is less than two right angles. Shew that a circle can be inscribed in the quadrilateral.

290. Two circles HPL , KPM , that touch each other externally, have the common tangents HK , LM ; HL and KM being joined, shew that a circle may be inscribed in the quadrilateral $HKML$.

291. Straight lines are drawn from the angles of a triangle to the centres of the opposite escribed circles: shew that these straight lines intersect at the centre of the inscribed circle.

292. Two sides of a triangle whose perimeter is constant are given in position: shew that the third side always touches a certain circle.

293. Given the base, the vertical angle, and the radius of the inscribed circle of a triangle, construct it.

IV. 5 to 9.

294. In IV. 5 shew that the perpendicular from F on BC will bisect BC .

295. If DE be drawn parallel to the base BC of a triangle ABC , shew that the circles described about the triangles ABC and ADE have a common tangent.

296. If the inscribed and circumscribed circles of a triangle be concentric, shew that the triangle must be equilateral.

297. Shew that if the straight line joining the centres of the inscribed and circumscribed circles of a triangle passes through one of its angular points, the triangle is isosceles.

298. The common chord of two circles is produced to any point P ; PA touches one of the circles at A , PBC is any chord of the other. Shew that the circle which passes through A , B , and C touches the circle to which PA is a tangent.

299. A quadrilateral $ABCD$ is inscribed in a circle, and AD , BC are produced to meet at E : shew that the circle described about the triangle ECD will have the tangent at E parallel to AB .

300. Describe a circle which shall touch a given straight line, and pass through two given points. ✓

301. Describe a circle which shall pass through two given points and cut off from a given straight line a chord of given length. ✓

302. Describe a circle which shall have its centre in a given straight line, and cut off from two given straight lines chords of equal given length. ✓

303. Two triangles have equal bases and equal vertical angles: shew that the radius of the circumscribing circle of one triangle is equal to that of the other. ✓

304. Describe a circle which shall pass through two given points, so that the tangent drawn to it from another given point may be of a given length. ✓

305. C is the centre of a circle; CA , CB are two radii at right angles; from B any chord BP is drawn cutting CA at N : a circle being described round ANP , shew that it will be touched by BA . ✓

306. AB and CD are parallel straight lines, and the straight lines which join their extremities intersect at E : shew that the circles described round the triangles ABE , CDE touch one another. ✓

307. Find the centre of a circle cutting off three equal chords from the sides of a triangle. ✓

308. If O be the centre of the circle inscribed in the triangle ABC , and AO be produced to meet the circumscribed circle at F , shew that FB , FO , and FC are all equal. ✓

309. The opposite sides of a quadrilateral inscribed in a circle are produced to meet at P and Q , and about the triangles so formed without the quadrilateral, circles are described meeting again at R : shew that P , R , Q are in one straight line. ✓

310. The angle ACB of any triangle is bisected, and the base AB is bisected at right angles, by straight lines which intersect at D : shew that the angles ACB , ADB are together equal to two right angles.

311. $ACDB$ is a semicircle, AB being the diameter, and the two chords AD , BC intersect at E : shew that if a circle be described round CDE it will cut the former at right angles. (?)

✓ 312. The diagonals of a given quadrilateral $ABCD$ intersect at O : shew that the centres of the circles described about the triangles OAB , OBC , OCD , ODA , will lie in the angular points of a parallelogram.

✓ 313. A circle is described round the triangle ABC ; the tangent at C meets AB produced at D ; the circle whose centre is D and radius DC cuts AB at E : shew that EC bisects the angle ACB .

314. AB , AC are two straight lines given in position; BC is a straight line of given length; D , E are the middle points of AB , AC ; DF , EF are drawn at right angles to AB , AC respectively. Shew that AF will be constant for all positions of BC .

✓ 315. A circle is described about an isosceles triangle ABC in which AB is equal to AC ; from A a straight line is drawn meeting the base at D and the circle at E : shew that the circle which passes through B , D , and E , touches AB .

✓ 316. AC is a chord of a given circle; B and D are two given points in the chord, both within or both without the circle: if a circle be described to pass through B and D , and touch the given circle, shew that AB and CD subtend equal angles at the point of contact.

✓ 317. A and B are two points within a circle: find the point P in the circumference such that if PAH , PBK be drawn meeting the circle at H and K , the chord HK shall be the greatest possible.

✓ 318. The centre of a given circle is equidistant from two given straight lines: describe another circle which shall touch these two straight lines and shall cut off from the given circle a segment containing an angle equal to a given angle.

319. O is the centre of the circle circumscribing a triangle ABC ; D , E , F the feet of the perpendiculars from A , B , C on the opposite sides: shew that OA , OB , OC are respectively perpendicular to EF , FD , DE .

148 ✓ 320. If from any point in the circumference of a given circle straight lines be drawn to the four angular points of an inscribed square, the sum of the squares on the four straight lines is double the square on the diameter.

321. Shew that no rectangle except a square can be described about a circle.

322. Describe a circle about a given rectangle.

323. If tangents be drawn through the extremities of two diameters of a circle the parallelogram thus formed will be a rhombus.

IV. 10.

324. Shew that the angle ACD in the figure of IV. 10 is equal to three times the angle at the vertex of the triangle.

325. Shew that in the figure of IV. 10 there are two triangles which possess the required property: shew that there is also an isosceles triangle whose equal angles are each one third part of the third angle.

326. Shew that the base of the triangle in IV. 10 is equal to the side of a regular pentagon inscribed in the smaller circle of the figure.

327. On a given straight line as base describe an isosceles triangle having the third angle treble of each of the angles at the base.

328. In the figure of IV. 10 suppose the two circles to cut again at E : then DE is equal to DC .

329. If A be the vertex and BD the base of the constructed triangle in IV. 10, D being one of the two points of intersection of the two circles employed in the construction, and E the other, and AE be drawn meeting BD produced at G , shew that GAB is another isosceles triangle of the same kind.

330. In the figure of IV. 10 if the two equal chords of the smaller circle be produced to cut the larger, and these points of section be joined, another triangle will be formed having the property required by the proposition.

331. In the figure of IV. 10 suppose the two circles to cut again at E ; join AE , CE , and produce AE , BD to meet at G : then $CDGE$ is a parallelogram.

332. Shew that the smaller of the two circles employed in the figure of IV. 10 is equal to the circle described round the required triangle.

✓ 333. In the figure of IV. 10 if AF be the diameter of the smaller circle, DF is equal to a radius of the circle which circumscribes the triangle BCD .

IV. 11 to 16.

✓ 334. The straight lines which connect the angular points of a regular pentagon which are not adjacent intersect at the angular points of another regular pentagon.

✓ 335. $ABCDE$ is a regular pentagon; join AC and BE , and let BE meet AC at F ; shew that AC is equal to the sum of AB and BF .

✓ 336. Shew that each of the triangles made by joining the extremities of adjoining sides of a regular pentagon is less than a third and greater than a fourth of the whole area of the pentagon.

✓ 337. Shew how to derive a regular hexagon from an equilateral triangle inscribed in a circle, and from the construction shew that the side of the hexagon equals the radius of the circle, and that the hexagon is double of the triangle.

✓ 338. In a given circle inscribe a triangle whose angles are as the numbers 2, 5, 8.

✓ 339. If $ABCDEF$ is a regular hexagon, and AC , BD , CE , DF , EA , FB be joined, another hexagon is formed whose area is one third of that of the former.

✓ 340. Any equilateral figure which is inscribed in a circle is also equiangular.

VI. 1, 2.

✓ 341. Shew that one of the triangles in the figure of IV. 10 is a mean proportional between the other two.

→ 342. Through D , any point in the base of a triangle ABC , straight lines DE , DF are drawn parallel to the sides AB , AC , and meeting the sides at E , F : shew that the triangle AEF is a mean proportional between the triangles FBD , EDC .

343. Perpendiculars are drawn from any point within an equilateral triangle on the three sides: shew that their sum is invariable.

344. Find a point within a triangle such that if straight lines be drawn from it to the three angular points the triangle will be divided into three equal triangles. ✓

345. From a point E in the common base of two triangles ACB , ADB , straight lines are drawn parallel to AC , AD , meeting BC , BD at F , G : shew that FG is parallel to CD . ✓

346. From any point in the base of a triangle straight lines are drawn parallel to the sides: shew that the intersection of the diagonals of every parallelogram so formed lies in a certain straight line.

347. In a triangle ABC a straight line AD is drawn perpendicular to the straight line BD which bisects the angle B : shew that a straight line drawn from D parallel to BC will bisect AC .

348. ABC is a triangle; any straight line parallel to BC meets AB at D and AC at E ; join BE and CD meeting at F : shew that the triangle ADF is equal to the triangle AEF .

349. ABC is a triangle; any straight line parallel to BC meets AB at D and AC at E ; join BE and CD meeting at F : shew that if AF be produced it will bisect BC .

350. If two sides of a quadrilateral figure be parallel to each other, any straight line drawn parallel to them will cut the other sides, or those sides produced, proportionally.

351. ABC is a triangle; it is required to draw from a given point P , in the side AB , or AB produced, a straight line to AC , or AC produced, so that it may be bisected by BC .

VI. 3, A.

352. The side BC of a triangle ABC is bisected at D , and the angles ADB , ADC are bisected by the straight lines DE , DF , meeting AB , AC at E , F respectively: shew that EF is parallel to BC .

353. AB is a diameter of a circle, CD is a chord at right angles to it, and E is any point in CD ; AE and BE

are drawn and produced to cut the circle at F and G : shew that the quadrilateral $CFDG$ has any two of its adjacent sides in the same ratio as the remaining two.

354. Apply VI. 3 to solve the problem of the trisection of a finite straight line.

355. In the circumference of the circle of which AB is a diameter, take any point P ; and draw PC , PD on opposite sides of AP , and equally inclined to it, meeting AB at C and D : shew that AC is to BC as AD is to BD .

356. AB is a straight line, and D is any point in it: determine a point P in AB produced such that PA is to PB as DA is to DB .

357. From the same point A straight lines are drawn making the angles BAC , CAD , DAE each equal to half a right angle, and they are cut by a straight line $BCDE$, which makes BAE an isosceles triangle: shew that BC or DE is a mean proportional between BE and CD .

358. The angle A of a triangle ABC is bisected by AD which cuts the base at D , and O is the middle point of BC : shew that OD bears the same ratio to OB that the difference of the sides bears to their sum.

359. AD and AE bisect the interior and exterior angles at A of a triangle ABC , and meet the base at D and E ; and O is the middle point of BC : shew that OB is a mean proportional between OD and OE .

360. Three points D , E , F in the sides of a triangle ABC being joined form a second triangle, such that any two sides make equal angles with the side of the former at which they meet: shew that AD , BE , CF are at right angles to BC , CA , AB respectively.

VI. 4 to 6.

361. If two triangles be on equal bases and between the same parallels, any straight line parallel to their bases will cut off equal areas from the two triangles.

362. AB and CD are two parallel straight lines; E is the middle point of CD ; AC and BE meet at F , and AE and BD meet at G : shew that FG is parallel to AB .

363. A , B , C are three fixed points in a straight line; any straight line is drawn through C ; shew that the perpendiculars on it from A and B are in a constant ratio.

364. If the perpendiculars from two fixed points on a straight line passing between them be in a given ratio, the straight line must pass through a third fixed point.

365. Find a straight line such that the perpendiculars on it from three given points shall be in a given ratio to each other.

366. Through a given point draw a straight line, so that the parts of it intercepted between that point and perpendiculars drawn to the straight line from two other given points may have a given ratio.

367. A tangent to a circle at the point A intersects two parallel tangents at B, C , the points of contact of which with the circle are D, E respectively; and BE, CD intersect at F : shew that AF is parallel to the tangents BD, CE .

368. P and Q are fixed points; AB and CD are fixed parallel straight lines; any straight line is drawn from P to meet AB at M , and a straight line is drawn from Q parallel to PM meeting CD at N : shew that the ratio of PM to QN is constant, and thence shew that the straight line through M and N passes through a fixed point.

369. Shew that the diagonals of a quadrilateral, two of whose sides are parallel and one of them double of the other, cut one another at a point of trisection.

370. A and B are two points on the circumference of a circle of which C is the centre; draw tangents at A and B meeting at T ; and from A draw AN perpendicular to CB : shew that BT is to BC as BN is to NA .

371. In the sides AB, AC of a triangle ABC are taken two points D, E , such that BD is equal to CE ; DE, BC are produced to meet at F : shew that AB is to AC as EF is to DF .

372. If through the vertex and the extremities of the base of a triangle two circles be described intersecting each other in the base or base produced, their diameters are proportional to the sides of the triangle.

373. Find a point the perpendiculars from which on the sides of a given triangle shall be in a given ratio.

374. On AB, AC , two adjacent sides of a rectangle, two similar triangles are constructed, and perpendiculars are drawn to AB, AC from the angles which they subtend, intersecting at the point P . If AB, AC be homologous

sides, shew that P is in all cases in one of the diagonals of the rectangle.

375. In the figure of I. 43 shew that if EG and FH be produced they will meet on AC produced.

376. APB and CQD are parallel straight lines, and AP is to PB as DQ is to QC : shew that the straight lines PQ , AC , BD , produced if necessary, will meet at a point: shew also that the straight lines PQ , AD , BC , produced if necessary, will meet at a point.

377. ACB is a triangle, and the side AC is produced to D so that CD is equal to AC , and BD is joined: if any straight line drawn parallel to AB cuts the sides AC , CB , and from the points of section straight lines be drawn parallel to DB , shew that these straight lines will meet AB at points equidistant from its extremities.

378. If a circle be described touching externally two given circles, the straight line passing through the points of contact will intersect the straight line passing through the centres of the given circles at a fixed point.

379. D is the middle point of the side BC of a triangle ABC , and P is any point in AD ; through P the straight lines BPE , CPF are drawn meeting the other sides at E , F : shew that EF is parallel to BC .

380. AB is the diameter of a circle, E the middle point of the radius OB ; on AE , EB as diameters circles are described; PQL is a common tangent meeting the circles at P and Q , and AB produced at L : shew that BL is equal to the radius of the smaller circle.

381. $ABCDE$ is a regular pentagon, and AD , BE intersect at O : shew that a side of the pentagon is a mean proportional between AO and AD .

382. $ABCD$ is a parallelogram; P and Q are points in a straight line parallel to AB ; PA and QB meet at R , and PD and QC meet at S ; shew that RS is parallel to AD .

383. A and B are two given points; AC and BD are perpendicular to a given straight line CD ; AD and BC intersect at E , and EF is perpendicular to CD : shew that AF and BF make equal angles with CD .

384. From the angular points of a parallelogram $ABCD$ perpendiculars are drawn on the diagonals meeting them at E , F , G , H respectively: shew that $EFGH$ is a parallelogram similar to $ABCD$.

385. If at a given point two circles intersect, and their centres lie on two fixed straight lines which pass through that point, shew that whatever be the magnitude of the circles their common tangents will always meet in one of two fixed straight lines which pass through the given point.

VI. 7 to 18.

386. If two circles touch each other, and also touch a given straight line, the part of the straight line between the points of contact is a mean proportional between the diameters of the circles.

387. Divide a given arc of a circle into two parts, so that the chords of these parts shall be to each other in a given ratio.

388. In a given triangle draw a straight line parallel to one of the sides, so that it may be a mean proportional between the segments of the base.

389. ABC is a triangle, and a perpendicular is drawn from A to the opposite side, meeting it at D between B and C : shew that if AD is a mean proportional between BD and CD the angle BAC is a right angle.

390. ABC is a triangle, and a perpendicular is drawn from A on the opposite side, meeting it at D between B and C : shew that if BA is a mean proportional between BD and BC , the angle BAC is a right angle.

391. C is the centre of a circle, and A any point within it; CA is produced through A to a point B such that the radius is a mean proportional between CA and CB : shew that if P be any point on the circumference, the angles CPA and CBP are equal.

392. O is a fixed point in a given straight line OA , and a circle of given radius moves so as always to be touched by OA ; a tangent OP is drawn from O to the circle, and in OP produced PQ is taken a third proportional to OP and the radius: shew that as the circle moves along OA , the point Q will move in a straight line.

393. Two given parallel straight lines touch a circle, and SPT is another tangent cutting the two former tangents at S and T , and meeting the circle at P : shew

that the rectangle SP, PT is constant for all positions of P .

394. Find a point in a side of a triangle, from which two straight lines drawn, one to the opposite angle, and the other parallel to the base, shall cut off towards the vertex and towards the base, equal triangles.

395. ACB is a triangle having a right angle at C ; from A a straight line is drawn at right angles to AB , cutting BC produced at E ; from B a straight line is drawn at right angles to AB , cutting AC produced at D : shew that the triangle ECD is equal to the triangle ACB .

396. The straight line bisecting the angle ABC of the triangle ABC meets the straight lines drawn through A and C , parallel to BC and AB respectively, at E and F : shew that the triangles CBE, ABF are equal.

397. Shew that the diagonals of any quadrilateral figure inscribed in a circle divide the quadrilateral into four triangles which are similar two and two; and deduce the theorem of III. 35.

398. AB, CD are any two chords of a circle passing through a point O ; EF is any chord parallel to OB ; join CE, DF meeting AB at the points G and H , and DE, CF meeting AB at the points K and L : shew that the rectangle OG, OH is equal to the rectangle OK, OL .

399. $ABCD$ is a quadrilateral in a circle; the straight lines CE, DE which bisect the angles ACB, ADB cut BD and AC at F and G respectively: shew that EF is to EG as ED is to EC .

400. From an angle of a triangle two straight lines are drawn, one to any point in the side opposite to the angle, and the other to the circumference of the circumscribing circle, so as to cut from it a segment containing an angle equal to the angle contained by the first drawn line and the side which it meets: shew that the rectangle contained by the sides of the triangle is equal to the rectangle contained by the straight lines thus drawn.

401. The vertical angle C of a triangle is bisected by a straight line which meets the base at D , and is produced to a point E , such that the rectangle contained by CD and CE is equal to the rectangle contained by AC and CB : shew that if the base and vertical angle be given, the position of E is invariable.

402. A square is inscribed in a right-angled triangle ABC , one side DE of the square coinciding with the hypotenuse AB of the triangle: shew that the area of the square is equal to the rectangle AD, BE .

403. $ABCD$ is a parallelogram; from B a straight line is drawn cutting the diagonal AC at F , the side DC at G , and the side AD produced at E : shew that the rectangle EF, FG is equal to the square on BF .

404. If a straight line drawn from the vertex of an isosceles triangle to the base, be produced to meet the circumference of a circle described about the triangle, the rectangle contained by the whole line so produced, and the part of it between the vertex and the base shall be equal to the square on either of the equal sides of the triangle.

405. Two straight lines are drawn from a point A to touch a circle of which the centre is E ; the points of contact are joined by a straight line which cuts EA at H ; and on HA as diameter a circle is described: shew that the straight lines drawn through E to touch this circle will meet it on the circumference of the given circle.

VI. 19 to D .

406. An isosceles triangle is described having each of the angles at the base double of the third angle: if the angles at the base be bisected, and the points where the lines bisecting them meet the opposite sides be joined, the triangle will be divided into two parts in the proportion of the base to the side of the triangle.

407. Any regular polygon inscribed in a circle is a mean proportional between the inscribed and circumscribed regular polygons of half the number of sides.

408. In the figure of VI. 24 shew that EG and KH are parallel.

409. Divide a triangle into two equal parts by a straight line at right angles to one of the sides.

410. If two isosceles triangles are to one another in the duplicate ratio of their bases, shew that the triangles are similar.

411. Through a given point draw a chord in a given circle so that it shall be divided at the point in a given ratio.

412. From a point without a circle draw a straight line cutting the circle, so that the two segments shall be equal to each other.

413. In the figure of II. 11 shew that four other straight lines, besides the given straight line are divided in the required manner.

414. Construct a triangle, having given the base, the vertical angle, and the rectangle contained by the sides.

415. A circle is described round an equilateral triangle, and from any point in the circumference straight lines are drawn to the angular points of the triangle: shew that one of these straight lines is equal to the other two together.

416. From the extremities B , C of the base of an isosceles triangle ABC , straight lines are drawn at right angles to AB , AC respectively, and intersecting at D : shew that the rectangle BC , AD is double of the rectangle AB , DB .

417. ABC is an isosceles triangle, the side AB being equal to AC ; F is the middle point of BC ; on any straight line through A perpendiculars FG and CE are drawn: shew that the rectangle AC , EF is equal to the sum of the rectangles FC , EG and FA , FG .

XI. 1 to 12.

418. Shew that equal straight lines drawn from a given point to a given plane are equally inclined to the plane.

419. If two straight lines in one plane be equally inclined to another plane, they will be equally inclined to the common section of these planes.

420. From a point A a perpendicular is drawn to a plane meeting it at B ; from B a perpendicular is drawn on a straight line in the plane meeting it at C : shew that AC is perpendicular to the straight line in the plane.

421. ABC is a triangle; the perpendiculars from A and B on the opposite sides meet at D ; through D a straight line is drawn perpendicular to the plane of the triangle, and E is any point in this straight line: shew that

the straight line joining E to any angular point of the triangle is at right angles to the straight line drawn through that angular point parallel to the opposite side of the triangle.

422. Straight lines are drawn from two given points without a given plane meeting each other in that plane: find when their sum is the least possible.

423. Three straight lines not in the same plane meet at a point, and a plane cuts these straight lines at equal distances from the point of intersection: shew that the perpendicular from that point on the plane will meet it at the centre of the circle described about the triangle formed by the portion of the plane intercepted by the planes passing through the straight lines.

424. Give a geometrical construction for drawing a straight line which shall be equally inclined to three straight lines meeting at a point.

425. From a point E draw EC , ED perpendicular to two planes CAB , DAB which intersect in AB , and from D draw DF perpendicular to the plane CAB meeting it at F : shew that the straight line CF , produced if necessary, is perpendicular to AB .

426. Perpendiculars are drawn from a point to a plane, and to a straight line in that plane: shew that the straight line joining the feet of the perpendiculars is perpendicular to the former straight line.

XI. 13 to 21.

427. BCD is the common base of two pyramids, whose vertices A and E lie in a plane passing through BC ; and AB , AC are respectively perpendicular to the faces BED , CED : shew that one of the angles at A together with the angles at E make up four right angles.

428. Within the area of a given triangle is inscribed another triangle: shew that the sum of the angles subtended by the sides of the interior triangle at any point not in the plane of the triangles is less than the sum of the angles subtended at the same point by the sides of the exterior angle.

429. From the extremities of the two parallel straight

lines AB, CD parallel straight lines Aa, Bb, Cc, Dd are drawn meeting a plane at a, b, c, d : shew that AB is to CD as ab to cd .

430. Shew that the perpendicular drawn from the vertex of a regular tetrahedron on the opposite face is three times that drawn from its own foot on any of the other faces.

431. A triangular pyramid stands on an equilateral base and the angles at the vertex are right angles: shew that the sum of the perpendiculars on the faces from any point of the base is constant.

432. Three straight lines not in the same plane intersect at a point, and through their point of intersection another straight line is drawn within the solid angle formed by them: shew that the angles which this straight line makes with the first three are together less than the sum, but greater than half the sum, of the angles which the first three make with each other.

433. Three straight lines which do not all lie in one plane, are cut in the same ratio by three planes, two of which are parallel: shew that the third will be parallel to the other two, if its intersections with the three straight lines are not all in the same straight line.

434. Draw two parallel planes, one through one straight line, and the other through another straight line which does not meet the former.

435. If two planes which are not parallel be cut by two parallel planes, the lines of section of the first two by the last two will contain equal angles.

436. From a point A in one of two planes are drawn AB at right angles to the first plane, and AC perpendicular to the second plane, and meeting the second plane at B, C : shew that BC is perpendicular to the line of intersection of the two planes.

437. Polygons formed by cutting a prism by parallel planes are equal.

438. Polygons formed by cutting a pyramid by parallel planes are similar.

439. The straight line $PBbp$ cuts two parallel planes at B, b , and the points P, p are equidistant from the planes; PAa, pcC are other straight lines drawn from P, p to cut the planes: shew that the triangles ABC, abc are equal.

440. Perpendiculars AE, BF are drawn to a plane

from two points A, B above it; a plane is drawn through A perpendicular to AB : shew that its line of intersection with the given plane is perpendicular to EF .

I. 1 to 48.

441. ABC is a triangle, and P is any point within it: shew that the sum of PA, PB, PC is less than the sum of the sides of the triangle.

442. From the centres A and B of two circles parallel radii AP, BQ are drawn; the straight line PQ meets the circumferences again at R and S : shew that AR is parallel to BS .

443. If any point be taken within a parallelogram the sum of the triangles formed by joining the point with the extremities of a pair of opposite sides is half the parallelogram.

444. If a quadrilateral figure be bisected by one diagonal the second diagonal is bisected by the first.

445. Any quadrilateral figure which is bisected by both of its diagonals is a parallelogram.

446. In the figure of I. 5 if the equal sides of the triangle be produced upwards through the vertex, instead of downwards through the base, a demonstration of I. 15 may be obtained without assuming any proposition beyond I. 5.

447. A is a given point, and B is a given point in a given straight line: it is required to draw from A to the given straight line, a straight line AP , such that the sum of AP and PB may be equal to a given length.

448. Shew that by superposition the first case of I. 26 may be immediately demonstrated, and also the second case with the aid of I. 16.

449. A straight line is drawn terminated by one of the sides of an isosceles triangle, and by the other side produced, and bisected by the base: shew that the straight lines thus intercepted between the vertex of the isosceles triangle and this straight line, are together equal to the two equal sides of the triangle.

450. Through the middle point M of the base BC of a triangle a straight line DME is drawn, so as to cut off equal parts from the sides AB , AC , produced if necessary: shew that BD is equal to CE .

451. Of all parallelograms which can be formed with diameters of given lengths the rhombus is the greatest.

452. Shew from I. 18 and I. 32 that if the hypotenuse BC of a right-angled triangle ABC be bisected at D , then AD , BD , CD are all equal.

453. If two equal straight lines intersect each other any where at right angles, the quadrilateral formed by joining their extremities is equal to half the square on either straight line.

454. Inscribe a parallelogram in a given triangle, in such a manner that its diagonals shall intersect at a given point within the triangle.

455. Construct a triangle of given area, and having two of its sides of given lengths.

456. Construct a triangle, having given the base, the difference of the sides, and the difference of the angles at the base.

457. AB , AC are two given straight lines: it is required to find in AB a point P , such that if PQ be drawn perpendicular to AC , the sum of AP and AQ may be equal to a given straight line.

458. The distance of the vertex of a triangle from the bisection of its base, is equal to, greater than, or less than half of the base, according as the vertical angle is a right, an acute, or an obtuse angle.

459. If in the sides of a given square, at equal distances from the four angular points, four other points be taken, one on each side, the figure contained by the straight lines which join them, shall also be a square.

460. On a given straight line as base, construct a triangle, having given the difference of the sides and a point through which one of the sides is to pass.

461. ABC is a triangle in which BA is greater than CA ; the angle A is bisected by a straight line which meets BC at D ; shew that BD is greater than CD .

462. If one angle of a triangle be triple another the triangle may be divided into two isosceles triangles.

463. If one angle of a triangle be double another, an isosceles triangle may be added to it so as to form together with it a single isosceles triangle.

464. Let one of the equal sides of an isosceles triangle be bisected at D , and let it also be doubled by being produced through the extremity of the base to E , then the distance of the other extremity of the base from E is double its distance from D .

465. Determine the locus of a point whose distance from one given point is double its distance from another given point.

466. A straight line AB is bisected at C , and on AC and CB as diagonals any two parallelograms $ADCE$ and $CFBG$ are described; let the parallelogram whose adjacent sides are CD and CF be completed, and also that whose adjacent sides are CE and CG : shew that the diagonals of these latter parallelograms are in the same straight line.

467. $ABCD$ is a rectangle of which A, C are opposite angles; E is any point in BC and F is any point in CD : shew that twice the area of the triangle AEF , together with the rectangle BE, DF is equal to the rectangle $ABCD$.

468. ABC, DBC are two triangles on the same base, and ABC has the side AB equal to the side AC ; a circle passing through C and D has its centre E on CA , produced if necessary; a circle passing through B and D has its centre F on BA , produced if necessary: shew that the quadrilateral $AEDF$ has the sum of two of its sides equal to the sum of the other two.

469. Two straight lines AB, AC are given in position:

it is required to find in AB a point P , such that a perpendicular being drawn from it to AC , the straight line AP may exceed this perpendicular by a proposed length.

470. Shew that the opposite sides of any equiangular hexagon are parallel, and that any two sides which are adjacent are together equal to the two to which they are parallel.

471. From D and E , the corners of the square $BDEC$ described on the hypotenuse BC of a right-angled triangle ABC , perpendiculars DM , EN are let fall on AC , AB respectively: shew that AM is equal to AB , and AN equal to AC .

472. AB and AC are two given straight lines, and P is a given point: it is required to draw through P a straight line which shall form with AB and AC the least possible triangle.

473. ABC is a triangle in which C is a right angle: shew how to draw a straight line parallel to a given straight line, so as to be terminated by CA and CB , and bisected by AB .

474. ABC is an isosceles triangle having the angle at B four times either of the other angles; AB is produced to D so that BD is equal to twice AB , and CD is joined: shew that the triangles ACD and ABC are equiangular to one another.

475. Through a point K within a parallelogram $ABCD$ straight lines are drawn parallel to the sides: shew that the difference of the parallelograms of which KA and KC are diagonals is equal to twice the triangle BKD .

476. Construct a right-angled triangle, having given one side and the difference between the other side and the hypotenuse.

477. The straight lines AD , BE bisecting the sides BC , AC of a triangle intersect at G : shew that AG is double of GD .

478. BAC is a right-angled triangle; one straight line is drawn bisecting the right angle A , and another bisecting the base BC at right angles; these straight lines intersect at E : if D be the middle point of BC , shew that DE is equal to DA .

479. On AC the diagonal of a square $ABCD$, a rhombus $AEFC$ is described of the same area as the square,

and having its acute angle at A : if AF be joined, shew that the angle BAC is divided into three equal angles.

480. AB, AC are two fixed straight lines at right angles; D is any point in AB , and E is any point in AC ; on DE as diagonal a half square is described with its vertex at G : shew that the locus of G is the straight line which bisects the angle BAC .

481. Shew that a square is greater than any other parallelogram of the same perimeter.

482. Inscribe a square of given magnitude in a given square.

483. ABC is a triangle; AD is a third of AB , and AE is a third of AC ; CD and BE intersect at F : shew that the triangle BFC is half the triangle BAC , and that the quadrilateral $ADFE$ is equal to either of the triangles CFE or BDF .

484. ABC is a triangle, having the angle C a right angle; the angle A is bisected by a straight line which meets BC at D , and the angle B is bisected by a straight line which meets AC at E ; AD and BE intersect at O : shew that the triangle AOB is half the quadrilateral $ABDE$.

485. Shew that a scalene triangle cannot be divided by a straight line into two parts which will coincide.

486. $ABCD, ACED$ are parallelograms on equal bases BC, CE , and between the same parallels AD, BE ; the straight lines BD and AE intersect at F : shew that BF is equal to twice DF .

487. Parallelograms $AFGC, CBKH$ are described on AC, BC outside the triangle ABC ; FG and KH meet at Z ; ZC is joined, and through A and B straight lines AD and BE are drawn, both parallel to ZC , and meeting FG and KH at D and E respectively: shew that the figure $ADEB$ is a parallelogram, and that it is equal to the sum of the parallelograms FC, CK .

488. If a quadrilateral have two of its sides parallel shew that the straight line drawn parallel to these sides through the intersection of the diagonals is bisected at that point.

489. Two triangles are on equal bases and between the same parallels: shew that the sides of the triangles intercept equal lengths of any straight line which is parallel to their bases.

490. In a right-angled triangle, right-angled at A , if the side AC be double of the side AB , the angle B is more than double of the angle C .

491. Trisect a parallelogram by straight lines drawn through one of its angular points.

492. AHK is an equilateral triangle; $ABCD$ is a rhombus, a side of which is equal to a side of the triangle, and the sides BC and CD of which pass through H and K respectively: shew that the angle A of the rhombus is ten-ninths of a right angle.

493. Trisect a given triangle by straight lines drawn from a given point in one of its sides.

494. In the figure of I. 35 if two diagonals be drawn to the two parallelograms respectively, one from each extremity of the base, and the intersection of the diagonals be joined with the intersection of the sides (or sides produced) in the figure, shew that the joining straight line will bisect the base.

II. 1 to 14.

495. Produce one side of a given triangle so that the rectangle contained by this side and the produced part may be equal to the difference of the squares on the other two sides.

496. Produce a given straight line so that the sum of the squares on the given straight line and on the part produced may be equal to twice the rectangle contained by the whole straight line thus produced and the part produced.

497. Produce a given straight line so that the sum of the squares on the given straight line and on the whole straight line thus produced may be equal to twice the rectangle contained by the whole straight line thus produced and the part produced.

498. Produce a given straight line so that the rectangle contained by the whole straight line thus produced and the part produced may be equal to the square on the given straight line.

499. Describe an isosceles obtuse-angled triangle such that the square on the largest side may be equal to three times the square on either of the equal sides.

500. Find the obtuse angle of a triangle when the

square on the side opposite to the obtuse angle is greater than the sum of the squares on the sides containing it, by the rectangle of the sides.

501. Construct a rectangle equal to a given square when the sum of two adjacent sides of the rectangle is equal to a given quantity.

502. Construct a rectangle equal to a given square when the difference of two adjacent sides of the rectangle is equal to a given quantity.

503. The least square which can be inscribed in a given square is that which is half of the given square.

504. Divide a given straight line into two parts so that the squares on the whole line and on one of the parts may be together double of the square on the other part.

505. Two rectangles have equal areas and equal perimeters: shew that they are equal in all respects.

506. $ABCD$ is a rectangle; P is a point such that the sum of PA and PC is equal to the sum of PB and PD : shew that the locus of P consists of the two straight lines through the centre of the rectangle parallel to its sides.

III. 1 to 37.

507. Describe a circle which shall pass through a given point and touch a given straight line at a given point.

508. Describe a circle which shall pass through a given point and touch a given circle at a given point.

509. Describe a circle which shall touch a given circle at a given point and touch a given straight line.

510. AD , BE are perpendiculars from the angles A and B of a triangle on the opposite sides; BF is perpendicular to ED or ED produced: shew that the angle FBD is equal to the angle EBA .

511. If ABC be a triangle, and BE , CF the perpendiculars from the angles on the opposite sides, and K the middle point of the third side, shew that the angles FEK , EFK are each equal to A .

512. AB is a diameter of a circle; AC and AD are two chords meeting the tangent at B at E and F respectively: shew that the angles FCE and FDE are equal.

513. Shew that the four straight lines bisecting the angles of any quadrilateral form a quadrilateral which can be inscribed in a circle.

514. Find the shortest distance between two circles which do not meet.

515. Two circles cut one another at a point A : it is required to draw through A a straight line so that the extreme length of it intercepted by the two circles may be equal to that of a given straight line.

516. If a polygon of an even number of sides be inscribed in a circle, the sum of the alternate angles together with two right angles is equal to as many right angles as the figure has sides.

517. Draw from a given point in the circumference of a circle, a chord which shall be bisected by its point of intersection with a given chord of the circle.

518. When an equilateral polygon is described about a circle it must necessarily be equiangular if the number of sides be odd, but not otherwise.

519. AB is the diameter of a circle whose centre is C , and DCE is a sector having the arc DE constant; join AE , BD intersecting at P ; shew that the angle APB is constant.

520. If any number of triangles on the same base BC , and on the same side of it have their vertical angles equal, and perpendiculars, intersecting at D , be drawn from B and C on the opposite sides, find the locus of D ; and shew that all the straight lines which bisect the angle BDC pass through the same point.

521. Let O and C be any fixed points on the circumference of a circle, and OA any chord; then if AC be joined and produced to B , so that OB is equal to OA , the locus of B is an equal circle.

522. From any point P in the diagonal BD of a parallelogram $ABCD$, straight lines PE , PF , PG , PH are drawn perpendicular to the sides AB , BC , CD , DA : shew that EF is parallel to GH .

523. Through any fixed point of a chord of a circle other chords are drawn; shew that the straight lines from the middle point of the first chord to the middle points of the others will meet them all at the same angle.

524. ABC is a straight line, divided at any point B

into two parts; ADB and CDB are similar segments of circles, having the common chord BD ; CD and AD are produced to meet the circumferences at F and E respectively, and AF , CE , BF , BE are joined: shew that ABF and CBE are isosceles triangles, equiangular to one another.

525. If the centres of two circles which touch each other externally be fixed, the common tangent of those circles will touch another circle of which the straight line joining the fixed centres is the diameter.

526. A is a given point: it is required to draw from A two straight lines which shall contain a given angle and intercept on a given straight line a part of given length.

527. A straight line and two circles are given: find the point in the straight line from which the tangents drawn to the circles are of equal length.

528. In a circle two chords of given length are drawn so as not to intersect, and one of them is fixed in position; the opposite extremities of the chords are joined by straight lines intersecting within the circle: shew that the locus of the point of intersection will be a portion of the circumference of a circle, passing through the extremities of the fixed chord.

529. A and B are the centres of two circles which touch internally at C , and also touch a third circle, whose centre is D , externally and internally respectively at E and F : shew that the angle ADB is double of the angle ECF .

530. C is the centre of a circle, and CP is a perpendicular on a chord APB : shew that the sum of CP and AP is greatest when CP is equal to AP .

531. AB , BC , CD are three adjacent sides of any polygon inscribed in a circle; the arcs AB , BC , CD are bisected at L , M , N ; and LM cuts BA , BC respectively at P and Q : shew that BPQ is an isosceles triangle; and that the angles ABC , BCD are together double of the angle LMN .

532. In the circumference of a given circle determine a point so situated that if chords be drawn to it from the extremities of a given chord of the circle their difference shall be equal to a given straight line less than the given chord.

533. Construct a triangle, having given the sum of the

sides, the difference of the segments of the base made by the perpendicular from the vertex, and the difference of the base angles.

534. On a straight line AB as base, and on the same side of it are described two segments of circles; P is any point in the circumference of one of the segments, and the straight line BP cuts the circumference of the other segment at Q : shew that the angle PAQ is equal to the angle between the tangents at A .

535. AKL is a fixed straight line cutting a given circle at K and L ; APQ , ARS are two other straight lines making equal angles with AKL , and cutting the circle at P , Q and R , S : shew that whatever be the position of APQ and ARS , the straight line joining the middle points of PQ and RS always remains parallel to itself.

536. If about a quadrilateral another quadrilateral can be described such that every two of its adjacent sides are equally inclined to that side of the former quadrilateral which meets them both, then a circle may be described about the former quadrilateral.

537. Two circles touch one another internally at the point A : it is required to draw from A a straight line such that the part of it between the circles may be equal to a given straight line, which is not greater than the difference between the diameters of the circles.

538. $ABCD$ is a parallelogram; AE is at right angles to AB , and CE is at right angles to CB : shew that ED , if produced, will cut AC at right angles.

539. From each angular point of a triangle a perpendicular is let fall on the opposite side: shew that the rectangles contained by the segments into which each perpendicular is divided by the point of intersection of the three are equal to each other.

540. The two angles at the base of a triangle are bisected by two straight lines on which perpendiculars are drawn from the vertex: shew that the straight line which passes through the feet of these perpendiculars will be parallel to the base and will bisect the sides.

541. In a given circle inscribe a rectangle equal to a given rectilineal figure.

542. In an acute-angled triangle ABC perpendiculars AD , BE are let fall on BC , CA respectively; circles

described on AC , BC as diameters meet BE , AD respectively at F , G and H , K : shew that F , G , H , K lie on the circumference of a circle.

543. Two diameters in a circle are at right angles; from their extremities four parallel straight lines are drawn; shew that they divide the circumference into four equal parts.

544. E is the middle point of a semicircular arc AEB , and CDE is any chord cutting the diameter at D , and the circle at C : shew that the square on CE is twice the quadrilateral $AEB C$.

545. AB is a fixed chord of a circle, AC is a moveable chord of the same circle; a parallelogram is described of which AB and AC are adjacent sides: find the locus of the middle points of the diagonals of the parallelogram.

546. AB is a fixed chord of a circle, AC is a moveable chord of the same circle; a parallelogram is described of which AB and AC are adjacent sides: determine the greatest possible length of the diagonal drawn through A .

547. If two equal circles be placed at such a distance apart that the tangent drawn to either of them from the centre of the other is equal to a diameter, shew that they will have a common tangent equal to the radius.

548. Find a point in a given circle from which if two tangents be drawn to an equal circle, given in position, the chord joining the points of contact is equal to the chord of the first circle formed by joining the points of intersection of the two tangents produced; and determine the limit to the possibility of the problem.

549. AB is a diameter of a circle, and AF is any chord; C is any point in AB , and through C a straight line is drawn at right angles to AB , meeting AF , produced if necessary at G , and meeting the circumference at D : shew that the rectangle FA , AG , and the rectangle BA , AC , and the square on AD are all equal.

550. Construct a triangle, having given the base, the vertical angle, and the length of the straight line drawn from the vertex to the base bisecting the vertical angle.

551. A , B , C are three given points in the circumference of a given circle: find a point P such that if AP , BP , CP meet the circumference at D , E , F respectively, the arcs DE , EF may be equal to given arcs.

552. Find the point in the circumference of a given circle, the sum of whose distances from two given straight lines at right angles to each other, which do not cut the circle, is the greatest or least possible.

553. On the sides of a triangle segments of a circle are described *internally*, each containing an angle equal to the excess of two right angles above the opposite angle of the triangle: shew that the radii of the circles are equal, that the circles all pass through one point, and that their chords of intersection are respectively perpendicular to the opposite sides of the triangle.

IV. 1 to 16.

554. From the angles of a triangle ABC perpendiculars are drawn to the opposite sides meeting them at D, E, F respectively: shew that DE and DF are equally inclined to AD .

555. The points of contact of the inscribed circle of a triangle are joined; and from the angular points of the triangle so formed perpendiculars are drawn to the opposite sides: shew that the triangle of which the feet of these perpendiculars are the angular points has its sides parallel to the sides of the original triangle.

556. Construct a triangle having given an angle and the radii of the inscribed and circumscribed circles.

557. Triangles are constructed on the same base with equal vertical angles; shew that the locus of the centres of the escribed circles, each of which touches one of the sides externally and the other side and base produced, is an arc of a circle, the centre of which is on the circumference of the circle circumscribing the triangles.

558. From the angular points A, B, C of a triangle perpendiculars are drawn on the opposite sides, and terminated at the points D, E, F on the circumference of the circumscribing circle: if L be the point of intersection of the perpendiculars, shew that LD, LE, LF are bisected by the sides of the triangle.

559. $ABCDE$ is a regular pentagon; join AC and BD intersecting at O : shew that AO is equal to DO , and that the rectangle AC, CO is equal to the square on BC .

560. A straight line PQ of given length moves so that its ends are always on two fixed straight lines CP, CQ ; straight lines from P and Q at right angles to CP and CQ respectively intersect at R ; perpendiculars from P and Q on CQ and CP respectively intersect at S : shew that the loci of R and S are circles having their common centre at C .

561. Right-angled triangles are described on the same hypotenuse: shew that the locus of the centres of the inscribed circles is a quarter of the circumference of a circle of which the common hypotenuse is a chord.

562. On a given straight line AB any triangle ACB is described; the sides AC, BC are bisected and straight lines drawn at right angles to them through the points of bisection to intersect at a point D ; find the locus of D .

563. Construct a triangle, having given its base, one of the angles at the base, and the distance between the centre of the inscribed circle and the centre of the circle touching the base and the sides produced.

564. Describe a circle which shall touch a given straight line at a given point, and bisect the circumference of a given circle.

565. Describe a circle which shall pass through a given point and bisect the circumferences of two given circles.

566. Within a given circle inscribe three equal circles, touching one another and the given circle.

567. If the radius of a circle be cut as in II. 11, the greater segment will be the side of a regular decagon inscribed in the circle.

568. If the radius of a circle be cut as in II. 11, the square on its greater segment, together with the square on the radius, is equal to the square on the side of a regular pentagon inscribed in the circle.

569. From the vertex of a triangle draw a straight line to the base so that the square on the straight line may be equal to the rectangle contained by the segments of the base.

570. Four straight lines are drawn in a plane forming four triangles; shew that the circumscribing circles of these triangles all pass through a common point.

571. The perpendiculars from the angles A and B of a triangle on the opposite sides meet at D ; the circles described round ADC and DBC cut AB or AB produced at the points E and F : shew that AE is equal to BF .

572. The four circles each of which passes through the centres of three of the four circles touching the sides of a triangle are equal to one another.

573. Four circles are described so that each may touch internally three of the sides of a quadrilateral: shew that a circle may be described so as to pass through the centres of the four circles.

574. A circle is described round the triangle ABC , and from any point P of its circumference perpendiculars are drawn to BC , CA , AB , which meet the circle again at D , E , F : shew that the triangles ABC and DEF are equal in all respects, and that the straight lines AD , BE , CF are parallel.

575. With any point in the circumference of a given circle as centre, describe another circle, cutting the former at A and B ; from B draw in the described circle a chord BD equal to its radius, and join AD , cutting the given circle at Q : shew that QD is equal to the radius of the given circle.

576. A point is taken without a square, such that straight lines being drawn to the angular points of the square, the angle contained by the two extreme straight lines is divided into three equal parts by the other two straight lines: shew that the locus of the point is the circumference of the circle circumscribing the square.

577. Circles are inscribed in the two triangles formed by drawing a perpendicular from an angle of a triangle on the opposite side; and analogous circles are described in relation to the two other like perpendiculars: shew that the sum of the diameters of the six circles together with the sum of the sides of the original triangle is equal to twice the sum of these perpendiculars.

578. Three concentric circles are drawn in the same plane: draw a straight line, such that one of its segments between the inner and outer circumference may be bisected at one of the points at which the straight line meets the middle circumference.

VI. 1 to D.

579. AB is a diameter, and P any point in the circumference of a circle; AP and BP are joined and produced if necessary; from any point C in AB a straight line is drawn at right angles to AB meeting AP at D and BP at E , and the circumference of the circle at F ; shew that CD is a third proportional to CE and CF .

580. A, B, C are three points in a straight line, and D a point at which AB and BC subtend equal angles: shew that the locus of D is the circumference of a circle.

581. If a straight line be drawn from one corner of a square cutting off one-fourth from the diagonal it will cut off one-third from a side. Also if straight lines be drawn similarly from the other corners so as to form a square, this square will be two-fifths of the original square.

582. The sides AB, AC of a given triangle ABC are produced to any points D, E , so that DE is parallel to BC . The straight line DE is divided at F so that DF is to FE as BD is to CE : shew that the locus of F is a straight line.

583. A, B, C are three points in order in a straight line: find a point P in the straight line so that PB may be a mean proportional between PA and PC .

584. A, B are two fixed points on the circumference of a given circle, and P is a moveable point on the circumference; on PB is taken a point D such that PD is to PA in a constant ratio, and on PA is taken a point E such that PE is to PB in the same ratio: shew that DE always touches a fixed circle.

585. ABC is an isosceles triangle, the angle at A being four times either of the others: shew that if BC be bisected at D and E , the triangle ADE is equilateral.

586. Perpendiculars are let fall from two opposite angles of a rectangle on a diagonal: shew that they will divide the diagonal into equal parts, if the square on one side of the rectangle be double that on the other.

587. A straight line AB is divided into any two parts at C , and on the whole straight line and on the two parts of it equilateral triangles ADB, ACE, BCF are described, the two latter being on the same side of the straight

line, and the former on the opposite side; G, H, K are the centres of the circles inscribed in these triangles: shew that the angles AGH, BGK are respectively equal to the angles ADC, BDC , and that GH is equal to GK .

588. On the two sides of a right-angled triangle squares are described: shew that the straight lines joining the acute angles of the triangle and the opposite angles of the squares cut off equal segments from the sides, and that each of these equal segments is a mean proportional between the remaining segments.

589. Two straight lines and a point between them are given in position: draw two straight lines from the given point to terminate in the given straight lines, so that they shall contain a given angle and have a given ratio.

590. With a point A in the circumference of a circle ABC as centre, a circle PBC is described cutting the former circle at the points B and C ; any chord AD of the former meets the common chord BC at E , and the circumference of the other circle at O : shew that the angles EPO and DPO are equal for all positions of P .

591. AEC, ABF are triangles on the same base in the ratio of two to one; AF and BF produced meet the sides at D and E ; in FB a part FG is cut off equal to FE , and BG is bisected at O : shew that BO is to BE as DF is to DA .

592. A is the centre of a circle, and another circle passes through A and cuts the former at B and C ; AD is a chord of the latter circle meeting BC at E , and from D are drawn DF and DG tangents to the former circle: shew that G, E, F lie on one straight line.

593. In AB, AC , two sides of a triangle, are taken points D, E ; AB, AC are produced to F, G such that BF is equal to AD , and CG equal to AE ; BG, CF are joined meeting at H : shew that the triangle FHG is equal to the triangles BHC, ADE together.

594. In any triangle ABC if BD be taken equal to one-fourth of BC , and CE one-fourth of AC , the straight line drawn from C through the intersection of BE and AD will divide AB into two parts, which are in the ratio of nine to one.

595. Any rectilineal figure is inscribed in a circle: shew that by bisecting the arcs and drawing tangents to the points of bisection parallel to the sides of the recti-

lineal figure, we can form a similar rectilineal figure circumscribing the circle.

596. Find a mean proportional between two similar right-angled triangles which have one of the sides containing the right angle common.

597. In the sides AC , BC of a triangle ABC points D and E are taken, such that CD and CE are respectively the third parts of AC and BC ; BD and AE are drawn intersecting at O : shew that EO and DO are respectively the fourth parts of AE and BD .

598. CA , CB are diameters of two circles which touch each other externally at C ; a chord AD of the former circle, when produced, touches the latter at E , while a chord BF of the latter, when produced, touches the former at G : shew that the rectangle contained by AD and BF is four times that contained by DE and FG .

599. Two circles intersect at A , and BAC is drawn meeting them at B and C ; with B , C as centres are described two circles each of which intersects one of the former at right angles: shew that these circles and the circle whose diameter is BC meet at a point.

600. $ABCDEF$ is a regular hexagon: shew that BF divides AD in the ratio of one to three.

601. ABC , DEF are triangles, having the angle A equal to the angle D ; and AB is equal to DF : shew that the areas of the triangles are as AC to DE .

602. If M , N be the points at which the inscribed and an escribed circle touch the side AC of a triangle ABC ; shew that if BM be produced to cut the escribed circle again at P , then NP is a diameter.

603. The angle A of a triangle ABC is a right angle, and D is the foot of the perpendicular from A on BC ; DM , DN are perpendiculars on AB , AC : shew that the angles BMC , BNC are equal.

604. If from the point of bisection of any given arc of a circle two straight lines be drawn, cutting the chord of the arc and the circumference, the four points of intersection shall also lie in the circumference of a circle.

605. The side AB of a triangle ABC is touched by the inscribed circle at D , and by the escribed circle at E : shew that the rectangle contained by the radii is equal to the rectangle AD , DB and to the rectangle AE , EB .

606. Shew that the locus of the middle points of straight lines parallel to the base of a triangle and terminated by its sides is a straight line.

607. A parallelogram is inscribed in a triangle, having one side on the base of the triangle, and the adjacent sides parallel to a fixed direction: shew that the locus of the intersection of the diagonals of the parallelogram is a straight line bisecting the base of the triangle.

608. On a given straight line AB as hypotenuse a right-angled triangle is described; and from A and B straight lines are drawn to bisect the opposite sides: shew that the locus of their intersection is a circle.

609. From a given point outside two given circles which do not meet, draw a straight line such that the portions of it intercepted by each circle shall be respectively proportional to their radii.

610. In a given triangle inscribe a rhombus which shall have one of its angular points coincident with a point in the base, and a side on that base.

611. ABC is a triangle having a right angle at C ; $ABDE$ is the square described on the hypotenuse; F, G, H are the points of intersection of the diagonals of the squares on the hypotenuse and sides: shew that the angles DCE, GFH are together equal to a right angle.

MISCELLANEOUS.

612. O is a fixed point from which any straight line is drawn meeting a fixed straight line at P ; in OP a point Q is taken such that the rectangle OP, OQ is constant: shew that the locus of Q is the circumference of a circle.

613. O is a fixed point on the circumference of a circle, from which any straight line is drawn meeting the circumference at P ; in OP a point Q is taken such that the rectangle OP, OQ is constant: shew that the locus of Q is a straight line.

614. The opposite sides of a quadrilateral inscribed in a circle when produced meet at P and Q : shew that the square on PQ is equal to the sum of the squares on the tangents from P and Q to the circle.

615. $ABCD$ is a quadrilateral inscribed in a circle; the opposite sides AB and DC are produced to meet at F ; and the opposite sides BC and AD at E : shew that the circle described on EF as diameter cuts the circle $ABCD$ at right angles.

616. From the vertex of a right-angled triangle a perpendicular is drawn on the hypotenuse, and from the foot of this perpendicular another is drawn on each side of the triangle: shew that the area of the triangle of which these two latter perpendiculars are two of the sides cannot be greater than one-fourth of the area of the original triangle.

617. If the extremities of two intersecting straight lines be joined so as to form two vertically opposite triangles, the figure made by connecting the points of bisection of the given straight lines, will be a parallelogram equal in area to half the difference of the triangles.

618. AB, AC are two tangents to a circle, touching it at B and C ; R is any point in the straight line which joins the middle points of AB and AC ; shew that AR is equal to the tangent drawn from R to the circle.

619. AB, AC are two tangents to a circle; PQ is a chord of the circle which, produced if necessary, meets the straight line joining the middle points of AB, AC at R ; shew that the angles RAP, AQR are equal to one another.

620. Shew that the four circles each of which passes through the middle points of the sides of one of the four triangles formed by two adjacent sides and a diagonal of any quadrilateral all intersect at a point.

621. Perpendiculars are drawn from any point on the three straight lines which bisect the angles of an equilateral triangle: shew that one of them is equal to the sum of the other two.

622. Two circles intersect at A and B , and CBD is drawn through B perpendicular to AB to meet the circles; through A a straight line is drawn bisecting either the interior or exterior angle between AC and AD , and meeting the circumferences at E and F : shew that the tangents to the circumferences at E and F will intersect in AB produced.

623. Divide a triangle by two straight lines into three

parts, which, when properly arranged, shall form a parallelogram whose angles are of given magnitude.

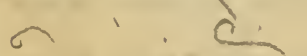
624. $ABCD$ is a parallelogram, and P is any point: shew that the triangle PAC is equal to the difference of the triangles PAB and PAD , if P is within the angle BAD or that which is vertically opposite to it; and that the triangle PAC is equal to the sum of the triangles PAB and PAD , if P has any other position.

625. Two circles cut each other, and a straight line $ABCDE$ is drawn, which meets one circle at A and D , the other at B and E , and their common chord at C : shew that the square on BD is to the square on AE as the rectangle BC, CD is to the rectangle AC, CE .

IV v VII. mcd2.

If a st line be divided into 2
pts either internally or
externally the diff. bet

the  THE END. 

 on two segments
is equal to twice the
rect. cont. by two
segments.

GENERAL INDEX OF GREEK WORDS AND FORMS.

[The references are to volumes and pages.]

- ἀγώνιον, *angle-less* (figure) I. 187
 ἀδύνατον: ἡ εἰς τὸ ἀδ. ἀπαγωγή, ἡ διὰ τοῦ ἀδ.
 δείξις, ἡ εἰς τὸ ἀδ. ἀγούσα ἀπόδειξις I. 136
 ἀκιδοειδής, *barb-like* I. 188
 ἄκρος, *extreme* (of numbers in a series) II. 328, 367: (εἰς) ἄκρον καὶ μέσον λόγον *τεμῆσθαι*, "to be cut in extreme and mean ratio" II. 189
 ἄλογος, having no ratio, *irrational* II. 117-8: a relative term, resting on assumption or convention (Pythagoreans) III. 1, 11: use of term restricted in Euclid III. 12
 ἀμβλεία (γωνία), *obtuse* (angle) I. 181
 ἀμβλυγώνιος, *obtuse-angled* I. 187
 ἀμερής, *indivisible* I. 41, 268
 ἀμφίκυκλος (of curvilinear angles) I. 178
 ἀμφίκυκτος (of curvilinear angles) I. 178
 ἀναγράφειν ἀπὸ, to *describe on*, contrasted with to *construct* (συστήσασθαι) I. 348: peculiar use of active participle; αἱ ἴσα τετράγωνα ἀναγράφουσι = *straight lines on which equal squares are described* III. 13
 ἀναλογία, *proportion*: definitions of, interpolated II. 119
 ἀνάλογον = ἀνὰ λόγον, *proportional* or in *proportion*: used as indeclinable adj. and as adv. II. 129, 165: μέση ἀνάλογον, *mean proportional* (of straight line) II. 129, similarly μέσος ἀνάλογον of numbers II. 295, 363 etc.: τρίτη (τρίτος) ἀνάλογον, *third proportional* II. 214, 407-8: τέταρτη (τέταρτος) ἀνάλογον, *fourth proportional* II. 215, 409: ἑξῆς ἀνάλογον, in continued proportion II. 346
 ἀναλυόμενος (τόπος), *Treasury of Analysis*, I. 8, 10, 11, 138
 ἀνάπαιιν (λόγος), *inverse* (ratio), *inversely* II. 134
 ἀναστρέψαντι, *convertendo*, in proportions II. 135: analogous use otherwise than in proportions III. 164
 ἀναστροφή λόγον, "conversion" of a ratio II. 135
 ἀναστροφικός (species of locus) I. 330
 ἀνίσάκεις ἀνισάκεις ἴσος, *unequal by unequal by equal* (of solid numbers) = *scalene*, σφηνίσκος, σφηκίσκος or βωμίσκος II. 290
 ἀνομοιομερής, *non-uniform* I. 40, 161-2
 ἀνομίως τεταγμένων τῶν λόγων (of *perturbed proportion*) in Archimedes II. 136
 ἀντανάλξεις, ἡ αὐτῇ, definition of *same ratio* in Aristotle (ἀνθυφαίσεις Alexander) II. 120: terms explained II. 121
 ἀντιπεπονθότα σχήματα, *reciprocal* (=reciprocally related) figures, interpolated def. of, II. 189
 ἀντιστροφή, *conversion* I. 256-7: *leading variety*, ἡ προηγουμένη or ἡ κυρίως, *ibid.*
 ἀνύπαρκτος, *non-existent* I. 129
 ἄξων, *axis* III. 269
 ἀόριστος, *indeterminate* (of lines or curves) I. 160: (of problems) I. 129
 ἀπαγωγή, *reduction* I. 135: εἰς τὸ ἀδύνατον I. 136
 ἄπειρος, *infinite*: ἡ ἐπ' ἄπ. ἐκβαλλομένη of line or curve extending without limit and not "forming a figure" I. 160-1: ἐπ' ἄπ. or εἰς ἄπ. adverbial I. 190: ἐπ' ἄπ. διαιρέσθαι I. 268: Aristotle on τὸ ἄπειρον I. 232-4
 ἀπλατής, *breadthless*: in definition of a line, μήκος ἀπλατές, *breadthless length* I. 158: (of prime numbers) II. 285
 ἀπλοῦς, *simple*: (of lines or curves) I. 161-2: (of surfaces) I. 170
 ἀπόδειξις, *proof* (one of necessary divisions of a proposition) I. 129, 130
 ἀποκαταστατικός, *recurrent* (=spherical), of numbers II. 291
 ἀποτομή, *apotomy*, a compound irrational, difference of two terms III. 7: defined III. 158-9: μέσης ἀποτομή πρώτη (δευτέρα), first (second) apotome of a medial (straight line) III. 7, defined III. 159-60
 ἄπτεσθαι, to *meet*, occasionally to *touch* (instead of ἐφάπτεσθαι) I. 57, II. 2: also = to *pass through*, to *lie on* II. 79
 ἀριθμός, *number*, definitions of, II. 280
 ἄρρητος, *inexpressible; irrational*: of λόγος

- I. 137: ἄρρητος διάμετρος τῆς πεμπτὰδος, "irrational diameter of 5" (Plato) = $\sqrt{50}$,
 I. 399, III. 12, 525
 ἀρτιακὸς ἀρτιοδύναμον (Nicomachus) II. 282
 ἀρτιακὸς ἀρτιος, *even-times even* II. 281-2
 ἀρτιακὸς περισσός, *even-times odd* II. 282-4
 ἀρτιοπέριττος *even-odd* (Nicomachus etc.) II. 282
 ἀρτιος (ἀριθμός), *even (number)* II. 281
 ἀσύμβατος, *incompatible* I. 129
 ἀσύμμετρος, *incommensurable*: ἄ. μήκει (μόνον) *incommensurable in length (only)*, δυνάμει "in square" III. 11
 ἀσύμπτωτος, *not-meeting, non-secant, asymptotic* I. 40, 161, 203: (of parallel planes) III. 265
 ἀσύνθετος, *incomposite*: (of lines) I. 160, 161: (of surfaces) I. 170: (prime and) *incomposite* (of numbers) II. 284
 ἀτακτος, *unordered*: (of problems) I. 128: (of irrationals) I. 115, III. 10
 ἄστοι γράμμαι, "indivisible lines" I. 268
 βάθος, *depth* I. 158-9
 βάσις, *base* I. 248-9
 βεβηκέναι, *to stand* (of angle standing on circumference) II. 4
 βωμιακός, *altar-shaped* (of "scalene" solid numbers) II. 290
 γεγονένω (in constructions), "let it be (*have been*) made" II. 248
 γεγονός ἂν εἴη τὸ ἐπιταχθέν, "what was enjoined will have been done" II. 80, 261
 γεγράφθω, "let it be (*lit. have been*) drawn" I. 242
 γενόμενος, ὁ ἐξ αὐτῶν, "their *product*" II. 316, 326 etc.: ὁ ἐκ τοῦ ἐνὸς γενόμενος = "the *square* of the one" II. 327
 γνώμων, *gnomon q.v.*: Democritus περὶ διαφορῆς γνῶμονος (γνώμης or γωνίας?) ἢ περὶ ψαύσιος κύκλου καὶ σφαίρης II. 40: (of numbers) II. 289
 γραμμή, *line (or curve) q.v.*
 γραμμικός, *linear* (of numbers in one dimension) II. 287: (of prime numbers) II. 285: γραμμικῶς, *graphically* I. 400
 γράφεσθαι, "to be *proved*" (Aristotle) II. 120
 δεδομένος, *given*, different senses I. 132-3: Euclid's δεδομένα or *Data q.v.*
 δειγματα, *illustrations*, of Stoics I. 329
 δεῖ δὴ, "thus it is required" (or "is necessary"), introducing διορισμούς I. 293
 δεύτερος, *secondary* (of numbers): in Nicomachus and Iamblichus a subdivision of *odd* II. 286, 287
 δεχόμενον, "admitting" (of segment of circle admitting or containing an angle) II. 5
 διάγραμμα = *proposition* (Aristotle) I. 252
 διαίρεσθαι (used of "separation" of ratios): διαίρεθέντα, *separando*, opp. to συγκεμένα, *componendo* II. 168
 διαίρεσις, *point of division* (Aristotle) I. 165, 170, 171: *method of division* (exhaustion) I. 285: Euclid's περὶ διαίρεσεων, *On divisions (of figures)* I. 8, 9, 18, 87, 110: διαίρεσις λόγου, *separation*, literally *division*, of ratio II. 135
 διάμετρος, *diameter*: of a circle, parallelogram etc. I. 185, 325: of sphere III. 270
 διαστάσεις, almost = "dimensions" I. 157, 158, III. 262: Aristotle speaks of *six* III. 263
 διαστατῶν, *extended*, ἐφ' ἓν *one way*, ἐπὶ δύο *two ways*, ἐπὶ τρία *three ways* (of lines, surfaces and solids respectively) I. 158, 170, III. 263
 διάστημα, *distance* I. 166, 167, 207: (of radius of circle) I. 199: (of an angle) = *divergence* I. 176-7
 διεξενγμένη (ἀναλογία), *disjoined = discrete (proportion)* II. 293
 διελόντι, *separando*, literally *dividendo* (of proportions) II. 135
 διεξοδικός (of a class of loci) I. 330
 διηρημένη (ἀναλογία), *discrete (proportion)*, i.e. in four terms, as distinct from *continuous* (συνεχής, *synnymmenē*) in *three* terms II. 131, 293
 διήχθω, "let it be drawn through" (=produced) or "across" I. 280, II. 7
 δι' ἴσου, *ex aequali* (of ratios) II. 136: δι' ἴσου ἐν τεταραγμένη ἀναλογίᾳ, "*ex aequali* in perturbed proportion" II. 136
 δικόλυνος, *twice-truncated* (of pyramidal numbers) II. 291
 διορισμός = (1) particular statement or definition, one of the formal divisions of a proposition I. 129: (2) statement of condition of possibility I. 128, 129, 130, 131, 234, 243, 293
 διπλάσιος λόγος, *double ratio*: διπλάσιων λόγος, *duplicate ratio*, contrasted with, II. 133
 δύναμις, *power*: = actual value of a submultiple in units (Nicomachus) II. 282: = side of number not a complete square (i.e. *root* or *surd*) in Plato II. 288, 290, III. 1, 2, 3: = *square* in Plato II. 294-5
 δύνασθαι, "to be side of square equal to" III. 13: αἱ δυνάμεναι αὐτά, *sides of squares equal to them* III. 13: ἡ ΒΓ τῆς Α μείζον δύναται τῇ ΔΖ, "the square on BC is greater than the square on A by the square on DF," literally "BC is *in power* greater than A by DF" III. 43
 εἶδος, *figure* II. 234: = *form* II. 254
 εἰσαγωγὴ ἀρμονικὴ, *Introduction to Harmony*, by Cleonides I. 17
 ἕκαστος, each: curious use of, II. 79
 ἑκατέρα ἑκατέρα, meaning *respectively* I. 248, 350
 ἐκβεβλήσθωσαν, *use of*, I. 244
 ἐκείνος = *Euclid* I. 400
 ἐκθεσις, *setting-out*, one of formal divisions of proposition I. 129: may sometimes be omitted I. 130
 ἐκτός, κατὰ τὸ (of an *exterior* angle in sense of re-entrant) I. 263: ἡ ἐκτός γωνία, the *exterior* angle I. 280

- ἐλάσων, *minor* (irrational) straight line III. 7 etc.
 ἐλικοειδής, spiral-shaped I. 159
 ἔλλειμμα, *defect* (in application of areas) II. 262
 ἐλλείπειν, "fall short" (in application of areas) II. 262
 ἔλλειψις, *falling-short* (in application of areas) I. 36, 343-5, 383-4
 ἐλλίπες πρόβλημα, a deficient (= indeterminate) problem I. 129
 ἐμπίπτειν, *fall in* (= be interpolated) II. 358
 ἐναλλάξ, alternately or (adjectivally) alternate I. 308: ἐναλλάξ λόγος, alternate ratio, *alternando* II. 134
 ἕνα πλείω, "several ones" (def. of number) II. 280
 ἐναρμόζω, to fit in (active) Book IV. Def. 7 and Prop. I, II. 79, 80, 81
 ἔννοια, *notion*, use of, I. 221
 ἔνστασις, *objection* I. 135
 ἐντός, within: (of internal contact of circles) II. 13: κατὰ τὸ ἐντός or ἡ ἐντός (γωνία), of an interior angle I. 263, 280: ἡ ἐντός καὶ ἀπεναντίον γωνία, the interior and opposite angle I. 280
 ἐξῆς ἀνάλογον, in continued proportion (of terms in geometrical progression) II. 346
 ἐπεξεύχθωσαν (ἐπιεύνυμι, join) I. 242
 ἐπιμόριος λόγος, *superparticularis ratio* = ratio $(n+1):n$, II. 295
 ἐπίπεδον, *plane* in Euclid, used for *surface* also in Plato and Aristotle I. 169, III. 263
 ἐπίπεδος (ἀριθμός), *plane* (number) II. 287-8
 ἐπιπροσθεῖν, ἐπιπροσθεν εἶναι, to stand in front of (hiding from view), in Plato's definitions of straight line and plane I. 165, 166
 ἐπιφάνεια, *surface*: in Euclid I. 169: in Aristotle III. 263
 ἐπόμενα, consequences (= "following" terms) in a proportion II. 134, 238
 ἑτερομήκης, oblong: ἑτερόμηκες, oblong (figure) I. 151, 188: (of numbers) in Plato = *προμήκης*, which however is distinguished from *ετερομήκης* by Nicomachus etc. II. 289-90, 293
 εὐθύ, τὸ, the straight I. 159: εὐθεῖα (γραμμὴ), straight line I. 165-9
 εὐθύγραμμός, rectilinear (term for prime numbers) II. 285
 εὐθύγραμμος, rectilinear I. 187: neuter as substantive I. 346
 εὐθυμετρικός, *euthymetric* (of prime numbers) II. 285
 ἐφάπτεσθαι, to touch I. 57
 ἐφαρμόζειν, to coincide, ἐφαρμόζεσθαι, to be applied to I. 168, 224-5, 249
 ἐφεκτικός (of a class of loci) I. 330
 ἐφεξῆς, "in order" I. 181: of adjacent angles I. 181, 278
 ἡγούμενα, antecedents ("leading" terms) in a proportion II. 134
 ἥπερ, than: construction after διπλασιῶν etc. II. 133
 θεωρήμα, *theorem*, *g.v.*
 θυρεός (shield) = ellipse I. 165
 ἰδιομήκης, of square number (Iamblichus) II. 293
 ἵππου πέδη, *hippopede* (horse-fetter), name for a certain curve I. 162-3, 176
 ἰσάκις ἰσάκις ἴσος, equal multiplied by equal and again by equal (of a cube number) II. 290, 291
 ἰσάκις ἴσος, equal multiplied by equal (of a square number) II. 291
 ἰσάκις ἴσος ἑλαττονάκις (μειζονάκις), species of solid numbers, = *πλυνθίς* (δοκίς or στηλῆς) II. 291
 ἰσομέτρων σχημάτων, περὶ, *On isometric figures* (Zenodorus) I. 26, 27, 333
 κάθετος (εὐθεῖα γραμμὴ), perpendicular I. 181-2, 271: "plane" and "solid" perpendicular I. 272
 καθηγρητής I. 20
 καλίσθω, "let it be called," indicating originality of a definition II. 129
 καμπύλος, curved (of lines) I. 159
 καταμετρεῖν, measure II. 115: without remainder, "completely" (πληροῦντως) II. 280
 κατασκευάζω, construct: τῶν αὐτῶν κατασκευασθέντων, "with the same construction" II. 11
 κατασκευή, *construction*, or *machinery*, one of the divisions of a proposition I. 129: sometimes unnecessary I. 130
 κατατομή κανόνος, *Sectio canonis* attributed to Euclid I. 17, II. 295
 κείσθω, "let it be made" I. 269
 κεκαμμένη, bent (of lines) I. 159, 176
 κέντρον, centre I. 183, 184, 199: of sphere III. 270: ἡ ἐκ τοῦ κέντρου = radius I. 190, II. 2
 κερατοειδής (γωνία), *horn-like* (angle) I. 177, 178, 182, II. 4, 39, 40
 κλᾶν, to break off, deflect, or inflect: κεκλάσθαι, def. of, alluded to by Aristotle I. 118, 150, 176, 178, II. 47: κεκλασμένη γραμμὴ, defined by Heron I. 150, 159: κεκλάσθω δὴ πάλιν II. 47
 κλάσις, *breaking* (of lines) I. 176
 κλίσις, inclination: (of line to line) I. 176: (of straight line to plane or of plane to plane) III. 263-4: ὁμοίως κεκλίσθαι, to be similarly inclined III. 265
 κοίλωνγωνιον, *hollow-angled* (figure), in Zenodorus I. 27, 188
 κοινὰ ἔννοια, Common Notions (= axioms) I. 221-2: called also τὰ κοινὰ, κοινὰ δόξαι (Aristotle) I. 120, 221
 κοινὴ προσκείσθω, ἀφηρήσθω, "let there be added to, subtracted from, each" I. 276
 κοινὴ τομὴ, common section (of planes) III. 263
 κλόρυπος, *truncated* (of pyramidal number minus vertex) II. 241
 κορυφή, vertex: κορυφὴν κορυφῆν, *vertical* (angles) I. 278
 κρίκος, ring (Heron) I. 163

- κυκλικός, *cyclic*, a particular species of square number II. 291
 κύλινδρος, cylinder III. 271
 κώνος, cone III. 270
- λήμμα, *lemma* (= something assumed, λαμβανόμενον) I. 133-4
 λόγος, ratio: meaning II. 117: definition of, II. 116-9: original meaning (of something expressed) accounts for use of λόγος, *having no ratio, irrational* II. 117
 λοιπός, remaining: λοιπή ἢ ἈΔ λοιπή τῇ ΒΗ ἴση ἐστίν I. 245
- μεῖζων, *major* (irrational straight line) III. 7, 87-8 etc.
 μεμονώσθαι, to be *isolated*, of μονάς, unit (Theon of Smyrna) II. 279
 μέρος, part: two meanings II. 115: generally = submultiple II. 280: μέρη, *parts* (= proper fraction) II. 115, 280: μέρη (= direction) I. 190, 308, 323: (=side) I. 271
 μέσος ἀνάλογον (εὐθεία), μέσος ἀνάλογον (ἀριθμός), mean proportional (straight line or number) II. 129, 295, 363 etc.
 μέσος, "medial" (of a certain irrational straight line or area) III. 49, 50: ἡ ἐκ δύο μέσων πρώτη (δευτέρα), "the first (second) bimedral (straight line)" III. 7, 84-6: μέσος ἀποτομή πρώτη (δευτέρα), "first (second) apotome of a medial (straight line)" III. 7, 159-62: ῥητὸν καὶ μέσον δυναμένη, "side of (square equal to) the sum of a rational and a medial area" III. 7, 88-9: δύο μέσα δυναμένη, "side of the sum of two medial areas" III. 7, 89-90: ἡ μετὰ ῥητοῦ (μέσου) μέσον τὸ ὅλον ποιούσα, "side of (square equal to) the difference between a medial and a rational (medial) area III. 7, 164-7
 μετέωρος, elevated (above a plane) III. 272
 μὴ γάρ, "suppose it is not" II. 7
 μήκος, length I. 158-9: in Plato = side of complete square or length commensurable with unit of length II. 288, III. 3: more generally, of number in *one dimension* II. 287-8
 μηνοειδής, *lune-like* (of angle) I. 26, 201: τὸ μηνοειδὲς (σχῆμα), lune I. 187
 μικτός, "mixed" (of lines or curves) I. 161, 162: (of surfaces) I. 170
 μονάς, unit, monad: supposed etymological connexion with μόνος, solitary, μονή, rest II. 279: μονὰς προσαβοῦσα θέσιν, definition of a *point* I. 155
 μονόστροφος ἑλῆξ, "single-turn spiral" I. 122-3 n., 164-5: in Pappus = cylindrical helix I. 165
- νεύσεις, *inclinations*, a class of problems I. 150-1: νέειν, to *verge* I. 118, 150
- ξυστροειδής, *scraper-like* (of angle) I. 178
- ὁμοειδής, "of the same form" I. 250
- ὁμοιομερής, *uniform* (of lines or curves) I. 40, 161-2
 ὅμοιος, similar: (of rectilinear figures) II. 188: (of angles) = equal (Thales, Aristotle) I. 252: (of segments of circles) II. 5; (of plane and solid numbers) I. 357, II. 293
 ὁμοιότης λόγων, "similarity of ratios" (interpolated def. of proportion) II. 119
 ὁμόλογος, homologous, corresponding II. 134: exceptionally "in the same ratio with" II. 238
 ὄνομα, name or *term*, in such expressions as ἡ ἐκ δύο ὀνομάτων, the binomial (straight line) III. 7 etc.
 ὀξεία (γωνία), acute (angle) I. 181
 ὀξυγώνιος, acute-angled I. 187
 ὅπερ ἔδει δεῖξαι (or ποιῆσαι) Q.E.D. (or F.) I. 57
 ὀρθογώνιος, right-angled: as used of quadrilaterals = *rectangular* I. 188-9
 ὀρισμός, definition I. 143
 ὅρος, definition I. 143: original meaning of, I. 143: = boundary, limit I. 182: = *term* in a proportion II. 131
 ὀψις, visual ray I. 166
- πάντη μεταλαμβάνομαι, "taken together in any manner" I. 282
 παραβάλλειν, to *apply* (an area): παραβάλλειν ἀπὸ used, exceptionally, instead of παραβάλλειν παρὰ or ἀναγράφειν ἀπὸ II. 262
 παραβολή τῶν χωρίων, *application of areas* I. 36, 343-5: contrasted with ὑπερβολή (*exceeding*) and ἔλλειψις (*falling-short*) I. 343: παραβολή contrasted with σύστασις (*construction*) I. 343: application of terms to conics by Apollonius I. 344-5
 παράδοξος τόπος, ὁ, *The Treasury of Paradoxes* I. 329
 παρὰλάττω, "fall beside," "sideways" or "awry" I. 262, II. 54
 παραλληλεπίπεδος (adj.), *parallelepipedal* = "with parallel planes or faces": στερεὸν παραλληλεπίπεδον = "parallelepipedal solid," not "solid parallelepiped" III. 326
 παραλληλόγραμμος, *parallelogrammic* (= parallel-lined): παραλληλόγραμμον χωρίον "parallelogrammic area," shortened to παραλληλόγραμμον, *parallelogram* I. 325
 παραπλήρωμα, *complement* (of a parallelogram) *g.v.*
 πεντάγραμμον II. 99
 περαινούσα ποσότης, "limiting quantity" (Thymaridas' definition of unit) II. 279
 πέρας, extremity I. 165, 182: πέρας συγκλείον (Posidonius' definition of *figure*) I. 183
 περιεχομένη (of angle), περιεχόμενον (of rectangle), *contained* I. 370: τὸ δις περιεχόμενον, *twice* the rectangle contained I. 380: (of figure) contained or bounded I. 182, 183, 184, 186, 187
 περισσάκις ἄρτιος, *odd-times even* II. 282-4
 περισσάκις περισσός, *odd-times odd* II. 284
 περισσάρτιος, *odd-even* (Nicomachus etc.) II. 283
 περισσός, *odd* (number) II. 281

- περιφέρεια**, circumference (includes *arc*) I. 184
περιφερής, circular I. 159
περιφερόγραμμα, contained by a circumference of a circle or by arcs of circles I. 182, 184
πῆλικος, how great: refers to *continuous* (geometrical) magnitude as **ποσός** to *discrete* (multitude) II. 116-7
πῆλικότης, used in v. Def. 3 and vi. Def. 5: = *size* (not *quantuplicity* as it is translated by De Morgan) II. 116-7, 189-90: supposed multiplication of **πῆλικότητες** (vi. Def. 5) II. 132: distinction between **πῆλικότης** and **μέγεθος** II. 117
πλάτος, breadth I. 158-9: (of numbers) II. 288
πλεονάζον (**πρόβλημα**), " (problem) in excess " I. 129
πλευρά, side: (of factors of "plane" and "solid" numbers) II. 288
πλήθος **ὡρισμένον** or **πεπερασμένον**, defined or finite multitude (definition of number) II. 280: **ἐκ μονάδων συγκείμενον πλήθος** (Euclid's def.) II. 280
πολλαπλασιάζειν, multiply: defined II. 287
πολλαπλασιασμός, multiplication: **καθ' ὅπου** **πολλαπλασιασμοῦ**, " (arising) from any multiple whatever " II. 120
πολλαπλάσιος, multiple: **ισάκις πολλαπλάσια**, equimultiples II. 120 etc.
πόλις, a mathematical instrument I. 370
πολύπλευρον, multilateral, many-sided figure I. 187: excludes **τετράπλευρον**, quadrilateral II. 239
πορίσασθαι, to "find" or "furnish" I. 125, II. 248
πόρισμα, porism *q.v.*
ποσάκις ποσάκις ποσολ, "so many times so many times so many" (of solid numbers, in Aristotle) II. 286, 290
ποσάκις ποσολ, "so many times so many" (of plane numbers, in Aristotle) II. 286
ποσόν, *quantity*, in Aristotle II. 115: refers to multitude as **πῆλικόν** to magnitude II. 116-7
πρίσμα, prism III. 268
πρόβλημα, problem *q.v.*
προηγούμενος, *leading*: (of conversion) = complete I. 256-7: **προηγούμενον** (**θεώρημα**), *leading* (theorem), contrasted with converse I. 257
προμήκης, **ὀδlong** (of numbers): in Plato = **ἐτερομήκης**, but distinguished from it by Nicomachus etc. II. 289-90, 293
πρός, in geometry, various meanings of, I. 277
προσαναγράψαι, to *draw on to*: (of a circle) to *complete*, when segment is given II. 56
προσαρμόζουσα (**εὐθεία**) = "*anncx*," the straight line which, when added to a compound irrational straight line formed by subtraction, makes up the greater "term," i.e. the negative "term" III. 159
προσσευεῖν, to find in addition (of finding third and fourth proportionals) II. 214
πρότασις, *enunciation* I. 129-30
προτείνω, to propound I. 128
προτιθέναι, to propose: ἡ **προτεθεισα εὐθεία**, any assigned straight line III. 11
πρώτοι **πρὸς ἀλλήλους**, (numbers) prime to one another II. 285-6
πρώτος, prime: two senses of, I. 146: II. 284-5
πτῶσις, *case* I. 134
πυραμῖς, pyramid III. 268
ῤητός, rational (literally "expressible") I. 137, II. 117, III. 1: a *relative* term, unlike **ἀσύμμετρος** (incommensurable) which is a *natural* kind (Pythagoreans) III. 1: **ῤητὴ διάμετρος τῆς πεμπτὰδος**, "rational diameter of 5" (=7, as approximation to $\sqrt{50}$) I. 399, III. 12, 525: **ῤητὸν καὶ μέσον δυναμένη** (=side of square equal to sum of a rational and a medial area) etc. III. 7
σημεῖον, point I. 155-6
στάθμη, a mathematical instrument I. 371
στερεός, solid III. 262-3: of solid numbers II. 290-1: **στερεὰ γωνία**, solid angle III. 267-8: **ὅμοια στερεὰ σχήματα**, similar solid figures III. 265-7
στιγμή, point I. 156
στοιχείον, *element* I. 114-6
στρογγύλον, τὸ, the round (circular), in Plato I. 159, 184
στρογγυλότης, roundness I. 182
σύμμετρος, commensurable: **μήκει**, in length, **δυνάμει μόνον**, in square only III. 11
συμπεράσμα. conclusion (of a proposition) I. 129, 130
σύνεσις, convergence I. 282
συνεχής, continuous: **συνεχὴς ἀναλογία**, "continuous proportion" (in three terms) II. 131
συνημένη ἀναλογία, *connected* (i.e. continuous) proportion II. 131, 293: **συνημένος** of *compound* ratio in Archimedes II. 133
συνθέντι, *componendo* II. 134-5
σύνθεσις λόγου, "composition of a ratio," distinct from *compounding* of ratios II. 134-5
σύνθετος, composite: (of lines or curves) I. 160: (of surfaces) I. 170: (of numbers), in Nicomachus and Iamblichus a subdivision of *odd* II. 286
συνίστασθαι, *construct*: special connotation I. 259, 289: with **ἐντός** I. 289: contrasted with **παραβάλλειν** (*apply*) I. 243: **οὐ συσταθήσεται, συσταθήσονται**, "there cannot be constructed" I. 259, II. 53
συντίθημι, σύγκειμαι (of ratios) II. 135, 189-90: **συγκείμενα** and **διαιερέντα** (*componendo* and *separando*) used relatively to one another II. 168, 170
σύστημα μονάδων, "collection of units" (def. of number) II. 280
συστηματικός, collective II. 279
σφαῖρα, sphere III. 269
σφαιρικός, *spherical* (of a particular species of cube number) II. 291
σφηκίσκος or **σφηγίσκος**, of solid number with all three sides unequal (=scalene) II. 290

- σχέσις, "relation": ποιά σχέσις, "a sort of relation" (in def. of ratio) II. 116-7
 σχηματογραφεῖν, σχηματογραφία, representing (numbers) by figures of like shape I. 359
 σχηματοποιούσα or σχῆμα ποιούσα, "forming a figure" (of a line or curve) I. 160-1
- ταυτομήκης, of square number (Nicomachus) II. 293
 ταυτότης λόγων, "sameness of ratios" II. 119
 τέλειος, *perfect* (of a class of numbers) II. 293-4
 τεταγμένος, "ordered": τεταγμένον πρόβλημα, "ordered" problem I. 128: τεταγμένη ἀναλογία, "ordered" proportion II. 137
 τεταραγμένη ἀναλογία, *perturbed proportion* II. 136
 τετραγωνισμός, *squaring*, definitions of, I. 149-50, 410
 τετράγωνον, square: sometimes (but not in Euclid) any four-angled figure I. 188
 τετράπλευρον, quadrilateral I. 187: not a "polygon" II. 239
 τμήμα κύκλου, segment of a circle: τμήματος γωνία, angle of a segment II. 4: ἐν τμήματι γωνία, angle *in* a segment II. 4
 τομεὺς (κύκλου), sector (of a circle): σκνυτομομὸς τομεύς, "shoemaker's knife" II. 5
 τομή, section, = point of section I. 170, 171, 278: κοινὴ τομή, "common section" III. 263
 τομοειδής (of figure), *sector-like* II. 5
 τοπικὸν θεώρημα, *locus-theorem* I. 329
 τόπος, *locus* I. 329-31: = room or space I. 23 n.: place (where things may be found), thus τόπος ἀναλυόμενος, *Treasury of Analysis* I. 8, 10, παράδοξος τόπος, *Treasury of Paradoxes*, I. 329
- τόρνος, instrument for drawing a circle I. 371
 τοσαυταπλάσιον, "the same multiple" II. 146
 τρίγωνον, triangle: τὸ τριπλοῦν, τὸ δι' ἀλλήλων, triple, interwoven triangle, = pentagram II. 99
 τριπλάσιος, triple, τριπλασίων, triplicate (of ratios) II. 133
 τρίπλευρον, three-sided figure I. 187
 τυγχάνειν, happen: τυχὸν σημείον, any point *at random* I. 252: τυχοῦσα γωνία, "any angle" II. 212: ἄλλα, ἃ ἐτυχεν, ἰσάκεις πολλαπλάσια, "other, chance, equimultiples" II. 143-4
- ὑπερβολή, *exceeding*, with reference to method of application of areas I. 36, 343-5, 386-7
 ὑπερτελής or ὑπερτέλειος, "over-perfect" (of a class of numbers) II. 293-4
 ὑπὸ, in expressions for an angle (ἡ ὑπὸ ΒΑΓ γωνία) I. 249, and a rectangle I. 370
 ὑποδιπλάσιος, *sub-duplicate*, = half (Nicomachus) II. 280
 ὑποκείμενος, laid down or assumed: τὸ ὑποκείμενον ἐπίπεδον, the plane of reference III. 272
 ὑπόκειται, "is by hypothesis" I. 303, 312
 ὑποπολλαπλάσιος, submultiple (Nicomachus) II. 280
 ὑποτείνειν, *subtend*, with acc. or ὑπὸ and acc. I. 249, 283, 350
 ὕψος, height II. 189
- χωρίον, area II. 254
- ὠρισμένη γραμμή, *determinate line* (curve), "forming a figure" I. 160

GENERAL INDEX.

[The references are to volumes and pages.]

- al-'Abbās b. Sa'id al-Jauharī I. 85
 "Abthiniathus" (or "Anthisathus") I. 203
 Abū 'l 'Abbās al-Faḍl b. Ḥātim, *see* an-Nairizī
 Abū 'Abdallāh Muḥ. b. Mu'adh al-Jayyānī I. 90
 Abū 'Alī al-Baṣrī I. 88
 Abū 'Alī al-Ḥasan b. al-Ḥasan b. al-Haitham I. 88, 89
 Abū Dā'ūd Sulaimān b. 'Uqba I. 85, 90
 Abū Ja'far al-Khāzin I. 77, 85
 Abū Ja'far Muḥ. b. Muḥ. b. al-Ḥasan Naṣīraddīn at-Tūsī, *see* Naṣīraddīn
 Abū Muḥ. b. Abdalbāqī al-Baḡdādī al-Faraḍī I. 8 n., 90
 Abū Muḥ. al-Ḥasan b. 'Ubaidallāh b. Sulaimān b. Waḥb I. 87
 Abū Naṣr Gars al-Na'ma I. 90
 Abū Naṣr Mansūr b. 'Alī b. 'Irāq I. 90
 Abū Naṣr Muḥ. b. Muḥ. b. Tarkhān b. Uzlaḡ al-Fārābī I. 88
 Abū Sahl Wījan b. Rustam al-Kūhī I. 88
 Abū Sa'id Sinān b. Thābit b. Qurra I. 88
 Abū 'Uthmān ad-Dimashqī I. 25, 77
 Abū 'l Wafā al-Būzjānī I. 77, 85, 86
 Abū Yūsuf Ya'qūb b. Ishāq b. aṣ-Ṣabbāḥ al-Kindī I. 86
 Abū Yūsuf Ya'qūb b. Muḥ. ar-Rāzī I. 86
Adjacent (ἐφεξῆς), meaning I. 181
 Adrastus II. 292
 Aeneas (or Aigeias) of Hierapolis I. 28, 311
 Aganis I. 27-8, 191
 Ahmad b. al-Husain al-Ahwāzī al-Kātib I. 89
 Ahmad b. 'Umar al-Karābisī I. 85
 al-Ahwāzī I. 89
 Aigeias (? Aeneas) of Hierapolis I. 28, 311
 Alcinoüs II. 98
 Alexander Aphrodisiensis I. 7 n., 29, II. 120
 Algebra, geometrical I. 372-4: classical method was that of Eucl. II. (cf. Apollonius) I. 373: preferable to semi-algebraical method I. 377-8: semi-algebraical method due to Heron I. 373, and favoured by Pappus I. 373: geometrical equivalents of algebraical operations I. 374: algebraical equivalents of propositions in Book II., I. 372-3: equivalents in Book X. of propositions in algebra, $\sqrt[n]{k} \cdot \sqrt[n]{\lambda}$ cannot be equal to k' , III. 58-60: if $a \pm \sqrt[n]{b} = x \pm \sqrt[n]{y}$, then $a = x$, $b = y$, III. 93-4, 167-8
 'Alī b. Ahmad Abū 'l Qāsim al-Antakī I. 86
 Allman, G. J. I. 135 n., 318, 352, III. 18-9, 439
 Alternate: (of angles) I. 308: (of ratios), *alternately* II. 134
 Alternative proofs, interpolated I. 58, 59: cf. III. 9 and following II. 22: that in III. 10 claimed by Heron II. 23-4
 Amaldi, Ugo I. 175, 179-80, 193, 201, 313, 328, II. 30, 126
Ambiguous case I. 306-7: in VI. 7, II. 208-9
 Amphinomus I. 125, 128, 150 n.
 Amyclas of Heraclea I. 117
 Analysis (and synthesis) I. 18: definitions of, interpolated, I. 138, III. 442: described by Pappus I. 138-9: mystery of Greek analysis III. 246: modern studies of Greek analysis I. 139: theoretical and problematical analysis I. 138: *Treasury of Analysis* (τὸ τὸς ἀναλυόμενος) I. 8, 10, II. 138: method of analysis and precautions necessary to, I. 139-40: analysis and synthesis of problems I. 140-2: two parts of analysis (a) *transformation*, (b) *resolution*, and two parts of synthesis, (a) *construction*, (b) *demonstration* I. 141: example from Pappus I. 141-2: analysis should also reveal *διορισμοὶς* (conditions of possibility) I. 142: interpolated alternative proofs of XIII. 1-5 by analysis and synthesis I. 137, III. 442-3
 Analytical method I. 36: supposed discovery of, by Plato I. 134, 137
 Anaximander I. 370, II. 111
 Anaximenes II. 111
 Anchor-ring I. 163
 Andron I. 126
 Angle: curvilinear and rectilinear, Euclid's definition of, I. 176 sq.: definition criticised by Syrianus I. 176: Aristotle's notion of angle as κλῆσις I. 176: Apollonius' view of, as *contraction* I. 176, 177: Plutarch and Carpus on, I. 177: to which category does it belong? *quantum*, Plutarch, Carpus, "Aganis" I. 177, Euclid I. 178; *quale*, Aristotle and Eudemus I. 177-8: *relation*,

- Euclid I. 178: Syrianus' compromise I. 178: treatise on *the Angle* by Eudemus I. 34, 38, 177-8: classification of angles (Geminus) I. 178-9: curvilinear and "mixed" angles I. 26, 178-9, *horn-like* (*κερατοειδής*) I. 177, 178, 182, 265, II. 4, 39, 40, *lune-like* (*μηνωειδής*) I. 26, 178-9, *scraper-like* (*ξυστροειδής*) I. 178: angle of a segment I. 253, II. 4: angle of a semicircle I. 182, 253, II. 4: controversies about "angle of semicircle" and *hornlike* angle II. 39-42: definitions of angle classified I. 179: recent Italian views I. 179-81: angle as cluster of straight lines or rays I. 180-1, defined by Veronese I. 180: as part of a plane ("angular sector") I. 179-80: *flat* angle (Veronese etc.) I. 180-1, 269: three kinds of angles, which is prior (Aristotle)? I. 181-2: angles not less than two right angles not recognised as angles (cf. Heron, Proclus, Zenodorus) II. 47-9: did Euclid extend "angle" to angles greater than two right angles in VI. 33? II. 275-6: *adjacent* angles I. 181: *alternate* I. 308: *similar* (=equal) I. 178, 182, 252: *vertical* I. 278: *exterior* and *interior* (to a figure) I. 263, 280: *exterior* when re-entrant I. 263, in which case we have a *hollow-angled* figure I. 27, 188, II. 48: *interior* and *opposite* I. 280: construction by Apollonius of angle equal to angle I. 296: angle in a semicircle, theorem of, I. 317-9: trisection of angle, by conchoid of Nicomedes I. 265-6, by quadratrix of Hippias I. 266, by spiral of Archimedes I. 267: dihedral angle III. 264-5: solid angle III. 261, 267-8
- Annex* (*προσάρμυζοντα*) = the straight line which, when added to a compound irrational straight line formed by subtraction, makes up the greater "term," i.e. the negative "term" III. 159
- al-Antākī I. 86
- Antecedents* (leading terms in proportion) II. 134
- "Anthisathus" (or "Abthiniathus") I. 203
- Antiparallels: may be used for construction of VI. 12, II. 215
- Antiphon I. 7 n., 35
- Apastamba-Sulba-Sutra I. 352: evidence in, as to early discovery of Eucl. I. 47 and use of gnomon I. 360-4: Bürk's claim that Indians had discovered the irrational I. 363-4: approximation to $\sqrt{2}$ and Thibaut's explanation I. 361, 363-4: inaccurate values of π in, I. 364
- Apollodorus "Logisticus" I. 37, 319, 351
- Apollonius: disparaged by Pappus in comparison with Euclid I. 3: supposed by some Arabians to be author of the *Elements* I. 5: a "carpenter" I. 5: on elementary geometry I. 42: on the *line* I. 159: on the *angle* I. 176: general definition of *diameter* I. 325: tried to prove axioms I. 42, 62, 222-3: his "general treatise" I. 42: constructions by, for bisection of straight line I. 268, for a perpendicular I. 270, for an angle equal to an angle I. 296: on parallel-axiom (?) I. 42-3: adaptation to conics of theory of application of areas I. 344-5: geometrical algebra in, I. 373: *Plane Loci*, I. 14, 259, 330, theorem from (arising out of Eucl. VI. 3), also found in Aristotle II. 198-200: *Plane velours* I. 151, problem from, II. 81, lemma by Pappus on, II. 64-5: comparison of dodecahedron and icosahedron I. 6, III. 439, 512, 513: on the *cochlias* I. 34, 42, 162: on "unordered" irrationals I. 42, 115, III. 3, 10, 246, 255-9: general definition of oblique (circular) cone III. 270: I. 138, 188, 221, 222, 246, 259, 370, 373, II. 75, 190, 258, III. 264, 267
- Apotome*: compound irrational straight line (difference between two "terms") III. 7: defined III. 158-9: connected by Theaetetus with harmonic mean III. 3, 4: biquadratic from which it arises III. 7: uniquely formed III. 167-8: *first*, *second*, *third*, *fourth*, *fifth* and *sixth* apotomes, quadratics from which arising III. 5-6, defined III. 177, and found respectively (X. 85-90) III. 178-90: apotome equivalent to square root of *first* apotome III. 190-4: *first*, *second*, *third*, *fourth*, *fifth* and *sixth* apotomes equivalent to squares of *apotome*, *first apotome of a medial* etc. III. 212-29: *apotome* cannot be *binomial* also III. 240-2: different from *medial* (straight line) and from other irrationals of same series with itself III. 242: used to rationalise binomial with proportional terms III. 243-8, 252-4
- Apotome of a medial* (straight line): *first* and *second*, and biquadratics of which they are roots III. 7: *first* apotome of a medial defined III. 159-60, uniquely formed III. 168-9, equivalent to square root of *second* apotome III. 194-8: *second* apotome of a medial, defined III. 161-2, uniquely formed III. 170-2, equivalent to square root of *third* apotome III. 199-202
- Application of areas I. 36, 343-5: contrasted with *exceeding* and *falling-short* I. 343: complete method equivalent to geometrical solution of mixed quadratic equation I. 344-5, 383-5, 386-8, II. 187, 258-60, 263-5, 266-7: adaptation to conics (Apollonius) I. 344-5: *application* contrasted with *construction* (Proclus) I. 343
- Approximations: $\frac{7}{5}$ as approximation to $\sqrt{2}$ (Pythagoreans and Plato) II. 119: approximations to $\sqrt{3}$ in Archimedes and (in sexagesimal fractions) in Ptolemy II. 119: to π (Archimedes) II. 119: to $\sqrt{4500}$ (Theon of Alexandria) II. 119: remarkably close approximations (stated in sexagesimal fractions) in scholia to Book X., I. 74 n.
- "Aqaton" I. 88
- Arabian editors and commentators I. 75-90

- Arabic numerals in scholia to Book x., 11th c., I. 71
- Archibald, R. C. I. 9 n., 10
- Archimedes: "postulates" in, I. 120, 123: "porisms" in, I. 11 n., 13: on *straight line* I. 166: on *plane* I. 171-2: *Liber assumptorum*, proposition from, II. 65: approximations to $\sqrt{3}$, square roots of large numbers and to π , II. 119: extension of a proportion between commensurables to cover incommensurables II. 193: "Axiom" of (called however "lemma," assumption, by A. himself) I. 234: relation of "Axiom" to X. I, III. 15-6: "Axiom" already used by Eudoxus and mentioned by Aristotle III. 16: proved by means of Dedekind's Postulate (Stolz) III. 16: on discovery by Eudoxus of method of exhaustion III. 365-6, 374: new fragment of, "Method (*ἔφοδος*) of Archimedes about mechanical theorems," or *ἐφῶδιον*, discovered by Heiberg and published by him II. 40, III. 366-8, adds new chapter to history of integral calculus, which the method actually is, III. 366-7: application to area of parabolic segment, *ibid.*: spiral of Archimedes I. 26, 267: I. 20, 21, 116, 142, 225, 249, 370, II. 136, 190, III. 246, 270, 375
- Archytas I. 20: proof that there is no numerical geometric mean between n and $n + 1$ II. 295
- Areskong, M. E. I. 113
- Arethas, Bishop of Caesarea I. 48: owned Bodleian ms. (B) I. 47-8: had famous Plato ms. of Patmos (Cod. Clarkianus) written I. 48.
- Argyrus, Isaak I. 74
- Anstaeus I. 138: on conics I. 3: *Solid Loci* I. 16, 329: comparison of five (regular solid) figures I. 6, III. 438-9, 513
- Aristotelian *Problems* I. 166, 182, 187
- Aristotle: on nature of *elements* I. 116: on first principles I. 117 sqq.: on definitions I. 117, 119-20, 143-4, 146-50: on distinction between hypotheses and definitions I. 119, 120, between hypotheses and postulates I. 118, 119, between hypotheses and axioms I. 120: on axioms I. 119-21: axioms indemonstrable I. 121: on definition by negation I. 156-7: on *points* I. 155-6, 165: on *lines*, definitions of, I. 158-9, classification of, I. 159-60: quotes Plato's definition of *straight line* I. 166: on definitions of *surface* I. 170: definition of "body" as that which has three dimensions or as "depth" III. 262: body "bounded by surfaces" (*ἐμπέδους*) III. 263: speaks of *six* "dimensions" III. 263: definition of sphere III. 269: on the *angle* I. 176-8: on priority as between right and acute angles I. 181-2: on *figure* and definition of, I. 182-3: definitions of "squaring" I. 149-50, 410: on parallels I. 190-2, 308-9: on *gnomon* I. 351, 355, 359: on attributes *κατὰ παντός* and *πρώτον καθόλου* I. 319, 320, 325: on the *objection* I. 135: on *reduction* I. 135: on *reductio ad absurdum* I. 136: on the *infinite* I. 232-4: supposed postulate or axiom about divergent lines taken by Proclus from, I. 45, 207: gives pre-Euclidean proof of Eucl. I. 5, I. 252-3: on theorem of angle in a semicircle I. 149: has proof (pre-Euclidean) that angle in semicircle is right II. 63: on sum of angles of triangle I. 319-21: on sum of exterior angles of polygon I. 322: on def. of *same ratio* (= *same antravalpeus*) II. 120-1: on proportion as "equality of ratios" II. 119: on theorem in proportion (*alternando*) not proved generally till his time II. 113: on proportion in three terms (*συνεχής*, continuous), and in four terms (*διηρημένη*, discrete) II. 131, 293: on *alternate ratios* II. 134: on *inverse ratio* II. 134, 149: on similar rectilinear figures II. 188: has locus-theorem (arising out of Eucl. VI. 3) also given in Apollonius' *Plane Loci* II. 198-200: on *unit* II. 279: on *number* II. 280: on non-applicability of arithmetical proofs to magnitudes if these are not numbers II. 113: on definitions of odd and even by one another II. 281: on prime numbers II. 284-5: on composite numbers as plane and solid II. 286, 288, 290: on representation of numbers by pebbles forming figures II. 288: gives proof (no doubt Pythagorean) of incommensurability of $\sqrt{2}$, III. 2: I. 38, 45, 117, 150 n., 181, 184, 185, 187, 188, 195, 202, 203, 221, 222, 223, 226, 259, 262-3, 283, 411, II. 2, 4, 22, 79, 112, 135, 149, 159, 160, 165, 184, 188, 189, III. 4
- Arithmetic, *Elements* of, anterior to Euclid II. 295
- Arithmetical calculations in scholia to Book x., I. 71, 74
- al-Arjāni, Ibn Rāhawaihi I. 86
- Ashkāl at-ta'sis I. 5 n.
- Ashraf Shamsaddin as-Samarqandī, Muḥ. b. I. 5 n., 89
- Astaroff, Ivan I. 113
- Asymptotic (non-secant): of lines I. 40, 161, 203: of parallel planes III. 265
- Athelhard of Bath I. 78, 93-6
- Athenaeus of Cyzicus I. 117
- Athenaeus I. 20
- August, E. F. I. 103, II. 23, 25, 149, 238, 250, 412, III. 2, 48
- Austin, W. I. 103, III. 11, 172, 188, 211, 259
- Autolycus, *On the moving sphere*, I. 17
- Avicenna, I. 77, 89
- "Axiom of Archimedes" III. 15-6: already used by Eudoxus, III. 15, and mentioned by Aristotle, III. 16: relation of, to Eucl. x. 1, III. 15-6
- Axioms, distinguished from postulates by Aristotle I. 118-9, by Proclus (Geminus and "others") I. 50, 121-3: Proclus on difficulties in distinctions I. 123-4: distin-

- guished from hypotheses, by Aristotle I. 120-1, by Proclus I. 121-2: indemonstrable I. 121: attempt by Apollonius to prove I. 223-3: = "common (things)" or "common opinions" in Aristotle I. 120, 221: common to all sciences I. 119, 120: called "common notions" in Euclid I. 121, 221: which are genuine? I. 221 sqq.: Proclus recognises five I. 222, Heroⁿ three I. 222: interpolated axioms I. 224, 232: Pappus' additions to axioms I. 25, 223, 224, 232: axioms of congruence, (1) Euclid's Common Notion 4, I. 224-7, (2) modern systems (Pasch, Veronese and Hilbert) I. 228-31: "axiom" with Stoics = every simple declaratory statement I. 41, 221: axioms tacitly assumed, in Book V., II. 137, in Book VII., II. 294
- Axis: of sphere III. 261, 269: of cone III. 261, 271: of cylinder III. 262, 271
- Babylonians: supposed discoverers of "*harmonic* proportion" II. 112
- Bacon, Roger I. 94, 416
- Baermann, G. F. II. 213
- Balbus, *de mensuris* I. 91
- Baltzer, R. II. 30
- Barbarin I. 219
- Barlaam, arithmetical commentary on Eucl. II., I. 74
- Barrow: on Eucl. v. Def. 3, II. 117: on v. Def. 5, II. 121: I. 103, 105, 110, 111, II. 56, 186, 238
- Base: meaning I. 248-9: of cone III. 262: of cylinder III. 262
- Basel *editio princeps* of Eucl., I. 100-1
- Basilides of Tyre I. 5, 6, III. 512
- Bāudhāyana Sulba-Sūtra I. 360
- Bayfius (Baif, Lazare) I. 100
- Becker, J. K. I. 174
- Beez I. 176
- Behā-ad-dīn I. 417
- Beltrami, E. I. 219
- Benjamin of Lesbos I. 113
- Bergh, P. I. 400-1
- Bernard, Edward I. 102
- Besthorn and Heiberg, edition of al Hajjāj's translation and an-Nairizi's commentary I. 22, 27 n., 79 n.
- Bhāskara I. 355, 418
- Billingsley, Sir Henry, I. 100-10, 418, II. 56, 238, III. 48
- Bimedial* (straight line): first and second, and biquadratic equations of which they are roots III. 7: first bimedral defined III. 84-5, equivalent to square root of second binomial III. 84, 120-3, uniquely divided III. 94-5: second bimedral defined III. 85-7, equivalent to square root of third binomial III. 84, 124-5, uniquely divided III. 95-7
- Binomial* (straight line): compound irrational straight line (sum of two "terms") III. 7: defined III. 83, 84: connected by Theaetetus with arithmetic mean III. 3, 4: biquadratic of which binomial is a positive root III. 7: *first, second, third, fourth, fifth and sixth* binomials, quadratics from which arising III. 5-6, defined III. 101-2, and found respectively (X. 48-53) III. 102-15, are equivalent to squares of binomial, first bimedral etc. III. 132-45: *binomial* equivalent to square root of *first* binomial III. 116-20: binomial uniquely divided, and algebraical equivalent of this fact III. 92-4: cannot be *apotome* also III. 240-2: different from *medial* (straight line) and from other irrationals (first bimedral etc.) of same series with itself III. 242: used to rationalise *apotome* with proportional terms III. 248-52, 252-4
- al-Bīrūnī I. 90
- Björnbo, Axel Anthon I. 17 n., 93
- Boccaccio I. 96
- Bodleian MS. (B) I. 47, 48
- Boeckh I. 351, 371
- Boethius I. 92, 95, 184, II. 295
- Bologna MS. (b) I. 49
- Bolyai, J. I. 219
- Bolyai, W. I. 174-5, 219, 328
- Bolzano I. 167
- Boncompagni I. 93 n., 104 n.
- Bonola, R. I. 202, 219
- Borelli, Giovanni Alfonso I. 106, 194, II. 2, 84
- Boundary* (δpos) I. 182, 183
- Brākenhjelm, P. R. I. 113
- Breadth (of numbers)=second dimension or factor II. 288
- Breitkopf, Joh. Gottlieb Immanuel I. 97
- Bretschneider I. 136 n., 137, 295, 304, 344, 354, 358, III. 439, 442
- Bricconnet, François I. 100
- "*Bride, Theorem of the*"=Eucl. I. 47. I. 417-8
- "*Bride's Chair*," name for Eucl. I. 47. I. 417-8
- Briggs, Henry I. 102, II. 143
- Brit. Mus. palimpsest, 7th-8th c., I. 50
- Bryson, I. 8 n.
- Bürk, A. I. 352, 360-4
- Bürklen I. 179
- Buteo (Borrel), Johannes I. 104
- Cabasilas, Nicolaus and Theodorus I. 72
- Caiani, Angelo I. 101
- Camerarius, Joachim I. 101
- Camerer, J. G. I. 103, 293, II. 22, 25, 28, 33, 34, 40, 67, 121, 131, 189, 213, 244
- Çamorano, Rodrigo, I. 112
- Campanus, Johannes I. 3, 78, 94-6, 104, 106, 110, 407, II. 28, 41, 56, 90, 116, 119, 121, 146, 189, 211, 234, 235, 253, 275, 320, 322, 328
- Candalla, Franciscus Flussates (François de Foix, Comte de Candale) I. 3, 104, 110, II. 189
- Cantor, Moritz I. 272, 304, 318, 320, 333, 352, 355, 357-8, 360, 401, II. 5, 40, 97, III. 8, 15, 438
- Cardano, Hieronimo II. 41, III. 8

- Carduchi, L. I. 112
 Carpus, on Astronomy, I. 34, 43: I. 45, 127, 128, 177
Case, technical term I. 134: cases interpolated I. 58, 59: Greeks did not *infer* limiting cases but proved them separately II. 75
 Casey, J. II. 227
 Casiri, I. 4 n., 17 n.
 Cassiodorus, Magnus Aurelius I. 92
 Catalan, II. 426
 Cataldi, Pietro Antonio, I. 106
Catoptrica, attributed to Euclid, probably Theon's I. 17: *Catoptrica* of Heron I. 21, 253
 Cauchy III. 267: proof of Eucl. XI. 4, III. 280
 "Cause": consideration of, omitted by commentators I. 19, 45: definition should state cause (Aristotle) I. 149: cause = middle term (Aristotle) I. 149: question whether geometry should investigate cause (Gemini), I. 45, 150 n.
 Censorinus I. 91
 Centre, *κέντρον* I. 184-5
Ceria Aristotelica I. 35
 Cesàro, E. II. 426
 "Chance equimultiples" in phrase "other, chance, equimultiples" II. 143-4
 Chasles on *Porisms* of Euclid I. 10, II. 14, 15
 Chaucer, *Dulcarnon* in, I. 416-7, 418
 Chinese, knowledge of triangle 3, 4, 5, I. 352, 362: "Chou pei" I. 355
 Chou Kung I. 362
 Christensen III. 8
 Chrysippus I. 330
 Chrystal, G. III. 19
 Cicero I. 91, 351
 Circle: definition of, I. 183-5: = *round*, *στρογγύλον* (Plato), I. 184: = *περιφερ-γραμμιον* (Aristotle) I. 184: a *plane figure* I. 183-4: exceptionally in sense of "circumference" II. 23: centre of, I. 184-5: pole of, I. 185: bisected by diameter (Thales) I. 185, (Saccheri) I. 185-6: intersections with straight line I. 237-8, 272-4, with another circle I. 238-40, 242-3, 293-4: definition of "equal circles" II. 2: circles *touching*, meaning of definition, II. 3: circles intersecting and touching, difficulties in Euclid's treatment of, II. 25-7, 28-9, modern treatment of, II. 30-2
 Circumference, *περιφέρεια*, I. 184
 Cissoid, I. 161, 164, 176, 330
 Clairaut I. 328
 Clavius (Christoph Klau?) I. 103, 105, 194, 232, 381, 391, 407, II. 2, 41, 42, 47, 49, 53, 56, 67, 70, 73, 130, 170, 190, 231, 238, 244, 271, III. 273, 331, 341, 350, 359, 433
 Claymundus, Joan. I. 101
 Cleonides, *Introduction to Harmony*, I. 17
Cochlias or *cochlion* (cylindrical helix) I. 162
 Codex Leidensis 399, I: I. 22, 27 n., 79 n.
 Coets, Hendrik, I. 109
 Commandinus I. 4, 102, 103, 104-5, 106, 110, III. 407, II. 47, 130, 190: scholia included in translation of *Elements* I. 73: edited (with Dee) *De divisionibus* I. 8, 9, 110
 Commensurable: defined III. 10: commensurable *in length*, commensurable *in square*, and commensurable *in square only* defined III. 10, 11: symbols used in notes for these terms III. 34
 Commentators on Eucl. criticised by Proclus I. 19, 26, 45
Common Notions: = axioms I. 62, 120-1, 221-2: which are genuine? I. 221 sq.: meaning and appropriation of term I. 221: called "axioms" by Proclus I. 221
Complement, *παραπλήρωμα*: meaning of, I. 341: "about diameter" I. 341: not necessarily parallelograms I. 341: use for application of areas I. 342-3
Componendo (*συνθέντι*), denoting "composition" of ratios *q.v.*: *componendo* and *separando* used relatively to each other II. 168, 170
 Composite, *σύνθετος*: (of lines) I. 160: (of surfaces) I. 170: (of numbers) II. 286: with Eucl. and Theon of Smyrna composite numbers may be even, but with Nicom. and Iamblichus are a subdivision of *odd* II. 286, plane and solid numbers are species of, II. 286
 "Composite to one another" (of numbers) II. 286-7
 Composition of ratio (*σύνθεσις λόγων*), denoted by *componendo* (*συνθέντι*), distinct from compounding ratios II. 134-5
 Compound ratio: explanation of, II. 132-3: (interpolated?) definition of, II. 189-90: compounded ratios in v. 20-3, II. 176-8
 Conchoids I. 160-1, 265-6, 330
Conclusion, *συντέρασμα*: necessary part of a proposition I. 129-30: particular conclusion immediately made general I. 131: definition merely stating *conclusion* I. 149
 Cone: definitions of, by Euclid III. 262, 270, by Apollonius III. 270: distinction between right-angled, obtuse-angled and acute-angled cones a relic of old theory of conics III. 270: similar cones, definition of, III. 262, 271
 Congruence-Axioms or Postulates: Common Notion 4 in Euclid I. 224-5: modern systems of (Pasch, Veronese, Hilbert) I. 228-31
 Congruence theorems for triangles, recapitulation of, I. 305-6
Conics, of Euclid, I. 3, 16: of Aristaetus, I. 3, 16: of Apollonius I. 3, 16: fundamental property as proved by Apollonius equivalent to Cartesian equation I. 344-5: focus-directrix property proved by Pappus I. 15
Consequents ("following" terms in a proportion) II. 134, 238
 Constantinus Lascaris I. 3
Construct (*συντάσθαι*) contrasted with *describe* on I. 343, with *apply to* I. 343: special connotation I. 259, 289

- Construction*, κατασκευή, one of formal divisions of a proposition I. 129: sometimes unnecessary I. 130: turns nominal into real definition I. 146: mechanical constructions I. 151, 387
- Continuity*, Principle of, I. 234 sq., 242, 272, 294
- Continuous proportion* (συνεχής or συνημμένη ἀναλογία) in three terms II. 131
- Conversion*, geometrical: distinct from *logical* I. 256: "leading" and partial varieties of, I. 256-7, 337
- Conversion of ratio* (ἀναστροφή λόγου), denoted by *convertendo* (ἀναστρέψαντι) II. 135: *convertendo* theorem not established by v. 19, Por. II. 174-5, but proved by Simson's Prop. E. II. 175, III. 38: Euclid's roundabout substitute III. 38
- Convertendo* denoting "conversion" of ratios, q.v.
- Copernicus I. 101
- Cordonis, Mattheus I. 97
- Corresponding magnitudes* II. 134
- Cosmic figures (= five regular solids) I. 413-4
- Cossali III. 8
- Cratistus I. 133
- Crelle, on the *plane* I. 172-4, III. 263
- Ctesibius I. 20, 21, 39 n.
- Cube*: defined III. 262: problem of inscribing in sphere, Euclid's solution III. 478-80, Pappus' solution III. 480: duplication of cube reduced by Hippocrates of Chios to problem of two mean proportionals I. 135, II. 133: cube number, defined II. 291: two mean proportionals between two cube numbers II. 294, 364-5
- Cunn, Samuel I. 111
- Curtze, Maximilian, editor of an-Nairizi I. 22, 78, 92, 94, 96, 97 n., II. 426
- Curves*, classification of: *see* line
- Cyclic*, of a particular kind of square number II. 291
- Cyclomathia* of Leotaud II. 42
- Cylinder*: definition of, III. 262: similar cylinders defined III. 262
- Cylindrical helix* I. 161, 162, 329, 330
- Czech, Jo. I. 113
- Dasypodius (Rauchfuss) Conrad I. 73, 102
- Data* of Euclid: I. 8, 132, 141, 385, 391: Def. 2, II. 248: Prop. 8, II. 249-50: Prop. 24, II. 246-7: Prop. 55, II. 254: Props. 56 and 68, II. 249: Prop. 58, II. 263-5: Props. 59 and 84, II. 266-7: Prop. 67 assumes part of converse of Simson's Prop. B (Book VI.) II. 224: Prop. 70, II. 250: Prop. 85, II. 264: Prop. 87, II. 228: Prop. 93, II. 227
- Deahna I. 174
- Dechales, Claude François Milliet I. 106, 107, 108, 110, II. 259
- Dedekind's theory of irrational numbers corresponds exactly to Eucl. V. Def. 5, II. 124-6; Dedekind's Postulate and applications of, I. 235-40, III. 16
- Dee, John I. 109, 110; discovered *De divisionibus* I. 8, 9
- Definition*, in sense of "closer statement" (διορισμός), one of formal divisions of a proposition I. 129: may be unnecessary I. 130
- Definitions*: Aristotle on, I. 117, 119, 120, 143: a class of *thesis* (Aristotle) I. 120: distinguished from hypotheses I. 119, but confused therewith by Proclus I. 121-2: must be assumed I. 117-9, but say nothing about *existence* (except in the case of a few primary things) I. 119, 143: terms for, ὅρος and ὁρισμός I. 143: *real* and *nominal* definitions (real=nominal *plus* postulate or proof), Mill anticipated by Aristotle, Saccheri and Leibniz I. 143-5: Aristotle's requirements in, I. 146-50, exceptions I. 148: should state *cause* or middle term and be *genetic* I. 149-50: Aristotle on unscientific definitions (ἐκ μὴ προτέρων) I. 148-9: Euclid's definitions agree generally with Aristotle's doctrine I. 146: interpolated definitions I. 61, 62: definitions of technical terms in Aristotle and Heron, not in Euclid I. 150
- De levi et ponderoso*, tract I. 18
- Demetrius Cydonius I. 72
- Democritus I. 38: "On difference of gnomon" etc. (? on "angle of contact") II. 40: on parallel and infinitely near sections of cone, II. 40, III. 368: stated, without proving, propositions about volumes of cone and pyramid, II. 40, III. 366: was evidently on the track of the infinitesimal calculus III. 368: treatise on irrationals (περὶ ἀλόγων γραμμῶν καὶ ναστῶν β') I. 413, III. 4
- De Morgan, A.: I. 246, 260, 269, 284, 291, 298, 300, 309, 313, 314, 315, 369, 376, II. 5, 7, 9-10, 11, 15, 20, 22, 29, 56, 76-7, 83, 101, 104, 116-9, 120, 130, 139, 145, 197, 202, 217-8, 232, 233, 234, 272, 275: on definition of ratio II. 116-7: on extension of meaning of *ratio* to cover incommensurables II. 118: means of expressing ratios between incommensurables by approximation to any extent II. 118-9: defence and explanation of v. Def. 5, II. 121-4: on necessity of proof that tests for greater and less, or greater and equal, ratios cannot coexist II. 130-1, 157: on compound ratio II. 132-3, 234: sketch of proof of existence of fourth proportional (assumed in v. 18) II. 171; proposed lemma about duplicate ratios as alternative means of proving VI. 22, II. 246-7: on Book X., III. 8
- Dercyllides II. 111
- Desargues I. 193
- Describe* on (ἀναγράφειν ἀπὸ) contrasted with *construct* I. 348
- De Zolt I. 328
- Diagonal* (διαγώνιος) I. 185
- "Diagonal" numbers: *see* "Side." and "diagonal." numbers
- Diameter* (διάμετρος), of circle or parallelo-

- gram I. 185: of sphere III. 261, 269, 270: as applied to figures generally I. 325; "rational" and "irrational" diameter of 5 (Plato) I. 399, 413, taken from Pythagoreans I. 399-400, III. 12
- Dickson, L. E. II. 426
- Diels, H. I. 412
- Dihedral angle = inclination of plane to plane, measured by a plane angle III. 264-5
- Dimensions* (cf. *διαστάσεις*), I. 157, 158: Aristotle's view of, I. 158-9, III. 262-3, speaks of *six* III. 263
- Dinostratus I. 117, 266
- Diocles I. 164
- Diodorus I. 203
- Diogenes Laertius I. 37, 305, 317, 351, III. 4
- Dionysius, friend of Heron, I. 21
- Diophantus I. 86
- Diorismus* (*διορισμός*) = (a) "definition" or "specification," a formal division of a proposition I. 129: (b) condition of possibility I. 128, determines how far solution possible and in how many ways I. 130-1, 243: *diorismi* said to have been discovered by Leon I. 116: revealed by *analysis* I. 142: introduced by *δεῖ δὴ* I. 203: first instances in *Elements* I. 234, 293: for solution of quadratic II. 259
- Dippe I. 108
- Direction, as primary notion, discussed I. 179: direction-theory of parallels I. 191-2
- Discrete* proportion, *διηρημένη* or *διεξευγμένη ἀναλογία*, in four terms, II. 131, 293
- "*Dissimilarly ordered*" proportion (*ἀνομοίως τετραγμένον τῶν λόγων*) in Archimedes = "perturbed proportion" II. 136
- Distance*, *διάστημα*: = radius I. 199: in Aristotle has usual general sense and = dimension I. 199
- Dividendo* (of ratios): see Separation, *separando*
- Division (method of), Plato's I. 134
- Divisions* (of figures), treatise by Euclid, I. 8, 9: translated by Muhammad al-Baġdādī I. 8: found by Woepcke in Arabic I. 9, and by Dee in Latin translation I. 8, 9: I. 110: proposition from, II. 5
- Dodecahedron: in sphere I. 411: decomposition of faces into elementary triangles II. 98: definition of, III. 262: dodecahedra found, apparently dating from centuries before Pythagoras III. 438; though said to have been discovered by Pythagoreans *ibid.*: problem of inscribing in sphere, Euclid's solution III. 493, Pappus' solution III. 501-3
- Dodgson, C. L. I. 194, 254, 261, 313, II. 48, 275
- Dou, Jan Pieterszoon I. 108
- Duhamel, J. M. C. I. 139, 328
- Dulcarnon*, name for Eucl. I. 47, I. 416, 418
- Duplicate ratio II. 133; *διπλασίων*, duplicate, distinct from *διπλάσιος*, double (=ratio 2:1), though use of terms not uniform II. 133: "duplicate" of given ratio found by VI. 11, II. 214: lemma on duplicate ratio as alternative to method of VI. 22 (De Morgan and others) II. 242-7
- Duplication of cube; reduction of, by Hippocrates, to problem of finding two mean proportionals I. 135, II. 133: wrongly supposed to be alluded to in *Timaeus* 32 A, B, II. 294-5 *n.*
- Egyptians II. 112: knowledge of $3^2 + 4^2 = 5^2$, I. 352: view of number II. 280
- Eibe, Thyra I. 113
- Elefuga*, name for Eucl. I. 5, I. 416-7
- Elements*: pre-Euclidean Elements, by Hippocrates of Chios, Leon I. 116, Theudius I. 117: contributions to, by Eudoxus I. 1, 37, II. 112, III. 15, 305-6, 374, 441, Theaetetus I. 1, 37, III. 3, 438, Hermotimus of Colophon I. 117; Euclid's *Elements*, ultimate aims of, I. 2, 115-6; commentators on, I. 19-45, Proclus I. 19, 29-45 and *passim*, Heron I. 20-4, an-Nairizī I. 21-4, Porphyry I. 24, Pappus I. 24-7, Simplicius I. 28, Aeneas (Aigeias) I. 28: mss. of, I. 46-51: Theon's changes in text I. 54-8: means of comparing Theonine with ante-Theonine text I. 51-3: interpolations before Theon's time I. 58-63: scholia I. 64-74: external sources throwing light on text, Heron, Taurus, Sextus Empiricus, Proclus, Iamblichus I. 62-3: Arabic translations (1) by al-Hajjāj I. 75, 76, 79, 80, 83-4, (2) by Ishāq and Thābit b. Qurra I. 75-80, 83-4, (3) Naṣīraddin at-Tūsī I. 77-80, 84: Hebrew translation by Moses b. Tibbon or Jakob b. Machir I. 76: Arabian versions compared with Greek text I. 79-83, with one another I. 83, 84: translation by Boethius I. 92: old translation of 10th c. I. 92; translations by Athelhard I. 93-6, Gherard of Cremona I. 93-4, Campanus I. 94-6, 97-100 etc., Zamberti I. 98-100, Commandinus I. 104-5: introduction into England, 10th c., I. 95: translation by Billingsley I. 109-10: Greek texts, *editio princeps* I. 100-1, Gregory's I. 102-3, Peyrard's I. 103, August's I. 103, Heiberg's *passim*: translations and editions generally I. 97-113: writers on Book X, III. 8-9: on the nature of *elements* (Proclus) I. 114-6, (Menaechmus) I. 114, (Aristotle) I. 116: Proclus on advantages of Euclid's *Elements* I. 115: immediate recognition of, I. 116: first principles of, definitions, postulates, common notions (axioms) I. 117-24: technical terms in connexion with, I. 125-42: no definitions of such technical terms I. 150: sections of Book I., I. 308
- Elinuam* I. 95
- Engel and Stäckel I. 219, 321
- Enriques, F. I. 113, 157, 175, 193, 195, 201, 313, II. 30, 126
- Enunciation* (*ἔνστασις*), one of formal divisions of a proposition I. 129-30

- Epicureans, objection to Eucl. I. 20, I. 41, 287: Savile on, I. 287
- Equality, in sense different from that of congruence (= "equivalent," Legendre) I. 327-8: two senses of equal (1) "divisibly-equal" (Hilbert) or "equivalent by sum" (Amaldi), (2) "equal in content" (Hilbert) or "equivalent by difference" (Amaldi) I. 328: modern definition of, I. 228
- Equimultiples: "any equimultiples whatever," *ισάκεις πολλαπλάσια καθ' ὁποιοῦν πολλαπλάσιασμέν* II. 120: stereotyped phrase "other, chance, equimultiples" II. 143-4: should include *once* each magnitude II. 145
- Eratosthenes: I. 1: contemporary with Archimedes I. 1, 2: Archimedes' "Method" addressed to, III. 366: measurement of obliquity of ecliptic ($23^{\circ} 51' 20''$) II. 111
- Errard, Jean, de Bar-le-Duc I. 108
- Erycinus I. 27, 290, 329
- Escribed circles of triangle II. 85, 86-7
- Euclid: account of, in Proclus' summary I. 1: date I. 1-2: allusions to, in Archimedes I. 1: (according to Proclus) a Platonist I. 2: taught at Alexandria I. 2: Pappus on personality of, I. 3: story of (in Stobaeus) I. 3: not "of Megara" I. 3, 4: supposed to have been born at Gela I. 4: Arabian traditions about, I. 4, 5: "of Tyre" I. 4-6: "of Tūs" I. 4, 5 n.: Arabian derivation of name ("key of geometry") I. 6: *Elements*, ultimate aim of, I. 2, 115-6: other works, *Conics* I. 16, *Pseudaria* I. 7, *Data* I. 8, 132, 141, 385, 391, *On divisions* (of figures) I. 8, 9, *Porisms* I. 10-15, *Surface-loci* I. 15, 16, *Phaenomena* I. 16, 17, *Optics* I. 17, *Elements of Music* or *Section Canonis* I. 17, II. 295: on "three- and four-line locus" I. 3: Arabian list of works I. 17, 18: bibliography I. 91-113
- Eudemus I. 29: *On the Angle* I. 34, 38, 177-8: *History of Geometry* I. 34, 35-8, 278, 295, 304, 317, 320, 387, 412, II. 99, 111, III. 3, 366
- Eudoxus I. 1, 37, 74, 116, II. 40, 99, 280, 295: discoverer of theory of proportion covering incommensurables as expounded generally in Bks. V., VI., I. 137, 351, II. 112: on the *golden section* I. 137: discoverer of method of exhaustion I. 234, III. 365-6, 374: used "Axiom of Archimedes" III. 15: first to prove theorems about volume of pyramid (Eucl. XII. 7 Por.) and cone (Eucl. XII. 10), also theorem of Eucl. XII. 2, III. 15: theorems of Eucl. XIII. 1-5 probably due to, III. 441: inventor of a certain curve, the *hippope*, horse-fetter I. 163: possibly wrote *Sphaerica* I. 17: III. 442
- Euler, Leonhard I. 401, II. 426
- Eutocius: I. 25, 35, 39, 142, 161, 164, 259, 317, 329, 330, 373: on "VI. Def. 5" and meaning of *πηλοκότης* II. 116, 132, 189-90: gives locus-theorem from Apollonius' *Plane Loci* II. 198-200
- Even (number): definitions by Pythagoreans and in Nicomachus II. 281: definitions of odd and even by one another unscientific (Aristotle) I. 148-9, II. 281: Nicom. divides even into three classes (1) *even-times even* and (2) *even-times odd* as extremes, and (3) *odd-times even* as intermediate II. 282-3
- Even-times even*: Euclid's use differs from use by Nicomachus, Theon of Smyrna and Iamblichus II. 281-2
- Even-times odd* in Euclid different from *even-odd* of Nicomachus and the rest II. 282-4
- Ex aequali*, of ratios, II. 136: *ex aequali* propositions (v. 20, 22), and *ex aequali* "in perturbed proportion" (v. 21, 23) II. 176-8
- Exhaustion, method of: discovered by Eudoxus I. 234, III. 365-6: evidence of Archimedes III. 365-6: III. 374-7
- Exterior* and *interior* (of angles) I. 263, 280
- Extreme and mean ratio (line cut in): defined, II. 188: known to Pythagoreans I. 403, II. 99, III. 19: irrationality of segments of (*apoteles*) III. 19, 449-51
- Extremity*, *πέρας*, I. 182, 183
- Faifofer II. 126
- Falk, H. I. 113
- al-Farāḍī I. 8 n., 90
- Fauquembergue, E. II. 426
- Fermat II. 425, 426
- Figure*, as viewed by Plato I. 182, by Aristotle I. 182-3, by Euclid I. 183: according to Posidonius is *confining boundary* only I. 41, 183: figures bounded by two lines classified I. 187: *angle-less* (*ἀγώνιον*) figure I. 187
- Figures, printing of, I. 97
- Fihrist* I. 4 n., 5 n., 17, 21, 24, 25, 27; list of Euclid's works in, I. 17, 18
- Finaeus, Orontius (Oronce Fine) I. 101, 104
- Flauti, Vincenzo I. 107
- Florence MS. Laurent. XXVIII. 3 (F) I. 47
- Flussates, *see* Candalla
- Forcadet, Pierre I. 108
- Fourier: definition of plane based on Eucl. XI. 4, I. 173-4, III. 263
- Fourth proportional: assumption of existence of, in v. 18, and alternative methods for avoiding (Saccheri, De Morgan, Simson, Smith and Bryant) II. 170-4: Clavius made the assumption an axiom II. 170: sketch of proof of assumption by De Morgan II. 171: condition for existence of number which is a fourth proportional to three numbers II. 409-11
- Francisci tunica*, "Franciscan's cowl," name for Eucl. I. 47, I. 418
- Frankland, W. B. I. 173, 199
- Frischauf, J. I. 174
- Galileo Galilei: on *angle of contact* II. 42
- Gartz I. 17
- Gauss I. 172, 193, 194, 202, 219, 321
- Geminus: name Latin or Greek? I. 38-9: title of work (*φιλοκαλία*) quoted from by

- Proclus I. 39, and by Schol., I. 74: elements of astronomy I. 38: comm. on Posidonius I. 39: Proclus' obligations to, I. 39-42: on postulates and axioms I. 122-3: on theorems and problems I. 128: two classifications of lines (or curves) I. 160-2: on homoeomeric (*uniform*) lines I. 162: on "mixed" lines (curves) and surfaces I. 162: classification of surfaces I. 170, of angles I. 178-9: on parallels I. 191: on Postulate 4, I. 200: on stages of proof of theorem of I. 32, I. 317-20: I. 27-8, 37, 44, 45, 74, 133 n., 203, 265, 330
- Geometrical algebra I. 372-4: Euclid's method in Book II. evidently the classical method I. 373: preferable to semi-algebraical method I. 377-8
- Geometrical progression II. 346 sqq.: summation of n terms of (IX. 35) II. 420-1
- Geometric means II. 357 sqq.: one mean between square numbers II. 294, 363, or between similar plane numbers II. 371-2: two means between cube numbers II. 294, 364-5, or between similar solid numbers II. 373-5
- Georgius Pachymeres I. 417
- Gherard of Cremona, translator of *Elements* I. 93-4: of an-Nairizi's commentary I. 22, 27 n., 94, II. 47: of tract *De divisionibus* I. 9, 10 n.
- Giordano, Vitale I. 106, 176
- Givēs*, *δεδομένως*, different senses, I. 132-3
- Gnomon*: literally "that enabling (something) to be known" I. 64, 370: successive senses of, (1) upright *marker* of sundial, I. 181, 185, 271-2, introduced into Greece by Anaximander I. 370, (2) carpenter's square for drawing right angles I. 371, (3) figure placed round square to make larger square I. 351, 371, Indian use of *gnomon* in this sense I. 362, (4) use extended by Euclid to parallelograms I. 371, (5) by Heron and Theon to any figures I. 371-2: Euclid's method of denoting in figure I. 383: arithmetical use of, I. 358-60, 371, II. 289
- "Gnomon-wise" (*κατὰ γνῶμονα*), old name for perpendicular (*κάθετος*) I. 36, 181, 272
- Görland, A. I. 233, 234
- Golden section* (section in extreme and mean ratio), discovered by Pythagoreans I. 137, 403, II. 99: connexion with theory of irrationals I. 137, III. 19: theory carried further by Plato and Eudoxus II. 99: theorems of Eucl. XIII. 1-5 on, probably due to Eudoxus III. 441
- "Goose's foot" (*pes anseris*), name for Eucl. III. 7, I. 99, 418
- Gow, James I. 135 n.
- Gracilis, Stephanus I. 101-2
- Grandi, Guido I. 107
- Greater ratio: Euclid's criterion not the only one II. 130: arguments from greater to less ratios etc. unsafe unless they go back to original definitions (Simson on v. 10) II. 156-7: test for, cannot coexist with test for equal or less ratio II. 130-1
- Greatest common measure: Euclid's method of finding corresponds exactly to ours II. 118, 299, III. 18, 21-2: Nicomachus gives the same method II. 300: method used to prove incommensurability III. 18-9; for this purpose often unnecessary to carry it far (cases of extreme and mean ratio and of $\sqrt{2}$) III. 18-9
- Greenhill, Sir George, I. 415, 418
- Gregory, David I. 102-3, II. 116, 143, III. 32
- Gregory of St Vincent I. 401, 404
- Gromatici* I. 91 n., 95
- Grynaeus I. 100-1
- Häbler, Th. II. 294 n.
- al-Haitham I. 88, 89
- al-Hajjāj b. Yūsuf b. Maṭar, translator of the *Elements* I. 22, 75, 76, 79, 80, 83, 84
- Halifax, William I. 108, 110
- Halliwell (-Phillips) I. 95 n.
- Hankel, H. I. 139, 141, 232, 234, 344, 354, II. 116, 117, III. 8
- Harmonica* of Ptolemy, Comm. on, I. 17
- Harmony*, *Introduction to*, not by Euclid, I. 17
- Hārūn ar-Rashīd I. 75
- al-Hasan b. 'Ubaidallāh b. Sulaimān b. Wāḥb I. 87
- Hauber, C. F. II. 244
- Hauff, J. K. F. I. 108
- "Heavy and Light," tract on, I. 18
- Heiberg, J. L. *passim*
- Helix, cylindrical I. 161, 162, 329, 330
- Helmholtz I. 226, 227
- Henrici and Treutlein I. 313, 404, II. 30
- Henrion, Denis I. 108
- Hérigone, Pierre I. 108
- Herlin, Christian I. 100
- Hermotimus of Colophon I. I, 117
- Herodotus I. 37 n., 370
- "Heromides" I. 158
- Heron of Alexandria, *mechanicus*, date of, I. 20-1: Heron and Vitruvius I. 20-1; commentary on Euclid's *Elements* I. 20-4: direct proof of I. 25, I. 301: comparison of areas of triangles in I. 24, I. 334-5: addition to I. 47, I. 366-8: apparently originated semi-algebraical method of proving theorems of Book II., I. 373, 378: Eucl. III. 12 interpolated from, II. 28: extends III. 20, 21 to angles in segments less than semicircles II. 47-8: does not recognise angles equal to or greater than two right angles II. 47-8: proof of formula for area of triangle, $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, II. 87-8: I. 137 n., 159, 163, 168, 170; 171-2, 176, 183, 184, 185, 188, 189, 222, 223, 243, 253, 285, 287, 299, 351, 369, 371, 405, 407, 408, II. 5, 16-7, 24, 28, 33, 36, 44, 47, 48, 116, 189, 302, 320, 383, 395, III. 24, 263, 265, 267, 268, 269, 270, 366, 404, 442
- Heron, Proclus' instructor I. 29
- "Herundes" I. 156
- Hieronymus of Rhodes I. 305

- Hilbert, D. I. 157, 193, 201, 228-31, 249, 313, 328
 Hipparchus I. 4 n., 20, 30 n., 74 n.
 Hippasus I. 411, II. 97, III. 438
 Hippias of Elis I. 42, 265-6, 413
 Hippocrates of Chios I. 8 n., 29, 35, 38, 116, 135, 136 n., 386-7, 413, II. 133: first proved that circles (and similar segments of circles) are to one another as the squares on their diameters III. 366, 374
Hippopede (ἵππου πῆδη), a certain curve used by Eudoxus I. 162-3, 176
 Hoffmann, Heinrich I. 107
 Hoffmann, Joh. Jos. Ign. I. 108, 365
 Holgate, T. F. III. 284, 303, 331
 Holtzmann, Wilhelm (Xylander) I. 107
Homoeomeria (uniform) lines I. 40, 161, 162
Hornlike angle (κεραιοειδὴς γωνία) I. 177, 178, 182, 265, II. 4, 39, 40: *hornlike angle* and *angle of semicircle*, controversies on, II. 39-42: Proclus on, II. 39-40: Democritus may have written on *hornlike angle* II. 40: view of Campanus ("not angles in same sense") II. 41: of Cardano (*quantities of different orders or kinds*): of Peletier (*hornlike angle* no angle, no quantity, nothing; angles of *all semicircles right angles* and equal) II. 41: of Clavius II. 42: of Vieta and Galileo ("angle of contact no angle") II. 42: of Wallis (angle of contact not *inclination* at all but *degree of curvature*) II. 42
 Horsley, Samuel I. 106
 Hottel, J. I. 219
 Hudson, John I. 102
 Hultsch, F. I. 17 n., 74, 329, 400, II. 133, 190, III. 4
 Hunain b. Ishāq al-'Ibādī I. 75
 Hypotheses, in Plato I. 122: in Aristotle I. 118-20: confused by Proclus with definitions I. 121-2: geometer's hypotheses not false (Aristotle) I. 119
 Hypothetical construction I. 199
 Hypsicles I. 5: author of Book XIV. I. 5, 6, III. 438-9, 512
 Iamblichus I. 63, 83, 417, II. 97, 116, 279, 280, 281, 283, 284, 285, 286, 287, 288, 289, 290, 291, 292, 293, 419, 425, 426
 Ibn al-'Amid I. 86
 Ibn al-Haitham I. 88, 89
 Ibn al-Lubūdī I. 90
 Ibn Rāhawaihi al-Arjānī I. 86
 Ibn Sīnā (Avicenna) I. 77, 89
 Icosahedron II. 98: defined III. 262: discovery of, attributed to Theaetetus III. 438: problem of inscribing in sphere, Euclid's solution III. 481-9, Pappus' solution III. 489-91: Mr H. M. Taylor's construction III. 491-2
 "Iflaton" I. 88
 Inclination (κλίσις) of straight line to plane, defined III. 260, 263-4: of plane to plane (=dihedral angle) III. 260, 264
 Incommensurables: discovered by Pythagoreans III. 1, 2, 3, and with reference to $\sqrt{2}$, I. 351, III. 1, 2, 19: incommensurable a *natural* kind, unlike *irrational* which depends on convention or assumption (Pythagoreans) III. 1: proof of incommensurability of $\sqrt{2}$ no doubt Pythagorean III. 2, proof in Chrystal's Algebra III. 19-20: incommensurable *in length* and incommensurable *in square* defined III. 10, 11: symbols for, used in notes III. 34: method of testing incommensurability (process of finding G.C.M.) II. 118, III. 18-9: means of expression consist in power of approximation without limit (De Morgan) II. 119: approximations to $\sqrt{2}$ by means of *side- and diagonal-numbers* I. 399-401, II. 119, by means of sexagesimal fractions I. 74 n., to $\sqrt{3}$, I. 74 n., II. 119: to $\sqrt{4500}$ by means of sexagesimal fractions II. 119: to π , II. 119
 Incomposite: (of lines) I. 160-1, (of surfaces) I. 170: (of number)=prime II. 284
 Indivisible lines (ἀτομοὶ γραμμαί), theory of, rebutted I. 268
 Infinite, Aristotle on the, I. 232-4: infinite division not assumed, but proved, by geometers I. 268
 Infinity, parallels meeting at, I. 192-3
 Ingram, G. I. 175, 193, 195, 201, 227-8, II. 30, 126
 Integral calculus, in new fragment of Archimedes III. 366-7
Interior and exterior (of angles) I. 263, 280: *interior and opposite angle* I. 280
 Interpolations in the *Elements* before Theon's time I. 58-63: by Theon I. 46, 55-6: Eucl. I. 40 interpolated I. 338: other propositions interpolated, (III. 12) II. 28, (proposition after XI. 37) III. 360, (XIII. 6) III. 449-51: cases in XI. 23, III. 319-21: defs. of analysis and synthesis, and proofs of XIII. 1-5 by III. 442-3
 Inverse (ratio), inversely (ἀνάπαλιν) II. 134: inversion is subject of v. 4, Por. (Theon) II. 144, and of v. 7, Por. II. 149, but is not properly put in either place II. 149: Simson's Prop. B on, directly deducible from v. Def. 5, II. 144
 Irrational: discovered by Pythagoreans I. 351, 411, 412-3, III. 1-2, 3, and with reference to $\sqrt{2}$, I. 351, III. 1, 2, 19: depends on assumption or convention, unlike *incommensurable* which is a *natural* kind (Pythagoreans) III. 1: claim of India to priority of discovery I. 363-4; "irrational diameter of 5" (Pythagoreans and Plato) I. 399-400, 413, III. 12: approximation to $\sqrt{2}$ by means of "side-" and "diagonal-" numbers I. 399-401, II. 119, to $\sqrt{2}$ and $\sqrt{3}$ in sexagesimal fractions I. 74 n.: Indian approximation to $\sqrt{2}$, I. 361, 363-4: *unordered irrationals* (Apollonius) I. 42, 115, III. 3, 10, 246, 255-9: irrational ratio (ἄρρητος λόγος) I. 137: an irrational straight line is so relatively to *any straight*

- line* taken as rational III. 10, 11: irrational area incommensurable with rational area or square on rational straight line III. 10, 12: Euclid's irrationals, object of classification of, III. 4, 5: Book X. a repository of results of solution of different types of quadratic and biquadratic equations III. 5: types of equations of which Euclid's irrationals are positive roots III. 5-7: actual use of Euclid's irrationals in Greek geometry III. 9-10: compound irrationals in Book X. all different III. 242-3
- Isaacus Monachus (or Argyrus) I. 73-4, 407
- Ishāq b. Hunain b. Ishāq al-'Ibādī, Abū Ya'qūb, translation of *Elements* by, I. 75-80, 83-4
- Isidorus of Miletus III. 520
- Isma'il b. Bulbul I. 88
- Isoperimetric* (or isometric) figures: Pappus and Zenodorus on, I. 26, 27, 333
- Isosceles* (ἰσοσκελῆς) I. 187: of numbers (= even) I. 188: isosceles right angled triangle I. 352: isosceles triangle of IV. 10, construction of, due to Pythagoreans I. 414, II. 97-9
- Jacobi, C. F. A. II. 188
- Jakob b. Machir I. 76
- Jan, C. I. 17
- al-Jauhārī, al-'Abbās b. Sa'id I. 85
- al-Jayyānī I. 90
- Joannes Pediasimus I. 72-3
- Johannes of Palermo III. 8
- Junge, G., on attribution of theorem of I. 47 and discovery of irrationals to Pythagoras I. 351, 411, 413
- Kästner, A. G. I. 78, 97, 101
- al-Karābīsī I. 85
- Kātyāyana Sulba-Sūtra I. 360
- Keill, John I. 105, 110-11
- Kepler I. 193
- al-Khāzin, Abū Jāfar I. 77, 85
- Killing, W. I. 194, 219, 225-6, 235, 242, 272, III. 276
- al-Kindī I. 5 n., 86
- Klamroth, M. I. 75-84
- Klau (?), Christoph = Clavius *q.v.*
- Klūgel, G. S. I. 212
- Kluge III. 520
- Knesa, Jakob I. 112
- Knoche I. 32 n., 33 n., 73
- Kroll, W. I. 399-400
- al-Kūhī I. 88
- Lachlan, R. II. 226, 227, 245-6, 247, 256, 272
- Lambert, J. H. I. 212-3
- Lardner, Dionysius I. 112, 246, 250, 298, 404, II. 58, 259, 271
- Lascaris, Constantinus 3
- Leading* theorems (as distinct from *converse*) I. 257: *leading* variety of conversion I. 256-7
- Least common multiple II. 336-41
- Leeke, John I. 110
- Lefèvre, Jacques I. 100
- Legendre, Adrien Marie I. 112, 169, 213-9, II. 30, III. 263, 264, 265, 266, 267, 268, 273, 275, 298, 309, 356, 436: proves VI. 1 and similar propositions in two parts (1) for commensurables, (2) for incommensurables II. 193-4: proof of Eucl. XI. 4, III. 280, of XI. 6, 8, III. 284, 289, of XI. 15, III. 299, of XI. 19, III. 305: definition of planes at right angles III. 303: alternative proofs of theorems relating to prisms III. 331-3: on equivalent parallelepipeds III. 335-6: proof of Eucl. XII. 2, III. 377-8: propositions on volumes of pyramids III. 389-91, of cylinders and cones III. 422-3
- Leibniz I. 145, 169, 176, 194
- Leiden MS. 399, I of al-Hajjāj and an-Nairizī I. 22, 27 n., 79 n.
- Lemma* I. 114: meaning (=assumption) I. 133-4: lemmas interpolated I. 59-60, especially from Pappus I. 67: lemma assumed in VI. 22, II. 242-3: alternative propositions on duplicate ratios and ratios of which they are duplicate (De Morgan and others) II. 242-7: lemmas interpolated, (after x. 9) III. 30-1, (after x. 59) III. 97, 131-2: lemmas suspected, (those added to x. 18, 23) III. 48, (that after XII. 2) III. 375, (that after XIII. 2) III. 444-5
- Length, μήκος (of numbers in one dimension) II. 287: Plato restricts term to side of integral square number II. 287
- Leodamas of Thasos I. 36, 134
- Leon I. 116
- Leonardo of Pisa I. 9 n., 10, III. 8
- Leotaud, Vincent II. 42
- Leucippus I. 413
- Linderup, H. C. I. 113
- Line*: Platonic definition I. 158: objection of Aristotle I. 158: "magnitude extended one way" (Aristotle, "Heromides") I. 158: "divisible or continuous one way" (Aristotle) I. 158-9: "flux of point" I. 159: Apollonius on, I. 159: classification of lines, Plato and Aristotle I. 159-60, Heron I. 159-60, Geminus, first classification I. 160-1, second I. 161: straight (εὐθεῖα), curved (καμπύλη), circular (περιφερής), spiral-shaped (ἐλικοειδής), bent (κεκαμμένη), broken (κεκλασμένη), round (τὸ στρογγύλον) I. 159, composite (σύνθετος), incomposite (ἀσύνθετος), "forming a figure" (σχηματοποιούσα), determinate (ὁρισμένη), indeterminate (ἀόριστος) I. 160: "asymptotic" or non-secant (ἀσύμπτωτος), secant (συμπτωτός) I. 161: simple, "mixed" I. 161-2: *homoeometric* (uniform) I. 161-2: Proclus on lines without extremities I. 165: *loci on lines* I. 329, 330
- Linear*, *loci* I. 330: problems I. 330: numbers=(1) in one dimension II. 287, (2) ptime II. 285
- Lionardo da Vinci, proof of I. 47, I. 365-6
- Lippert I. 88 n.
- Lobachewsky, N. I. I. 174-5, 213, 219
- Locus-theorems* (τοπικά θεωρήματα) and *loci*

- (τόποι): locus defined by Proclus I. 329: loci likened by Chrysippus to Platonic *ideas* I. 330-1: locus-theorems and loci (1) *on lines* (a) *plane* loci (straight lines and circles) (b) *solid* loci (conics), (2) *on surfaces* I. 329: corresponding distinction between *plane* and *solid* problems, to which Pappus adds *linear* problems I. 330: further distinction in Pappus between (1) *ἐφεκτικοί* (2) *διεξοδικοί* (3) *ἀναστροφικοί τόποι* I. 330: Proclus regards locus in I. 35, III. 21, 31 as an *area* which is locus of area (parallelogram or triangle) I. 330
- Logical conversion, distinct from geometrical I. 256
- Logical deductions I. 256, 284-5, 300: not made by Euclid II. 22, 29: logical equivalents I. 309, 314-5
- Lorenz, J. F. I. 107-8, III. 34
- Loria, Gino I. 7 *n.*, 10 *n.*, 11 *n.*, 12 *n.*, II. 425, III. 8, 9
- Luca Paciolo I. 98-9, 100, 418, III. 8
- Lucas, E. I. 418, II. 426
- Lucian II. 99
- Lundgren, F. A. A. I. 113
- Machir, Jakob b. I. 76
- Magni, Domenico I. 106
- Magnitude: common definition vicious I. 148
- al-Māhānī I. 85
- Major* (irrational) straight line: biquadratic of which it is positive root III. 7: defined III. 87-8: equivalent to square root of fourth binomial III. 84, 125-7: uniquely divided III. 98: extension of meaning to irrational straight line of *three* terms III. 258
- al-Ma'mūn, Caliph I. 75
- Manitius, C. I. 38
- Mansion, P. I. 219
- al-Manšūr, Caliph I. 75
- Manuscripts of *Elements* I. 46-51
- "Marriage, Figure of" (Plutarch), name for Pythagorean triangle (3, 4, 5), I. 417
- Martianus Capella I. 91, 155
- Martin, T. H. I. 20, 29 *n.*, 30 *n.*
- Mas'ūd b. al-Qass al-Baġdādī I. 90
- Maximus Planudes, scholia and lectures on *Elements* I. 72
- Means: three kinds, arithmetic, geometric and harmonic II. 292-3: geometric mean is "proportion *par excellence*" (*κυπλως*) II. 292-3: one geometric mean between two square numbers, two between two cube numbers (Plato) II. 294, 363-5: one geometric mean between similar plane numbers, two between similar solid numbers II. 371-5: no numerical geometric mean between *n* and *n* + 1 (Archytas and Euclid) II. 295
- Medial* (straight line): connected by Theaetetus with *geometric* mean III. 3, 4: defined III. 49, 50: *medial area* III. 54-5: an unlimited number of irrationals can be derived from medial straight line, III. 254-5
- Meguar* = axis I. 93
- Mehler, F. G. I. 404, III. 268, 284-5
- Menaechmus: story of M. and Alexander I. 1: on *elements* I. 114: I. 117, 125, 133 *n.*
- Menelaus I. 21, 23: direct proof of I. 25, I. 300
- Menge, H. I. 16 *n.*, 17
- Middle term, or cause, in geometry, illustrated by Eucl. III. 31, I. 149
- Mill, J. S. I. 144
- Minor* (irrational) straight line: biquadratic of which it is root III. 7: defined III. 163-4: uniquely formed III. 172-3: equivalent to square root of fourth apotome III. 203-6
- "Mixed" (lines) I. 161-2: (surfaces) I. 162, 170: different meanings of "mixed" I. 162
- Mocenigo, Prince I. 97-8
- Moderatus, a Pythagorean II. 280
- Mollweide, C. B. I. 108
- Mondoré (Montaureus), Pierre I. 102
- Moses b. Tibbon I. 76
- Motion, in mathematics I. 226: *motion without deformation* considered by Helmholtz necessary to geometry I. 226-7, but shown by Veronese to be *petitio principii* I. 226-7
- Müller, J. H. T. I. 189
- Müller, J. W. I. 365
- Muḥammad (b. 'Abdalbaqī) al-Baġdādī, translator of *De divisionibus* I. 8*n.*, 90, 110
- Muḥ. b. Aḥmad Abū 'r-Raiḥān al-Bīrūnī I. 90
- Muḥ. b. Ashraf Shamsaddīn as-Samarqandī I. 89
- Muḥ. b. 'Isā Abū 'Abdallāh al-Māhānī I. 85
- Multinomial* (straight line): an extension from *binomial*, probably investigated by Apollonius, III. 256
- Multiplication, definition of, II. 287
- Munich ms. of enunciations (R) I. 94-5
- Mūsā b. Muḥ. b. Maḥmūd Qāḍizāde ar-Rūmī I. 5 *n.*, 90
- Music, Elements of* (*Sectio Canonis*), attributed to Euclid, I. 17, II. 295, III. 33
- al-Musta'ṣim, Caliph I. 90
- al-Mutawakkil, Caliph I. 75
- an-Nairizī, Abū 'l-'Abbās al-Faḍl b. Ḥātim, I. 21-4, 85, 184, 190, 191, 195, 223, 232, 258, 270, 285, 299, 303, 326, 364, 367, 369, 373, 405, 408, II. 5, 16, 28, 34, 36, 44, 47, 302, 320, 383
- Napoleon I. 103
- Naṣīraddīn at-Tūsī I. 4, 5 *n.*, 77, 84, 89, 208-10, II. 28
- Nazīf b. Yūmn (Yaman) al-Qass I. 76, 77, 87
- Neide, J. G. C. I. 103
- Nesselmann, G. H. F. II. 287, 293, III. 8
- Nicomachus I. 92, 417, II. 116, 119, 131, 279, 280, 281, 282, 283, 284, 285, 286, 287, 288, 289, 290, 291, 292, 293, 294, 300, 363, 425
- Nicomedes I. 42, 160-1, 265-6
- Nipsus, Marcus Junius I. 305
- Nixon, R. C. J. II. 16
- Nominal* and *real* definitions: see *Definitions*
- Number: defined by Thales, Eudoxus, Moderatus, Aristotle, Euclid II. 280:

- Nicomachus and Iamblichus on, II. 280 : represented by lines II. 287, and by points or dots II. 288-9
- "Nuptial Number" = Plato's Geometrical Number in *Republic*, I. 417
- Objection* (ἐνστάσις), technical term, in geometry I. 135, 257, 260, 265 : in logic (Aristotle) I. 135
- Oblong : (of geometrical figure) I. 62, 151, 188 : (of number) in Plato either *προμήκης* or *ἐτερομήκης* II. 288 : but these terms denote two distinct divisions of plane numbers in Nicomachus, Theon of Smyrna and Iamblichus II. 289-90
- Octahedron II. 98 : definition of, III. 262 : discovery of, attributed to Theaetetus III. 438 : problem of inscribing in sphere, Euclid's solution III. 474-7, Pappus' solution III. 477
- Odd (number) : defs. of in Nicomachus II. 281 : Pythagorean definition II. 281 : def. of odd and even by one another unscientific (Aristotle) I. 148-9, II. 281 : Nicom. and Iambl. distinguish three classes of odd numbers (1) prime and incomposite, (2) secondary and composite, as extremes, (3) secondary and composite in themselves but prime and incomposite to one another, which is intermediate II. 287
- Odd-times even* (number) : definition in Eucl. spurious II. 283-4, and differs from definitions by Nicomachus etc. *ibid.*
- Odd-times odd* (number) : defined in Eucl. but not in Nicom. and Iambl. II. 284 : Theon of Smyrna applies term to prime numbers II. 284
- Oenopides of Chios I. 34, 36, 126, 271, 295, 371, 414, II. 111
- Ofterdinger, L. F. I. 9
- Olympiodorus I. 29
- Oppermann I. 151
- Optics* of Euclid I. 17
- "Ordered" proportion (τεταγμένη ἀναλογία), interpolated definition of, II. 137
- Oresme, N. I. 97
- Orontius Finaeus (Oronce Fine) I. 101, 104
- Ozanam, Jacques I. 107, 108
- Paciuolo, Luca I. 98-9, 100, 418, III. 8
- Pamphile I. 317, 319
- Pappus : contrasts Euclid and Apollonius I. 3 : on Euclid's *Porisms* I. 10-14, *Surface-loci* I. 15, 16, *Data* I. 8 : on *Treasury of Analysis* I. 8, 10, 11, 138 : commentary on *Elements* I. 24-7, partly preserved in scholia I. 66 : evidence of scholia as to Pappus' text I. 66-7 : commentary on Book X. survives in Arabic III. 3 : quotations from it, III. 3-4, 255-9 : lemmas in Book X. interpolated from, I. 67 : on *Analysis and Synthesis* I. 138-9, 141-2 : additional axioms by, I. 25, 223, 224, 232 : on converse of Post. 4, I. 25, 201 : proof of I. 5 by, I. 254 : extension of I. 47, I. 366 : semi-algebraical methods in, I. 373, 378 : on loci I. 329, 330 : on conchoids I. 161, 266 : on quadratrix I. 266 : on isoperimetric figures I. 26, 27, 333 : on paradoxes of Erycinus I. 27, 290 : lemma on Apollonius' *Plane νεύσεις* II. 64-5 : problem from same work II. 81 : assumes case of VI. 3 where external angle bisected (Simson's VI. Prop. A) II. 197 : theorem from Apollonius' *Plane Loci* II. 198 : theorem that ratio compounded of ratios of sides (of equiangular parallelograms) is equal to ratio of rectangles contained by sides II. 250 : use of Euclid's irrationals III. 9, 10 : methods of inscribing regular solids in sphere, tetrahedron III. 472-3, octahedron III. 477, cube III. 480, icosahedron III. 489-91, dodecahedron III. 501-3 : I. 20, 39, 133 *z.*, 137, 151, 225, 388, 391, 401 : II. 4, 27, 29, 67, 79, 81, 113, 133, 211, 250, 251, 292
- Papyrus, Herculensis No. 1061, I. 50, 184 : Oxyrhynchus I. 50 : Fayūm I. 51, 337, 338 : Rhind I. 304, 352
- Paradoxes, in geometry I. 188 : of Erycinus I. 27, 290, 329 : an ancient "Budget of Paradoxes" I. 329
- "Parallelepipedal" = with parallel planes or faces : "parallelepipedal solid" (not "solid parallelepiped") or "parallelepiped" III. 326 : generally has six faces but sometimes more ("parallelepipedal prism") III. 401, 404 : "parallelepipedal" (solid) numbers in Nicomachus have two of sides differing by unity II. 290
- Parallelogram (= parallelogrammic area), first introduced I. 325 : rectangular parallelogram I. 370
- Parallels : Aristotle, on I. 190, 191-2 : definitions, by "Aganis" I. 191, Geminus I. 191, Posidonius I. 190, Simplicius I. 190 : as equi-distants I. 190-1, 194 : direction-theory of, I. 191-2, 194 : definitions classified I. 192-4 : Veronese's definition and postulate I. 194 : Parallel Postulate, see Postulate 5 : Legendre's attempt to establish theory of, I. 213-9 : parallel planes, definition of, III. 260, 265
- Paris MSS. of *Elements*, (p) I. 49, (q) I. 50
- Pasch, M. I. 157, 228, 250
- "Peacock's tail," name for Eucl. III. 8, I. 99, 418
- Pediasimus, Joannes I. 72-3
- Peet, T. Eric I. 352
- Peithon I. 203
- Peletarius (Jacques Peletier) I. 103, 104, 249, 407, II. 47, 56, 84, 146, 190 : on *angle of contact* and *angle of semicircle* II. 41
- Pena I. 104
- Pentagon : decomposition of regular pentagon into 30 elementary triangles II. 98 : relation to pentagram II. 99
- Pentagonal numbers II. 289
- "Perfect" (of a class of numbers) II. 293-4, 421-6 : Pythagoreans applied term to 10,

- II. 294: 3 also called "perfect" I. 417, II. 294
- Perpendicular (*καθέτος*): definition I. 181: "plane" and "solid" I. 272: perpendicular and obliques I. 291: perpendicular to plane, III. 260, 263: perpendicular to two straight lines not in one plane III. 306-7
- Perseus I. 42, 162-3
- Perturbed proportion* (*τεταραγμένη αναλογία*) II. 136, 176-7
- Pesch, J. G. van, *De Procli fontibus* I. 235qq., 29 n.
- Petrus Montaneus (Pierre Mondoré) I. 102
- Peyrard and Vatican MS. 190 (P) I. 46, 47, 103: I. 108
- Pfleiderer, C. F. I. 168, 208, II. 2
- Phaenomena* of Euclid I. 16, 17
- Philippus of Medma I. 1, 116
- Phillips, George I. 112
- Philo of Byzantium I. 20, 23: proof of I. 8, I. 263-4
- Philolaus I. 34, 351, 371, 399, 411
- Philoponus I. 45, 191-2, II. 234, 282
- Pirckenstein, A. E. Burkh. von, I. 107
- Plane (or plane surface): Plato's definition of, I. 171, III. 272-3: Proclus' and Simplicius' interpretation of Euclid's def. I. 171: possible origin of Euclid's def. I. 171: Archimedes' assumption I. 171, 172: other ancient definitions of, in Proclus, Heron, Theon of Smyrna, an-Nairizi I. 171-2: "Simson's" definition ("axiom of the plane") I. 172-3, III. 273, and Gauss on, I. 172-3: Crelle's tract on, I. 172-4: other definitions by Fourier I. 173, Deahna I. 174, J. K. Becker I. 174, Leibniz I. 176, Beez I. 176: evolution of, by Bolyai and Lobachewsky I. 174-5: Enriques and Amaldi, Ingrams, Veronese and Hilbert on, I. 175: plane at right angles to plane, Euclid's definition of, III. 260, 263, and alternative definition making it a particular case of "inclination" III. 303-4: parallel planes defined III. 260, 265
- "Plane loci" I. 329-30: *Plane Loci* of Apollonius I. 14, 259, 330, II. 198-200
- Plane numbers, product of two factors ("sides" or "length" and "breadth") II. 287-8: in Plato either square or oblong II. 287-8: similar plane numbers II. 293: one mean proportional between similar plane numbers II. 371-2
- "Plane problems" I. 329
- Planudes, Maximus I. 72
- Plato I. 1, 2, 3, 137, 155-6, 159, 184, 187, 203, 221, 411, 417, III. 1, 3: supposed invention of Analysis by, I. 134: def. of straight line I. 165-6: def. of plane surface I. 171: on *golden section* II. 99: on art of stereometry (length, breadth, and depth) as one of three *μαθηματικά*, next to geometry but commonly put after astronomy because little advanced III. 262: generation of cosmic figures by putting together triangles, I. 226, 413-4, II. 97-8, III. 267: rule for rational right-angled triangles I. 356, 357, 359, 360, 385: "rational diameter of 5" I. 399, 413, gives $7/5$ as approximation to $\sqrt{2}$, II. 119: passage of *Theaetetus* on *δυναμεις* (square roots or surds) I. 411, 412-3, II. 288, 290, III. 1-3: on square and oblong numbers II. 288, 290: theorem that between square numbers one mean suffices, between cube numbers two means are necessary II. 294, 364
- "Platonic" figures I. 2, 413-4: scholium on, III. 438
- Playfair, John I. 103, 111: "Playfair's" Axiom I. 220: used to prove Eucl. I. 29, I. 312, and Eucl. Post. 5, I. 313: comparison of Axiom with Post. 5, I. 313-4: II. 2
- Pliny I. 333
- Plutarch I. 29, 37, 177, 343, 351, 417, II. 98, 254, III. 368
- Point: Pythagorean definition of, I. 155: interpretation of Euclid's definition I. 155: Plato's view of, and Aristotle's criticism I. 155-6: attributes of, according to Aristotle I. 156: terms for (*στιγμή*, *σημείον*) I. 156: other definitions by "Herundes," Posidonius I. 156, Simplicius I. 157: negative character of Euclid's def. I. 156: is it sufficient? I. 156: motion of, produces line I. 157: an-Nairizi on, I. 157: modern explanations by abstraction I. 157
- Polybius I. 331
- Polygon: sum of interior angles (Proclus' proof) I. 322: sum of exterior angles I. 322
- Polygonal numbers II. 289
- Polyhedral angles, extension of XI. 21 to, III. 310-1
- "*Pons Asinorum*" I. 415-6: *Pont aux ânes*, *ibid.*
- Porism: two senses I. 13: (1)=corollary I. 134, 278-9: as corollary to proposition precedes "Q.E.D." or "Q.E.F." II. 8, 64: Porism to IV. 15 mentioned by Proclus II. 109: Porism to VI. 19, II. 234: interpolated Porisms (corollaries) I. 60-1, 381: (2) as used in *Porisms* of Euclid, distinguished from theorems and problems I. 10, II. 11: account of the *Porisms* given by Pappus I. 10-13: modern restorations by Simson and Chasles I. 14: views of Heiberg I. 11, 14, and of Zeuthen I. 15
- Porphyrus I. 17: commentary on Euclid I. 24: *Symmikta* I. 24, 34, 44: I. 136, 277, 283, 287
- Poselger III. 8
- Posidonius, the Stoic I. 20, 27, 28 n., 39, 189, 197: book directed against the Epicurean Zeno I. 34, 43: on parallels I. 40, 190: definition of *figure* I. 41, 183
- Postulate, distinguished from axiom, by Aristotle I. 118-9, by Proclus (Geminus and "others") I. 121-3: from hypothesis, by Aristotle I. 120-1, by Proclus I. 121-2: postulates in Archimedes I. 120, 123: Euclid's view of, reconcileable with Aristotle's I. 119-20, 124: postulates do not confine us to ruler and compasses I. 124:

- Postulates 1, 2, significance of, I. 195-6: famous "Postulate" or "Axiom of Archimedes" I. 234, III. 15-6
- Postulate 4: significance of, I. 200: proofs of, resting on other postulates I. 200-1, 231: converse true only when angles rectilinear (Pappus) I. 201
- Postulate 5: probably due to Euclid himself I. 202: Proclus on, I. 202-3: attempts to prove, Ptolemy I. 204-6, Proclus I. 206-8, Naṣiraddin at-Tūsī I. 208-10, Wallis I. 210-1, Saccheri I. 211-2, Lambert I. 212-3: substitutes for, "Playfair's" axiom (in Proclus) I. 220, others by Proclus I. 207, 220, Posidonius and Geminus I. 220, Legendre I. 213, 214, 220, Wallis I. 220, Carnot, Laplace, Lorenz, W. Bolyai, Gauss, Worpitzky, Clairaut, Veronese, Ingrami I. 220: Post. 5 proved from, and compared with, "Playfair's" Axiom I. 313-4: I. 30 is *logical* equivalent of, I. 220
- Potts, Robert I. 112, 246
- Powers, R. E. II. 426
- Prestet, Jean II. 426
- Prime (number): definitions of, II. 284-5: Aristotle on two senses of "prime" I. 146, II. 285: 2 admitted as prime by Eucl. and Aristotle, but excluded by Nicomachus, Theon of Smyrna and Iamblichus, who make prime a subdivision of *odd* II. 284-5: "prime and incomposite (ἀσύνθετος)" II. 284: different names for prime, "odd-times odd" (Theon), "linear" (Theon), "rectilinear" (Thymaridas), "euthymetric" (Iamblichus) II. 285: prime absolutely or in themselves as distinct from prime to one another (Theon) II. 285: definitions of "prime to one another" II. 285-6
- Principles, First, I. 117-24
- Prism, definition of, by Euclid I. 261, by others III. 268-9: "parallelepipedal prisms" III. 404
- Problem, distinguished from theorem I. 124-8: problems classified according to number of solutions (a) one solution, *ordered* (τεταγμένα) (b) a definite number, *intermediate* (μέσα) (c) an infinite number of solutions, *unordered* (ἀτακτα) I. 128: in widest sense anything *propounded* (possible or not) but generally a *construction* which is possible I. 128-9: another classification (1) problem *in excess* (πλεονάζον), asking too much I. 129, (2) *deficient* problem (ἐλλειπές πρόβλημα), giving too little I. 129
- Proclus: details of career I. 29-30: remarks on earlier commentators I. 19, 33, 45: commentary on Eucl. I, sources of, I. 29-45, object and character of, I. 31-2: commentary probably not continued, though continuation intended I. 32-3: books quoted by name in, I. 34: famous "summary" I. 37-8: list of writers quoted I. 44: his own contributions I. 44-5: character of ms. used by, I. 62, 63: on the nature of *elements* and things *elementary* I. 114-6: on advantages of Euclid's *Elements*, and their object I. 115-6: on first principles, hypotheses, postulates, axioms I. 121-4: on difficulties in three distinctions between postulates and axioms I. 123: on theorems and problems I. 124-9: on formal divisions of proposition I. 129-31, II. 100: attempt to prove Postulate 5, I. 206-8: commentary on Plato's *Republic*, allusion in, to "side-" and "diagonal-" numbers in connexion with Eucl. II. 9, 10, I. 399-400: on use of "quindecagon" for astronomy II. 111: I. 412, II. 4, 39, 40, 193, 247, 269, III. 10, 264, 267, 273, 310, 441
- Proof* (ἀπόδειξις), necessary part of proposition I. 129-30
- Proportion: complete theory applicable to incommensurables as well as commensurables is due to Eudoxus I. 137, 351, II. 112: old (Pythagorean) theory practically represented by arithmetical theory of Eucl. VII., II. 113: in giving older theory as well Euclid simply followed tradition II. 113: Aristotle on general proof (new in his time) of theorem (*alterando*) in proportion II. 113: X. 5 as connecting two theories II. 113: De Morgan on extension of meaning of *ratio* to cover incommensurables II. 118: power of *expressing* incommensurable ratio is power of approximation without limit II. 119: interpolated definitions of proportion as "sameness" or "similarity of ratios" II. 119: definition in v. Def. 5 substituted for that of VII. Def. 20 because latter found inadequate, not *vice versa* II. 121: De Morgan's defence of v. Def. 5 as necessary and sufficient II. 122-4: v. Def. 5 corresponds to Weierstrass' conception of number in general and to Dedekind's theory of irrationals II. 124-6: alternatives for v. Def. 5 by a geometer-friend of Saccheri, by Faifofer, Ingrami, Veronese, Enriques and Amaldi II. 126: proportionals of VII. Def. 20 (numbers) a particular case of those of v. Def. 5 (Simson's Props. C, D and notes) II. 126-9, III. 25: proportion in three terms (Aristotle makes it four) the "least" II. 131: "continuous" proportion (συνεχής or συνεχόμενη ἀναλογία, in Euclid ἐξῆς ἀνάλογον) II. 131, 293: three "proportions" II. 292, but proportion *par excellence* or primary is continuous or geometric II. 292-3: "discrete" or "disjoined" (διηρημένη, διεσπασμένη) II. 131, 293: "ordered" proportion (τεταγμένη), interpolated definition of, II. 137: "perturbed" proportion (τεταραμμένη) II. 136, 176-7: extensive use of proportions in Greek geometry II. 187: proportions enable any quadratic equation with real roots to be solved II. 187: supposed use of propositions of Book v. in arithmetical Books II. 314, 320
- Proposition, formal divisions of, I. 129-31
- Protarchus I. 5, III. 512

- Psellus, Michael, scholia by, I. 70, 71, II. 234
Pseudaria of Euclid I. 7: *Pseudographemata* I. 7n.
 Pseudoboethius I. 92
 Ptolemy I.: I. 1, 2: story of Euclid and Ptolemy I. 1
 Ptolemy II. Philadelphus I. 20
 Ptolemy VII. (Euergetes II.), Physcon, I. 20
 Ptolemy, Claudius I. 21, 30n.: *Harmonica* of, and commentary on, I. 17: on Parallel-Postulate I. 28n., 34, 43, 45: attempt to prove it I. 204-6: lemma about quadrilateral in circle (Simson's VI. Prop. D) II. 225-7: II. 111, 117, 119
Punch on Pons Asinorum I. 416
 Pyramid, definitions of, by Euclid III. 261, by others III. 268
 Pyramidal numbers II. 290: pyramids truncated, twice-truncated etc. II. 291
 Pythagoras I. 4n., 36: supposed discoverer of the irrational I. 351, 411, 412, of *application of areas* I. 343-4, of theorem of I. 47, I. 343-4, 350-4, 411, 412, of construction of five regular solids I. 413-4, II. 97: story of sacrifice I. 37, 343, 350: probable method of discovery of I. 47 and proof of, I. 352-5; suggestions by Bretschneider and Hankel I. 354, by Zeuthen I. 355-6: rule for forming right-angled triangles in rational numbers I. 351, 356-9, 385: construction of figure equal to one and similar to another rectilinear figure II. 254: introduced "the most perfect proportion in four terms and specially called 'harmonic'" into Greece II. 112
 Pythagoreans I. 19, 36, 155, 188, 279, 411-4: theorem for *surface* (ὑπόδα) I. 169: angles of triangle equal to two right angles, theorem and proof I. 317-20: three polygons which in contact fill space round point I. 318, II. 98: method of *application of areas* (including *exceeding* and *falling-short*) I. 343, 384, 403, II. 187, 258-60, 263-5, 266-7: *gnomon* Pythagorean I. 351: "rational" and "irrational diameter of 5" I. 399-400, 413: story of Pythagorean who, having divulged the irrational, perished by shipwreck I. 411, III. 1: $7/5$ as approximation to $\sqrt{2}$, II. 119: approximation to $\sqrt{2}$ by "side-" and "diagonal-" numbers I. 398-400, III. 2, 20: proof of incommensurability of $\sqrt{2}$, III. 2: construction of isosceles triangle of Eucl. IV. 10, and of regular pentagon, I. 414, II. 97-8: possible method of discovery of latter II. 97-9: distinguished three sorts of *means*, arithmetic, geometric and harmonic II. 112: had theory of proportion applicable to commensurables only II. 112: construction of dodecahedron in sphere II. 97, and of other regular solids I. 413-4, III. 438: definitions of unit II. 279, of even and odd II. 281: called *ro* "perfect" II. 294
 Qāḍizāde ar-Rūmī I. 5n., 90
 Q.E.D. (or F.) I. 57
 al-Qiftī I. 4n., 94
 Quadratic equations: solution assumed by Hippocrates I. 386-7: geometrical solution of particular quadratics I. 383-5, 386-8: solution of general quadratic by means of proportions II. 187, 263-5, 266-7: *διορισμός* or condition of possibility of solving equation of Eucl. VI. 28, II. 259: one solution only given, for obvious reasons II. 260, 264, 267: but method gives both roots if real II. 258: exact correspondence of geometrical to algebraical solution, II. 263-4, 266-7: indication that Greeks solved them numerically III. 43-4
Quadratrix I. 265-6, 330, 413
 Quadrature (τετραγωνισμός), definitions of, I. 149
 Quadrilateral: varieties of, I. 188-90: inscribing in circle of quadrilateral equiangular to another II. 91-2: condition for inscribing circle in, II. 93, 95: quadrilateral in circle, Ptolemy's lemma (Simson's VI. Prop. D) II. 225-7: quadrilateral not a "polygon" II. 239
Quadrinomial (straight line), compound irrational (extension from *binomial*) III. 256
 "Quindecagon" (fifteen-angled figure): useful for astronomy II. 111
 Quintilian I. 333
 Qustā b. Liqā al-Ba'labakkī, translator of "Books XIV, XV" I. 76, 87, 88
 Radius, no Greek word for, I. 199, II. 2
 Ramus, Petrus (Pierre de la Ramée) I. 104, II. 121
 Ratdolt, Erhard I. 78, 97
 Ratio: definition of, II. 116-9, no sufficient ground for regarding it as spurious II. 117, Barrow's defence of it II. 117: method of transition from arithmetical to more general sense covering incommensurables II. 118: means of *expressing* ratio of incommensurables is by approximation to any degree of accuracy II. 119: def. of greater ratio only *one* criterion (there are others) II. 130: tests for greater, equal and less ratios mutually exclusive II. 130-1: test for greater ratio easier to apply than that for equal ratio II. 129-30: arguments about greater and less ratios unsafe unless they go back to original definitions (Simson on V. 10) II. 156-7: *compound* ratio II. 132-3, 189-90, 234: operation of compounding ratios II. 234: "ratio compounded of their sides" (careless expression) II. 248: *duplicate*, *triplicate* etc. ratio as distinct from *double*, *triple* etc. II. 133: *alternate* ratio, *alternando* II. 134: *inverse* ratio, *inversely* II. 134: *composition* of ratio, *componendo*, different from *compounding* ratios II. 134-5: *separation* of ratio, *separando* (commonly *dividendo*) II. 135: *conversion* of ratio, *convertendo* II. 135: ratio *ex aequali* II. 136, *ex aequali in perturbed proportion* II. 136: *division* of ratios used in *Data* as

- general method alternative to compounding II. 249-50: names for particular arithmetical ratios II. 292
- Rational (*ῥητός*): (of ratios) I. 137: "rational diameter of ζ " I. 399-400, 413: rational right-angled triangles, *see* right-angled triangles: *any* straight line may be taken as rational and the *irrational* is irrational in relation thereto III. 10: rational straight line is still rational if incommensurable with rational straight line *in square only* (extension of meaning by Euclid) III. 10, 11-12
- Rationalisation of fractions with denominator of form $a \pm \sqrt{B}$ or $\sqrt{A \pm \sqrt{B}}$, III. 243-52
- Rauchfuss, *see* Dasypodius
- Rausenberger, O. I. 157, 175, 313, III. 307, 309
- ar-Rāzī, Abū Yūsuf Ya'qūb b. Muḥ. I. 86
- Reciprocal or reciprocally-related figures: definition spurious II. 189
- Rectangle: = rectangular parallelogram I. 370: "rectangle contained by" I. 370
- Rectilinear angle: definitions classified I. 179-81: rectilinear figure I. 187: "rectilinear segment" I. 196
- Reductio ad absurdum* I. 134: described by Aristotle and Proclus I. 136: synonyms for, in Aristotle I. 136: a variety of Analysis I. 140: by exhaustion I. 285, 293: nominal avoidance of, I. 369: the only possible method of proving Eucl. III, I, II. 8
- Reduction* (*ἀπαγωγή*), technical term, explained by Aristotle and Proclus I. 135: first "reduction" of a difficult construction due to Hippocrates I. 135, II. 133
- Regiomontanus (Johannes Müller of Königsberg) I. 93, 96, 100
- Reyher, Samuel I. 107
- Rhaeticus I. 101
- Rhind Papyrus I. 304, 352
- Rhomboid I. 62, 151, 189
- Rhombus I. 62, 151, meaning and derivation I. 189
- Riccardi, P. I. 96, 112, 202
- Riemann, B. I. 219, 273, 274, 280
- Right angle: definition I. 181: drawing straight line at right angles to another, Apollonius' construction for, I. 270: construction when drawn at extremity of second line (Heron) I. 270
- Right-angled triangles, rational: rule for finding, by Pythagoras I. 356-9, by Plato I. 356, 357, 359, 360, 385, by Euclid III. 63-4: discovery of rules by means of gnomons I. 358-60: connexion of rules with Eucl. II. 4, 8, I. 360: rational right-angled triangles in Āpastamba I. 361, 363
- Röth I. 357-8
- Rouché and de Comberousse I. 313
- Rudd, Capt. Thos. I. 110
- Ruellius, Joan. (Jean Ruel) I. 100
- "Rule of three": Eucl. VI. 12 equivalent to, II. 215
- Russell, Bertrand I. 227, 249
- Saccheri, Gerolamo I. 106, 144-5, 167-8, 185-6, 194, 197-8, 200-1, II. 126, 130: proof of existence of fourth proportional by Eucl. VI. 1, 2 and 12, II. 170
- Ša'id b. Mas'ūd b. al-Qass I. 90
- Sāthapatha-Brāhmaṇa I. 362
- Savile, Henry I. 105, 166, 245, 250, 262, II. 190
- Scalene* (*σκαληνός* or *σκαληνής*) I. 187-8: of numbers (= odd) I. 188: a class of solid numbers II. 290: of cone (Apollonius) I. 188
- Schessler, Chr. I. 107
- Scheubel, Joan. I. 101, 107
- Schiapparelli, G. V. I. 163
- Schmidt, Max C. P. I. 304, 319
- Scholia to *Elements* and MSS. of, I. 64-74: historical information in, I. 64: evidence in, as to text I. 64-5, 66-7: sometimes interpolated in text I. 67: classes of, "Schol. Vat." I. 65-9, "Schol. Vind." I. 69-70, miscellaneous I. 71-4: "Schol. Vat." partly derived from Pappus' commentary I. 66: many scholia partly extracted from Proclus on Bk. I, I. 66, 69, 72: many from Geminus solely I. 74: numerical illustrations in, in Greek and Arabic numerals I. 71, 74 n.: scholia by Psellus I. 70-1, by Maximus Planudes I. 72, Joannes Pediasimus I. 72-3: scholia in Latin published by G. Valla, Commandinus, Conrad Dasypodius I. 73: scholia on Eucl. II. 13, I. 407: Scholium IV. No. 2 ascribes Book IV. to Pythagoreans I. 414, II. 97: Scholium V. No. 1 attributes Book V. to Eudoxus II. 112: Scholium X. No. 1 attributes discovery of irrational and incommensurable to Pythagoreans III. 1: scholium published later by Heiberg attributes Scholium X. No. 62 to Proclus I. 32 n.
- Scholias to *Clouds* of Aristophanes II. 99
- Schooten, Franz van I. 108
- Schopenhauer I. 227, 354
- Schotten, H. I. 167, 174, 179, 192-3, 202
- Schultze, A. and Sevenoak, F. L. III. 284, 303, 331
- Schumacher I. 321
- Schur, F. I. 328
- Schweikart, F. K. I. 219
- Scipio Végus I. 99
- Secio Canonis* attributed to Euclid I. 17, II. 295, III. 33
- Section (*τομή*) = *point* of section I. 170, 171, 383: "the section" = "golden section" *q.v.*
- Sector (of circle): explanation of name: two kinds (1) with vertex at centre, (2) with vertex at circumference II. 5
- Sector-like (figure) II. 5: bisection of such a figure by straight line II. 5
- Seelhoff, P. II. 426
- Segment of circle: angle of, I. 253, II. 4: *similar* segments II. 5: segment less than semicircle called *ἀψίς* I. 187
- Semicircle: I. 186: centre of, I. 186: angle of, I. 182, 253, II. 4, 39-41 (*see* Angle): angle in semicircle a right angle, pre-Euclidean proof II. 63

- Separation* of ratio, διαρσεις λόγων, and *separando* (διελόντι) II. 135: *separando* and *componendo* used relatively to one another, not to original ratio II. 168, 170
- Seqt* I. 304
- Serenus of Antinoeia I. 203
- Serle, George I. 110
- Servais, C. II. 426
- Setting-out* (ἐκθεσις), one of formal divisions of a proposition I. 129: may be omitted I. 130
- Sexagesimal fractions in scholia I. 74
- Sextus Empiricus I. 62, 63, 184
- Shamsaddin as-Samarqandi I. 5 n., 89
- "Side-" and "diagonal-" numbers, described I. 398-400: due to Pythagoreans I. 400, III. 2, 20: connexion with Eucl. II. 9, 10, I. 398-400: use for approximation to $\sqrt{2}$, I. 399
- "Side of a medial minus a medial area" (in Euclid "that which produces with a medial area a medial whole"), a compound irrational straight line: biquadratic of which it is a root III. 7: defined III. 165-6: uniquely formed III. 174-7: equivalent to square root of *sixth apotome* III. 209-11
- "Side of a medial minus a rational area" (in Euclid "that which produces with a rational area a medial whole"), a compound irrational straight line: biquadratic of which it is a root III. 7: defined III. 164: uniquely formed III. 173-4: equivalent to square root of *fifth apotome* III. 206-8
- "Side of a rational plus a medial area," a compound irrational straight line: biquadratic equation of which it is a root III. 7: defined III. 88-9: uniquely divided III. 99: equivalent to square root of *fifth binomial* III. 84, 128-9
- "Side of the sum of two medial areas," a compound irrational straight line: biquadratic of which it is a root III. 7: defined III. 89-91: uniquely divided III. 99-101: equivalent to square root of *sixth binomial* III. 84, 130-1
- "Side" used in translation of Book x. for ἡ δυναμένη (τὸ χωρίον), "side of a square equal to (the area)" III. 13, 119
- Sides* of plane and solid numbers, II. 287-8
- Sigboto I. 94
- "Similar" (=equal) angles I. 182, 252
- "Similarly inclined" (of planes) III. 260, 265
- Similar plane and solid numbers I. 357, II. 293: one mean between two similar plane numbers II. 294, 371-2, two means between two similar solid numbers II. 294, 373-5
- Similar rectilinear figures: def. of, given in Aristotle II. 188: def. gives at once too little and too much II. 188: similar figures on straight lines which are proportional are themselves proportional and conversely (VI. 22), alternatives for proposition II. 242-7
- Similar segments of circles II. 5
- Similar solids: definitions of, III. 261, 265-7
- Simon, Max I. 108, 155, 157-8, 167, 202, 328, II. 124, 134
- Simplicius: commentary on Euclid I. 27-8: on lunes of Hippocrates I. 29, 35, 386-7: on Eudemus' style I. 35, 38: on parallels I. 190-1: I. 22, 167, 171, 184, 185, 197, 203, 223, 224, 413, III. 366
- Simpson, Thomas, II. 121, III. 274
- Simson, Robert: on Euclid's *Porisms* I. 14: on "vitiations" in *Elements* due to Theon I. 46, 103, 104, 106, 111, 148: definition of plane I. 172-3: Props. C, D (Bk. v.) connecting proportionals of VII. Def. 20 as particular case with those of v. Def. 5, II. 126-9, III. 25: Axioms to Bk. v., II. 137: Prop. B (inversion) II. 144: Prop. E (*convertendo*) II. 175: shortens v. 8 by compressing two cases into one II. 152-3: important note showing flaw in v. 10 and giving alternative II. 156-7: Bk. VI. Prop. A extending VI. 3 to case where external angle bisected II. 197: Props. B, C, D II. 222-7: remarks on VI. 27-9, II. 258-9: Prop. D, Book XI., III. 345: I. 185, 186, 255, 259, 287, 293, 296, 322, 328, 384, 387, 403, II. 2, 3, 8, 22, 33, 34, 37, 43, 49, 53, 70, 73, 79, 90, 117, 131, 132, 140, 143-4, 145, 146, 148, 154, 161, 162, 163, 165, 170-2, 177, 179, 180, 182, 183, 184, 185, 186, 189, 193, 195, 209, 211, 212, 230-1, 238, 252, 269, 270, 272-3, III. 265, 266, 273-4, 275, 276, 286-7, 289, 295, 301, 309, 314, 321, 324, 327, 331, 334, 340, 341, 349, 351, 359, 362, 375, 433, 434
- Sind b. Ali Abū 't-Taiyib I. 86
- Size*, proper translation of *πληκότης* in v. Def. 3, II. 116-7; 189-90
- Smith and Bryant, alternative proofs of v. 16, 17, 18 by means of VI. 1, where magnitudes are straight lines or rectilinear areas II. 165-6, 169, 173-4: I. 404, III. 268, 275, 284, 303, 307
- Smith, D. E. I. 362, 417
- Solid: definition of, III. 260, 262-3: similar solids, definitions of, III. 261, 265-7: equal and similar solids, *ibid.*
- Solid angle: definitions of, III. 261, 267-8: solid "angle" of "quarter of sphere," of cone, or of half-cone III. 268
- "Solid loci" I. 329, 330: *Solid Loci* of Aristaeus I. 16, 329
- Solid numbers, three varieties according to relative lengths of sides II. 290-1
- "Solid problems" I. 329, 330
- Speusippus I. 125
- Sphaerica*, early treatise on, I. 17
- Sphere: definitions of, by Euclid III. 261, 269, by others III. 269
- Spherical* number, a particular species of cube number II. 291
- Spiral, "single-turn," I. 122-3 n., 164-5: in Pappus=cylindrical helix I. 165
- Spiral of Archimedes I. 26, 267
- Spire* (tore) or *Spiric surface* I. 163, 170; varieties of, I. 163

- Spiric curves or sections, discovered by Perseus I. 161, 162-4
- Square number, product of equal numbers II. 289, 291: one mean between square numbers II. 294, 363-4
- Staudt, Ch. von III. 276
- Steenstra, Pybo I. 109
- Steiner, Jakob I. 193
- Steinmetz, Moritz I. 101
- Steinschneider, M. I. 8 n., 76 sqq.
- Stephanus Gracilis I. 101-2
- Stephen Clericus I. 47
- Stevin, Simon III. 8
- Stifel, Michael III. 8
- Stobaeus I. 3, II. 280
- Stoic "axioms" I. 41, 221: *illustrations* (*δελ-
ματα*) I. 329
- Stolz, O. I. 328, III. 16
- Stone, E. I. 105
- Straight line: pre-Euclidean (Platonic) definition I. 165-6: Archimedes' *assumption* respecting, I. 166: Euclid's definition, interpreted by Proclus and Simplicius I. 166-7: language and construction of, I. 167, and conjecture as to origin I. 168: other definitions I. 168-9, in Heron I. 168, by Leibniz I. 169, by Legendre I. 169: two straight lines cannot enclose a space I. 195-6, cannot have a common segment I. 196-9, III. 273: one or two cannot make a figure I. 169, 183: division of straight line into any number of equal parts (an-Nairizi) I. 326: straight line at right angles to plane, definition of, III. 260, alternative constructions for, III. 293-4
- Strömer, Mårten I. 113
- Studemund, W. I. 92 n.
- St Vincent, Gregory of, I. 401, 404
- Subduplicate of any ratio found by Eucl. VI. 13, II. 216
- Subtend*, meaning and construction I. 249, 283, 350
- Suidas I. 370, 413, III. 366, 438
- Sulaimān b. 'Uşma (or 'Uqba) I. 85, 90
- Superposition: Euclid's dislike of method of, I. 225, 249: apparently assumed by Aristotle as legitimate I. 226: used by Archimedes I. 225: objected to by Peletarius I. 249: no use theoretically, but merely furnishes *practical* test of equality I. 227: Bertrand Russell on, I. 227, 249
- Surface: Pythagorean term for, *χρoιά* (=colour, or skin) I. 169: terms for, in Plato and Aristotle I. 169: *ἐπιφάνεια* in Euclid (not *ἐπιτρεδω*) I. 169: alternative definition of, in Aristotle I. 170: produced by motion of line I. 170: divisions or sections of solids are surfaces I. 170, 171: classifications of surfaces by Heron and Geminus I. 170: composite, incomposite, simple, mixed I. 170: *spiric* surfaces I. 163, 170: *homoeo-
meric* (uniform) surfaces I. 170: spheroids I. 170: plane surface, *see* plane: loci on surfaces I. 329, 330
- Surface-loci* of Euclid I. 15, 16, 330: Pappus' lemmas on, I. 15, 16
- Suter, H. I. 8 n., 17 n., 18 n., 25 n., 78 n., 85-90, III. 3
- Suvoroff, Pr. I. 113
- Swinden, J. H. van I. 169, II. 188
- Sylvester, J. II. 426
- Synthesis, *see* Analysis and Synthesis
- Syrianus I. 30, 44, 176, 178
- Tacquet, André I. 103, 105, 111, II. 121, 258
- Tāitiriya-Samhitā I. 362
- Tannery, P. I. 7 n., 37-40, 44, 160, 163, 221, 223, 224, 225, 232, 305, 353, 412, 417, II. 112, 113, III. 1, 5
- Ta'-rikh al-Hukamā* I. 4 n.
- Tartaglia, Niccolò I. 3, 103, 106, II. 2, 47
- Taurinus, F. A. I. 219
- Taurus I. 62, 184
- Taylor, H. M. I. 248, 377-8, 404, II. 16, 22, 29, 56, 75, 102, 227, 244, 247, 272, III. 268, 275, 303, 491-2, 498
- Taylor, Th. I. 259
- Tetrahedron, regular: II. 98: problem of inscribing in given sphere, Euclid's solution III. 407-72, Pappus' solution III. 472-3
- Thābit b. Qurra, translator of *Elements* I. 42, 75-80, 82, 84, 87, 94: proof of I. 47, I. 364-5
- Thales I. 36, 37, 185, 252, 253, 278, 317, 318, 319, II. 111, 227, 244: on distance of ship from shore I. 304-5
- Theaetetus I. 1, 37: contributions to theory of incommensurables III. 3: Eucl. X. 9 attributed to, III. 3, 30: supposed to have discovered octahedron and icosahedron III. 438: was the first to write a treatise on regular solids III. 438: III. 447
- Theodorus Antiochita I. 71
- Theodorus Cabasilas I. 72
- Theodorus of Cyrene: proved incommensurability of $\sqrt{3}$, $\sqrt{5}$ etc. up to $\sqrt{17}$, I. 411, 412-3, III. 1, 2
- Theodorus Metochita, I. 3
- Theodosius II. 37, III. 269, 366, 472
- Theognis I. 371
- Theon of Alexandria: edition of *Elements* I. 46: changes made by, I. 46: Simson on "vitiations" by, I. 46: principles for detecting his alterations, by comparison of P, ancient papyri and "Theonine" MSS. I. 51-3: character of changes by, I. 54-8: interpolation in v. 13 and Porism II. 144: interpolated Porism to VI. 20, II. 239: additions to VI. 33 (about *sectors*) II. 274-6: II. 43, 109, 117, 119, 149, 152, 161, 186, 190, 234, 235, 240, 242, 256, 262, 311, 322, 412
- Theon of Smyrna: I. 172, 357, 358, 371, 398, II. 111, 119, 279, 280, 281, 284, 285, 286, 288, 289, 290, 291, 292, 293, 294, 425, III. 2, 263, 273
- Theorem and problem, distinguished by Speusippus I. 125, Amphinomus I. 125, 128, Menaechmus I. 125, Zenodotus, Posi-

- donius I. 126, Euclid I. 126, Carpus I. 127, 128: views of Proclus I. 127-8, and of Geminus I. 128: "general" and "not-general" (or partial) theorems (Proclus) I. 325
- Theudius of Magnesia I. 117
- Thibaut, B. F. I. 321
- Thibaut, C.: On *Sulvasūtras* I. 360, 363-4
- Thompson, Thomas Perronet I. 112
- Thrasylus II. 292
- Thucydides I. 333
- Thymaridas II. 279, 285
- Tibbon, Moses b. I. 76
- Timaeus* of Plato I. 413-4, II. 97-8, 294-5, 263
- Tiraboschi I. 94 n.
- Tittel, K. I. 38, 39
- Todhunter, I. I. 112, 189, 246, 258, 277, 283, 293, 298, 307, II. 3, 7, 22, 49, 51, 52, 67, 73, 90, 99, 172, 195, 202, 204, 208, 259, 271, 272, 300
- Tonstall, Cuthbert I. 100
- Tore* I. 163
- Transformation of areas I. 346-7, 410
- Trapezium: Euclid's definition his own I. 189: further division into trapezia and trapezoids (Posidonius, Heron) I. 189-90: a theorem on area of parallel-trapezium I. 338-9: name applied to truncated pyramidal numbers (Theon of Smyrna) II. 291
- Treasury of Analysis* (*ἀναλυόμενος τόπος*) I. 8, 10, 11, 138
- Trendelenburg, F. A. I. 146 n., 148, 149
- Treutlein, P. I. 358-60
- Triangle: seven species of, I. 188: "four-sided" triangle, called also "barb-like" (*ἀκιδοειδές*) and (by Zenodorus) *κοιλογώνιον* I. 27, 188: construction of isosceles and scalene triangles I. 243: Heron's proof of expression for area in terms of sides, $\sqrt{s(s-a)(s-b)(s-c)}$, II. 87-8: right-angled triangle which is half of equilateral triangle used for construction of tetrahedron, octahedron and icosahedron (*Timaeus* of Plato) II. 98
- Triangular numbers II. 289
- Trihedral angles: conditions of equality III. 311-2: symmetrical trihedral angles III. 312
- Trimedial* (straight line), first and second, corresponding to *bimedial* III. 257-8
- Trinomial* (straight line), extension from binomial III. 256
- Triplicate*, distinct from *triple*, ratio II. 133
- Trisection of an angle I. 265-7
- at-Ṭūsi, *see* Naṣīraddin
- Unger, E. S. I. 108, 169
- Unit: definitions of, by Thymaridas, "some Pythagoreans," Chrysippus, Aristotle and others II. 279: Euclid's definition that of the "more recent" writers II. 279: *μονάς* connected etymologically by Theon of Smyrna and Nicomachus with *μόνος* (solitary) or *μονή* (rest) II. 279
- Vacca, Giovanni I. 113
- Vachtchenko-Zakhartchenko I. 113
- Vailati, G. I. 144 n., 145 n.
- Valerius Maximus I. 3
- Valla, G., *De expetendis et fugiendis rebus* I. 73, 98
- Van Swinden, J. H. I. 169, II. 188
- Vatican MS. 190 (P) I. 46, 47
- Vaux, Carra de, I. 20
- Verona palimpsest I. 91
- Veronese, G. I. 157, 168, 175, 180, 193-4, 195, 201, 226-7, 228, 249, 328, II. 30, 126
- Vertical* (angles) I. 278
- Vettius Valens III. 3
- Viennese MS. (V) I. 48, 49
- Vieta: on *angle of contact* II. 42
- Vinci, Lionardo da I. 365-6
- Vitali, G. I. 237
- Vitruvius I. 352: Vitruvius and Heron I. 20, 21
- Viviani, Vincenzo I. 107, 401
- Vogt, Heinrich, I. 360, 364, 411-4
- Vooght, C. J. I. 108
- Wachsmuth, C. I. 32 n., 73
- Walker, John II. 204, 208, 259
- Wallis, John I. 103: edited Comm. on Ptolemy's *Harmonica* I. 17: attempt to prove Post. 5, I. 210-11: on *angle of contact* ("degree of curvature") II. 42
- Weber (H.) and Wellstein (J.) I. 157
- Weierstrass II. 124
- Weissenborn, H. I. 78 n., 92 n., 94 n., 95, 96, 97 n., 418
- Whiston, W. I. 111
- Williamson, James I. 111, 293
- Witt, H. A. I. 113
- Woepcke, F., discovered *De divisionibus* in Arabic and published translation I. 9: on Pappus' commentary on *Elements* I. 25, 66, 77: I. 85 n., 86, 87, II. 5, III. 3, 255, 256, 258, 259
- Xenocrates I. 268, 413
- Ximenes, Leonardo I. 107
- Xylander I. 107
- Yahyā b. Khālid b. Barmak I. 75
- Yahyā b. Muḥ. b. 'Abdān b. 'Abdalwāhid (Ibn al-Lubūdī) I. 90
- Yrinius = Heron I. 22
- Yūhannā b. Yūsuf b. al-Ḥārith b. al-Bīṭriq al-Qass I. 76, 87
- Zamberti, Bartolomeo I. 98-100, 101, 104, 106
- Zeno the Epicurean I. 34, 196, 197, 199, 242
- Zenodorus I. 26, 27, 188, 333, II. 276
- Zenodotus I. 126
- Zeuthen, H. G. I. 15, 113, 139, 141, 146 n., 151, 355-6, 360, 363, 387, 398, 399, III. 4